Journal of
Nonlinear
Science
Springer
332 J. Nonlinear Sci. ISSN 0938-8974
AUGUST 2019
29(4) 1247–1864 (2019)

Journal of Nonlinear Science (2019) 29:1813–1863
https://doi.org/10.1007/s00332-019-09531-w

Journal of Nonlinear Science

Nonlinear mechanics of accretion

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Received: 27 November 2018 / Accepted: 16 January 2019 / Published online: 6 February 2019

Abstract
We formulate a geometric nonlinear theory of the mechanics of accretion. In this theory, the material manifold of an accreting body is represented by a time-dependent Riemannian manifold with a time-independent metric that at each point depends on the state of deformation at that point at its time of attachment to the body, and on the way the new material is added to the body. We study the incompatibilities induced by accretion through the analysis of the material metric and its curvature in relation to the foliated structure of the accreted body. Balance laws are discussed and the initial boundary value problem of accretion is formulated. The particular cases where the growth surface is either fixed or traction-free are studied and some analytical results are provided. We numerically solve several accretion problems and calculate the residual stresses in nonlinear elastic bodies induced from accretion.

Keywords Accretion · Surface growth · Nonlinear elasticity · Residual stress · Foliations · Material metric

Mathematics Subject Classification 74B20 · 74A05 · 74A10 · 53Z05

1 Introduction

Consider a body undergoing finite deformations while new material is being added to its boundary. This is called accretion or surface growth. Examples of accretion processes in Nature are the growth of biological tissues and crystals, the build-up of volcanic and sedimentary rocks, the formation of planets, etc. Examples in techno-

Communicated by Dr. Alain Goriely.

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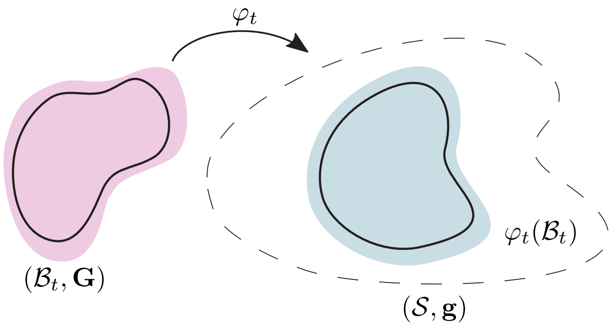

Fig. 1 Motion of a nonlinear elastic body undergoing accretion. The material manifold is a time-dependent set and the material metric $\mathbf{G}$ is not known a priori

logical applications are additive manufacturing (3D printing), metal solidification, the build-up of concrete structures in successive layers, and the deposition of thin films. The goal of this paper is to formulate the mechanics of nonlinear elastic bodies that grow as a result of addition of new material on their boundary while deforming at the same time. (See Fig. 1.) We do not assume any symmetries and formulate the most general accretion assuming the absence of ablation. To our best knowledge, such a general theory does not exist in the literature. In particular, calculation of stresses and deformation during accretion and after the completion of accretion (residual stresses) is necessary for the design of accreting structures.

In a deforming body, undergoing growth mass is not a conserved quantity and the body after unloading may be residually stressed. There are two types of growth, namely bulk and surface growth [accretion or boundary growth (Epstein 2010)]. In bulk growth, in the language of continuum mechanics, material points are preserved; only the mass density and the natural (stress-free) configuration of the body evolve. Due to its similarities with finite plasticity, the decomposition of the deformation gradient into elastic and growth parts has been assumed in most of the works in the literature (see Sadik and Yavari 2017 for a historical perspective). There are several theoretical and computational works in the literature of (bulk) growth mechanics, e.g., Takamizawa and Matsuda (1990), Takamizawa (1991), Rodriguez et al. (1994), Epstein and Maugin (2000), Garikipati et al. (2004), Ben Amar and Goriely (2005), Klarbring et al. (2007), Yavari (2010), Sadik et al. (2016). The recent book by Goriely (2017) summarizes the recent developments in this exciting field.

In accretion (or surface growth), instead, new material points are added to the boundary of a deforming body, meaning that the set of material points is time dependent. (See Fig. 1.) Moreover, the relaxed (natural) configuration of the body explicitly depends not only on the accretion characteristics (accretion flux, accretion velocity, etc.), but also on the history of loading and deformation during accretion. The mechanics of accretion is much less developed mainly because of the complexities involved in modeling the kinematics of accretion and the intrinsic incompatible nature of accreting bodies. For a history of accretion mechanics, see Naumov (1994) and Sozio and Yavari (2017). Recently, Tomassetti et al. (2016) modeled a spherically symmetric accretion of a hollow spherical ball made of an incompressible nonlinear elastic solid coupling nonlinear elasticity and diffusion. Accretion occurs on the inner boundary with fixed radius as the new material is diffused from the surface of a rigid sphere, while the outer boundary is time dependent. In particular, they defined a material manifold that is the Cartesian product of the inner boundary of the sphere and a finite time interval. This is understood as a submanifold of the Euclidean space $\mathbb{R}^4$ with a material metric induced from $\mathbb{R}^4$. A theory of surface growth coupled with diffusion was presented in Abi-Akl et al. (2018). Lychev et al. (2018) presented a geometric theory for the mechanics of "layer-by-layer structures," which is relevant to our present work in so far as an accreting body can be seen as a family of infinitely many two-dimensional layers. In Sozio and Yavari (2017), we introduced a geometric theory of nonlinear accretion mechanics for symmetric surface growth of cylindrical and spherical bodies. We used this theory for the analysis of several model problems. Our objective in the present paper is to develop a nonlinear theory of accretion mechanics without any symmetry assumptions. This generalization is quite non-trivial.

In classical nonlinear elasticity, one uses a stress-free reference configuration and motion is a time-dependent mapping from this configuration into an Euclidean ambient space. In anelasticity (in the sense of Eckart (1948)), the body is residually stressed, and hence, a global stress-free configuration is non-Euclidean, in general, which cannot be isometrically embedded in the Euclidean ambient space. However, local relaxed configurations can be defined using a multiplicative decomposition of the deformation gradient. In other words, a global stress-free configuration of the body is incompatible with the Euclidean geometry of the ambient space. This idea in the mechanics of defects is due to Kondo (1955a, b) and Bilby et al. (1955). In a geometric formulation of anelasticity, the geometry of the material manifold depends on the source of anelasticity. In the case of point and line defects, the geometry depends on the (areal or volumetric) density of defects (Yavari and Goriely 2012a, b, 2013a, 2014). In thermoelasticity, the material metric depends on both the temperature distribution and the thermal properties of the solid, e.g., (temperature dependent) coefficient of thermal expansion (Ozakin and Yavari 2010; Sadik and Yavari 2016). In bulk growth, material metric is explicitly time dependent; it depends on the mass flux through the balance of mass (Yavari 2010; Sadik et al. 2016). For inclusions (or inhomogeneities with eigenstrains) material metric explicitly depends on the distribution of (finite) eigenstrains (Yavari and Goriely 2013b, 2015; Golgoon et al. 2016). The geometry of material manifold of accreting bodies is slightly more complicated. Accretion can be seen as the study of the formation of non-Euclidean solids (Poincaré 1905) through a continuous joining of infinitely many two-dimensional layers (Zurlo and Truskinovsky 2017, 2018). Therefore, a fundamental requirement for an accretion model consists of expressing the geometry of a deformable 3D body with respect to its layer-wise structure. The material manifold of the accreted body depends on the way the layers are put together, that is described by the growth velocity. Note that in general, at different points on the boundary new material is added at different directions and with different speeds. One should note that the material manifold also depends on the deformation the body is experiencing at the time at which each layer is added. The formulation of the nonlinear mechanics of accretion introduced in this paper explicitly uses a Riemannian material manifold with a priori unknown metric that describes this material structure. In particular, the continuous addition of two-dimensional layers will be modeled using the geometry of foliations following the ADM formalism of general relativity presented in Arnowitt et al. (1959).

This paper is organized as follows. In Sect. 2 a model for the kinematics of an accreting body is presented. In Sect. 2.1 a continuously accreted body is defined. The kinematics of an accreting body is defined as a pair of a spatial motion of a time-dependent domain in Sect. 2.2, and a material motion of the layers in the material ambient manifold in Sect. 2.3. In Sect. 2.4, the spatial growth velocity is defined and then used to construct the material metric through the introduction of the accretion tensor $\mathbf{Q}$. In Sect. 3 we present a study of the curvature of the material manifold following the procedure of the ADM formalism of general relativity. In Sect. 3.1 we decompose the material metric according to the material foliated structure, and identify the first fundamental form of the layers. In Sect. 3.2 we focus on the second fundamental form and on the curvature tensor. In Sect. 3.3 all the foliation quantities are expressed in terms of the growth velocity, and then are used to construct the three-dimensional non-Euclidean geometry of the accreting solid. In Sect. 4 the balance laws and the initial boundary value problem of accreting bodies are discussed. In Sect. 5 two classes of problems are investigated: accretion through a fixed growth surface in Sect. 5.1, and accretion through a traction-free growth surface in Sect. 5.2. Several numerical examples are presented in Sect. 5.3. In the appendices, we tersely review some elements of nonlinear anelasticity and the differential geometry of submanifolds, surfaces in Riemannian manifolds, and foliated manifolds.

2 Kinematics of Accretion

In this section we formulate the kinematics of accretion. An accreting body is seen as an evolving subset of a material manifold (Segev 1996; Ong and O'Reilly 2004) endowed with a metric that at any point depends on the state of deformation of the body at its time of attachment, and on the way the new particles are added to the body. Unlike most anelastic problems, in accretion mechanics, the material metric is coupled with deformation and is calculated after solving the accretion initial boundary value problem (IBVP). In other words, the metric of the material manifold is an unknown field to be determined as part of the solution of the accretion IBVP.

2.1 An Accreting Body

Definition 2.1 An accreting body is a connected, orientable 3-manifold $\mathcal{M}$ (possibly with corners) that is embeddable in the ambient space $\mathbb{R}^3$ together with a smooth map $\tau : \mathcal{M} \to [0, T]$, called the time of attachment map, with T being the final time, satisfying the following properties:

(i) The sets $\mathcal{B}_t = \{X \in \mathcal{M} \mid \tau(X) \leq t\}$ are connected 3-manifolds $\forall t \in [0,\ T]$;
(ii) The differential $\mathrm{d}\tau$ of the time of attachment map never vanishes, so that each level surface $\Omega_t = \tau^{-1}(t)$ is a smooth 2-manifold;
(iii) All Ω_t's are diffeomorphic to each other.

We call $\mathcal{M}$ the material ambient space and $\mathcal{B}_t$ the body at time t. The boundary $\partial\mathcal{B}_t$ is the union of Ω_t, the growth surface at time t, and $\Pi_t = \partial\mathcal{B}_t \setminus \Omega_t$, the non-active boundary (see Fig. 2).

As subsets of the boundary of an orientable manifold, all the Ω_t's and Π_t's are orientable surfaces. Notice that $\mathcal{B}_{t_1} \subset \mathcal{B}_{t_2}$ for $t_1 \leq t_2$, which expresses the monotonicity of growth, or equivalently, the absence of ablation. Of course, $\mathcal{M} = \mathcal{B}_T$. Property (ii) expresses the continuity of growth, meaning that the layer added during the time interval $[t, t+\mathrm{d}t]$ has an infinitesimal thickness. It is straightforward to check that for each $t \in \mathbb{R}$ one has

$$\mathcal{B}_t = \bigcup_{0\leq\tau\leq t} \Omega_\tau. \tag{2.1}$$

Therefore, property (ii) implies that the body is a union of 2-dimensional layers, a foliation in the language of differential geometry (see Appendix C and the developments in Sect. 3). Finally, property (iii) is a regularity requirement, which will turn out to be useful in the following developments, allowing us to define material motions. Note that this requirement implies that one single layer defines the topology of the entire family, so if one starts with a Ω_0 with some topological characteristics, these will be preserved throughout the whole accretion process. The non-active boundary represents that part of the boundary that is not involved in the accretion process.

Remark 2.2 The requirements enunciated for the pair $(\mathcal{M}, \tau)$ may be violated for some accretion problems. For example, one may need to include the presence of an initial body $\mathring{\mathcal{B}}$ that works as a substrate on which the new material starts being deposited. This would violate property (ii), as $\mathring{\mathcal{B}} = \Omega_0 = \tau^{-1}(0)$ would not be a surface. Another example is the case of accretion problems in which the growth surface changes in a discontinuous manner, e.g., when accretion occurs on one side of the body and suddenly stops and starts on a different side. This would violate the smoothness of τ. Another example is a change of topology of the layers Ω_t that violates property (iii). In all these cases one can divide the problem into more regular pieces $(\mathcal{M}_i, \tau_i)$, with $1 \leq i \leq n$, that satisfy all the requirements. Then, the total body $\mathcal{B}$ at the end of accretion will be the union of the initial body $\mathring{\mathcal{B}}$ with all the material manifolds defined for all these regular pieces, viz. $\mathcal{B} = \mathring{\mathcal{B}} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n$. Nevertheless, the

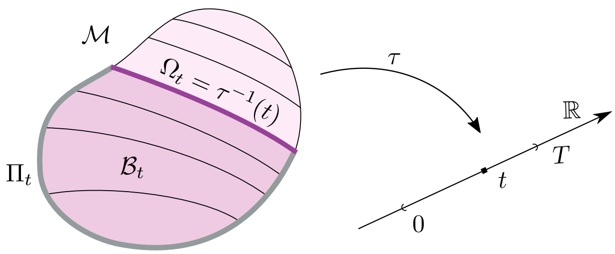

Fig. 2 The material manifold of an accreting body. A schematic picture with labels for the material ambient space $\mathcal{M}$, the body at current time $\mathcal{B}_t$, the growth surface Ω_t, and the non-active surface Π_t

material structure that will be defined in the following sections would not necessarily be continuous through the interface of the different bodies.

2.2 Motion of an Accreting Body

The motion of a body subject to accretion is a time-dependent map with a time-dependent domain, i.e., a family of maps $\varphi_t : \mathcal{B}_t \to \mathcal{S}$, with $t \in [0, T]$, where $\mathcal{S}$ is the ambient space manifold. For accretion applications we assume that $\mathcal{S}$ is a three-dimensional Euclidean space. More precisely, indicating with $\mathcal{C}_t$ the set of all embeddings of $\mathcal{B}_t$ in $\mathcal{S}$, a motion is a map $\varphi : \mathbb{R} \to \{\mathcal{C}_t \mid 0 \le t \le T\}$ such that $\varphi_t \in \mathcal{C}_t$. We indicate with ω_t the deformed configuration of each layer at its time of attachment, i.e., $\omega_t = \varphi_t(\Omega_t)$, and with π_t the deformed non-active boundary, i.e., $\pi_t = \varphi_t(\Pi_t)$. Therefore, as φ_t is a homeomorphism for every t, one has

$$\partial\varphi_t(\mathcal{B}_t) = \varphi_t(\partial\mathcal{B}_t) = \varphi_t(\Omega_t \cup \Pi_t) = \varphi_t(\Omega_t) \cup \varphi_t(\Pi_t) = \omega_t \cup \pi_t. \tag{2.2}$$

As usual, the material and spatial velocity fields $\mathbf{V}_t$ and $\mathbf{v}_t$ are defined as $\mathbf{V}_t(X) = \frac{\partial}{\partial t}\varphi(X, t)$, and $\mathbf{v}_t \circ \varphi_t = \mathbf{V}_t$. The material and spatial acceleration fields $\mathbf{A}_t$ and $\mathbf{a}_t$ are defined as $\mathbf{A}_t = \frac{\partial}{\partial t}\mathbf{V}_t$, and $\mathbf{a}_t \circ \varphi_t = \mathbf{A}_t$. The so-called deformation gradient $\mathbf{F}$ is the derivative map of φ_t defined as $\mathbf{F}_t(X) = T\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{S}$.

We define a map $\bar{\varphi} : \mathcal{M} \to \mathcal{S}$, $\bar{\varphi}(X) = \varphi(X, \tau(X))$, that records the placement of each point at its time of attachment. Note that for each layer Ω_t one has $\bar{\varphi}|_{\Omega_t} = \varphi_t|_{\Omega_t}$, as when $\tau(X) = t$ one has $\bar{\varphi}(X) = \varphi(X, t)$. Hence, $\bar{\varphi}$ records the deformed configuration $\omega_t = \varphi_t(\Omega_t) = \bar{\varphi}(\Omega_t)$ of each layer at its time of attachment. Therefore, one writes

$$\bar{\varphi}(\mathcal{B}_t) = \bar{\varphi}\Bigg(\bigcup_{0\le\tau\le t} \Omega_\tau\Bigg) = \bigcup_{0\le\tau\le t} \bar{\varphi}\,(\Omega_\tau) = \bigcup_{0\le\tau\le t} \omega_\tau. \tag{2.3}$$

Note that the same position in the ambient space $\mathcal{S}$ may be occupied by different material points at different times, i.e., one may have $\varphi(X_1, \tau(X_1)) = \varphi(X_2, \tau(X_2))$ for $X_1 \neq X_2$. This means that, in general, ω_{t_1} and ω_{t_2} are not disjoint for $t_1 \neq t_2$, and therefore, the maps $\bar{\varphi}$ and $T\bar{\varphi}$ need not be injective, and hence, in general $\bar{\varphi}$ is not an embedding, although it is smooth by construction. See Fig. 3 for a schematic sketch of the mappings φ_t and $\bar{\varphi}$.

Finally, the two-point tensor $\bar{\mathbf{F}}$ is defined as

$$\bar{\mathbf{F}}(X) = \mathbf{F}(X, \tau(X)), \tag{2.4}$$

recording the deformation gradient of each point X at its time of attachment $\tau(X)$.

Remark 2.3 Even when $\bar{\varphi}$ is an embedding, its tangent map $T\bar{\varphi}$ is invertible but is not equal to $\bar{\mathbf{F}}$. In fact, in components one has

$$(T\bar{\varphi})^i{}_J = \frac{\partial\bar{\varphi}^i}{\partial X^J} = \frac{\partial\varphi^i(X, \tau(X))}{\partial X^J} = \bar{F}^i{}_J + V^i\frac{\partial\tau}{\partial X^J}, \tag{2.5}$$

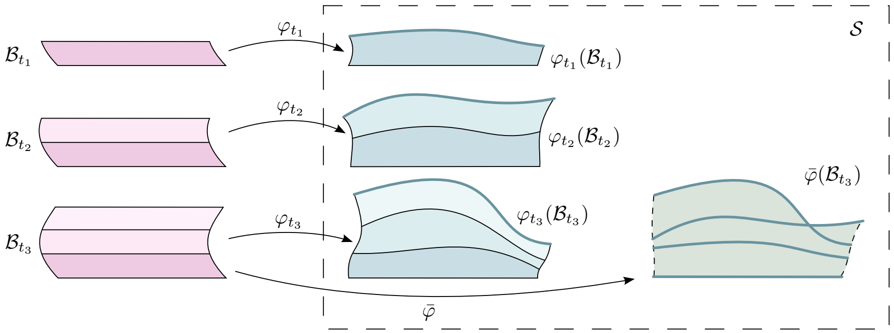

Fig. 3 The motion of an accreting body. On the left the evolving material manifold at three different times $t_1 < t_2 < t_3$ is shown. In the middle the deformed configurations at t_1, t_2, and t_3 are shown. On the right $\mathcal{B}_{t_3}$ is mapped by $\bar{\varphi}$ to the union of layers considered at their time of attachment. Note that $\bar{\varphi}$ may not be injective

i.e., $T\bar{\varphi} = \bar{\mathbf{F}} + \mathbf{V} \otimes \mathrm{d}\tau$. In general, $\bar{\mathbf{F}}$ is not the tangent map of any embedding, or in other words, $\bar{\mathbf{F}}$ is not compatible. Nevertheless, since $\bar{\mathbf{F}}|_{T\Omega_t} = T\bar{\varphi}|_{T\Omega_t} = \mathbf{F}_t|_{T\Omega_t}$ for each $t \in \mathbb{R}$, it is compatible on each single layer.[1]

2.3 The Material Motion

In the material manifold, the pointwise rate at which the material is added is not uniquely defined. In fact, in order to define a velocity with which the growth surface Ω_t is moving in $\mathcal{M}$, one needs to identify points on different layers. This can be achieved by defining a motion of layers in the material manifold.

Definition 2.4 A material motion is a 1-parameter family of diffeomorphisms $\Phi : \Omega_0 \times [0, T] \to \mathcal{M}$ such that $\Phi(\Omega_0, t) = \Omega_t$. Its trajectories are called material trajectories. We indicate with $\mathbf{U}$ the velocity of the material motion, viz.

$$\mathbf{U}(X, t) = \frac{\partial}{\partial t}\Phi(X, t), \tag{2.6}$$

and we call it the material growth velocity.

The existence of a material motion is guaranteed by property (iii) of Sect. 2.1. Clearly it is not unique. On the other hand, the condition for a given nowhere-vanishing vector field $\mathbf{W}$ on $\mathcal{M}$ to be a velocity for some material motion can be obtained by the requirement $\tau(\Phi(X, t)) = t$. Differentiating with respect to time one finds that $\mathbf{W}$ is a material growth velocity for some material motion if and only if $\langle \mathrm{d}\tau, \mathbf{W} \rangle = 1$. This means that all the growth velocity fields one can define on $\mathcal{M}$ must have their τ-component equal to 1 at every point (see Fig. 4).

[1] The tangent bundle $T\Omega_t$ of $\Omega_t \subset \mathcal{M}$, is the union of all $T_X\Omega_t$'s for $X \in \Omega_t$, i.e., the set of all vectors that are tangent to Ω_t. Therefore, with $\bar{\mathbf{F}}|_{T\Omega_t}$ we indicate the restriction of $\bar{\mathbf{F}}$ operating on tangent vectors on Ω_t. $T\bar{\varphi}|_{T\Omega_t}$ and $\mathbf{F}_t|_{T\Omega_t}$ are defined similarly. We use the same notation for $T\omega_t$.

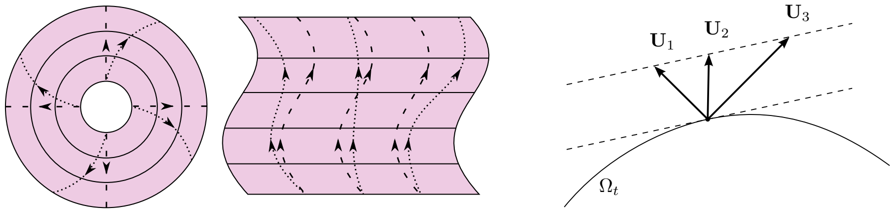

Fig. 4 Left: examples of material trajectories induced by different material motions on the same material manifold. Right: the velocities of the three different material motions must have the same τ-component

A material motion can also be defined starting from a foliation atlas on $\mathcal{M}$ made of charts (Ξ^1, Ξ^2, τ), with the third coordinate defined globally and given by the time of attachment map (see Appendix C). The material motion determined by such coordinates is the one that preserves the "layer coordinates" (Ξ^1, Ξ^2) of each particle. This means that if a point $X_0 \in \Omega_0$ has coordinates $(\Xi^1, \Xi^2, 0)$, then Φ_t maps it to the point X with coordinates (Ξ^1, Ξ^2, t). Therefore, the trajectories of the motion are the curves with constant layer coordinates, or τ-lines. In this sense, the third vector of the basis induced by (Ξ^1, Ξ^2, τ), i.e.,

$$\mathbf{U} = \left.\frac{\partial}{\partial \tau}\right|_{\Xi^1, \Xi^2}, \tag{2.7}$$

is the material growth velocity associated with Φ. Note that $\langle \mathrm{d}\tau, \frac{\partial}{\partial \tau} \rangle = 1$.

Remark 2.5 In general, two different foliation charts induce different material motions. Let us consider two sets of foliation coordinates (Ξ^1, Ξ^2, τ) and $(\Theta^1, \Theta^2, \tau)$. On the overlapping region, one has the following change of coordinates

$$\Theta^1 = \hat{\Theta}^1(\Xi^1, \Xi^2, \tau), \quad \Theta^2 = \hat{\Theta}^2(\Xi^1, \Xi^2, \tau), \quad \tau = \tau. \tag{2.8}$$

This means that

$$\begin{aligned} \mathbf{U}_2 = \left.\frac{\partial}{\partial \tau}\right|_{\Theta=\text{const}} &= \frac{\partial \Xi^1}{\partial \tau}\frac{\partial}{\partial \Xi^1} + \frac{\partial \Xi^2}{\partial \tau}\frac{\partial}{\partial \Xi^2} + \left.\frac{\partial}{\partial \tau}\right|_{\Xi=\text{const}} \\ &= \frac{\partial \Xi^1}{\partial \tau}\frac{\partial}{\partial \Xi^1} + \frac{\partial \Xi^2}{\partial \tau}\frac{\partial}{\partial \Xi^2} + \mathbf{U}_1. \end{aligned} \tag{2.9}$$

This implies that when the change of the layer coordinates depends on τ, in general, $\mathbf{U}_1 \neq \mathbf{U}_2$. On the other hand, its dual co-vector $\mathrm{d}\tau$ remains unchanged since the third coordinate τ is the same in both coordinate systems.

One should note that a material motion induces a foliation atlas, in the same way that foliation coordinates define a material motion. Given a material motion Φ, one can fix coordinates (Ξ^1, Ξ^2) on Ω_0, and extend them to $\mathcal{M}$ such that the coordinates of a point $X \in \mathcal{M}$ are $\Xi^1(\Phi^{-1}(X, \tau(X)))$, $\Xi^2(\Phi^{-1}(X, \tau(X)))$, and $\tau(X)$. In short, we have shown that

$$\text{Material motion} \quad \longleftrightarrow \quad \text{Foliation atlas}.$$

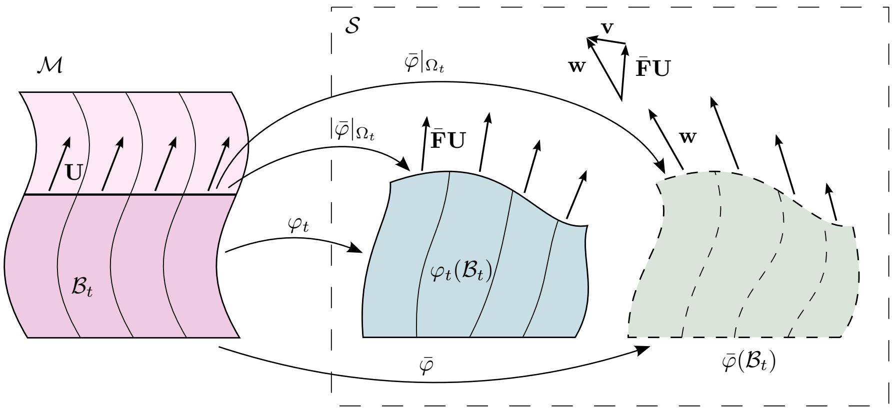

Fig. 5 Configurations, motions, and growth velocities. On the left the material growth velocity $\mathbf{U}$ is shown. It is defined on the material ambient space $\mathcal{M}$ and is tangent to the trajectories of the material motion. The middle configuration is the deformed configuration at time t. The velocity $\bar{\mathbf{F}}\mathbf{U}$ is defined on the growth surface and is tangent to the φ_t image of the trajectories of Φ (or the τ-lines of the foliation coordinates induced by the material motion Φ). The right configuration is the union of all layers at their time of attachment until time t (in general $\bar{\varphi}$ is not an embedding). The total velocity $\mathbf{w}$ is defined on each layer at its time of attachment and describes how the growth surface moves. It is tangent to the $\bar{\varphi}$ image of the trajectories of Φ

It should be emphasized that in the present accretion model what is given is the map τ, which partitions the material manifold into a union of surfaces, and respects the topology of the problem. The material motion Φ is any flow on $\mathcal{M}$ that is compatible with the foliation given by τ, i.e., such that $\langle \mathrm{d}\tau, \mathbf{U}\rangle = 1$, with $\mathbf{U}$ being the generator of Φ.

The total velocity of the growth surface ω in the deformed configuration has two contributions: one due to accretion, and one due to deformation. In particular, noting that $\Omega_t = \Phi(\Omega_0, t)$, one can write $\omega_t = \varphi(\Omega_t, t) = \bar{\varphi}(\Omega_t) = \bar{\varphi}(\Phi(\Omega_0, t))$, which represents the motion of ω_t in terms of the coordinates of the material layers, i.e., the map $\bar{\varphi} \circ \Phi_t : \Omega_0 \times \mathbb{R} \to \mathcal{S}$. Its velocity $\mathbf{w}_t$ is called the total velocity and can be obtained recalling (2.5):

$$\mathbf{w} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{\varphi} \circ \Phi_t\right) = (T\bar{\varphi})\,\mathbf{U} = \left(\bar{\mathbf{F}} + \mathbf{v} \otimes \mathrm{d}\tau\right)\mathbf{U} = \bar{\mathbf{F}}\mathbf{U} + \mathbf{v}. \tag{2.10}$$

The term $\bar{\mathbf{F}}\mathbf{U}$ represents the contribution of accretion, while $\mathbf{v}$ is the contribution of deformation. Figure 5 shows the three velocities $\mathbf{U}$, $\bar{\mathbf{F}}\mathbf{U}$ and $\mathbf{w}$, and their respective configurations.

2.4 The Material Metric

We assume that at each time t one can univocally identify a vector field $\mathbf{u}_t$ on ω_t called the growth velocity that describes the rate and direction at which new material is being

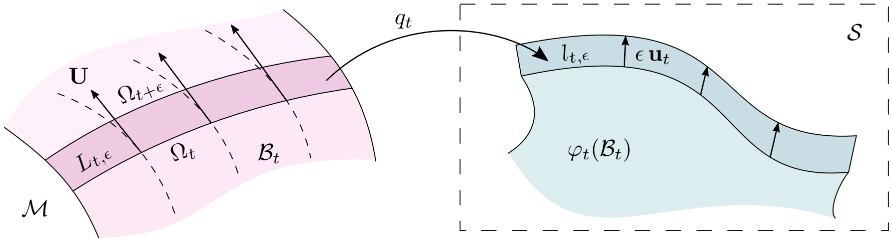

Fig. 6 Addition of an infinitesimally thin layer to the boundary of a deforming body. The two-point tensor $\mathbf{Q}$ is defined as $\mathbf{Q}|_{\Omega_t} = Tq_t|_{\Omega_t}$

added.[2] In other words, if one considers accretion as the addition of material from the exterior of the body, the velocity of a material point that is about to attach to the growth surface is $\mathbf{v}_t - \mathbf{u}_t$, and its velocity relative to the growth surface is $-\mathbf{u}_t$. We assume that $\mathbf{u}_t$ is nowhere purely tangential and that it points outside of the deformed body $\varphi_t(\mathcal{B}_t)$. However, in general, it does not need to be normal to ω (see Skalak et al. 1997), Ganghoffer (2011), and the example in Sect. 5.1).[3] The growth velocity describes the way in which the body would naturally evolve in the case of no deformation during accretion. So it provides information on the material structure of the body. As a matter of fact, we will see that the Riemannian material structure depends on the deformed configuration of each layer when it joins the body and on the growth velocity $\mathbf{u}$. Let us define the following two sets

$$L_{t,\epsilon} = \{X \in \mathcal{M} \mid t \le \tau(X) \le t+\epsilon\} \quad \text{and} \quad l_{t,\epsilon} = \{x + s\,\mathbf{u}(x,t) \mid x \in \omega_t,\ 0 \le s \le \epsilon\}, \tag{2.11}$$

for $0 \le t < T$ and $0 < \epsilon \le T - t$ (Fig. 6). The first set represents the material added during the time interval $[t, t+\epsilon]$ in the material manifold. The second set represents the layer added to the deformed configuration at time t. Given a point $X \in L_{t,\epsilon}$ (note that this point has time of attachment $t \le \tau(X) \le t+\epsilon$), we move it to Ω_t following its material trajectory. Let us denote this point by $X_t = \Phi_t(\Phi^{-1}_{\tau(X)}(X))$, and define the map

$$\begin{aligned} q_t : L_{t,\epsilon} &\to l_{t,\epsilon} \\ X &\mapsto \bar{\varphi}(X_t) + (\tau(X) - t)\,\mathbf{u}(\bar{\varphi}(X_t), t), \end{aligned} \tag{2.12}$$

where the linear structure of the Euclidean ambient space $\mathcal{S}$ allows one to use a translation operator. Choosing Cartesian coordinates in the ambient space,[4] the map q_t can be written as

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[2] It is related to the flux of mass in the way explained in Sect. 4.1.

[3] We assume that the growth velocity is a given vector field on the boundary of the deformed body. However, in a coupled theory of accretion it would be one of the unknown fields. In this paper we assume that this vector field is given and focus our attention on formulating the nonlinear elasticity problem. Extending the present theory to consider the coupling between mass transport and elasticity will be the subject of a future communication.

[4] The translation operation would force one to work with shifters when using curvilinear coordinates (see Marsden and Hughes 1983). Here, for the sake of simplicity, we use Cartesian coordinates.

$$q_t^i(X) = \bar{\varphi}^i(X_t) + (\tau(X) - t)\, u^i(\varphi(X_t, t), t). \tag{2.13}$$

Therefore, referring to foliation coordinates for material indices and to Cartesian coordinates for spatial indices, one has the following representation for the tangent map Tq_t

$$\begin{aligned} \frac{\partial q_t^i}{\partial \Xi^A} &= \frac{\partial \bar{\varphi}^i}{\partial \Xi^A} + (\tau - t)\frac{\partial u^i}{\partial \Xi^A} = \bar{F}^i{}_A + (\tau - t)\frac{\partial u^i}{\partial \Xi^A}, \quad A = 1, 2, \\ \text{and} \quad \frac{\partial q_t^i}{\partial \tau} &= u^i, \end{aligned} \tag{2.14}$$

as on the growth surface $T\bar{\varphi} = \bar{\mathbf{F}}$. Evaluating at $\tau = t$, one obtains

$$\left.\frac{\partial q_t^i}{\partial \Xi^A}\right|_{\tau=t} = \bar{F}^i{}_A, \quad A = 1, 2, \quad \text{and} \quad \left.\frac{\partial q_t^i}{\partial \tau}\right|_{\tau=t} = u^i. \tag{2.15}$$

Definition 2.6 The accretion tensor $\mathbf{Q}$ is a time-independent two-point tensor field defined as $\mathbf{Q}|_{\Omega_t} = Tq_t|_{\Omega_t}$, $\forall t$, which from (2.15) is written as

$$\mathbf{Q}(X) = \bar{\mathbf{F}}(X) + \left[\mathbf{u}(\bar{\varphi}(X), \tau(X)) - \bar{\mathbf{F}}(X)\mathbf{U}(X)\right] \otimes \mathrm{d}\tau(X), \tag{2.16}$$

for $X \in \mathcal{M}$.

Note that the accretion tensor can be understood as the tangent of a mapping that takes a layer of material that is about to be attached to the body from the material manifold and maps it to its current configuration right before attachment. It has been constructed layer by layer, however, in such a way that it captures the out-of-layer part of the mapping as well. Note that $\mathbf{QU} = \mathbf{u}$ because $\langle \mathrm{d}\tau, \mathbf{U}\rangle = 1$. With respect to the frame $\left\{\frac{\partial}{\partial \Xi^1}, \frac{\partial}{\partial \Xi^2}, \frac{\partial}{\partial \tau} = \mathbf{U}\right\}$ induced by the fixed foliation chart (Ξ^1, Ξ^2, τ) in the material manifold $\mathcal{M}$, and a frame $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$ induced by a local chart (x^1, x^2, x^3) in the ambient space $\mathcal{S}$, $\mathbf{Q}$ has the following representation

$$\left[Q^i{}_J(X)\right] = \begin{bmatrix} \frac{\partial \bar{\varphi}^1}{\partial \Xi^1}(X) & \frac{\partial \bar{\varphi}^1}{\partial \Xi^2}(X) & u^1(\bar{\varphi}(X), \tau(X)) \\ \frac{\partial \bar{\varphi}^2}{\partial \Xi^1}(X) & \frac{\partial \bar{\varphi}^2}{\partial \Xi^2}(X) & u^2(\bar{\varphi}(X), \tau(X)) \\ \frac{\partial \bar{\varphi}^3}{\partial \Xi^1}(X) & \frac{\partial \bar{\varphi}^3}{\partial \Xi^2}(X) & u^3(\bar{\varphi}(X), \tau(X)) \end{bmatrix}. \tag{2.17}$$

Note that the accretion tensor $\mathbf{Q}$, similar to $\bar{\mathbf{F}}$, is not the tangent map of any embedding, even though it is compatible on each single layer. In particular, $\mathbf{Q}|_\Omega = \bar{\mathbf{F}}|_\Omega = T\bar{\varphi}|_\Omega$.

Throughout this paper, we consider accretion as the addition of undeformed material. Under this assumption, each mapping q_t sending layer t in the material manifold to its natural state in the ambient space is an isometry. Therefore, in the limit $\epsilon \to 0$, the set $l_{t,\epsilon}$ represents the natural state of the material. For this reason, we define the material metric on $\mathcal{M}$ as the pull-back of the Euclidean ambient metric $\mathbf{g}$ using $\mathbf{Q}$, i.e.,

$$\mathbf{G}(X) = \mathbf{Q}^\star(X)\,\mathbf{g}(\bar{\varphi}(X))\,\mathbf{Q}(X), \tag{2.18}$$

which in components reads

$$G_{IJ}(X) = Q^i{}_I(X)\, g_{ij}(\bar{\varphi}(X))\, Q^j{}_J(X). \tag{2.19}$$

Note that the material metric $\mathbf{G}$ explicitly depends on the deformation at the time of attachment of each single layer, and on the growth velocity $\mathbf{u}$, that represents a physical characteristic of the accretion process. Notice that $\mathbf{G}$ also depends on the material motion Φ via the material growth velocity $\mathbf{U}$. As a matter of fact, if one considers a different material motion, and hence, a different material growth velocity $\mathbf{U}'$, the material metric defined through (2.16) and (2.18) changes. This means that the body will be represented by a different Riemannian manifold. One may think that the elastic response of the body will be altered by this change. Nevertheless, it turns out that this is not the case. This result, established next in Proposition 2.7, implies that our accretion model is well defined.

Proposition 2.7 *The elastic response of an accreting simple body is invariant with respect to the choice of the material motion.*

Proof Fixing $(\mathcal{M}, \tau)$, one can define a map $\mathcal{G}$ associating a material metric to any pair of material and spatial motions, i.e., $\mathbf{G} = \mathcal{G}[\Phi, \varphi]$. We claim that for any other material motion Φ' there exists a diffeomorphism $\lambda : \mathcal{M} \to \mathcal{M}$ that pulls $\mathcal{G}[\Phi', \varphi']$ back to $\mathcal{G}[\Phi, \varphi]$, where φ' is the motion such that $\varphi_t = \varphi'_t \circ \lambda$ for any $t \in [0, T]$.[5] Let us fix Φ'. Given Φ and Φ', one can define the diffeomorphism

$$\lambda : X \mapsto \Phi'_{\tau(X)}\left(\Phi^{-1}_{\tau(X)}(X)\right), \tag{2.20}$$

sending trajectories of Φ to trajectories of Φ'. Therefore, $\Phi' = \lambda \circ \Phi$. Note that $\lambda(\Omega_\tau) = \Omega_\tau$, or equivalently, $\tau(\lambda(X)) = \tau(X)$. Now one defines the motion φ' such that $\varphi_t = \varphi'_t \circ \lambda$ for any $t \in [0, T]$. Setting $\mathbf{\Lambda} = T\lambda$ and $\mathbf{F}' = T\varphi'$, one has $\mathbf{F}(X, t) = \mathbf{F}'(\lambda(X), t)\mathbf{\Lambda}(X)$. Thus, since $\tau(\lambda(X)) = \tau(X)$, one concludes that $\bar{\mathbf{F}}(X) = \bar{\mathbf{F}}'(\lambda(X))\mathbf{\Lambda}(X)$. Indicating with $\mathbf{U}'$ the material growth velocity induced by Φ', using (2.16) one can define

$$\begin{aligned}
\mathbf{Q}(X) &= \bar{\mathbf{F}}(X) + \left[\mathbf{u}(\bar{\varphi}(X), \tau(X)) - \bar{\mathbf{F}}(X)\mathbf{U}(X)\right] \otimes \mathrm{d}\tau(X),\\
\mathbf{Q}'(X) &= \bar{\mathbf{F}}'(X) + \left[\mathbf{u}(\bar{\varphi}'(X), \tau(X)) - \bar{\mathbf{F}}'(X)\mathbf{U}'(X)\right] \otimes \mathrm{d}\tau(X).
\end{aligned} \tag{2.21}$$

Note that $\mathbf{U}'(\lambda(X)) = \mathbf{\Lambda}(X)\mathbf{U}(X)$, and as $\tau(X) = \tau(\lambda(X))$, $\mathrm{d}\tau$ is not affected by this diffeomorphism, i.e., $\mathrm{d}\tau(X) = \mathrm{d}\tau(\lambda(X))\mathbf{\Lambda}(X)$. Hence, one has

$$\begin{aligned}
\mathbf{Q}(X) &= \bar{\mathbf{F}}(X) + \left[\mathbf{u}(\bar{\varphi}(X), \tau(X)) - \bar{\mathbf{F}}(X)\mathbf{U}(X)\right] \otimes \mathrm{d}\tau(X)\\
&= \bar{\mathbf{F}}'(\lambda(X))\mathbf{\Lambda}(X) + \left[\mathbf{u}(\bar{\varphi}'(\lambda((X)), \tau(X)) - \bar{\mathbf{F}}'(\lambda(X))\mathbf{\Lambda}(X)\mathbf{U}(X)\right] \otimes \mathrm{d}\tau(\lambda(X))\mathbf{\Lambda}(X)\\
&= \left\{\bar{\mathbf{F}}'(\lambda(X)) + \left[\mathbf{u}(\bar{\varphi}'(\lambda((X)), \tau(X)) - \bar{\mathbf{F}}'(\lambda(X))\mathbf{U}'(\lambda(X))\right] \otimes \mathrm{d}\tau(\lambda(X))\right\}\mathbf{\Lambda}(X)\\
&= \mathbf{Q}'(\lambda(X))\mathbf{\Lambda}(X).
\end{aligned} \tag{2.22}$$

[5] Recall that a motion φ' for an accreting body is a time-dependent map with a time-dependent domain, i.e., $\varphi'_t : \mathcal{B}_t \to \mathcal{S}$.

Setting $\mathbf{G}' = \mathcal{G}[\Phi', \varphi']$, from (2.18) it follows that

$$\begin{aligned}\mathbf{G}(X) &= \mathbf{\Lambda}^\star(X)\,\mathbf{Q}'^\star(\lambda(X))\,\mathbf{g}\left(\bar{\varphi}'(\lambda(X))\right)\,\mathbf{Q}'(\lambda(X))\,\mathbf{\Lambda}(X)\\ &= \mathbf{\Lambda}^\star(X)\,\mathbf{G}'(\lambda(X))\,\mathbf{\Lambda}(X).\end{aligned} \tag{2.23}$$

This means that $\mathbf{G} = \lambda^*\mathbf{G}'$, and hence, $(\mathcal{M}, \mathbf{G})$ and $(\mathcal{M}, \mathbf{G}')$ are isometric Riemannian manifolds (through λ). Since the body is simple by hypothesis, this isometry implies that if at a given time t the mapping φ_t satisfies the balance equations for $(\mathcal{B}_t, \mathbf{G}|_{\mathcal{B}_t})$, then $\varphi_t' = \varphi_t \circ \lambda^{-1}$ will satisfy the balance equations for $(\mathcal{B}_t, \mathbf{G}'|_{\mathcal{B}_t})$.[6] In other words, the elastic response of the accreting body is independent of the choice of the material metric. □

Remark 2.8 Despite what our intuition may suggest, in general $\mathbf{w} \neq \mathbf{u} + \mathbf{v}$. In fact, as shown in (2.10), one has $\mathbf{w} = \bar{\mathbf{F}}\mathbf{U} + \mathbf{v}$, and in general $\mathbf{u} = \mathbf{Q}\mathbf{U} \neq \bar{\mathbf{F}}\mathbf{U}$. This means that the growth velocity $\mathbf{u}$ is not enough to tell how the growth surface is moved by the addition of new material; it only provides information on the natural state of the material that is being added. However, in some cases, e.g., when the growth surface is subject to no external forces, one has $\mathbf{w} = \mathbf{u} + \mathbf{v}$ (see Proposition 5.4).

Remark 2.9 The material metric does not change in time. This means that the evolution of the material manifold simply consists of adding new material points, leaving the already existing points unaltered and without adding any other sources of incompatibility. This feature is referred to as the "non-rearrangement property" of accreting bodies (Manzhirov and Lychev 2014).[7] Of course this ceases to be true if one considers the thermal effects that characterize many accretion problems, e.g., additive manufacturing or solidification.

We denote by ∇ and $\nabla^{\mathbf{g}}$ the Levi–Civita connections associated with the Riemannian manifolds $(\mathcal{M}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$, respectively. The Riemann curvature associated with $(\mathcal{M}, \nabla)$ is denoted by $\boldsymbol{\mathcal{R}}$, while the curvature tensor for $(\mathcal{S}, \nabla^{\mathbf{g}})$ vanishes. An algorithm for the construction of the material manifold and all the related quantities of an accreting body are summarized in Table 1. For some details on Riemannian manifolds, see Appendix A.

Remark 2.10 In the case of addition of pre-stressed particles, the material metric is defined in a similar way. The only difference is that the natural distances in the body at the time of attachment are represented by a metric $h_{ij} \neq g_{ij}$. For instance, this metric can be obtained by transforming the Euclidean metric using a pre-deformation F (which in general is not compatible) so that $h_{ij} = \mathsf{F}^h{}_i \mathsf{F}^k{}_j g_{hk}$. Therefore, setting $G_{IJ} = Q^i{}_I\, h_{ij}\, Q^j{}_J$, one obtains $G_{IJ} = Q^i{}_I Q^j{}_J \mathsf{F}^h{}_i \mathsf{F}^k{}_j g_{hk}$. However, in this paper, we assume that only stress-free particles are attached to the growth surface.

[6] In simple bodies the mechanical response at each point only depends on the local state of deformation. Therefore, in the elasticity of simple bodies, when two material manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{B}', \mathbf{G}')$ are isometric through $\lambda : \mathcal{B} \to \mathcal{B}'$, if $\varphi : \mathcal{B} \to \mathcal{S}$ is a solution of the balance equations for $(\mathcal{B}, \mathbf{G})$, then $\varphi' = \varphi \circ \lambda^{-1} : \mathcal{B}' \to \mathcal{S}$ is a solution for $(\mathcal{B}', \mathbf{G}')$.

[7] In the linear analysis a similar assumption is used, e.g., in Kadish et al. (2005): "We assume that the material of the sphere behaves elastically once it has accreted." This means that incompatibility at a material point is caused only at the time of its attachment (or before), and therefore, one can write $\boldsymbol{\epsilon}(x,t) = \boldsymbol{\epsilon}^{\text{inc}}(x) + \boldsymbol{\epsilon}^{\text{comp}}(x,t)$ for $t > \tau(x)$ (see also Schwerdtfeger et al. 1998).

Table 1 Summary of the steps needed in the construction of the material manifold of an accreting body in terms of the history of deformation and the growth velocity

Construction of the material manifold of an accreting body
1. Choose a pair $(\mathcal{M}, \tau)$ representing the material manifold. This pair is problem dependent, e.g., it must respect the topology of the growing body and its layers
2. Define a material motion for the layers in $\mathcal{M}$. This can be done by fixing foliation coordinates (Ξ^1, Ξ^2, τ), from which the material growth velocity is calculated as $\mathbf{U} = \frac{\partial}{\partial \tau}$
3. Use $\mathbf{u}$ to build the accretion tensor $\mathbf{Q}$ using (2.17). Note that $\mathbf{Q}$ is expressed in terms of the configuration map so it is not known a priori; it is an unknown field in the accretion boundary value problem
4. Express the material metric $\mathbf{G}$ using (2.18). Material metric explicitly depends on the accretion tensor $\mathbf{Q}$, and hence, is not known a priori

Remark 2.11 The material manifold $\mathcal{M}$ can always be defined as a subset of the Euclidean space $\mathcal{S}$. Consider Cartesian coordinates $\{x^i\}$ on $\mathcal{S}$ with the associated frame $\{\mathbf{e}_i\}$. Note that, in general, these do not constitute a foliation chart for $\mathcal{M}$. Let $Q^i{}_j$ and $(Q^{-1})^j{}_i$ be the components of $\mathbf{Q}$ and $\mathbf{Q}^{-1}$ with respect to Cartesian coordinates in all the indices. By definition, one has $G_{ij} = Q^h{}_i \delta_{hk} Q^k{}_j$. Consider the moving frame $\bar{\mathbf{e}}_i(X) = (Q^{-1})^j{}_i(X)\mathbf{e}_j$ on $\mathcal{M}$. With respect to this frame one has

$$\begin{aligned} \mathbf{G}\left(\bar{\mathbf{e}}_i, \bar{\mathbf{e}}_j\right) &= \mathbf{G}\left((Q^{-1})^h{}_i \mathbf{e}_h, (Q^{-1})^k{}_j \mathbf{e}_k\right) = (Q^{-1})^h{}_i (Q^{-1})^k{}_j \mathbf{G}\left(\mathbf{e}_h, \mathbf{e}_k\right) \\ &= (Q^{-1})^h{}_i (Q^{-1})^k{}_j Q^m{}_h Q^m{}_k \delta_{mn} = \delta_{ij}. \end{aligned} \tag{2.24}$$

This means that the moving frame represents the natural distances in the body. In this sense, we can interpret accretion through a multiplicative decomposition of the deformation gradient, i.e., $\mathbf{F} = \mathbf{F}^{\mathrm{e}}\mathbf{Q}$, where the accretion tensor $\mathbf{Q}$ represents the anelastic part, and $\mathbf{F}^{\mathrm{e}} = \mathbf{F}\mathbf{Q}^{-1}$ represents its elastic part. The material metric $\mathbf{G}$ represents the right Cauchy–Green strain tensor associated with $\mathbf{Q}$.

3 An Accreting Body as a Foliation

An accreting body grows as a result of continuous addition of new layers of material to its boundary. Therefore, it is natural to describe the accreting body in its material manifold in terms of the geometry of the added layers, i.e., the geometry of foliations in the language of differential geometry.[8] In this section, we describe the geometric structure of the material manifold in terms of the growth quantities that were introduced in Sect. 2. We follow the ADM formalism of general relativity, for which we refer the reader to the seminal work (Arnowitt et al. 1959) or to the more recent review (Golovnev 2013). As for some basic concepts of Riemannian geometry, geometry of surfaces, and foliations, see the appendices.

[8] We should mention that the idea of decomposing reference configuration of a body into two-dimensional subbodies was investigated in the work of Wang (1968) on the so-called laminated bodies.

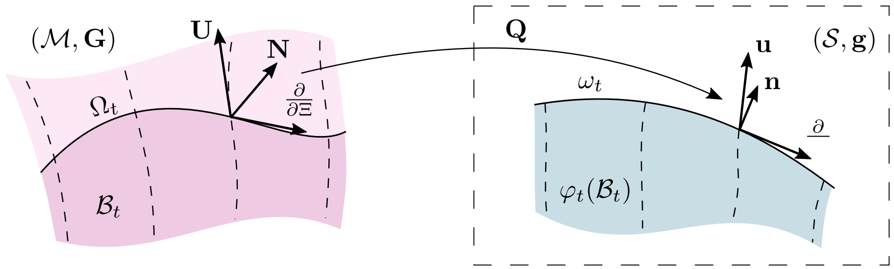

Fig. 7 A two-dimensional sketch of the local frames on $\mathcal{M}$. It should be noted that the material unit normal vector $\mathbf{N}$ is not the unit normal vector to Ω_t in the "standard $\mathbb{R}^3$ sense"; it is with respect to the material metric $\mathbf{G}$

3.1 The Metric of the Material Foliation

The material manifold representing an accreting body can be seen as a foliated Riemannian manifold, i.e., as a Riemannian manifold $(\mathcal{M}, \mathbf{G})$ partitioned into a collection of surfaces $(\Omega_\tau, \tilde{\mathbf{G}}_\tau)$, $\tau \in [0, T]$, with $\tilde{\mathbf{G}}_\tau$ being the first fundamental form induced by $\mathbf{G}$ on $T\Omega_\tau$. The unit vector normal to each layer Ω_τ pointing in the direction of increasing τ is denoted by $\mathbf{N}_\tau$. Note that Ω_τ is orientable being a subset of the boundary of an orientable three-dimensional manifold, and thus, it is possible to define $\mathbf{N}_\tau$ globally. The associated 1-form is denoted by $\mathbf{N}^\flat_\tau$. From here on we use letters from A to G to denote indices in the set $\{1, 2\}$ (layer indices), and letters from H to Z for indices in the set $\{1, 2, 3\}$ (3D indices). For the sake of simplicity, we drop the subscript τ.

Let us fix some foliation coordinates (Ξ^1, Ξ^2, τ). In the following developments, we will mainly consider two moving frames—or local bases—on $\mathcal{M}$ (see Fig. 7), viz.

$$\mathfrak{F} = \left\{ \frac{\partial}{\partial \Xi^1}, \frac{\partial}{\partial \Xi^2}, \frac{\partial}{\partial \tau} = \mathbf{U} \right\}, \text{ and } \breve{\mathfrak{F}} = \left\{ \frac{\partial}{\partial \Xi^1}, \frac{\partial}{\partial \Xi^2}, \mathbf{N} \right\}. \tag{3.1}$$

The first frame is holonomic and is the one induced by the foliation coordinates (Ξ^1, Ξ^2, τ), while the second frame is obtained by substituting $\mathbf{U}$ with $\mathbf{N}$, and hence, is in general anhonolomic. Their associated dual bases are

$$\mathfrak{F}^* = \left\{ \mathrm{d}\Xi^1, \mathrm{d}\Xi^2, \mathrm{d}\tau \right\}, \text{ and } \breve{\mathfrak{F}}^* = \left\{ \breve{\boldsymbol{\theta}}^1, \breve{\boldsymbol{\theta}}^2, \breve{\boldsymbol{\theta}}^3 = \mathbf{N}^\flat \right\}. \tag{3.2}$$

Note that $\mathrm{d}\Xi^1 \neq \breve{\boldsymbol{\theta}}^1$, and $\mathrm{d}\Xi^2 \neq \breve{\boldsymbol{\theta}}^2$. In fact, setting $\mathfrak{F} = \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and $\breve{\mathfrak{F}} = \{\breve{\mathbf{E}}_1, \breve{\mathbf{E}}_2, \breve{\mathbf{E}}_3\}$, one can write $\breve{\mathbf{E}}_{\breve{I}} = \mathbf{E}_I A^I{}_{\breve{I}}$, where

$$\left[A^I{}_{\breve{I}}\right] = \begin{bmatrix} 1 & 0 & N^1 \\ 0 & 1 & N^2 \\ 0 & 0 & N^3 \end{bmatrix}, \quad \left[(A^{-1})^{\breve{I}}{}_I\right] = \begin{bmatrix} 1 & 0 & -\frac{N^1}{N^3} \\ 0 & 1 & -\frac{N^2}{N^3} \\ 0 & 0 & \frac{1}{N^3} \end{bmatrix}. \tag{3.3}$$

The change of frame in terms of the co-frame fields is written as $\breve{\boldsymbol{\theta}}^{\breve{I}} = (A^{-1})^{\breve{I}}{}_I \boldsymbol{\theta}^I$, with $\boldsymbol{\theta}^I$ indicating the 1-forms in the basis $\mathfrak{F}^*$. Hence

$$\breve{\boldsymbol{\theta}}^1 = \mathrm{d}\Xi^1 - \frac{N^1}{N^3}\mathrm{d}\tau, \quad \breve{\boldsymbol{\theta}}^2 = \mathrm{d}\Xi^2 - \frac{N^2}{N^3}\mathrm{d}\tau, \quad \mathbf{N}^\flat = \frac{1}{N^3}\mathrm{d}\tau. \tag{3.4}$$

With respect to the moving co-frame field $\breve{\mathfrak{F}}$ the metric is written as

$$\mathbf{G} = \tilde{G}_{11}\breve{\boldsymbol{\theta}}^1 \otimes \breve{\boldsymbol{\theta}}^1 + \tilde{G}_{22}\breve{\boldsymbol{\theta}}^2 \otimes \breve{\boldsymbol{\theta}}^2 + \tilde{G}_{12}(\breve{\boldsymbol{\theta}}^1 \otimes \breve{\boldsymbol{\theta}}^2 + \breve{\boldsymbol{\theta}}^2 \otimes \breve{\boldsymbol{\theta}}^1) + \mathbf{N}^\flat \otimes \mathbf{N}^\flat. \tag{3.5}$$

Changing basis using (3.3)$_2$ (i.e., $G_{IJ} = (A^{-1})^{\breve{I}}{}_I G_{\breve{I}\breve{J}} (A^{-1})^{\breve{J}}{}_J$) one obtains the metric in the holonomic frame:

$$\begin{aligned}
\mathbf{G} = {} & \tilde{G}_{11}\mathrm{d}\Xi^1 \otimes \mathrm{d}\Xi^1 + \tilde{G}_{22}\mathrm{d}\Xi^2 \otimes \mathrm{d}\Xi^2 + \tilde{G}_{12}\left(\mathrm{d}\Xi^1 \otimes \mathrm{d}\Xi^2 + d\Xi^2 \otimes d\Xi^1\right) \\
& - \left(\frac{N^1}{N^3}\tilde{G}_{11} + \frac{N^2}{N^3}\tilde{G}_{12}\right)\left(\mathrm{d}\tau \otimes \mathrm{d}\Xi^1 + \mathrm{d}\Xi^1 \otimes \mathrm{d}\tau\right) - \left(\frac{N^1}{N^3}\tilde{G}_{12} + \frac{N^2}{N^3}\tilde{G}_{22}\right)\left(\mathrm{d}\tau \otimes \mathrm{d}\Xi^2 + \mathrm{d}\Xi^2 \otimes \mathrm{d}\tau\right) \\
& + \left[\left(\frac{N^1}{N^3}\right)^2 \tilde{G}_{11} + 2\frac{N^1N^2}{(N^3)^2}\tilde{G}_{12} + \left(\frac{N^2}{N^3}\right)^2 \tilde{G}_{22} + \frac{1}{(N^3)^2}\right]\mathrm{d}\tau \otimes \mathrm{d}\tau.
\end{aligned} \tag{3.6}$$

Therefore, the material metric has the following matrix representations in the two moving frames

$$[G_{\breve{I}\breve{J}}] = \left[\begin{array}{c|c} \tilde{G}_{AB} & 0 \\ \hline 0 & 1 \end{array}\right], \quad [G_{IJ}] = \left[\begin{array}{c|c} \tilde{G}_{AB} & \eta_A \\ \hline \eta_B & G_{33} \end{array}\right], \tag{3.7}$$

where the layer part coincides with the first fundamental form and has the same components in the two frames as they share the two layer basis vectors. The components η_A, coming from the change of co-frames from $\breve{\mathfrak{F}}^*$ to $\mathfrak{F}^*$, constitute a 1-form $\boldsymbol{\eta}$ representing the "shift" of the metric in the coordinates (Ξ^1, Ξ^2, τ). In particular, one has $\eta_A = G_{A3} = (A^{-1})^{\breve{I}}{}_A \breve{G}_{\breve{I}\breve{J}} (A^{-1})^{\breve{J}}{}_3$, i.e.,

$$\eta_A = -\tilde{G}_{AB}\frac{N^B}{N^3}. \tag{3.8}$$

As for the component G_{33}, one has $G_{33} = (A^{-1})^{\breve{I}}{}_3 \breve{G}_{\breve{I}\breve{J}} (A^{-1})^{\breve{J}}{}_3$, and it reads

$$G_{33} = \frac{1}{(N^3)^2}\left(N^A N^B \tilde{G}_{AB} + 1\right). \tag{3.9}$$

Using the same change of frame, one obtains the following representations of the inverse metric $\mathbf{G}^{-1}$ with respect to the two frames:

$$\left[G^{\breve{I}\breve{J}}\right] = \left[\begin{array}{c|c} \tilde{G}^{AB} & 0 \\ \hline 0 & 1 \end{array}\right], \quad \left[G^{IJ}\right] = \left[\begin{array}{c|c} \tilde{G}^{AB} + N^A N^B & N^A N^3 \\ \hline \text{sym} & (N^3)^2 \end{array}\right]. \tag{3.10}$$

3.2 The Curvature of the Material Foliation

Let us consider the following connections: (i) The Levi–Civita connection ∇ on $(\mathcal{M}, \mathbf{G})$, and (ii) the Levi–Civita connection $\tilde{\nabla}_\tau$ (or simply $\tilde{\nabla}$) on each layer $(\Omega_\tau, \tilde{\mathbf{G}}_\tau)$. Since every leaf of the foliation is a surface in $(\mathcal{M}, \mathbf{G})$, on each layer the two connections ∇ and $\tilde{\nabla}$ satisfy the Gauss formula:

$$\tilde{\nabla}_{\mathbf{Y}}\mathbf{Z} = \nabla_{\mathbf{Y}}\mathbf{Z} - \mathbf{G}(\nabla_{\mathbf{Y}}\mathbf{Z}, \mathbf{N})\mathbf{N}, \quad \forall \mathbf{Y}, \mathbf{Z} \in T\Omega. \tag{3.11}$$

The connections ∇ and $\tilde{\nabla}$ have Christoffel symbols $\Gamma^I{}_{JK}$ and $\tilde{\Gamma}^A{}_{BC}$, respectively.

Remark 3.1 As was explained in Sect. 2, $\mathbf{Q}|_{T\Omega_t} = \bar{\mathbf{F}}|_{T\Omega_t} = \mathbf{F}_t|_{T\Omega_t}$, so the first fundamental form of the layer Ω_t is the pull-back of the first fundamental form of ω_t through the configuration map φ_t, i.e., $\tilde{\mathbf{G}} = \varphi_t{}^*(\mathbf{g}|_{T\omega_t}) = \bar{\varphi}^*(\mathbf{g}|_{T\omega_t})$. In other words, Ω_t and ω_t are isometric. Therefore, since $\tilde{\nabla}$ is the Levi–Civita connection for $\tilde{\mathbf{G}}$, it can be shown that $\tilde{\nabla}_{\mathbf{W}}\mathbf{Z} = \tilde{\nabla}^{\tilde{\mathbf{g}}}{}_{\mathbf{FW}}(\mathbf{FZ})$ for any pair of tangent vectors $\mathbf{W}$, $\mathbf{Z}$ ($\tilde{\nabla}^{\tilde{\mathbf{g}}}$ is the covariant derivative induced on the surface ω_t). The connection $\tilde{\nabla}$ is called the pull-back connection of $\tilde{\nabla}^{\tilde{\mathbf{g}}}$. This property does not hold for the 3D connections ∇ and $\nabla^{\mathbf{g}}$ as $\mathbf{G}$ is obtained by "pulling back" $\mathbf{g}$ using $\mathbf{Q}$, which does not constitute a real pull-back because $\mathbf{Q}$ is not a tangent map (or a "deformation gradient").

Remark 3.2 A measure of the anholonomicity of the frame $\breve{\mathfrak{F}} = \left\{\frac{\partial}{\partial \Xi^1}, \frac{\partial}{\partial \Xi^2}, \mathbf{N}\right\}$ is given by the commutator coefficients $\Upsilon^I{}_{\breve{J}\breve{K}} = [\breve{\mathbf{E}}_{\breve{J}}, \breve{\mathbf{E}}_{\breve{K}}]^I$, with $\breve{\mathbf{E}}_{\breve{J}}$ being $\frac{\partial}{\partial \Xi^1}$, $\frac{\partial}{\partial \Xi^2}$, or $\mathbf{N}$.[9] So one has

$$\Upsilon^I{}_{AB} = 0\,, \quad \Upsilon^I{}_{AN} = -\Upsilon^I{}_{NA} = \frac{\partial N^I}{\partial \Xi^A}. \tag{3.12}$$

This means that the anholonomicity of $\breve{\mathfrak{F}}$ is given by the nonuniformity of the unit normal vector with respect to $\mathfrak{F}$.[10] Since the Levi–Civita connection is symmetric ($\Gamma^I{}_{JK} = \Gamma^I{}_{KJ}$ in a holonomic frame), one can write the commutator coefficients for the moving frame $\left\{\frac{\partial}{\partial \Xi^1}, \frac{\partial}{\partial \Xi^2}, \mathbf{N}\right\}$ as

$$\breve{\Upsilon}^{\breve{I}}{}_{\breve{J}\breve{K}} = \breve{\Gamma}^{\breve{I}}{}_{\breve{J}\breve{K}} - \breve{\Gamma}^{\breve{I}}{}_{\breve{K}\breve{J}}, \tag{3.14}$$

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9 The Lie brackets are defined in Appendix A.

10 The commutator can be calculated entirely with respect to the frame $\breve{\mathfrak{F}}$, i.e., $\Upsilon^{\breve{I}}{}_{\breve{J}\breve{K}} = [\breve{\mathbf{E}}_{\breve{J}}, \breve{\mathbf{E}}_{\breve{K}}]^{\breve{I}}$, which has the following components

$$\breve{\Upsilon}^{\breve{I}}{}_{AB} = 0\,, \quad \breve{\Upsilon}^B{}_{AN} = -\breve{\Upsilon}^B{}_{NA} = \frac{\partial N^B}{\partial \Xi^A} - \frac{N^B}{N^3}\frac{\partial N^3}{\partial \Xi^A}\,, \quad \breve{\Upsilon}^N{}_{AN} = -\breve{\Upsilon}^N{}_{NA} = \frac{1}{N^3}\frac{\partial N^3}{\partial \Xi^A}. \tag{3.13}$$

Note that these coefficients do not constitute the components of a tensor, as they vanish in some frames (the holonomic ones) and are nonzero in some other frames (the non-holonomic ones).

where $\breve{\Gamma}^{\breve{I}}{}_{\breve{J}\breve{K}}$ are the Christoffel symbols with respect to $\breve{\mathfrak{F}} = \{\frac{\partial}{\partial \Xi^1}, \frac{\partial}{\partial \Xi^2}, \mathbf{N}\}$.[11]

Since the leaves are the level surfaces of τ, one can write the outward normal vector $\mathbf{N}$ and the associated 1-form $\mathbf{N}^\flat$ in terms of the gradient of τ, i.e., in terms of $(\mathrm{Grad}\,\tau) = (\mathrm{d}\tau)^\sharp$. In particular, one has

$$\mathbf{N} = \frac{\mathrm{Grad}\,\tau}{\|\mathrm{Grad}\,\tau\|}, \quad \mathbf{N}^\flat = \frac{\mathrm{d}\tau}{\|\mathrm{Grad}\,\tau\|}. \tag{3.15}$$

These relations can be easily obtained from (3.10), as $(\mathrm{Grad}\,\tau)^I = G^{IJ}\tau_{,J} = G^{IJ}\delta^3{}_J = G^{I3}$. On each layer one defines the second fundamental form $\mathbf{K}$ as a $\binom{0}{2}$-tensor defined as

$$\mathbf{K}(\mathbf{Y},\mathbf{Z}) = \mathbf{G}(\nabla_{\mathbf{Y}}\mathbf{Z},\mathbf{N}), \quad \forall \mathbf{Y},\mathbf{Z} \in T\Omega, \tag{3.16}$$

so that Gauss formula (3.11) is written as

$$\nabla_{\mathbf{Y}}\mathbf{Z} - \tilde{\nabla}_{\mathbf{Y}}\mathbf{Z} = \mathbf{K}(\mathbf{Y},\mathbf{Z})\mathbf{N}. \tag{3.17}$$

It can be easily shown that $\mathbf{K}$ is well defined and is symmetric. Moreover, the Weingarten formula reads

$$\mathbf{K}(\mathbf{Y},\mathbf{Z}) = -\langle \nabla_{\mathbf{Y}}\mathbf{N}^\flat, \mathbf{Z}\rangle, \tag{3.18}$$

and in short $\mathbf{K} = -\nabla\mathbf{N}^\flat$. This allows one to obtain the components of the second fundamental form:

$$K_{AB} = -\frac{\partial N_A}{\partial X^B} + \Gamma^I{}_{AB}N_I = \Gamma^I{}_{AB}N_I = \frac{1}{N^3}\Gamma^3{}_{AB}. \tag{3.19}$$

Remark 3.3 While the material first fundamental form $\tilde{\mathbf{G}}$ is the pull-back of the first fundamental form $\mathbf{g}|_{T\omega_t}$ on ω_t, this property does not hold for the second fundamental form, i.e., $\mathbf{K} \neq \varphi_t{}^*\mathbf{k}$. This is due to the fact that unlike the first fundamental form, $\mathbf{K}$ is not an intrinsic object of a surface; it depends on the way the surface is embedded in its ambient space. In the case of accreting bodies, the two ambient spaces $\mathcal{M}$ and $\mathcal{S}$ (respectively for Ω and ω) are not isometric, as the 3D metric tensors $\mathbf{G}$ and $\mathbf{g}$ are mapped by the incompatible map $\mathbf{Q}$. Therefore, for each layer in general $\mathbf{K} \neq \varphi_t{}^*\mathbf{k}$, even though for the first fundamental form one has $\tilde{\mathbf{G}} = \varphi_t{}^*\tilde{\mathbf{g}}$ (see also Remark 3.1).

[11] Note that the Christoffel symbols do not constitute a tensorial object; they transform in the following way

$$\breve{\Gamma}^{\breve{I}}{}_{\breve{J}\breve{K}} = (A^{-1})^{\breve{I}}{}_I A^J{}_{\breve{J}} A^K{}_{\breve{K}} \Gamma^I{}_{JK} + (A^{-1})^{\breve{I}}{}_I A^I{}_{\breve{J},\breve{K}},$$

where $A^I{}_{\breve{I}}$ represent the change of basis given in (3.3), i.e., $\mathbf{e}_I = (A^{-1})^{\breve{I}}{}_I \breve{\mathbf{e}}_{\breve{I}}$.

The Gauss equation relates the tangent part of $\boldsymbol{\mathcal{R}}$, which is represented by only one independent component, say $\mathcal{R}_{1212}$, to $\mathbf{K}$ and $\tilde{\boldsymbol{\mathcal{R}}}$, that is represented by $\tilde{\mathcal{R}}_{1212}$, or by the Gaussian curvature $g = \det(\tilde{G}^{AC} K_{CB}) = \mathcal{R}_{1212}/\det\tilde{\mathbf{G}}$, or by the scalar curvature $\tilde{\mathcal{R}} = \tilde{G}^{AB}\mathcal{R}^{C}{}_{ACB} = 2g$. It reads

$$\mathcal{R}_{ABCD} = \tilde{\mathcal{R}}_{ABCD} + K_{AD}K_{BC} - K_{AC}K_{BD}. \tag{3.20}$$

The Codazzi–Mainardi equations, instead, relate $\boldsymbol{\mathcal{R}}$ to $\tilde{\nabla}\mathbf{K}$, viz.

$$N^{H}\mathcal{R}_{HBCD} = N_{H}\mathcal{R}^{H}{}_{BCD} = \tilde{\nabla}_{D}K_{BC} - \tilde{\nabla}_{C}K_{BD}. \tag{3.21}$$

These equations express three components of $\boldsymbol{\mathcal{R}}$, namely $\mathcal{R}_{1212}$, $\mathcal{R}_{1213}$, and $\mathcal{R}_{2123}$, in terms of $\tilde{\boldsymbol{\mathcal{R}}}$ and $\mathbf{K}$. In other words, they describe the geometry of the non-Euclidean 3D accreted body in terms of the geometry of its layers. Note that the other three components of the curvature tensor need more information than just the layer-wise geometry.

3.3 The Geometry of the Material Foliation in Terms of the Growth Velocity

The growth velocity $\mathbf{u}$ can be decomposed into its parallel and normal components, i.e., $\mathbf{u} = \mathbf{u}^{\parallel} + u^{n}\mathbf{n}$, where $\mathbf{n}$ is the unit normal vector to ω pointing out of the image of the body, and $\mathbf{u}^{\parallel}$ is the orthogonal projection of $\mathbf{u}$ onto $T\omega$. Recalling that by construction $\mathbf{QU} = \mathbf{u}$, for the material growth velocity $\mathbf{U}$ one can write

$$\mathbf{U} = \mathbf{Q}^{-1}\mathbf{u} = \mathbf{Q}^{-1}\mathbf{u}^{\parallel} + u^{n}\mathbf{Q}^{-1}\mathbf{n} = \mathbf{Q}^{-1}\mathbf{u}^{\parallel} + u^{n}\mathbf{N} = \mathbf{U}^{\parallel} + U^{N}\mathbf{N}, \tag{3.22}$$

where we define $\mathbf{U}^{\parallel} = \mathbf{Q}^{-1}\mathbf{u}^{\parallel}$ and $U^{N} = u^{n}\circ\varphi$. Note that we have used the fact that $\mathbf{Q}^{-1}\mathbf{n} = \mathbf{N}$ since the metric is transformed by $\mathbf{Q}$ itself.[12] Note also that $\mathbf{G}(\mathbf{U}^{\parallel}, \mathbf{N}) = 0$, and so (3.22) is a unique orthogonal decomposition with respect to the material metric $\mathbf{G}$ and the vector $\mathbf{N}$. One can write $\mathbf{U}^{\parallel} = U^{A}\frac{\partial}{\partial\Xi^{A}}$, where the U^{A}'s coincide with the first two components of $\mathbf{U}$ with respect to $\breve{\mathfrak{F}} = \left\{\frac{\partial}{\partial\Xi^{1}}, \frac{\partial}{\partial\Xi^{2}}, \mathbf{N}\right\}$. In particular, one has

$$\mathbf{U} = U^{A}\frac{\partial}{\partial\Xi^{A}} + U^{N}\mathbf{N}. \tag{3.23}$$

Comparing this with (3.3), one finds

$$\begin{aligned} U^{1} &= -\frac{N^{1}}{N^{3}}, \quad U^{2} = -\frac{N^{2}}{N^{3}}, \quad U^{N} = \frac{1}{N^{3}}, \\ N^{1} &= -\frac{U^{1}}{U^{3}}, \quad N^{2} = -\frac{U^{2}}{U^{3}}, \quad N^{3} = \frac{1}{U^{N}}. \end{aligned} \tag{3.24}$$

[12] Note that $\mathbf{G}(\mathbf{Q}^{-1}\mathbf{n}, \mathbf{Q}^{-1}\mathbf{n}) = (\mathbf{Q}^{-\star}\mathbf{G}\mathbf{Q}^{-1})(\mathbf{n}, \mathbf{n}) = \mathbf{g}(\mathbf{n}, \mathbf{n}) = 1$. Moreover, for any $\mathbf{Y}$ tangent to Ω there exists a $\mathbf{y} = \mathbf{QY}$ tangent to ω, so one has $\mathbf{G}(\mathbf{Q}^{-1}\mathbf{n}, \mathbf{Y}) = \mathbf{G}(\mathbf{Q}^{-1}\mathbf{n}, \mathbf{Q}^{-1}\mathbf{y}) = (\mathbf{Q}^{-\star}\mathbf{G}\mathbf{Q}^{-1})(\mathbf{n}, \mathbf{y}) = \mathbf{g}(\mathbf{n}, \mathbf{y}) = 0$. Therefore, $\mathbf{N} = \mathbf{Q}^{-1}\mathbf{n}$.

Hence

$$\mathbf{N} = -\frac{U^A}{U^N}\frac{\partial}{\partial \Xi^A} + \frac{1}{U^N}\mathbf{U}, \quad \mathbf{N}^\flat = U^N \mathrm{d}\tau. \tag{3.25}$$

Note that $G_{33} = \|\mathbf{U}\|^2_{\mathbf{G}} = \|\mathbf{u}\|^2_{\mathbf{g}}$. From (3.8), the shift part is written as[13]

$$\eta_A = \tilde{G}_{AB} U^B. \tag{3.26}$$

Therefore, the material metric and its inverse in the frame $\mathfrak{F}$ read

$$[G_{IJ}] = \left[\begin{array}{c|c} \tilde{G}_{AB} & \tilde{G}_{AC}U^C \\ \hline \text{sym} & \|\mathbf{U}\|^2 \end{array}\right], \left[G^{IJ}\right] = \left[\begin{array}{c|c} \tilde{G}^{AB} + \frac{U^A U^B}{(U^N)^2} & -\frac{U^A}{(U^N)^2} \\ \hline \text{sym} & \frac{1}{(U^N)^2} \end{array}\right]. \tag{3.27}$$

Using (3.27) to find the second fundamental form through (3.19), one obtains the following expression in terms of the growth velocity and the first fundamental form as

$$\begin{aligned} K_{AB} = U^N \Gamma^3{}_{AB} &= \frac{1}{2U^N}(\tilde{\nabla}_A \eta_B + \tilde{\nabla}_B \eta_A - \partial_\tau \tilde{G}_{AB}) \\ &= \frac{1}{2U^N}(\tilde{G}_{AC}\tilde{\nabla}_A U^C + \tilde{G}_{BC}\tilde{\nabla}_A U^C - \partial_\tau \tilde{G}_{AB}). \end{aligned} \tag{3.28}$$

One can now write the derivative of the first fundamental form of each layer with respect to τ in terms of the material growth velocity, the first, and the second fundamental forms as

$$\partial_\tau \tilde{G}_{AB} = \tilde{G}_{AC}\tilde{\nabla}_A U^C + \tilde{G}_{BC}\tilde{\nabla}_A U^C - 2U^N K_{AB}. \tag{3.29}$$

Note that $\partial_\tau \tilde{G}_{AB}$ are the components of the tensor

$$\mathfrak{L}_{\mathbf{U}}\tilde{\mathbf{G}}(X) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=\tau(X)} \left(\Phi_s \circ \Phi^{-1}_{\tau(X)}\right)^* \left\{\tilde{\mathbf{G}}\left(\Phi_s\left(\Phi^{-1}_{\tau(X)}(X)\right)\right)\right\}, \tag{3.30}$$

where Φ indicates the material motion. Note that the map $\Phi_s \circ \Phi^{-1}_{\tau(X)}$ is a diffeomorphism from $\{X \in \mathcal{M} | 0 \le \tau(X) \le T - s\}$ to $\{X \in \mathcal{M} | s \le \tau(X) \le T\}$. The Lie derivative of a $\binom{0}{2}$-tensor is given by $(\mathfrak{L}_{\mathbf{Z}}\mathbf{S})_{AB} = \partial_C S_{AB} Z^C + S_{AC}\partial_B Z^C + S_{BC}\partial_A Z^C$, and therefore, one obtains $(\mathfrak{L}_{\mathbf{U}}\tilde{\mathbf{G}})_{AB} = \partial_\tau \tilde{G}_{AB}$ as the layer components of $\mathbf{U}$ vanish.

[13] Indicating with $\tilde{g}_{ab}$ the components of the first fundamental form of ω (with respect to a coordinate chart (ξ^1, ξ^2) on the surface), one also has $\eta_A = Q^a{}_A Q^b{}_B \tilde{g}_{ab} (Q^{-1})^B{}_c u^c = Q^a{}_A \tilde{g}_{ac} u^c$.

Now it is possible to write the Christoffel symbols of the connection ∇ in terms of $\mathbf{U}$, U^N, and $\mathbf{K}$ (and of the coefficients of $\tilde{\nabla}$). In particular, one has the following expressions

$$\Gamma^A{}_{BC} = \tilde{\Gamma}^A_{BC} - \frac{U^A}{U^N} K_{BC}, \tag{3.31}$$

$$\Gamma^3{}_{3A} = \Gamma^3_{A3} = \frac{U^A}{U^N} + K_{AB}\frac{U^B}{U^N}, \tag{3.32}$$

$$\Gamma^A{}_{3B} = \Gamma^A{}_{B3} = -\frac{U^A}{U^N}\partial_B U^N - U^N\left[\tilde{G}^{AD} + \frac{U^A U^D}{(U^N)^2}\right] K_{DB} + \tilde{\nabla}_B U^A, \tag{3.33}$$

$$\Gamma^3{}_{33} = \frac{1}{U^N}\left(\partial_\tau U^N + U^A \partial_A U^N + K_{AB} U^A U^B\right). \tag{3.34}$$

Note that the coefficients (3.32), (3.33), and (3.34) are sums of terms that are tensor fields in the layer tangent space $T\Omega$, and hence, they are tensor fields. Note that $\Gamma^A{}_{BC}$'s are not the coefficients of $\tilde{\nabla}$, i.e., $\Gamma^A{}_{BC} \neq \tilde{\Gamma}^A{}_{BC}$, as $\mathbf{K} \neq \mathbf{0}$.

As was mentioned earlier, the Gauss equation (3.20) gives R_{1212} in terms of $\tilde{\mathcal{R}}$ and $\mathbf{K}$, while the two Codazzi–Mainardi equations (3.21) express R_{1213} and R_{2123} in terms of $\tilde{\nabla}\mathbf{K}$. Therefore, we still need to calculate the remaining three independent components of $\mathcal{R}$. In order to find them, one substitutes Eqs. (3.31)–(3.34) into (A.18) and obtains

$$N^I N^J R_{IAJB} = \frac{1}{U^N}\left(\partial_\tau K_{AB} - \tilde{\nabla}_A \tilde{\nabla}_B U^N + U^N K_{AC} K^C{}_B - \mathfrak{L}_{\mathbf{U}^\parallel} K_{AB}\right), \tag{3.35}$$

with an abuse of notation $\mathfrak{L}_{\mathbf{U}^\parallel} K_{AB} = (\mathfrak{L}_{\mathbf{U}^\parallel}\mathbf{K})_{AB}$. Note that from (3.20), (3.21), and (3.35) one can obtain all the components of the curvature tensor $\mathcal{R}$ of the Riemannian manifold $(\mathcal{M}, \mathbf{G})$ in the frame $\breve{\mathfrak{F}}^*$. Therefore, we are now able to recover the geometry of the 3D accreted body from the layer quantities $\tilde{\mathbf{G}}$ and $\mathbf{K}$, and from the material growth velocity $\mathbf{U}^\parallel$ and U^N (see Table 2).

Remark 3.4 If one assumes that the growth velocity is normal, i.e., $\mathbf{U}^\parallel = \mathbf{0}$, then, (3.35) is simplified to read

$$N^I N^J R_{IAJB} = \frac{1}{U^N}\left(\partial_\tau K_{AB} - \tilde{\nabla}_A \tilde{\nabla}_B U^N + U^N K_{AC} K^C{}_B\right), \tag{3.36}$$

with $K_{AB} = -\frac{1}{2U^N}\partial_\tau \tilde{G}_{AB}$. The expressions (3.20) and (3.21) remain unchanged. If one also assumes constant U^N, then one obtains

$$N^I N^J R_{IAJB} = \frac{1}{U^N}\partial_\tau K_{AB} + K_{AC} K^C{}_B, \tag{3.37}$$

with $\partial_\tau K_{AB} = -\frac{1}{2U^N}\partial_\tau \partial_\tau \tilde{G}_{AB}$.

Table 2 Summary of the steps needed in the construction of the Riemann curvature tensor of the material manifold $(\mathcal{M}, \mathbf{G})$

Analysis of the local compatibility of an accreted body

1. Solve the accretion initial boundary value problem. This gives the first fundamental form $\tilde{\mathbf{G}}$, and the unit normal vector field $\mathbf{N}$. Calculate $\mathbf{U}^{\parallel}$, U^N, and the in-layer Christoffel symbols $\tilde{\Gamma}^A{}_{BC}$. There are some cases, e.g., when the growth surface is fixed, in which one does not need to solve the accretion initial boundary value problem as the material metric is known a priori (see Sect. 5.1)

2. Calculate the second fundamental form $\mathbf{K}$ using (3.19)

3. Use Gauss's equation to write the tangent part of the curvature R_{ABCD}:

$$R_{ABCD} = \tilde{R}_{ABCD} + K_{AD}K_{BC} - K_{AC}K_{BD}$$

4. Use Codazzi–Mainardi equation to find the components $N^H R_{HABC}$:

$$N^H \mathcal{R}_{HBCD} = N_H \mathcal{R}^H{}_{BCD} = \tilde{\nabla}_D K_{BC} - \tilde{\nabla}_C K_{BD}$$

5. Use (3.35) to find the components $N^H N^K R_{HAKB}$:

$$N^I N^J R_{IAJB} = \tfrac{1}{U^N}\left(\partial_\tau K_{AB} - \tilde{\nabla}_A \tilde{\nabla}_B U^N + U^N K_{AC} K^C{}_B - \mathfrak{L}_{\mathbf{U}^{\parallel}} K_{AB}\right)$$

For the sake of completeness, we calculate the Ricci and scalar curvatures. From (3.20), (3.21), and (3.35) one can calculate the Ricci curvature tensor. In particular, we consider components in the frame $\left\{\frac{\partial}{\partial \Xi^1}, \frac{\partial}{\partial \Xi^2}, \mathbf{N}\right\}$, where the material metric has the representation (3.10). Notice that from the change of frame (3.3), for a 1-form $\boldsymbol{\alpha}$ one has $\alpha_N = A^I{}_N \alpha_I = N^I \alpha_I$. For the tangential components, one has

$$\begin{aligned} R_{AB} &= R_{ACBD}\tilde{G}^{CD} + N^I N^J R_{IAJB} \\ &= \tilde{R}_{AB} + K_{AC}K^C{}_B - K^C{}_C K_{AB} + N^I N^J R_{IAJB}, \end{aligned} \tag{3.38}$$

where (B.8) was used. The components (N, A) are calculated using (3.21):

$$N^I R_{IA} = N^I R_{ICAD}\tilde{G}^{CD} + N^I N^J N^K R_{IJAK} = \tilde{\nabla}_C K^C{}_A - \tilde{\nabla}_A K^C{}_C, \tag{3.39}$$

with the term $N^I N^J N^K R_{IJAK} = 0$ by virtue of the anti-symmetry of $\boldsymbol{\mathcal{R}}$. For the normal component R_{NN} one writes

$$\begin{aligned} N^I N^J R_{IJ} &= N^I N^J \mathcal{R}_{ICJD}\tilde{G}^{CD} + N^I N^J N^K N^L R_{IJLK} \\ &= \frac{1}{U^N}\left(\partial_\tau K^C{}_C - U^N K^C{}_D K^D{}_C - \tilde{\Delta} U^N - U^C \tilde{\nabla}_C U^N\right), \end{aligned} \tag{3.40}$$

where use was made of (3.29) and of the relation $(\mathfrak{L}_{\mathbf{U}^{\parallel}}\tilde{\mathbf{G}})_{AB} = \tilde{G}_{AC}\tilde{\nabla}_B U^C + \tilde{G}_{BC}\tilde{\nabla}_A U^C$. Expressions (3.38), (3.39), and (3.40) give the entire Ricci curvature tensor. Finally, one can calculate the scalar curvature $R = R_{\check{I}\check{J}} G^{\check{I}\check{J}} = R_{AB}\tilde{G}^{AB} + N^I N^J R_{IJ}$ of the material manifold, which reads

$$R = \tilde{R} - K^C{}_D K^D{}_C - K^C{}_C K^C{}_C + \frac{2}{U^N}\left(\partial_\tau K^C{}_C - \tilde{\Delta} U^N - U^C \tilde{\nabla}_C U^N\right). \tag{3.41}$$

4 Balance Laws and the Initial Boundary Value Problem of Accreting Bodies

In Sect. 2 we defined the notion of an accreting body and a family of material manifolds $\mathcal{B}_t$, $t \in [0, T]$. We also defined a material metric on $\mathcal{M}$, that was studied in detail in Sect. 3, and that for most accretion problems is an unknown a priori as it depends on the kinematics of the deforming body during the accretion process. Each of the material configurations $\mathcal{B}_t$ was therefore endowed with a metric obtained by restricting $\mathbf{G}$ to $\mathcal{B}_t$, resulting in the identification of the pair $(\mathcal{B}_t, \mathbf{G}|_{\mathcal{B}_t})$ at time $t \in [0, T]$. This constitutes the first step in formulating the accretion initial boundary value problem (IBVP). The next step is to discuss balance laws and the constitutive equations for an accreting body. These, together with initial and boundary conditions, define the IBVP of accretion for the unknown motion φ.

4.1 Balance of Mass

The material manifold is endowed with a volume form $\mathrm{d}V$, that in a coordinate chart $\{X^I\}$ is defined as $\mathrm{d}V_{IJK} = \sqrt{\det \mathbf{G}}\,\epsilon_{IJK}$, where ϵ_{IJK} is the Levi–Civita symbol in dimension three. Note that $\mathrm{d}V$ is well defined only for orientation-preserving changes of coordinates. The Jacobian $J(X, t)$ is a scalar on $\mathcal{B}_t$ given by

$$J(X,t) = \sqrt{\frac{\det \mathbf{g}(\varphi(X,t))}{\det \mathbf{G}(X)}}\,\det \mathbf{F}(X,t). \tag{4.1}$$

Evaluating at $\tau(X)$, one obtains

$$\begin{aligned}\bar{J}(X) &= \sqrt{\frac{\det \mathbf{g}(\bar{\varphi}(X))}{\det \mathbf{G}(X)}}\,\det \bar{\mathbf{F}}(X) = \sqrt{\frac{\det \mathbf{g}(\bar{\varphi}(X))}{\det\left[\mathbf{Q}^\star(X)\mathbf{g}(\bar{\varphi}(X))\mathbf{Q}(X)\right]}}\,\det \bar{\mathbf{F}}(X)\\ &= \frac{\det \bar{\mathbf{F}}}{\det \mathbf{Q}} = \det\left(\mathbf{Q}^{-1}\bar{\mathbf{F}}\right).\end{aligned} \tag{4.2}$$

For any point $X \in \mathcal{M}$, $\mathbf{U}(X)$ points in the direction of increasing τ, and therefore, points outside of $\mathcal{B}_{\tau(X)}$. Hence, the vector $\bar{\mathbf{F}}(X)\mathbf{U}(X)$ points outside of $\varphi_t(\mathcal{B}_t)$. By assumption $\mathbf{Q}(X)$ maps $\mathbf{U}(X)$ to a vector $\mathbf{Q}(X)\mathbf{U}(X) = \mathbf{u}(\bar{\varphi}(X), \tau(X))$ that points outside of $\varphi_{\tau(X)}(\mathcal{B}_{\tau(X)})$, just as $\bar{\mathbf{F}}(X)$ does. On $T_X\Omega_{\tau(X)}$ the two tensors coincide. Hence, since $\bar{\mathbf{F}}(X)$ preserves orientation, so does $\mathbf{Q}(X)$. Therefore, $\det \mathbf{Q}(X) > 0$. Since $\bar{\mathbf{F}}|_{T\Omega} = \mathbf{Q}|_{T\Omega}$, the tensor $\mathbf{F}\mathbf{Q}^{-1}$ has the following representation with respect to a foliation chart:

$$\left[(Q^{-1})^I{}_i \bar{F}^i{}_J\right] = \begin{bmatrix} 1 & 0 & (Q^{-1})^1{}_i \bar{F}^i{}_3 \\ 0 & 1 & (Q^{-1})^2{}_i \bar{F}^i{}_3 \\ 0 & 0 & (Q^{-1})^3{}_i \bar{F}^i{}_3 \end{bmatrix}, \tag{4.3}$$

and so $\bar{J} = (Q^{-1})^3{}_i \bar{F}^i{}_3 = \langle \mathrm{d}\tau, \mathbf{Q}^{-1}\bar{\mathbf{F}}\mathbf{U}\rangle$. The Jacobian relates the volume forms in the material and the ambient spaces as $\varphi_t^*\mathrm{d}v = J\mathrm{d}V$, where $\mathrm{d}v$ is the standard volume form of the Euclidean space $\mathcal{S}$. In this way, since $\int_{\varphi_t(\mathcal{B}_t)} f\,\mathrm{d}v = \int_{\mathcal{B}_t}(f\circ\varphi_t)(\varphi_t^*\mathrm{d}v)$, one writes

$$\int_{\varphi_t(\mathcal{B}_t)} f\,\mathrm{d}v = \int_{\mathcal{B}_t} J f\circ\varphi_t\,\mathrm{d}V. \tag{4.4}$$

Each layer is in turn endowed with an area form $\mathrm{d}A$, defined as $\mathrm{d}A_{IJ} = \sqrt{\det\tilde{\mathbf{G}}}\,\epsilon_{IJ}$. Note that the Jacobian of the map $\bar{\varphi}|_\Omega$ is equal to 1 as Ω_t and ω_t are isometric. Therefore, indicating with $\mathrm{d}a$ the area form on ω_t, one has $\mathrm{d}A = \bar{\varphi}^*\mathrm{d}a$, and hence

$$\int_{\omega_t} f\,\mathrm{d}a = \int_{\Omega_t} f\circ\varphi_t\,\mathrm{d}A. \tag{4.5}$$

For an accreting body mass is conserved only away from the growth surface ω_t. Unlike classical elasticity, mass is not conserved globally as new particles are joining the body on its boundary. We indicate with $\rho_0(X)$ the mass density field in the material manifold $\mathcal{M}$, and with $\rho(x,t)$ the mass density field in the deformed configuration $\varphi_t(\mathcal{B}_t)$. The local conservation of mass reads

$$\rho_0(X) = \rho(\varphi(X,t),t)J(X,t), \quad X\in\mathcal{M},\ \tau(X)\le t\le T. \tag{4.6}$$

The total mass of the body at time t is calculated as

$$M(t) = \int_{\varphi_t(\mathcal{B}_t)} \rho(x,t)\mathrm{d}v = \int_{\mathcal{B}_t}\rho_0(X)\mathrm{d}V, \tag{4.7}$$

where the equality holds by virtue of the local conservation of mass. Recalling the definition of the frame $\breve{\mathfrak{F}}^*$ given in (3.2), one writes

$$\begin{aligned} \mathrm{d}V &= \sqrt{\det[G_{IJ}]}\,\mathrm{d}\Xi^1\wedge\mathrm{d}\Xi^2\wedge\mathrm{d}\Xi^3 = \sqrt{\det\left[G_{\breve{I}\breve{J}}\right]}\,\breve{\boldsymbol{\theta}}^1\wedge\breve{\boldsymbol{\theta}}^2\wedge\breve{\boldsymbol{\theta}}^3 \\ &= \left(\sqrt{\det\left[\tilde{G}_{AB}\right]}\,\breve{\boldsymbol{\theta}}^1\wedge\breve{\boldsymbol{\theta}}^2\right)\wedge\mathbf{N}^\flat, \end{aligned} \tag{4.8}$$

since $\det[G_{\breve{I}\breve{J}}] = \det[\tilde{G}_{AB}]$ from the representation (3.7). Therefore, in the foliated structure, the material volume form can be written as

$$\mathrm{d}V = \mathrm{d}A\wedge\mathbf{N}^\flat = U^N\mathrm{d}A\wedge\mathrm{d}\tau, \tag{4.9}$$

as $\mathbf{N}^\flat = U^N\mathrm{d}\tau$ by (3.25). Invoking (4.9), one can now express the total mass as

$$M(t) = \int_0^t\left(\int_{\Omega_\tau}\rho_0 U^N\,\mathrm{d}A\right)\mathrm{d}\tau. \tag{4.10}$$

As was explained in Sect. 2.4, when stress-free material is added on the growth surface, the accretion tensor $\mathbf{Q}$ is a local isometry, i.e., its Jacobian is 1, and hence, mass density

Table 3 History of a particle X in terms of the local deformation, Jacobian and mass density in the deformed configuration right before attachment, at the time of attachment, and after attachment for $t \in [\tau(X), T]$

	Local deformation	Jacobian	Density
Right before attachment	$\mathbf{Q}(X)$	1	$\rho_0(X)$
At the time of attachment	$\bar{\mathbf{F}}(X)$	$\bar{J}(X) = \frac{\det \bar{\mathbf{F}}(X)}{\det \mathbf{Q}(X)}$	$\bar{\rho}(X)$
After attachment	$\mathbf{F}(X,t)$	$\sqrt{\frac{\det \mathbf{g}(\varphi(X,t))}{\det \mathbf{G}(X)}} \det \mathbf{F}(X,t)$	$\rho(X,t)$

Note that a particle, in general, experiences a discontinuity in the local deformation at its time of attachment

of a particle before attachment is ρ_0 (Table 3 recaps the history of deformation for a single particle). Therefore, the rate of change of the total mass is written as

$$\dot{M}(t) = \int_{\Omega_t} \rho_0 U^N \, \mathrm{d}A = \int_{\omega_t} \rho_0 u^n \, \mathrm{d}a, \tag{4.11}$$

as $u^n \circ \varphi = U^N$. This means that the rate of change of the total mass $\dot{M}(t)$ is equal to the flux of mass through the boundary of $\mathcal{B}_t$.

4.2 Constitutive Relations

We work in the context of hyperelasticity, i.e., we assume the existence of an energy function of some measure of strain. The $\binom{0}{2}$ right Cauchy–Green strain tensor is defined as $\mathbf{C}^\flat = \varphi^*\mathbf{g}$, and in components $C_{IJ} = F^i{}_I F^j{}_J g_{ij}$. Its $\binom{1}{1}$ variant $\mathbf{C}$ has components $C^I{}_J = G^{IH} C_{HJ} = G^{IH} F^i{}_H F^j{}_J g_{ij}$. One defines $\mathbf{F}^\top$ as $(F^\top)^I{}_j = G^{IH} F^h{}_H g_{hj}$, so one has $\mathbf{C} = \mathbf{F}^\top \mathbf{F}$. We assume the existence of an energy function $\mathcal{W}(X, \mathbf{C})$. Using this function, the second Piola–Kirchhoff stress tensor $\mathbf{S}$, the first Piola–Kirchhoff stress tensor $\mathbf{P}$, and the Cauchy stress tensor $\boldsymbol{\sigma}$ are written as

$$S^{IJ} = 2G^{IK} \frac{\partial \mathcal{W}}{\partial C^K{}_J}, \quad P^{iJ} = 2F^i{}_I G^{IK} \frac{\partial \mathcal{W}}{\partial C^K{}_J}, \quad \sigma^{ij} = \frac{2}{J} F^i{}_I G^{IK} \frac{\partial \mathcal{W}}{\partial C^K{}_J} F^j{}_J. \tag{4.12}$$

One can also express the strain energy density as a function of the deformation gradient, and the metric tensors, i.e., $W(X, \mathbf{F}, \mathbf{G}, \mathbf{g}) = \mathcal{W}(\mathbf{F}^\top \mathbf{F})$.[14] Using this function, the stress tensors are written as

$$S^{IJ} = (F^{-1})^I{}_i g^{ij} \frac{\partial W}{\partial F^j{}_J}, \quad P^{iJ} = g^{ij} \frac{\partial W}{\partial F^j{}_J}, \quad \sigma^{ij} = \frac{1}{J} g^{ij} \frac{\partial W}{\partial F^j{}_J}. \tag{4.13}$$

Therefore, one obtains the constitutive relations for the stress tensors, e.g., $\mathbf{P}(X) = \boldsymbol{\mathcal{P}}(X, \mathbf{F}, \mathbf{G}, \mathbf{g})$.

[14] Note that while a function $\mathcal{W}(X, \mathbf{C})$ is automatically objective, objectivity needs to be imposed on an energy function $W(X, \mathbf{F})$.

Remark 4.1 The right Cauchy–Green strain tensor **C** explicitly depends on the material metric **G**, which is determined by the accretion tensor **Q**, i.e.,

$$C^I{}_J = (Q^{-1})^I{}_i g^{ik} (Q^{-1})^H{}_k F^h{}_H g_{hj} F^j{}_J. \tag{4.14}$$

Thus, it is possible to define the energy as a function $\hat{W}(X, \mathbf{F}, \mathbf{Q}, \mathbf{g}) = W(X, \mathbf{F}, \mathbf{G}(\mathbf{Q}, \mathbf{g}), \mathbf{g})$, which has the same form as the one usually considered in the context of the multiplicative decomposition of the deformation gradient. As a matter of fact, as was mentioned in Remark 2.11, this is one of the possible interpretations of the accretion tensor. The tensor **Q** is in turn built from $\bar{\mathbf{F}}$ and **u**. Therefore, one can define a new energy function, viz. $\check{W}(X, \mathbf{F}, \bar{\mathbf{F}}, \mathbf{u}, \mathbf{g}) = \hat{W}\left(X, \mathbf{F}, \mathbf{Q}(\bar{\mathbf{F}}, \mathbf{u}), \mathbf{g}\right)$.

4.3 Balance of Linear and Angular Momenta

The local balance of linear momentum can be obtained using covariance arguments (Marsden and Hughes 1983; Yavari et al. 2006). For any point X in the interior of $\mathcal{B}_t$, there exists an open set $\mathcal{U} \subset \mathcal{B}_t$ such that $\partial\mathcal{U} \cap \partial\mathcal{B}_t = \emptyset$ and $X \in \mathcal{U}$. Balance of energy for such a subbody $\mathcal{U}$ is identical to that of a subbody in classical nonlinear elasticity with the only difference being that the material metric is not known a priori. However, this would not affect the covariance arguments. Covariance of energy balance for $\mathcal{U}$ results in all the balance laws, and in particular, the balance of linear momentum, which in terms of the first Piola–Kirchhoff stress and the Cauchy stress reads

$$\operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}, \quad \operatorname{div}^{\mathbf{g}} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}, \tag{4.15}$$

where **b** is the body force referred to spatial coordinates, $\mathbf{B} = \mathbf{b} \circ \varphi$ is the body force referred to material coordinates, and $\mathbf{A} = \mathbf{a} \circ \varphi$ is the acceleration referred to material coordinates. In components

$$\nabla_J P^{iJ} + \rho_0 B^i = \rho_0 A^i, \quad \nabla^{\mathbf{g}}_j \sigma^{ij} + \rho b^i = \rho a^i. \tag{4.16}$$

If an accretion process is slow, one can ignore the inertial effects. This is what we will assume in our numerical examples. However, it should be emphasized that the accretion theory introduced in this paper is not restricted to slow accretion processes. The local balance of angular momentum follows from covariance of energy balance as well and is expressed as $P^{iJ} F^j{}_J = P^{jJ} F^i{}_J$, or equivalently, $\sigma^{ij} = \sigma^{ji}$.

Invoking (4.4) and (4.9), the rate of change of linear momentum of an accreting body is written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\varphi_t(\mathcal{B}_t)} \rho \mathbf{v}\, \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{B}_t} \rho_0 \mathbf{V}\, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \left[\int_{\Omega_\tau} \rho_0 U^N \mathbf{V}\, \mathrm{d}A \right] \mathrm{d}\tau. \tag{4.17}$$

From Leibniz's integral rule $\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(s,t)\mathrm{d}s = f(t,t) + \int_0^t \frac{\partial f}{\partial t}(s,t)\mathrm{d}s$, (4.17) is simplified to read

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \left[\int_{\Omega_\tau} \rho_0 U^N \mathbf{V}\,\mathrm{d}A\right]\mathrm{d}\tau = \int_{\mathcal{B}_t} \rho_0 \mathbf{A}\,\mathrm{d}V + \int_{\Omega_t} \rho_0 U^N \mathbf{V}\,\mathrm{d}A. \tag{4.18}$$

Therefore, one can write the rate of change of linear momentum of an accreting body as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(\mathcal{B}_t)} \rho \mathbf{v}\,\mathrm{d}v = \int_{\varphi_t(\mathcal{B}_t)} \rho \mathbf{a}\,\mathrm{d}v + \int_{\omega_t} \rho_0 u^n \mathbf{v}\,\mathrm{d}a. \tag{4.19}$$

Recalling that $\rho\mathbf{a} = \mathrm{div}^{\mathbf{g}}\boldsymbol{\sigma} + \rho\mathbf{b}$, the rate of change of linear momentum reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(\mathcal{B}_t)} \rho \mathbf{v}\,\mathrm{d}v = \int_{\varphi_t(\mathcal{B}_t)} \rho \mathbf{b}\,\mathrm{d}v + \int_{\partial\varphi_t(\mathcal{B}_t)} \mathbf{t}\,\mathrm{d}a + \int_{\omega_t} \rho_0 u^n \mathbf{v}\,\mathrm{d}a. \tag{4.20}$$

Remark 4.2 On the growth surface ω_t, the traction vector can be decomposed as a sum of a non-accretion part $\mathbf{t}^{\mathrm{na}}$, due to external loads and constraints, and an accretion part $\mathbf{t}^{\mathrm{a}}$, due to the flux of the new particles. Consider a particle that is about to join the accreting body. From Table 3, one can write the mass 3-form of the particle before attachment as $\mathrm{d}m = \rho_0\mathrm{d}v = \rho_0 u^n \mathrm{d}a \wedge \mathrm{d}t$. Since for a particle before attachment $J = 1$, one has $\mathrm{d}m = \rho_0\mathrm{d}V = \rho_0 U^N \mathrm{d}A \wedge \mathrm{d}\tau$. Right before attaching the body this particle has velocity $-\mathbf{u}$ relative to the growth surface, and its absolute velocity is $\mathbf{v}-\mathbf{u}$. After it joins the body it will have velocity $\mathbf{v}$. Balance of linear momentum for this particle in the time interval $[t, t+\mathrm{d}t]$ reads

$$(\rho_0 u^N \mathrm{d}a \wedge \mathrm{d}t)(\mathbf{v}-\mathbf{u}) + (-\mathbf{t}^{\mathrm{a}}\mathrm{d}a)\wedge \mathrm{d}t = (\rho_0 u^N \mathrm{d}a \wedge \mathrm{d}t)\mathbf{v}, \tag{4.21}$$

where $-\mathbf{t}^{\mathrm{a}}\mathrm{d}a$ is a vector-valued 2-form representing the force that the body exerts on the particle. Therefore, the particle exerts the force $\mathbf{t}^{\mathrm{a}}\mathrm{d}a = -\rho_0 u^n \mathbf{u}\mathrm{d}a$ on the accreting body. Hence

$$\int_{\partial\varphi_t(\mathcal{B}_t)} \mathbf{t}\,\mathrm{d}a = \int_{\partial\varphi_t(\mathcal{B}_t)} \mathbf{t}^{\mathrm{na}}\,\mathrm{d}a - \int_{\omega_t} \rho_0 u^n \mathbf{u}\,\mathrm{d}a. \tag{4.22}$$

Therefore, substituting (4.22) into (4.20), one rewrites the rate of change of linear momentum of the accreting body as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(\mathcal{B}_t)} \rho \mathbf{v}\,\mathrm{d}v = \int_{\varphi_t(\mathcal{B}_t)} \rho \mathbf{b}\,\mathrm{d}v + \int_{\partial\varphi_t(\mathcal{B}_t)} \mathbf{t}^{\mathrm{na}}\,\mathrm{d}a + \int_{\omega_t} \rho_0 u^n (\mathbf{v}-\mathbf{u})\,\mathrm{d}a. \tag{4.23}$$

Note that when the growth velocity is small, the term $\mathbf{t}^{\mathrm{a}}$ can be neglected. If $\mathbf{t}^{\mathrm{na}}$ vanishes on ω_t, for small growth velocities, one can assume a traction-free growth surface.

4.4 The Accretion Initial Boundary Value Problem

The motion of an accreting body is determined by solving the governing equations, which consist of the equation defining the material metric **G**, the constitutive equations, and the balance of linear momentum. To complete this set of equations, one needs to impose initial and boundary conditions. At time $t = 0$ the accreting body $\mathcal{B}_0$ is represented by the surface Ω_0. We assume that the surface ω_0 on which accretion starts is known a priori, so one imposes $\varphi_0 = \mathring{\varphi}$, where $\mathring{\varphi} : \Omega_0 \to \mathcal{S}$ and $\mathring{\varphi}(\Omega_0) = \omega_0$. We indicate with $\partial_N \mathcal{B}_t$ that part of $\partial \mathcal{B}_t$ where traction $\mathbf{T}$ is assigned (see Fig. 8). We set $\partial_N \varphi_t(\mathcal{B}_t) = \varphi_t(\partial_N \mathcal{B}_t)$. The natural boundary conditions are given by

$$\begin{aligned} &P^{iJ} N_J = T^i \quad \text{on } \partial_N \mathcal{B}_t,\ \forall t \in [0, T], \\ &\text{or equivalently} \quad \sigma^{ij} n_j = t^i \quad \text{on } \partial_N \varphi_t(\mathcal{B}_t), \quad \forall t \in [0, T], \end{aligned} \tag{4.24}$$

where $T^i = (t^i J_\partial) \circ \varphi$. J_∂ is the Jacobian of the boundary motion, i.e., $\varphi_t^* da = J_\partial dA$, where $J_\partial = J\sqrt{C^{IJ} N_I N_J}$. Note that $P^{iJ} N_J = U^N P^{i3}$. We indicate with $\partial_D \mathcal{B}_t$ the part of $\partial \mathcal{B}_t$ where φ is specified and note that $\partial_D \varphi_t(\mathcal{B}_t) = \varphi_t(\partial_D \mathcal{B}_t)$. The Dirichlet (or essential) boundary conditions are given by

$$\varphi_t = \hat{\varphi}_t \quad \text{on } \partial_D \mathcal{B}_t,\ \forall t \in [0, T], \tag{4.25}$$

where $\hat{\varphi}_t : \partial_D \mathcal{B}_t \to \mathcal{S}$ is given. We assume that $\partial \mathcal{B}_t = \partial_N \mathcal{B}_t \cup \partial_D \mathcal{B}_t$. However, in general, there is no relation between the growth surface and the two portions of the boundary just defined. In other words, one can have $\Omega_t \cap \partial_D \mathcal{B}_t \neq \emptyset$, i.e., one can impose essential boundary conditions on a portion of the growth surface. In summary, the motion φ of an accreting body is a solution for the following accretion IBVP for slow accretion:

$$\begin{cases} \mathbf{G} = \mathbf{Q}^\star(\mathbf{g} \circ \bar{\varphi})\mathbf{Q} & X \in \mathcal{M}, \\ \mathbf{P} = \boldsymbol{\mathcal{P}}(\mathbf{F}, \mathbf{G}, \mathbf{g}) & 0 \le t \le T,\ X \in \mathcal{B}_t, \\ \operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} = \mathbf{0} & 0 \le t \le T,\ X \in \mathcal{B}_t, \\ \mathbf{P}\mathbf{N}^\flat = \mathbf{T} & 0 \le t \le T,\ X \in \partial_N \mathcal{B}_t, \\ \varphi_t = \hat{\varphi}_t & 0 \le t \le T,\ X \in \partial_D \mathcal{B}_t, \\ \varphi_0 = \mathring{\varphi} & X \in \omega_0. \end{cases} \tag{4.26}$$

In the incompressible case, one needs a pressure field p as the Lagrange multiplier associated with the internal incompressibility constraint $J = 1$. The accretion IBVP in this case reads:

$$\begin{cases} \mathbf{G} = \mathbf{Q}^\star(\mathbf{g} \circ \bar{\varphi})\mathbf{Q} & X \in \mathcal{M}, \\ \mathbf{P} = \boldsymbol{\mathcal{P}}(\mathbf{F}, \mathbf{G}, \mathbf{g}) + p\mathbf{g}^\sharp \mathbf{F}^{-\star} & 0 \le t \le T,\ X \in \mathcal{B}_t, \\ J = 1 & 0 \le t \le T,\ X \in \mathcal{B}_t, \\ \operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} = \mathbf{0} & 0 \le t \le T,\ X \in \mathcal{B}_t, \\ \mathbf{P}\mathbf{N}^\flat = \mathbf{T} & 0 \le t \le T,\ X \in \partial_N \mathcal{B}_t, \\ \varphi_t = \hat{\varphi}_t & 0 \le t \le T,\ X \in \partial_D \mathcal{B}_t, \\ \varphi_0 = \mathring{\varphi} & X \in \omega_0. \end{cases} \tag{4.27}$$

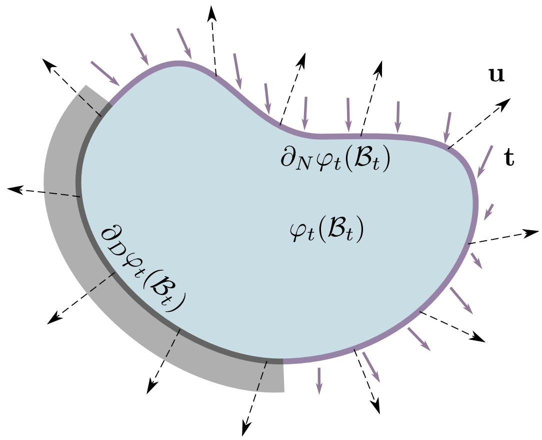

Fig. 8 An accreting body in its deformed configuration. Natural boundary conditions are assigned on $\partial_N \mathcal{B}_t$, while the essential boundary conditions are assigned on $\partial_D \mathcal{B}_t$

In Sect. 2.4 we showed that the Riemannian structure of an accreting body is independent of the choice of the material motion, as any choice would result in a material manifold in the same class of isometric Riemannian manifolds. It turns out that the accretion IBVP is invariant under reparametrizations of the initial and boundary data.

Proposition 4.3 *Let $\mathring{\varphi}' : \Omega_0 \to \mathcal{S}$ be an embedding such that $\mathring{\varphi}'(\Omega_0) = \Omega_0$. Let the displacement boundary conditions be modified such that on Ω_0 they read $\varphi_t = \hat{\varphi}_t \circ \mathring{\varphi}^{-1} \circ \mathring{\varphi}'$ and on Ω_t they read $\varphi_t = \hat{\varphi}_t \circ \Phi_t \circ \mathring{\varphi}^{-1} \circ \mathring{\varphi}' \circ \Phi_t^{-1}$. Then, the solution φ' of the accretion IBVP is such that $\varphi'_t(\Omega_t) = \varphi_t(\Omega_t)$ for all $t \in [0, T]$.*

Proof Let us define the diffeomorphism $\lambda_0 = \mathring{\varphi}^{-1} \circ \mathring{\varphi}'$ of Ω_0 into itself. This in turn can be extended to a map λ on the whole $\mathcal{M}$ such that $\lambda|_{\Omega_\tau} = \Phi_\tau \circ \lambda_0 \circ \Phi_\tau^{-1}$, with Φ indicating the material motion. It is now possible to define the motion φ' for the accreting body such that $\varphi_t = \varphi'_t \circ \lambda$ for any t. Note that this motion satisfies the modified boundary conditions, and for $t = 0$ one recovers the initial condition $\mathring{\varphi}'$. Moreover, one has

$$\varphi'_t(\Omega_t) = \varphi_t(\Omega_t), \quad \forall t \in [0, \tau], \tag{4.28}$$

i.e., φ and φ' map the body $\mathcal{B}_t$ into the same domain in the ambient space $\mathcal{S}$, and therefore, they define the same foliation on the same deformed configuration. Now one needs to check if φ' satisfies the accretion IBVP. The motion φ' defines a material metric $\mathbf{G}'$ through (2.16) and (2.18). Setting $T\lambda = \mathbf{\Lambda}$, since $\varphi_t = \varphi'_t \circ \lambda$ one has $\mathbf{F}(\lambda(X)) = \mathbf{F}'(X)\mathbf{\Lambda}(X)$. Moreover, note that λ does not change the family of material trajectories of Φ, as it is straightforward to check that $\lambda(\Phi_t(X_0)) = \Phi_t(\lambda(X_0))$ for any $X_0 \in \Omega_0$. Therefore, one has

$$\mathbf{U}'(\lambda(X)) = \mathbf{U}(\lambda(X)) = \mathbf{\Lambda}(X)\mathbf{U}(X). \tag{4.29}$$

Similar to the proof of Proposition 2.7, one can use $\mathbf{Q}$ to show that $\mathbf{G} = \lambda^* \mathbf{G}'$, and hence, φ' satisfies the accretion IBVP. $\square$

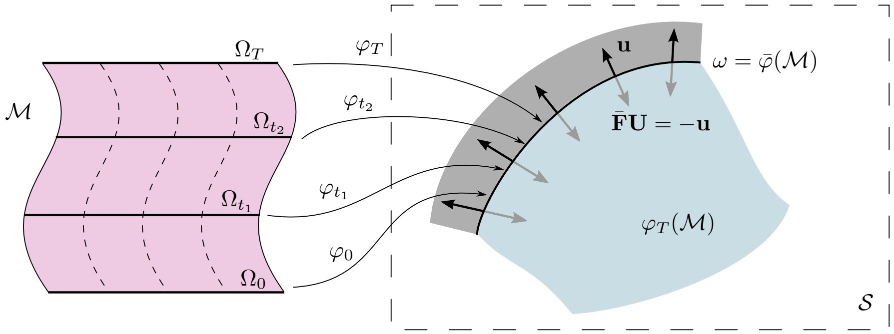

Fig. 9 Accretion through a fixed surface. Left: the material manifold with layers with the time of attachment equal to 0, t_1, t_2 and T. Right: the deformed configuration at time T. All the layers are mapped by $\bar{\varphi}$ to ω. The spatial growth velocity points outside of the body and $\mathbf{v} = -\bar{\mathbf{F}}\mathbf{U}$

Once the Riemannian manifold $(\mathcal{M}, \mathbf{G})$ has been determined, one can calculate the residual stress field by solving the boundary value problem (no time is involved) with $\mathbf{B} = \mathbf{0}$, and $\mathbf{T} = \mathbf{0}$ on $\partial\mathcal{M}$.

5 Examples of Surface Growth

In this section, we discuss several examples of surface growth. We start with two classes of accretion problems for which one can find some analytical results. We also present some numerical examples of nonlinear accretion.

5.1 Accretion Through a Fixed Surface

Let us assume that the growth surface is subject to a constraint on its position (an essential boundary condition), i.e., at time t the layer Ω_t is mapped to a time-independent surface $\omega \subset \mathcal{S}$, viz.

$$\varphi_t(\Omega_t) = \omega, \quad \forall\, t \in [0, T]. \tag{5.1}$$

This occurs, for example, when new material is generated by some tracking cells located on a rigid substrate (Skalak et al. 1997). An example of such a problem was discussed in Tomassetti et al. (2016), albeit in a different framework. Under this assumption, $\bar{\varphi}$ is not an embedding as it maps the entire three-dimensional manifold $\mathcal{M}$ to the surface ω (Fig. 9).

By virtue of the invariance under different choices of material motions (see Proposition 2.7), one can define a material motion Φ such that the mapping $\varphi_t \circ \Phi_t = \bar{\varphi} \circ \Phi_t$, that was defined in Sect. 2.2, does not depend on time. This means that given $X_0 \in \Omega_0$, all points on the material trajectory $\Phi_t(X_0)$ are mapped to the same point $x \in \omega$, $\forall\, t \in [0, T]$. With respect to such a material motion, the total velocity vanishes, $\mathbf{w} = \mathbf{0}$. Hence, from (2.10), one obtains $\mathbf{v} = -\bar{\mathbf{F}}\mathbf{U}$.

Proposition 5.1 *When the growth surface is fixed, the Riemannian material structure is independent of the history of deformation during accretion.*

Proof Since $\bar{\varphi}$ is given, $\mathbf{Q}$ is determined as $\mathbf{Q}|_{T\Omega} = T\bar{\varphi}|_{T\Omega}$, and of course, $\mathbf{QU} = \mathbf{u}$. Therefore, the material metric $\mathbf{G} = \mathbf{Q}^{\star}\mathbf{g}\mathbf{Q}$ is known before solving the accretion IBVP, and so are all the derived quantities, e.g., the material Christoffel symbols, and curvature tensor. □

Consider the definition of the Lie derivative of the first fundamental form of the layers in (3.30). As $\tilde{\mathbf{G}} = \bar{\varphi}^*(\mathbf{g}|_{T\omega_t})$ (see Remark 3.1), one can write

$$\mathfrak{L}_{\mathbf{U}}\tilde{\mathbf{G}}(X) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=\tau(X)} \left(\bar{\varphi}\circ\Phi_s\circ\Phi^{-1}_{\tau(X)}\right)^* \left\{\mathbf{g}|_{T\omega_t}\left(\left(\bar{\varphi}\circ\Phi_s\circ\Phi^{-1}_{\tau(X)}\right)(X)\right)\right\}. \quad (5.2)$$

Since the map $\bar{\varphi}\circ\Phi_s$ does not depend on s by assumption, one has $\mathfrak{L}_{\mathbf{U}}\tilde{\mathbf{G}} = \mathbf{0}$. In components, $\partial_\tau \tilde{G}_{AB} = 0$. Note that even for such a material motion, in general, $\partial_\tau K_{AB} \neq 0$ (see Remark 3.3).

Example 5.2 (Stationary growth surface with normal growth velocity). Let us assume that the growth surface is stationary and that the growth velocity is everywhere normal to the growth surface, i.e., $\mathbf{u}^{\parallel} = \mathbf{0}$. Therefore, one has $\tilde{\nabla}^{\tilde{\mathbf{g}}}\mathbf{u}^{\parallel} = \mathbf{0}$. Since $\tilde{\nabla}$ is the pull-back connection of $\tilde{\nabla}^{\tilde{\mathbf{g}}}$ through $\mathbf{Q}|_{T\Omega} = \bar{\mathbf{F}}|_{T\Omega}$ (see Remark 3.1) and since $\mathbf{U}^{\parallel} = \mathbf{Q}^{-1}\mathbf{u}^{\parallel}$, one has $\tilde{\nabla}\mathbf{U}^{\parallel} = \mathbf{0}$ as well. Therefore, from (3.28) one obtains $\mathbf{K} = \mathbf{0}$ on every material layer.[15] Note that vanishing of the second fundamental form of the material layers does not imply the flatness of Ω when the material ambient space is not Euclidean (this is clear from the Gauss equations). From (3.20), (3.21), and (3.35) one obtains

$$R_{ABCD} = \tilde{R}_{ABCD}, \quad N^I R_{IABC} = 0, \quad N^I N^J R_{IAJB} = -\frac{1}{U^N}\tilde{\nabla}_A\tilde{\nabla}_B U^N. \quad (5.3)$$

Knowing that $U^N = u^n\circ\varphi$, one has $\tilde{\nabla}_A\tilde{\nabla}_B U^N = \bar{F}^a{}_A\bar{F}^b{}_B\tilde{\nabla}^{\tilde{\mathbf{g}}}{}_a\tilde{\nabla}^{\tilde{\mathbf{g}}}{}_b u^n$ ($\tilde{\nabla}$ is the pull-back connection of $\tilde{\nabla}^{\tilde{\mathbf{g}}}$ using $\bar{\mathbf{F}}$). This means that under the hypothesis of stationary growth surface and uniform tangential growth velocity (or, in particular, pure normal growth velocity), the material manifold of the accreted body is locally flat if and only if ω has zero Gaussian curvature and $\tilde{\nabla}^{\tilde{\mathbf{g}}}{}_a\tilde{\nabla}^{\tilde{\mathbf{g}}}{}_b u^n = 0$.

Example 5.3 (Different growth velocities for a fixed planar growth surface). In this example we compare two problems with growth velocities with the same normal component and different tangential components. The stationary growth surface is an infinite plane. Let us consider a Cartesian coordinate chart (x, y, z) for the ambient space, with the growth surface parametrized by $(x, y, 0)$. We define the following two growth velocities:

[15] In fact, $\tilde{\nabla}\mathbf{U}^{\parallel} = \mathbf{0}$, and so the first two terms of (3.28) vanish. As for the third term, note that since every point on a τ-line is mapped to the same point on ω, the components of $\bar{\mathbf{F}}|_{T\Omega}$ are constant along τ, i.e., $\partial_\tau \bar{F}^i{}_A = 0$. Therefore, $\partial_\tau \tilde{G}_{AB} = \partial_\tau(\bar{F}^i{}_A\bar{F}^j{}_B g_{ij}) = 0$, as the components of the ambient metric on a fixed surface are constant along τ as well.

$$\left[u^i\right]=\begin{bmatrix}0\\0\\-u\end{bmatrix},\quad \left[(u')^i\right]=\begin{bmatrix}-\frac{X}{L}\mathrm{e}^{-\frac{X^2}{L^2}}u\\0\\-u\end{bmatrix},\tag{5.4}$$

where L is some characteristic length. From Example 5.2, we know that $\mathbf{u}$ induces zero curvature tensor. As for the case with $\mathbf{u}'$, we take the lower half space as material manifold, and fix coordinates X, Y and $\tau=-\frac{Z}{u}$. It is straightforward to compute the accretion tensor and the induced material metric:

$$\begin{aligned}\left[(Q')^i{}_J(X)\right]&=\begin{bmatrix}1&0&-\frac{X}{L}\mathrm{e}^{-\frac{X^2}{L^2}}u\\0&1&0\\0&0&-u\end{bmatrix},\\ \left[G'_{IJ}(X)\right](\tau)&=\begin{bmatrix}1&0&-\frac{X}{L}\mathrm{e}^{-\frac{X^2}{L^2}}u\\0&1&0\\-\frac{X}{L}\mathrm{e}^{-\frac{X^2}{L^2}}u&0&u^2+\frac{X^2}{L^2}\mathrm{e}^{-2\frac{X^2}{L^2}}u^2\end{bmatrix},\end{aligned}\tag{5.5}$$

and hence, the inverse material metric and the unit normal vector field read

$$\left[(G')^{IJ}(X)\right]=\begin{bmatrix}1+\frac{X^2}{L^2}\mathrm{e}^{-2\frac{X^2}{L^2}}&0&\frac{X}{L}\mathrm{e}^{-\frac{X^2}{L^2}}\frac{1}{u}\\0&1&0\\\frac{X}{L}\mathrm{e}^{-\frac{X^2}{L^2}}\frac{1}{u}&0&\frac{1}{u^2}\end{bmatrix},\quad \left[(N')^I(X)\right]=\begin{bmatrix}\frac{X}{L}\mathrm{e}^{-\frac{X^2}{L^2}}u\\0\\\frac{1}{u}\end{bmatrix}.\tag{5.6}$$

Thus, one obtains the following layer quantities:

$$\begin{aligned}&\left[\tilde{G}'_{AB}(X)\right]=\begin{bmatrix}1&0\\0&1\end{bmatrix},\left[(U')^A(X)\right]=\begin{bmatrix}-\frac{X}{L}\mathrm{e}^{-\frac{X^2}{L^2}}u\\0\end{bmatrix},\\ &(U')^N(\tau)=u,\quad [K_{AB}(X)]=\begin{bmatrix}\frac{L^2-2X^2}{L^3}\mathrm{e}^{-\frac{X^2}{L^2}}&0\\0&1\end{bmatrix}.\end{aligned}\tag{5.7}$$

Hence, using (3.20), (3.21), and (3.35), the curvature components read

$$\begin{aligned}&R'_{1212}=0,\quad N'^H R'_{HABC}=0,\\ &\left[N'^H N'^K R'_{HAKB}(X)\right]=\begin{bmatrix}-\frac{L^4-10L^2X^2+8X^4}{L^6}\mathrm{e}^{-2\frac{X^2}{L^2}}&0\\0&0\end{bmatrix}.\end{aligned}\tag{5.8}$$

In particular, $N'^H N'^K R'_{H1K1}(0)=-\frac{1}{L^2}$. Non-vanishing curvature components, revealing the presence of residual stress, imply that the two descriptions $\mathbf{u}$ and $\mathbf{u}'$ are not equivalent. This example shows that it is necessary to take into account the tangential component of growth velocity if the accretion process requires it. In other words, it is not possible to reduce a general accretion problem to a normal flux of mass.

5.2 Accretion Through a Traction-Free Surface

Let us now assume that the growth surface is traction-free. For this class of problems, we have the following result.

Proposition 5.4 *Let the energy function W be rank-one convex.*[16] *If the growth surface is traction free, then $\bar{\mathbf{F}} = \mathbf{Q}$. Moreover, on the growth surface the growth velocity $\mathbf{u}$ is tangent to the φ_t images of the material trajectories of Φ, and $\mathbf{w} = \mathbf{u} + \mathbf{v}$.*

Proof The two tensors $\bar{\mathbf{F}}$ and $\mathbf{Q}$ already have the same in-layer component, i.e., $\bar{\mathbf{F}}|_{T\Omega} = \mathbf{Q}|_{T\Omega}$. Hence, one only needs to determine the component $\bar{\mathbf{F}}\mathbf{U}$.[17] Note that $\mathbf{Q}$ and $\bar{\mathbf{F}}$ are rank-one connected as $\bar{\mathbf{F}} - \mathbf{Q} = (\bar{\mathbf{F}}\mathbf{U} - \mathbf{u}) \otimes d\tau = \mathbf{z} \otimes d\tau$, where $\mathbf{z} = \bar{\mathbf{F}}\mathbf{U} - \mathbf{u}$. Let us define $\mathbf{F}(s) = (1-s)\mathbf{Q} + s\,\bar{\mathbf{F}} = \mathbf{Q} + s\,\mathbf{z} \otimes d\tau$, where $\mathbf{Q} = \mathbf{F}(0)$ and $\bar{\mathbf{F}} = \mathbf{F}(1)$. Let us now define the function $\hat{\boldsymbol{\mathcal{P}}}(s) = \boldsymbol{\mathcal{P}}(\mathbf{F}(s))$ that evaluates the constitutive part of the first Piola–Kirchhoff stress tensor along the curve $\mathbf{F}(s)$. Then one can write

$$
\begin{aligned}
W(\bar{\mathbf{F}}) &= W(\mathbf{Q}) + \int_0^1 \frac{dW(\mathbf{F}(s))}{ds}\, ds = W(\mathbf{Q}) + \int_0^1 \hat{\mathcal{P}}_i{}^J(s) \frac{dF^i{}_J}{ds}\, ds \\
&= W(\mathbf{Q}) + \int_0^1 \hat{\mathcal{P}}_i{}^3(s) z^i\, ds.
\end{aligned} \tag{5.9}
$$

Rank-one convexity of W implies that the integrand $f(s) = \hat{\mathcal{P}}_i{}^3(s) z^i$ is monotone increasing. Note that

$$
f(s) = \hat{\mathcal{P}}_i{}^3(s)\, z^i = \hat{\mathcal{P}}_i{}^{\check{J}}(s) A^3{}_{\check{J}}\, z^i = \frac{1}{U^N} \hat{\mathcal{P}}_i{}^N(s)\, z^i, \tag{5.10}
$$

where use was made of the change of frame (3.3). Now one has to make a distinction between the compressible and the incompressible cases. In the compressible case, the hypothesis of traction-free growth surface implies $\hat{\mathcal{P}}_i{}^N(1) = 0$, and so $f(1) = 0$. In

[16] An energy function at a point X is rank-one convex if

$$
W(X, \mathbf{F} + s\mathbf{z} \otimes \boldsymbol{\zeta}) \leq W(X, \mathbf{F}) + sW(X, \mathbf{F} + \mathbf{z} \otimes \boldsymbol{\zeta}),
$$

for any $\mathbf{F}$, any spatial vector $\mathbf{z} \in T_X\mathcal{S}$, any linear form $\boldsymbol{\zeta} \in T_X^*\mathcal{B}$, and $0 \leq s \leq 1$. One can show that rank-one convexity is equivalent to the monotonicity of the function $f(s) = \boldsymbol{\mathcal{P}}(X, \mathbf{F} + s\mathbf{z} \otimes \boldsymbol{\zeta}) : (\mathbf{z} \otimes \boldsymbol{\zeta})$. Rank-one convexity is a necessary condition for quasiconvexity (and polyconvexity), which is related to the existence of minimizers of the total energy functional in hyperelasticity (Ball 1976).

[17] Note that by hypothesis $\mathbf{PN} = \mathbf{0}$, which in the compressible case reads $\boldsymbol{\mathcal{P}}(\mathbf{F})\mathbf{N} = \mathbf{0}$, and provides three scalar equations for the three unknowns represented by the components of $\bar{\mathbf{F}}\mathbf{U}$. A solution is clearly given by $\bar{\mathbf{F}}\mathbf{U} = \mathbf{u}$, which means that $\bar{\mathbf{F}} = \mathbf{Q}$ as the two tensors already agree on the layer. However, this does not need to be the only solution because of the nonlinearity of the equations. In the incompressible case, things are slightly more complicated due to the presence of an unknown pressure field p that changes the zero-traction condition to $\mathbf{PN} = \boldsymbol{\mathcal{P}}(\mathbf{F})\mathbf{N} + p(\mathbf{F}^{-1}\mathbf{N})^\sharp = \mathbf{0}$. We use an energetic argument and rank-one convexity of the energy function to show that $\bar{\mathbf{F}} = \mathbf{Q}$ is the only solution.

the incompressible case, the zero-traction condition does not apply to the integrand $f(s)$. However, one can write $f(s) = g(s) + h(s)$, where $g(s) = P_i{}^3(s)\,z^i$, and $h(s) = -p(F^{-1})^3{}_i(s)\,z^i$. The zero-traction condition is written as $g(1) = 0$, which implies that $f(1) = h(1)$. Note that $(F^{-1})^3{}_i(1) = (\bar{F}^{-1})^3{}_i$, and hence, one has

$$f(1) = -p\left(\bar{F}^{-1}\right)^3{}_i z^i = -p\left(\bar{F}^{-1}\right)^3{}_i\left(\bar{F}^i{}_3 - Q^i{}_3\right) = p\left[\left(\bar{F}^{-1}\right)^3{}_i Q^i{}_3 - 1\right]. \tag{5.11}$$

On the other hand, from incompressibility $\det\bar{\mathbf{F}}/\det\mathbf{Q} = 1$, or equivalently, $\det(\bar{\mathbf{F}}^{-1}\mathbf{Q}) = 1$. From (4.3) one obtains $(\bar{F}^{-1})^3{}_i Q^i{}_3 = \det(\bar{\mathbf{F}}^{-1}\mathbf{Q}) = 1$, and hence, $f(1) = 0$. Therefore, in both the compressible and incompressible cases one concludes that $f(s) \leq 0$ for any $0 \leq s \leq 1$, by virtue of the monotonicity of f. Thus, the integrand in (5.9) is non-positive and this means that $W(\bar{\mathbf{F}}) \leq W(\mathbf{Q})$. Note that since the material metric has been built such that $\mathbf{Q}$ is an isometry,[18] W attains its minimum only at $\mathbf{Q}$ and at those $\mathbf{F}$'s that are equal to $\mathbf{Q}$ up to rotations of the ambient space. Therefore, $W(\bar{\mathbf{F}}) = W(\mathbf{Q})$, and $\bar{\mathbf{F}} = \mathbf{R}\mathbf{Q}$, where $\mathbf{R}$ is a rotation of the ambient space. But since $\bar{\mathbf{F}}|_{T\Omega} = \mathbf{Q}|_{T\Omega}$, one concludes that $\bar{\mathbf{F}} = \mathbf{Q}$.

Finally note that at any time t the material trajectories are mapped by φ_t to curves with tangent vector $\mathbf{F}_t\mathbf{U}$. On the growth surface their tangent vector is simply $\bar{\mathbf{F}}\mathbf{U}$, that we showed is equal to $\mathbf{u}$. Moreover, by virtue of (2.10), when $\bar{\mathbf{F}} = \mathbf{Q}$ we observe that the total velocity can now be written as $\mathbf{w} = \mathbf{u} + \mathbf{v}$, which is sometimes called "accretion law" (Metlov 1985). This means that the accretion surface ω_t moves with velocity $\mathbf{u} + \mathbf{v}$, with $\mathbf{v}$ being the "standard" spatial velocity. □

Remark 5.5 In both the compressible and incompressible cases $\bar{\mathbf{F}} = \mathbf{Q}$ implies that the deformation gradient on the growth surface is an isometry. However, while in the compressible case, one can conclude that the growth surface is stress-free, in the incompressible case one has $\mathbf{P} = p\mathbf{g}^\sharp\mathbf{Q}^{-\star}$, which in general does not vanish.

One may ask if the accretion model is invariant with respect to different spatial growth velocities in some equivalence class. It turns out that for problems with traction-free growth surface the answer is yes. But first one needs to define equivalent growth velocities. If the growth surface has a boundary, we indicate with $\mathbf{t}_t$ a vector tangent to the curve $\partial\omega_t$. Two spatial growth velocities $\mathbf{u}$ and $\mathbf{u}'$ are said to be equivalent if

(i) $\langle \mathrm{d}(\tau\circ{\varphi_t}^{-1}), \mathbf{u}_t\rangle = \langle \mathrm{d}(\tau\circ{\varphi_t}^{-1}), \mathbf{u}'{}_t\rangle$, $\forall t\in[0,T]$;
(ii) If $\partial\omega_t \neq \emptyset$, then $\forall t$ the vectors $\mathbf{u}_t, \mathbf{u}'_t, \mathbf{t}_t$ on $\partial\omega_t$ are linearly dependent.

Note that the map $\tau\circ{\varphi_t}^{-1}$ defines a foliation on $\varphi_t(\mathcal{B}_t)$, and hence, property (i) implies that the two velocities must have the same out-of-layer component in order to be equivalent. Note that this component does not depend on the map φ_t; it is the same for any map φ'_t that preserves the foliation, viz.

[18] Equivalently, one can say that the material metric is the right Cauchy–Green strain tensor associated with the accretion tensor $\mathbf{Q}$ (see Remark 2.11).

$$\varphi_t(\Omega_t) = \varphi'_t(\Omega_t) \quad \Longrightarrow \quad \mathrm{d}(\tau \circ \varphi_t{}^{-1}) = \mathrm{d}(\tau \circ \varphi_t'^{-1}). \tag{5.12}$$

This property can also be expressed as $\mathbf{g}(\mathbf{u}_t, \mathbf{n}_t) = \mathbf{g}(\mathbf{u}'{}_t, \mathbf{n}_t)$. Property ii) implies that $\mathbf{u}'_t$ must lie in the plane spanned by $\mathbf{u}'_t$ and $\mathbf{t}_t$. It can also be expressed as $\mathbf{g}(\mathbf{t}_t \times \mathbf{u}_t, \mathbf{u}'_t) = 0$. In some cases, $\mathbf{u}$ admits an equivalent velocity $\mathbf{u}'$ that is normal to the growth surface. This is always the case when ω_t is boundaryless, e.g., when $\omega_t = \partial\varphi_t(\mathcal{B}_t)$, or when $\mathbf{u}$ is normal to ω_t on $\partial\omega_t$.

When a body grows without undergoing any deformation, one trivially has $\mathbf{Q} = \bar{\mathbf{F}}$, and $\mathbf{v} = \mathbf{0}$. Therefore, the growth surface moves with velocity $\mathbf{w} = \bar{\mathbf{F}}\mathbf{U} = \mathbf{Q}\mathbf{U} = \mathbf{u}$, i.e., only due to the addition of new material. In this case the description of the accretion process is not altered if one replaces the prescribed growth velocity $\mathbf{u}$ with an equivalent growth velocity $\mathbf{u}'$. Accretion with no deformation is a very special case, so now one may ask the following question: Is an accretion problem affected by the choice of equivalent spatial growth velocities? The answer is yes, as was shown in Example 5.3. Nevertheless, there are some cases in which two equivalent growth velocities give the same solution for the accretion IBVP. To this extent, the next result tells us that under the hypothesis $\mathbf{u} = \bar{\mathbf{F}}\mathbf{U}$ one can work with equivalent growth velocities. By virtue of Proposition 5.4, this means that when accretion occurs through a traction-free surface, equivalent growth velocities can be used interchangeably, and one can work with a normal growth velocity, provided that it is equivalent to the prescribed one.

Proposition 5.6 *When $\bar{\mathbf{F}} = \mathbf{Q}$, the accretion boundary value problem is invariant under different choices of equivalent spatial growth velocities.*

Proof Let $\mathbf{u}'$ be a spatial growth velocity that is equivalent to $\mathbf{u}$. Take a motion φ' such that

$$\text{(i) } \varphi_s(\Omega_t) = \varphi'_s(\Omega_t)\ \forall t \leq s\,, \qquad \text{(ii) } \bar{\mathbf{F}}'\mathbf{U} = \mathbf{u}'\,. \tag{5.13}$$

As a matter of fact, one can always choose a map φ' that satisfies (i). Note that for any time t the mappings φ_t and φ'_t define the same foliation on the deformed configuration. This implies that the two vectors $\bar{\mathbf{F}}\mathbf{U} = \mathbf{u}$ and $\bar{\mathbf{F}}'\mathbf{U}$ are equivalent because

$$\begin{aligned}\left\langle \mathrm{d}\left(\tau \circ \varphi_t^{-1}\right), \bar{\mathbf{F}}\mathbf{U}_t \right\rangle &= \left\langle \mathrm{d}(\tau \circ \varphi_t^{-1}), \mathbf{F}_t\mathbf{U}_t \right\rangle = \langle \mathrm{d}\tau, \mathbf{U}\rangle = \left\langle \mathrm{d}(\tau \circ \varphi_t'^{-1}), \mathbf{F}'_t\mathbf{U}_t \right\rangle \\ &= \left\langle \mathrm{d}\left(\tau \circ \varphi_t'^{-1}\right), \bar{\mathbf{F}}'\mathbf{U}_t \right\rangle,\end{aligned} \tag{5.14}$$

while on $\partial\omega_t$ the vectors $\bar{\mathbf{F}}\mathbf{U}$ and $\bar{\mathbf{F}}'\mathbf{U}$ necessarily differ by a vector tangent to $\partial\omega_t$. Therefore, $\bar{\mathbf{F}}'\mathbf{U}$ is equivalent to $\mathbf{u}$. Then one simply chooses φ' such that (ii) holds. The tensor $\mathbf{Q}'$ associated with the motion φ' is given by

$$\mathbf{Q}'(X) = \bar{\mathbf{F}}'(X) + [\mathbf{u}(\varphi'(X)) - \bar{\mathbf{F}}'(X)\mathbf{U}(X)] \otimes \mathrm{d}\tau. \tag{5.15}$$

Therefore, one has $\mathbf{u}(\varphi'(X)) = \bar{\mathbf{F}}'(X)\mathbf{U}(X)$. Next define the map

$$\begin{aligned}\lambda : \mathcal{M} &\to \mathcal{M} \\ X &\mapsto \left(\varphi'_{\tau(X)}\right)^{-1}\left(\varphi_{\tau(X)}(X)\right),\end{aligned} \tag{5.16}$$

which is well defined because of property (i). Note that if $\bar{\varphi}$ is invertible, one can write $\lambda = (\bar{\varphi}')^{-1} \circ \bar{\varphi}$, satisfying $\lambda(\Omega_t) = \Omega_t$. Since both $\bar{\varphi}$ and $\bar{\varphi}'$ are invertible on each layer, λ is invertible. Let us define $\mathbf{G}'$ to be the material metric associated with the motion φ'. We claim that $\mathbf{G}' = \lambda^* \mathbf{G}$. To show this we directly compute the tangent map $\boldsymbol{\Lambda} = T\lambda$:

$$
\begin{aligned}
\Lambda^I{}_J = \frac{\partial \lambda^I}{\partial X^J} &= \left[\frac{\partial \left(\varphi'^{-1}\right)^I}{\partial x^k} \left(\frac{\partial \varphi^k}{\partial X^J} + \frac{\partial \varphi^k}{\partial t} \frac{\partial \tau}{\partial X^J} \right) + \frac{\partial \left(\varphi'^{-1}\right)^I}{\partial t} \frac{\partial \tau}{\partial X^J} \right]_{t=\tau(X)} \\
&= \left(\bar{F}'^{-1}\right)^I{}_k \left(\bar{F}^k{}_J + v^k \frac{\partial \tau}{\partial X^J} \right) - \left(\bar{F}'^{-1}\right)^I{}_k v^k \frac{\partial \tau}{\partial X^J} \\
&= \left(\bar{F}'^{-1}\right)^I{}_k \bar{F}^k{}_J .
\end{aligned}
\tag{5.17}
$$

Therefore, one obtains $\bar{\mathbf{F}}(X) = \bar{\mathbf{F}}'(\lambda(X))\boldsymbol{\Lambda}(X)$. By hypothesis and property (ii) of (5.13), the material metrics are written as

$$
\mathbf{G}(X) = \bar{\mathbf{F}}(X)^\star \mathbf{g}(\varphi(X))\bar{\mathbf{F}}(X), \mathbf{G}'(\lambda(X)) = \bar{\mathbf{F}}'(\lambda(X))^\star \mathbf{g}(\varphi'(\lambda(X)))\bar{\mathbf{F}}'(\lambda(X)), \tag{5.18}
$$

and hence

$$
\begin{aligned}
\mathbf{G}(X) &= \boldsymbol{\Lambda}(X)^\star\, \bar{\mathbf{F}}'(\lambda(X))^\star\, \mathbf{g}(\varphi(\lambda(X)))\, \bar{\mathbf{F}}'(\lambda(X))\, \boldsymbol{\Lambda}(X) \\
&= \boldsymbol{\Lambda}(X)^\star\, \mathbf{G}'(\lambda(X))\, \boldsymbol{\Lambda}(X).
\end{aligned}
\tag{5.19}
$$

This means that $\mathbf{G} = \lambda^* \mathbf{G}'$, i.e., the two Riemannian manifolds are isometric. By the same argument used in the proof of Proposition 2.7, if φ is a solution for $(\mathcal{M}, \mathbf{G})$, then φ' will be a solution for $(\mathcal{M}, \mathbf{G}')$. $\square$

Example 5.7 (Radial growth of a cylinder). We now demonstrate how the accretion theory developed so far can be used in a simple example of accretion through a traction-free surface, namely the radial outer accretion of an infinitely long cylinder (see Fig. 10). The simplicity of this example is due to its symmetry, which makes it a one-dimensional problem. Following the procedure explained in Sect. 2, first one needs to define a material manifold with a material foliation $(\mathcal{M}, \tau)$. Let us consider cylindrical coordinates (Θ, Z, R), $R \geq R_0$, on $\mathbb{R}^3$. We define foliation coordinates (Θ, Z, τ) with $\tau = \tau(R)$, which is required to be invertible because of property (iii) of Sect. 2.1, such that its inverse $R(\tau)$ satisfies $\frac{\partial R}{\partial t}(t) = u(t)$, i.e., $R(t) = R_0 + \int_0^t u(\zeta)\mathrm{d}\zeta$.[19] Let (θ, z, r) be the cylindrical coordinates on the ambient space. We consider only those configurations for which $(\Theta, Z, \tau) \mapsto (\Theta, Z, r(\tau, t))$, i.e., those that preserve the axial symmetry. Therefore, one obtains the following tensors:

[19] This can be pictured as a reference cylinder that grows at the same rate as prescribed by the spatial growth velocity $u(t)$. This also represents the configuration of the body in the case where no deformation occurs during the accretion process.

$$\left[F^i{}_J(\tau,t)\right] = \operatorname{diag}\left(1, 1, \frac{\partial r}{\partial \tau}(\tau,t)\right), \quad \left[Q^i{}_J(\tau)\right] = \operatorname{diag}\left(1, 1, u(\tau)\right), \tag{5.20}$$

and so the metric, its inverse, and the unit normal field read

$$[G_{IJ}(\tau)] = \operatorname{diag}\left(\bar{r}^2(\tau), 1, u^2(\tau)\right), \quad \left[G^{IJ}(\tau)\right] = \operatorname{diag}\left(\frac{1}{\bar{r}^2(\tau)}, 1, \frac{1}{u^2(\tau)}\right),$$
$$\left[N^I(\tau)\right] = \left[0\ 0\ \tfrac{1}{u(\tau)}\right]. \tag{5.21}$$

With these one can calculate all the quantities involved in the accretion IBVP in terms of the unknown $\bar{r}(\tau)$ and the input $u(\tau)$ (see also Sozio and Yavari 2017), where we solve the IBVP for accretion problems with cylindrical and spherical symmetry). We next focus on the layer-wise geometry of the material body described in Sect. 3, and obtain the following objects:

$$\left[\tilde{G}_{AB}(\tau)\right] = \operatorname{diag}\left(\bar{r}^2(\tau), 1\right), \quad \left[\tilde{G}^{AB}(\tau)\right] = \operatorname{diag}\left(\frac{1}{\bar{r}^2(\tau)}, 1\right),$$
$$\left[U^A(\tau)\right] = \mathbf{0}, \quad U^N(\tau) = u(\tau). \tag{5.22}$$

The material second fundamental form is calculated using (3.28) and reads

$$[K_{AB}(\tau)] = -\frac{1}{2u(\tau)}\left[\partial_\tau \tilde{G}_{AB}(\tau)\right] = \operatorname{diag}\left(-\frac{1}{u(\tau)}\bar{r}(\tau)\bar{r}'(\tau), 0\right). \tag{5.23}$$

Note that all the layer Christoffel symbols $\tilde{\Gamma}^A{}_{BC}$ are zero because $\tilde{\mathbf{G}}$ does not depend on the layer coordinates (Θ, Z), and hence, $\tilde{R}_{ABCD} = 0$. Therefore, the tangential component of the curvature tensor vanishes, because

$$R_{1212} = \tilde{R}_{1212} + K_{12}K_{21} - K_{11}K_{22} = 0. \tag{5.24}$$

Moreover, from (3.21) one has

$$\frac{1}{u}R_{3ABC} = \tilde{\nabla}_D K_{BC} - \tilde{\nabla}_C K_{BD} = 0. \tag{5.25}$$

The only non-vanishing component is given by (3.35) and reads

$$\frac{1}{u^2}R_{3232} = \frac{1}{u}\partial_\tau K_{22} + K_{2C}K^C{}_2 = -\frac{\bar{r}}{u^3}\left(u'\bar{r}' - u\bar{r}''\right). \tag{5.26}$$

This means that the accreted solid is Euclidean if and only if $u'\bar{r}' - u\bar{r}'' = 0$, i.e., if and only if

$$\bar{r}(\tau) = r_0 + w_0 \int_0^\tau e^{\int_0^\zeta \frac{u'(\sigma)}{u(\sigma)} d\sigma} d\zeta = r_0 + \frac{w_0}{u_0}\int_0^\tau u(\xi)\, d\xi, \tag{5.27}$$

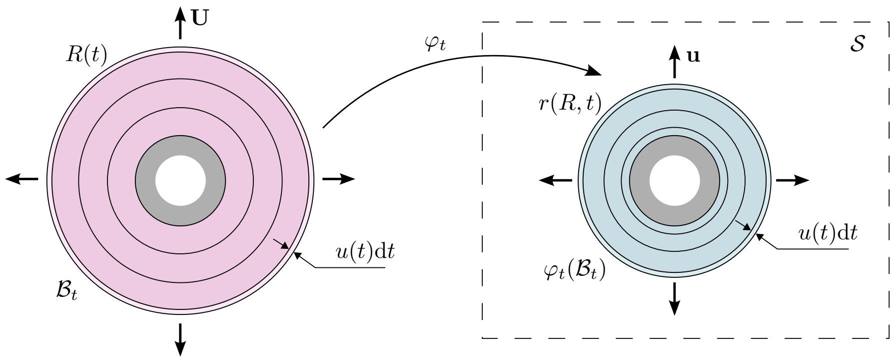

Fig. 10 Radial growth of an infinitely long cylinder. Left: the material manifold at time t. It is chosen such that the external radius grows at the rate u, the growth velocity in the physical space. Right: the deformed configuration at time t

where the integration constants are $r_0 = \bar{r}(0)$ and $w_0 = \bar{r}'(0)$, while $u_0 = u(0)$.[20] In the case of constant growth velocity $u(t) = u_0$, (5.27) is simplified to read $\bar{r}(\tau) = r_0 + w_0\tau$, and the accreted solid is Euclidean if and only if $\bar{r}$ is linear in time. This can be achieved when there is no deformation during the growth process, resulting in $\bar{r}(\tau) = r_0 + u_0\tau$. Note finally that in the case of non-simply connected bodies like this one, the vanishing of the curvature tensor alone does not guarantee the absence of residual stresses; additional conditions need to be satisfied (Yavari 2013).

5.3 Numerical Examples

Next we consider some examples of both fixed and traction-free growth surfaces, where we numerically solve the accretion initial boundary value problem. In the case of traction-free growth surface, one needs to solve the IBVP in order to determine the Riemannian material structure of the accreted body. We will also calculate the residual deformations and stresses when all the loads are released after completion of accretion. For solving the numerical examples in this section, we use a finite difference scheme that has recently been developed for accretion problems (Sozio et al. 2019). See also Sozio and Yavari (2017), where accretion problems with axial and spherical symmetry were numerically solved.

We restrict our calculations to neo-Hookean solids. Note that, as was mentioned in Sect. 4.3, the strain tensor $\mathbf{C}$ explicitly depends on the material metric $\mathbf{G}$, which for

[20] If the growth surface is traction-free, from Proposition 5.4 one obtains $\bar{\mathbf{F}} = \mathbf{Q}$, i.e., $\frac{\partial r}{\partial \tau}(\tau,\tau) = u(\tau)$. Then, one has

$$w(\tau) = \bar{r}'(\tau) = \frac{\partial r}{\partial \tau}(\tau,\tau) + \frac{\partial r}{\partial t}(\tau,\tau) = u(\tau) + v(\tau),$$

and so one recovers $\mathbf{w} = \mathbf{u} + \mathbf{v}$. In Sozio and Yavari (2017), we assumed $\frac{\partial r}{\partial \tau}(\tau,\tau) = u(\tau)$ a priori, which is now justified by Proposition 5.4 as we were dealing with a traction-free growth surface. However, we should emphasize that, in general, this does not hold when traction does not vanish on the growth surface.

accretion problems is an unknown that is calculated after solving the accretion IBVP. In particular, for the compressible case we choose the following energy function

$$\mathcal{W}(\mathbf{C}) = \frac{\mu}{2}\left[\operatorname{tr}\mathbf{C} - 3 - \ln(\det\mathbf{C})\right] + \frac{\lambda}{2}\left(\sqrt{\det\mathbf{C}} - 1\right)^2, \tag{5.28}$$

where λ and μ are two material parameters. Note that $\det\mathbf{C} = J^2$. From (4.12) one obtains the following constitutive equation for the first Piola–Kirchhoff stress tensor:

$$P^{iJ} = \mu\left[G^{IH}F^i{}_H - g^{ih}\left(F^{-1}\right)^J{}_h\right] + \lambda J\,(J-1)\,g^{ih}\left(F^{-1}\right)^J{}_h. \tag{5.29}$$

In the incompressible case, the energy function has the form $\mathcal{W}(\mathbf{C}) = \frac{\mu}{2}\left[\operatorname{tr}\mathbf{C} - 3\right]$, and the first Piola–Kirchhoff stress tensor then reads

$$P^{iJ} = \mu G^{IH}F^i{}_H + pJg^{ih}\left(F^{-1}\right)^J{}_h, \tag{5.30}$$

where p is the Lagrange multiplier associated with the internal incompressibility constraint $J = 1$.[21] For two-dimensional elasticity, all the equations for both compressible and incompressible solids are the same except for the term $-\frac{3}{2}\mu$ in the energy function (5.28), that becomes $-\mu$ in order to have a vanishing energy density for $\mathbf{C} = \mathbf{I}$. It should be emphasized that the Cauchy stress explicitly depends on the material metric $\mathbf{G}$. It should also be emphasized that we assume that the residually stressed body is isotropic in its stress-free state; in its current configuration the body may not be isotropic.

We work with dimensionless quantities normalized with respect to some length parameter L, the accretion time T, and the shear modulus μ, whose values are problem dependent. All the contour plots show the value of $\operatorname{tr}\mathbf{C}$. The lowest values are in purple, while the highest ones are in yellow.

Example 5.8 (Two-dimensional accretion through a fixed curve). In this example we consider the accretion of a two-dimensional body through a fixed curve (see Sect. 5.1). The body undergoes deformation and stress not because of the presence of external forces, but due to the geometry of accretion constraints. The growth curve ω is a half circle of unit radius, represented in Cartesian coordinates as the lower half of the circle $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = L,\ y \le 0\}$. The material ambient space $\mathcal{M}$ is a rectangle on which one defines material coordinates (X, τ) that map it to the unit square $[-\frac{L}{2}, \frac{L}{2}] \times [0, 1]$. The material growth surface Ω_t is represented by the line $\left[-\frac{L}{2}, \frac{L}{2}\right] \times \{t\}$. Since this is a fixed growth surface problem, one is able to calculate the material metric before solving the accretion IBVP. The map $\bar{\varphi}$ sends Ω_t to ω and can be represented by the time-independent pair $(x(X), y(X))$ given by

[21] The coordinate-free constitutive expression for the $\binom{1}{1}$ variant $\mathbf{P}^\flat$ of $\mathbf{P}$ with components $P_i{}^J = g_{ij}P^{jJ}$, is written as

$$\mathbf{P}^\flat = \mu\left(\mathbf{F}^\top - \mathbf{F}^{-1}\right) + \lambda J(J-1)\mathbf{F}^{-1}, \quad \mathbf{P}^\flat = \mu\mathbf{F}^\top + p\mathbf{F}^{-1},$$

for the compressible and the incompressible cases, respectively.

$$x(X) = L \sin \frac{\pi X}{L}, \quad y(X) = -L \cos \frac{\pi X}{L}. \tag{5.31}$$

Note that X can be seen as the angular coordinate on the half circle scaled by the factor $\frac{L}{\pi}$. We assume that the growth velocity, with components (v, w), is always normal, although not uniform, viz.

$$v(X) = -u(X) \sin \frac{\pi X}{L}, \quad w(X) = u(X) \cos \frac{\pi X}{L}, \tag{5.32}$$

with $u(X)$ being the norm of the growth velocity. This allows one to write

$$\begin{aligned} \left[Q^i{}_J(X)\right] &= \begin{bmatrix} \frac{\partial x}{\partial X} & v(X) \\ \frac{\partial y}{\partial X} & w(X) \end{bmatrix} = \begin{bmatrix} \pi \cos \frac{\pi X}{L} & -u(X) \sin \frac{\pi X}{L} \\ \pi \sin \frac{\pi X}{L} & u(X) \cos \frac{\pi X}{L} \end{bmatrix}, \\ [G_{IJ}(X)] &= \begin{bmatrix} \pi^2 & 0 \\ 0 & u^2(X) \end{bmatrix}. \end{aligned} \tag{5.33}$$

Note that the material Gaussian curvature reads $g(X) = -u''(X)/u(X)$, so the accreted body is Euclidean for growth velocities that are linear in X or in the angle on the circle. As the body is simply connected, the vanishing of the curvature implies the absence of residual stress in the relaxed configuration. In the case shown in Fig. 11, the material is incompressible neo-Hookean. The growth velocity is assumed to be $u(X) = 1.25L/T$. Since it is constant, the accreted body is Euclidean and the relaxed configuration is simply given by $[-\frac{\pi L}{2}, \frac{\pi L}{2}] \times [0, uT]$. The total area at the end of accretion is therefore $\pi u L T$.

Example 5.9 (Two-dimensional accretion through a fixed straight line). Another example is given by the accretion of a two-dimensional body through a fixed growth surface represented by a straight line with a general growth velocity. The fixed growth surface ω is represented by $[-\frac{L}{2}, \frac{L}{2}] \times \{0\}$. The material ambient space $\mathcal{M}$ is a rectangle on which one defines material coordinates (X, τ) in the set $[-\frac{L}{2}, \frac{L}{2}] \times [0, T]$. The map $\varphi_t|_{\Omega_t}$ can be represented by the identity on the interval $[-\frac{L}{2}, \frac{L}{2}]$. Therefore, indicating with (v, w) the components of the growth velocity, the accretion and material metric tensors are written as

$$\left[Q^i{}_J(X)\right] = \begin{bmatrix} 1 & v(X) \\ 0 & w(X) \end{bmatrix}, \quad [G_{IJ}(X)] = \begin{bmatrix} 1 & v(X) \\ v(X) & v(X)^2 + w(X)^2 \end{bmatrix}. \tag{5.34}$$

The Gaussian curvature reads

$$g(X) = \frac{v(X)v'(X)w'(X)}{w(X)^3} - \frac{v'(X)^2 + v(X)v''(X)}{w(X)^2} - \frac{w''(X)}{w(X)}. \tag{5.35}$$

When the growth velocity is normal to ω, i.e., when $v(X) = 0$, one has $g(X) = -w''(X)/w(X)$.

We consider two cases, one with a non-uniform normal growth velocity, and one with a growth velocity with a uniform normal component and a non-uniform tangential

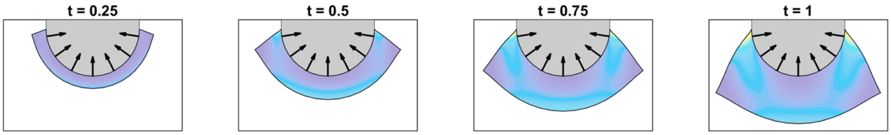

Fig. 11 Growth through a fixed half circle: The deformed configuration during accretion at four different times. The growth velocity is $u(X) = 1.0\frac{L}{T}$, and hence, the accreted body is Euclidean and the relaxed configuration is given by $[-\frac{\pi}{4}, \frac{\pi}{4}] \times [0, u]$. The material is incompressible neo-Hookean

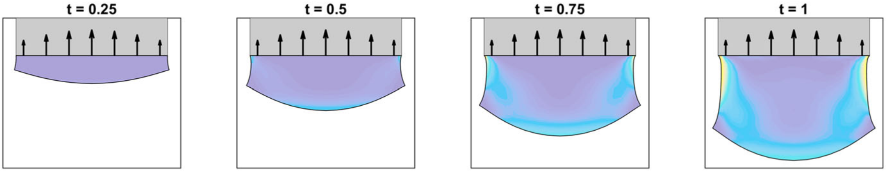

Fig. 12 Growth through a fixed straight line: The deformed configuration during accretion at four different times

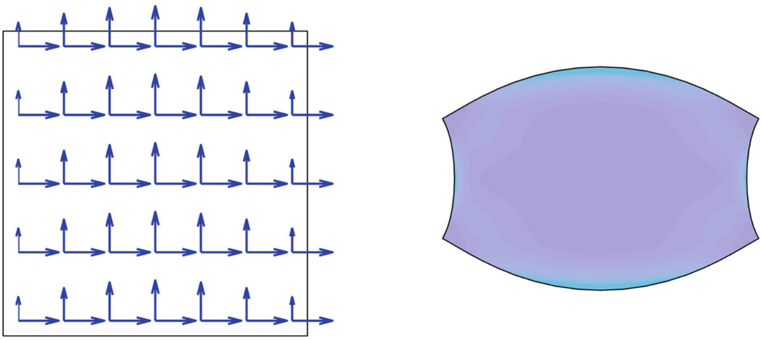

Fig. 13 Growth through a fixed straight line with a non-uniform normal growth velocity. Left: the material manifold with the coordinate frames $\{\partial/\partial X, \partial/\partial\tau\}$ distorted by $\mathbf{Q}$. Right: the residually stressed configuration (the lighter colors correspond to larger values of stress)

component (a two-dimensional simplified version of Example 5.3). In both cases, the material is incompressible neo-Hookean. The first case is shown in Fig. 12, where we consider $v(X) = 0$, $w(X) = a\cos\frac{X}{b}$, with the requirement $b > \frac{L}{\pi}$ as w has to be strictly positive. In particular, we choose $a = \frac{3}{4}\frac{L}{T}$, $b = \frac{3}{2\pi}L$, so that $\frac{3}{8}\frac{L}{T} < w(X) < \frac{3}{4}\frac{L}{T}$. Thus, one finds

$$[G_{IJ}(X)] = \begin{bmatrix} 1 & 0 \\ 0 & a^2\cos^2\frac{X}{b} \end{bmatrix}, \quad g(X) = \frac{1}{b^2}. \tag{5.36}$$

This means that the accreted body is isomorphic to a portion of a sphere of radius b. Therefore, the accreted body is not Euclidean and once the accreted body is released (no body forces and tractions) it will be residually stressed (see Fig. 13). The total area at the end of accretion is $0.620L^2$. The second case is shown in Fig. 14, and we consider

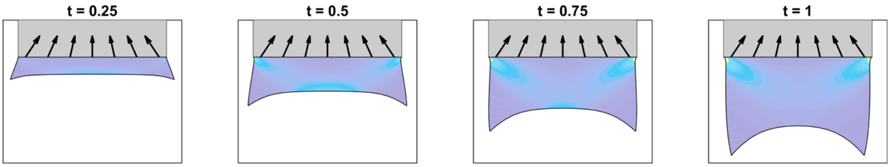

Fig. 14 Growth through a fixed straight line: the deformed configuration during accretion at four different times

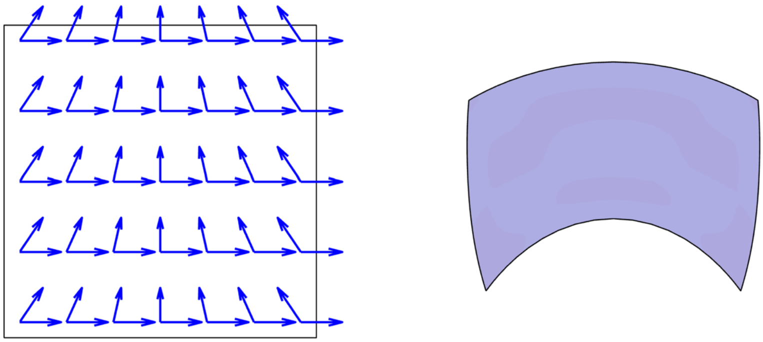

Fig. 15 Growth through a fixed straight line with a non-uniform normal growth velocity. Left: the material manifold with the coordinate frames $\{\partial/\partial X, \partial/\partial\tau\}$ distorted by $\mathbf{Q}$. Right: the residually stressed configuration (the lighter colors correspond to larger values of stress)

$v(X) = -\frac{aX}{b}$, $w(X) = a$. We choose $a = \frac{1}{2}\frac{L}{T}$, $b = \frac{2}{3}L$, and so $0 < v(X) < \frac{3}{8}\frac{L}{T}$. Hence

$$[G_{IJ}(X)] = \begin{bmatrix} 1 & -\frac{a}{b}X \\ -\frac{a}{b}X & \frac{a^2}{b^2}X^2 + a^2 \end{bmatrix}, \quad g(X) = -\frac{1}{b^2}. \tag{5.37}$$

This means that the accreted body is isomorphic to a portion of a pseudosphere of radius b. Since the Gaussian curvature is nonzero, the accreted body will be residually stressed (see Fig. 15). The total area at the end of accretion is $\frac{L^2}{2}$.

Example 5.10 (Two-dimensional accretion through a traction-free growth curve) In this example we consider the accretion of a two-dimensional body through a traction-free surface. We are interested in the effects of external loads on the material characteristics of the accreted body. We consider a body which is the result of a vertical deposition process. We assume that the growth surface at time $t = 0$ is given by $[-\frac{L}{2}, \frac{L}{2}] \times \{0\}$. The growth velocity is defined on either the lower or upper boundary of the body ω_t and is vertical and constant in time. We assume that throughout the accretion process the configuration of the accretion surface ω_t is given by points $(x, f(x,t))$ for some time-dependent function f so that the vertical addition of material on the entire upper boundary will always be possible. This allows us to take a simply connected material manifold, for example the rectangle $[-\frac{L}{2}, \frac{L}{2}] \times [0, t]$. The accreting body is subject to Dirichlet boundary conditions on either its upper or lower boundary.

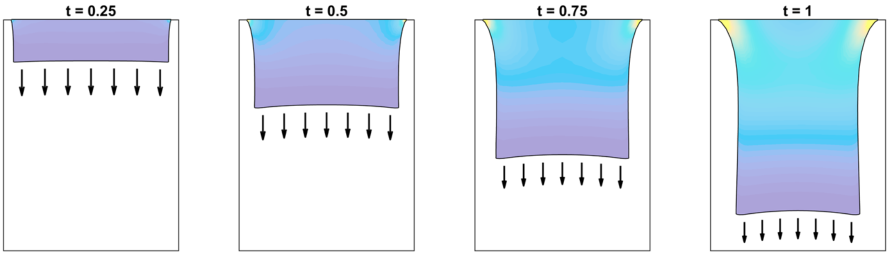

Fig. 16 Accretion through a traction-free line in the presence of a uniform body force. The growth velocity and the body force vectors are pointing in the same direction. The deformed configuration during accretion at four different times

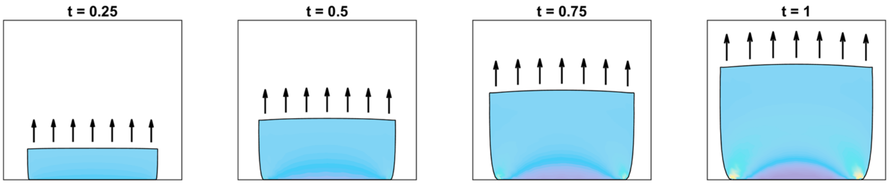

Fig. 17 Accretion through a traction-free line in the presence of a uniform body force. The growth velocity and body force have opposite directions. The deformed configuration during accretion at four different times

We consider the example of an accreting body subject to a vertical and uniform body force. We solve the problem for two cases: (i) when the body force and the growth velocity have the same direction (Figs. 16, 18), and (ii) when they have opposite directions (Figs. 17, 19). Note that since this is a traction-free accretion problem, the Riemannian geometry of the material manifold is not known a priori. In both cases, the material constitutive model is assumed to be nearly incompressible neo-Hookean, i.e., the energy function is given in (5.28). We choose $\lambda = 10\mu$, $u(X) = 1.0\frac{L}{T}$ and $\| \rho \mathbf{b} \| = 1.5\frac{\mu}{L}$ in tension and $0.95\frac{\mu}{L}$ in compression. Notice from Figs. 18 and 19 that the outcome of the process in the two cases is different. This is due to the fact that the load, and hence, the deformation histories are different. For example, the final area is $0.899L^2$ in case (i), and $1.08L^2$ in case (ii).

6 Conclusions

In this paper we introduced a geometric model for the nonlinear mechanics of accreting bodies. This problem was studied in the most general framework, without any symmetry assumptions and allowing finite deformations. This is a highly non-trivial problem as the final mechanical characteristics of an accreted body depend on the deformation the body is experiencing at the time at which each layer is added. In this formulation the stress-free (natural) state of an accreting body is a time-dependent Riemannian manifold with a time-independent metric that is non-flat, in general, and is defined through the introduction of a new object—the accretion tensor—that can be interpreted as the anelastic part of the deformation gradient. The material metric, and consequently the Riemannian structure of a material manifold, depends on the

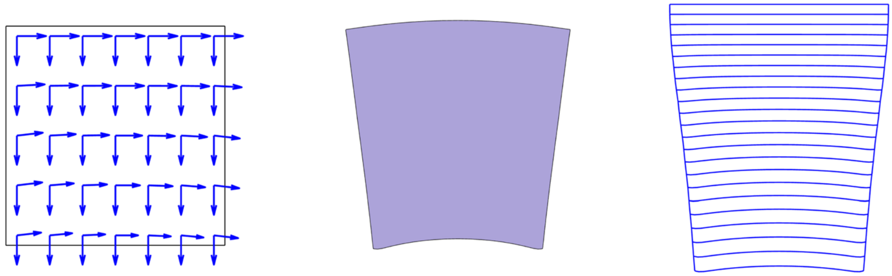

Fig. 18 Accretion through a traction-free line in the presence of a uniform body force. The growth velocity and the body force vectors are pointing in the same directions. Left: the material manifold with the coordinate frames $\{\partial/\partial X, \partial/\partial\tau\}$ distorted by $\mathbf{Q}$. Center: the residually stressed configuration (the lighter colors correspond to larger values of stress). Right: the set $\bar{\varphi}(\mathcal{M})$. Note that in this case the map $\bar{\varphi}$ is an embedding

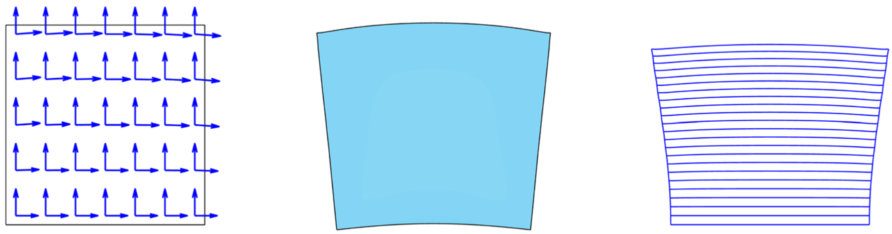

Fig. 19 Accretion through a traction-free line in the presence of a uniform body force. The growth velocity and the body force vectors have opposite directions. Left: the material manifold with the coordinate frames $\{\partial/\partial X, \partial/\partial\tau\}$ distorted by $\mathbf{Q}$. Center: the residually stressed configuration (the lighter colors correspond to larger values of stress). Right: the set $\bar{\varphi}(\mathcal{M})$. Note that in this case the map $\bar{\varphi}$ is an embedding

growth velocity and the state of deformation of the body at the time of attachment of the new material points. In order to express the material metric in terms of the growth velocity a material manifold is constructed using Riemannian geometry and the theory of foliations. In this accretion theory the Riemannian material structure is not known a priori; the material metric is calculated after solving the accretion initial boundary value problem. We formulated the accretion initial boundary value problem, which is defined on a time-dependent domain. We studied two classes of accretion problems where the growth surface is either fixed or traction free. In the first case, the material structure of the body is known a priori. In the second case, we provided some analytical results. Solving several numerical examples allowed us to show how non-Euclidean solids are generated through different types of accretion processes, and how the material geometrical structure of such bodies are related to the accretion characteristics.

We should emphasize that our geometric accretion model is not a coupled theory. We assume that the growth velocity is a given vector field on the boundary of the deformed body. However, in a coupled theory of accretion, it would be one of the unknown fields. In this paper we assumed that this vector field is given and focused our attention on formulating the nonlinear elasticity problem, and in particular, on the incompatibilities induced by accretion. Extending the present theory to consider the coupling between mass transport and elasticity will be the subject of a future communication.

Acknowledgements This research was partially supported by NSF—Grant Nos. CMMI 1130856 and ARO W911NF-16-1-0064.

Appendices

We tersely review some elements of Riemannian geometry, geometry of surfaces and of foliations, and the geometric theory of nonlinear elasticity and anelasticity. For more detailed discussions, see Marsden and Hughes (1983), Camacho and Neto (2013), Yavari (2010), Yavari and Goriely (2012a).

A Riemannian Geometry

For a smooth n-manifold $\mathcal{M}$, the tangent space to $\mathcal{M}$ at a point $p \in \mathcal{M}$ is denoted $T_p\mathcal{M}$. A smooth vector field $\mathbf{W}$ on $\mathcal{M}$ assigns a vector $\mathbf{W}_p$ to every $p \in \mathcal{M}$ and $p \mapsto \mathbf{W}_p \in T_p\mathcal{M}$ varies smoothly. A 1-form at $p \in \mathcal{M}$ is a linear map $\boldsymbol{\lambda} : T_p\mathcal{M} \to \mathbb{R}$. In a local coordinate chart $\{X^A\}$ for $\mathcal{M}$, $\langle \boldsymbol{\lambda}, \mathbf{U} \rangle = \lambda_K U^K$, where $\mathbf{U} \in T_p\mathcal{M}$. The vector space of 1-form at $p \in \mathcal{M}$ is denoted with $T_p^*\mathcal{M}$. A smooth differential 1-form assigns a 1-form $\boldsymbol{\lambda}_p$ to every $p \in \mathcal{M}$ and $\boldsymbol{\lambda}_p \in T_p^*\mathcal{M}$ varies smoothly. A type $\binom{r}{s}$-tensor at $p \in \mathcal{M}$ is a multilinear map

$$T : \overbrace{T_p^*\mathcal{M} \times \cdots \times T_p^*\mathcal{M}}^{r \text{ times}} \times \overbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}^{s \text{ times}} \to \mathbb{R}. \tag{A.1}$$

In a local chart one has

$$\mathbf{T}(\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r, \mathbf{U}_1, \ldots, \mathbf{U}_s) = T^{I_1 \ldots I_r}{}_{J_1 \ldots J_s} \lambda^1_{I_1} \ldots \lambda^r_{I_r} U_1^{J_1} \ldots U_s^{J_s}. \tag{A.2}$$

An $\binom{r}{s}$-tensor field is a smooth map $p \mapsto \mathbf{T}_p$. A Riemannian manifold is a pair $(\mathcal{M}, \mathbf{G})$ where $\mathbf{G}$, the metric tensor field, is a field of positive-definite $\binom{0}{2}$-tensors. If $\mathbf{U}$ and $\mathbf{W}$ are vector fields on $\mathcal{M}$, then $p \mapsto \mathbf{G}_p(\mathbf{U}_p, \mathbf{W}_p) =: \langle\!\langle \mathbf{U}_p, \mathbf{W}_p \rangle\!\rangle_{\mathbf{G}_p}$ is a smooth function. A metric tensor can be used to raise and lower indices. Given a 1-form $\boldsymbol{\lambda}$, we indicate with $\boldsymbol{\lambda}^\sharp$ the vector with components $G^{IJ}\lambda_J$ (the G^{IJ}'s being the components of the inverse metric, i.e., $G_{IJ}G^{JK} = \delta_I^K$), while given a vector $\mathbf{U}$, we indicate with $\mathbf{U}^\flat$ the 1-form with components $G_{IJ}U^J$.

Suppose $\mathcal{N}$ is another n-manifold and $\psi : \mathcal{M} \to \mathcal{N}$ is a smooth and invertible map. If $\mathbf{W}$ is a vector field on $\mathcal{M}$, then $\psi_*\mathbf{W} = T\psi \cdot \mathbf{W} \circ \psi^{-1}$ is a vector field on $\psi(\mathcal{M})$ called the push-forward of $\mathbf{W}$ by ψ. Similarly, if $\mathbf{w}$ is a vector field on $\psi(\mathcal{M}) \subset \mathcal{N}$, then $\psi^*\mathbf{w} = T(\psi^{-1}) \cdot \mathbf{w} \circ \psi$ is a vector field on $\mathcal{M}$ that is called the pull-back of $\mathbf{w}$ by ψ. Let us denote $\mathbf{F} = T\psi$. In the local charts $\{X^I\}$ and $\{x^i\}$ for $\mathcal{M}$ and $\mathcal{N}$, respectively, $\mathbf{F}$ (a two-point tensor) has the following representation (when ψ is a deformation mapping, $\mathbf{F}$ is the so-called deformation gradient of nonlinear elasticity)

$$\mathbf{F} = F^i{}_I \frac{\partial}{\partial x^i} \otimes \mathrm{d}X^I, \quad F^i{}_I = \frac{\partial \psi^i}{\partial X^I}, \tag{A.3}$$

where $\left\{\frac{\partial}{\partial X^I}\right\}$ is a basis for $T_p\mathcal{M}$ and $\{\mathrm{d}X^I\}$ is a basis for $T^*_{\psi(p)}\psi(\mathcal{M})$, the co-tangent space, i.e., the dual space of $T_{\psi(p)}\psi(\mathcal{M})$, or the space of co-vectors (1-forms). The push-forward and pull-back of vectors have the following coordinate representations $(\psi_*\mathbf{W})^a = F^a{}_A W^A$, and $(\psi^*\mathbf{w})^A = (F^{-1})^A{}_a w^a$. The push-forward and pull-back of 1-forms are defined as $(\psi_*\mathbf{\Lambda})_a = (F^{-1})^A{}_a \Lambda_A$, and $(\psi^*\boldsymbol{\lambda})_A = F^a{}_A \lambda_a$. We sometimes indicate them with $(\mathbf{F}^{-1})^\star\mathbf{\Lambda}$ and $\mathbf{F}^\star\boldsymbol{\lambda}$, where $\mathbf{F}^\star$ is the dual of $\mathbf{F}$ and is defined as

$$\mathbf{F}^\star = F^i{}_I \mathrm{d}X^I \otimes \frac{\partial}{\partial x^i}. \tag{A.4}$$

The push-forward and pull-back of an $\binom{r}{s}$-tensor field are given by

$$\begin{aligned}
(\psi_*\mathbf{T})^{i_1\ldots i_r}{}_{j_1\ldots j_s} &= F^{i_1}{}_{I_1}\ldots F^{i_r}{}_{I_r}\ T^{I_1\ldots I_r}{}_{J_1\ldots J_s}\ (F^{-1})^{J_1}{}_{j_1}\ldots(F^{-1})^{J_s}{}_{j_s},\\
(\psi^*\mathbf{t})^{I_1\ldots I_r}{}_{J_1\ldots J_s} &= (F^{-1})^{I_1}{}_{i_1}\ldots(F^{-1})^{I_r}{}_{i_r}\ t^{i_1\ldots i_r}{}_{j_1\ldots j_s}\ F^{j_1}{}_{J_1}\ldots F^{j_s}{}_{J_s}.
\end{aligned} \tag{A.5}$$

Suppose $(\mathcal{M}, \mathbf{G})$ and $(\mathcal{N}, \mathbf{g})$ are Riemannian manifolds and $\psi : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism (smooth map with smooth inverse). Push-forward of the metric $\mathbf{G}$ is a metric on $\psi(\mathcal{M}) \subset \mathcal{N}$, which is denoted by $\psi_*\mathbf{G}$ and is defined as

$$(\psi_*\mathbf{G})_{\psi(p)}\left(\mathbf{u}_{\psi(p)}, \mathbf{w}_{\psi(p)}\right) := \mathbf{G}_p\left((\psi^*\mathbf{u})_p, (\psi^*\mathbf{w})_p\right). \tag{A.6}$$

In components, $(\psi_*\mathbf{G})_{ij} = (F^{-1})^I{}_i (F^{-1})^J{}_j G_{IJ}$. Similarly, pull-back of the metric $\mathbf{g}$ is a metric in $\psi^{-1}(\mathcal{N}) \subset \mathcal{M}$, which is denoted by $\psi^*\mathbf{g}$ and is defined as

$$\left(\psi^*\mathbf{g}\right)_p\left(\mathbf{U}_p, \mathbf{W}_p\right) := \mathbf{g}_{\psi(p)}\left((\psi_*\mathbf{U})_{\psi(p)}, (\psi_*\mathbf{W})_{\psi(p)}\right). \tag{A.7}$$

In components

$$(\psi^*\mathbf{g})_{AB} = F^a{}_A F^b{}_B g_{ab}. \tag{A.8}$$

If $\mathbf{g} = \psi_*\mathbf{G}$, or equivalently, $\mathbf{G} = \psi^*\mathbf{g}$, ψ is called an isometry and the Riemannian manifolds $(\mathcal{M}, \mathbf{G})$ and $(\mathcal{N}, \mathbf{g})$ are isometric. Note that an isometry preserves distances.

A linear (affine) connection on a manifold $\mathcal{B}$ is an operation $\nabla : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ is the set of vector fields on $\mathcal{B}$, such that $\forall\, \mathbf{X}, \mathbf{Y}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{X}(\mathcal{B})$, $\forall\, f, f_1, f_2 \in C^\infty(\mathcal{B})$, $\forall\, a_1, a_2 \in \mathbb{R}$:

$$i)\quad \nabla_{f_1\mathbf{X}_1 + f_2\mathbf{X}_2}\mathbf{Y} = f_1\nabla_{\mathbf{X}_1}\mathbf{Y} + f_2\nabla_{\mathbf{X}_2}\mathbf{Y}, \tag{A.9}$$

$$ii)\quad \nabla_{\mathbf{X}}(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2) = a_1\nabla_{\mathbf{X}}(\mathbf{Y}_1) + a_2\nabla_{\mathbf{X}}(\mathbf{Y}_2), \tag{A.10}$$

$$iii)\quad \nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y}. \tag{A.11}$$

$\nabla_{\mathbf{X}}\mathbf{Y}$ is called the covariant derivative of $\mathbf{Y}$ along $\mathbf{X}$. In a local chart $\{X^I\}$, $\nabla_{\partial_I}\partial_J = \Gamma^K{}_{IJ}\partial_K$, where $\Gamma^K{}_{IJ}$ are Christoffel symbols of the connection and $\partial_K = \frac{\partial}{\partial x^K}$ are natural bases for the tangent space corresponding to a coordinate chart $\{x^A\}$. In components the covariant derivative of a tensor field reads

$$(\nabla_{\mathbf{U}}\mathbf{W})^I = \frac{\partial W^I}{\partial X^H} U^H + \Gamma^I{}_{HK} W^K U^H, \tag{A.12}$$

while for a field of 1-forms one has

$$(\nabla_{\mathbf{U}}\boldsymbol{\lambda})_I = \frac{\partial \lambda_I}{\partial X^H} U^H - \Gamma^K{}_{HI} \lambda_K U^H. \tag{A.13}$$

The previous relations can be generalized to tensors as

$$\begin{aligned}(\nabla_{\mathbf{U}}\mathbf{T})^I = \frac{\partial T^{I_1 \ldots I_r}{}_{J_1 \ldots J_s}}{\partial X^H} U^H + \Gamma^{I_1}{}_{HK} T^{K \ldots I_r}{}_{J_1 \ldots J_s} U^H + \cdots \\ -\Gamma^K{}_{HJ_1} T^{I_1 \ldots I_r}{}_{K \ldots J_s} U^H.\end{aligned} \tag{A.14}$$

A linear connection is said to be compatible with a metric $\mathbf{G}$ of the manifold if

$$\nabla_{\mathbf{X}} \langle\!\langle \mathbf{Y}, \mathbf{Z} \rangle\!\rangle_{\mathbf{G}} = \langle\!\langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle\!\rangle_{\mathbf{G}} + \langle\!\langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z} \rangle\!\rangle_{\mathbf{G}}, \tag{A.15}$$

where $\langle\!\langle ., . \rangle\!\rangle_{\mathbf{G}}$ is the inner product induced by the metric $\mathbf{G}$. It can be easily shown that ∇ is compatible with $\mathbf{G}$ if and only if $\nabla\mathbf{G} = \mathbf{0}$. A linear connection is said to be symmetric if $\nabla_{\mathbf{U}}\mathbf{W} - \nabla_{\mathbf{W}}\mathbf{U} - [\mathbf{U}, \mathbf{W}] = \mathbf{0}$, with $[\mathbf{U}, \mathbf{W}]$ indicating the Lie brackets of $\mathbf{U}$ and $\mathbf{W}$, i.e., $([\mathbf{U}, \mathbf{W}])^I = W^I{}_{,J}\, U^J - U^I{}_{,J}\, W^J$. The symmetric connection that is compatible with the metric is called the Levi–Civita connection and its Christoffel coefficients read

$$\Gamma^I{}_{JK} = G^{IH} \left(\frac{\partial G_{HJ}}{\partial X^K} + \frac{\partial G_{HK}}{\partial X^J} - \frac{\partial G_{JK}}{\partial X^H} \right). \tag{A.16}$$

The Riemann curvature $\boldsymbol{\mathcal{R}}$ associated with $(\mathcal{M}, \nabla)$ is a $\binom{1}{3}$-tensor defined as

$$\boldsymbol{\mathcal{R}}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} \quad \forall \mathbf{X}, \mathbf{Y}, \mathbf{Z} \in T\mathcal{M}, \tag{A.17}$$

where $\boldsymbol{\mathcal{R}}(\mathbf{X}, \mathbf{Y})\mathbf{Z}$ stands for the contraction $R^I{}_{JLM} X^L Y^M Z^J$. In a local coordinate chart $\{X^A\}$, the components of the Riemman curvature tensor read

$$\mathcal{R}^I{}_{JLM} = \frac{\partial \Gamma^I{}_{MJ}}{\partial X^L} - \frac{\partial \Gamma^I{}_{LJ}}{\partial X^M} + \Gamma^I{}_{LH}\Gamma^H{}_{MJ} - \Gamma^I{}_{MH}\Gamma^H{}_{LJ}. \tag{A.18}$$

The Ricci curvature $\mathbf{R}$ is defined as $R_{IJ} = \mathcal{R}^H{}_{IHJ}$ and is a symmetric $\binom{0}{2}$-tensor. The scalar curvature R is defined as $R = R_{HK} G^{HK}$. In dimension three only six components of $\boldsymbol{\mathcal{R}}$ are independent and the Ricci curvature fully determines the Riemannian curvature. In dimension two $\boldsymbol{\mathcal{R}}$ contains only one independent component so it is fully determined by the scalar curvature. A metric whose Levi–Civita connection has zero curvature is said to be flat and is locally isometric to $\mathbb{R}^n$ (see the different versions of the Test Case Theorem in Spivak 1999).

B Geometry of Surfaces

For our purposes, a surface is a two-dimensional submanifold Ω of a three-dimensional manifold $\mathcal{M}$ (for details see do Carmo 1992). The ambient manifold $\mathcal{M}$ is endowed with a metric $\mathbf{G}$ that, in general, is not flat. The surface Ω inherits a metric $\tilde{\mathbf{G}}$ — the first fundamental form — from $\mathbf{G}$, such that

$$\tilde{\mathbf{G}}(\mathbf{Y}, \mathbf{Z}) = \mathbf{G}(\mathbf{Y}, \mathbf{Z}), \quad \forall \mathbf{Y}, \mathbf{Z} \in T\Omega. \tag{B.1}$$

On an open set one can define the unit vector $\mathbf{N}$ normal to Ω, i.e., a vector field such that i) $\mathbf{G}(\mathbf{Y}, \mathbf{N}) = 0$ for each tangent vector $\mathbf{Y}$, ii) $\mathbf{G}(\mathbf{N}, \mathbf{N}) = 1$. It is globally defined only when Ω is orientable. We also define the associated 1-form $\mathbf{N}^\flat$ such that $\langle \mathbf{N}^\flat, \mathbf{X} \rangle = 0$ for any $\mathbf{X} \in T\Omega$, and $\langle \mathbf{N}^\flat, \mathbf{N} \rangle = 1$, which can be obtained by lowering the index of $\mathbf{N}$, i.e., $N_I = G_{IJ} N^J$.

The two Levi–Civita connections ∇ on $(\mathcal{M}, \mathbf{G})$, and $\tilde{\nabla}$ on $(\Omega, \tilde{\mathbf{G}})$ satisfy the *Gauss formula*

$$\tilde{\nabla}_{\mathbf{Y}} \mathbf{Z} = \nabla_{\mathbf{Y}} \mathbf{Z} - \mathbf{G}(\nabla_{\mathbf{Y}} \mathbf{Z}, \mathbf{N}) \mathbf{N}, \quad \forall \mathbf{Y}, \mathbf{Z} \in T\Omega. \tag{B.2}$$

This means that $\tilde{\nabla}$ is the tangent projection of ∇. The second fundamental form $\mathbf{K}$ is a $\binom{0}{2}$-tensor defined as

$$\mathbf{K}(\mathbf{Y}, \mathbf{Z}) = \mathbf{G}(\nabla_{\mathbf{Y}} \mathbf{Z}, \mathbf{N}), \quad \forall \mathbf{Y}, \mathbf{Z} \in T\Omega. \tag{B.3}$$

Then, Eq. (B.2) can be rewritten as

$$\nabla_{\mathbf{Y}} \mathbf{Z} - \tilde{\nabla}_{\mathbf{Y}} \mathbf{Z} = \mathbf{K}(\mathbf{Y}, \mathbf{Z}) \mathbf{N}. \tag{B.4}$$

It can be easily shown that $\mathbf{K}$ is well defined and symmetric. Moreover, since $\nabla_{\mathbf{Y}} (\mathbf{G}(\mathbf{Z}, \mathbf{N})) = \nabla_{\mathbf{Y}} \mathbf{0} = \mathbf{0}$ for any pair $\mathbf{Y}, \mathbf{Z} \in T\Omega$, one has

$$\mathbf{K}(\mathbf{Y}, \mathbf{Z}) = \mathbf{G}(\nabla_{\mathbf{Y}} \mathbf{Z}, \mathbf{N}) = -\mathbf{G}(\mathbf{Z}, \nabla_{\mathbf{Y}} \mathbf{N}) = -\langle \nabla_{\mathbf{Y}} \mathbf{N}^\flat, \mathbf{Z} \rangle, \tag{B.5}$$

which is known as the *Weingarten formula*, and in short it reads $\mathbf{K} = -\nabla \mathbf{N}^\flat$.

An adapted chart Ξ on $\mathcal{U} \subset \mathcal{M}$ has the property $\Xi(\mathcal{U} \cap \Omega) = V \times \{0\}$ with $V \subset \mathbb{R}^2$. In other words, the surface is locally given by $\Xi^3 = 0$ and (Ξ^1, Ξ^2) are coordinate functions on $\Omega \cap \mathcal{U}$. We use letters from A to G to denote indices that span $\{1, 2\}$ (layer indices), and letters from H to Z for indices that span $\{1, 2, 3\}$ (3D indices). The components of the second fundamental form can now be obtained using the Weingarten formula:

$$K_{AB} = -\frac{\partial N_A}{\partial X^B} + \Gamma^I{}_{AB} N_I = N_I \Gamma^I{}_{AB} = \frac{1}{N^3} \Gamma^3{}_{AB}, \tag{B.6}$$

where all quantities are referred to an adapted chart.

The Riemann curvature tensor $\boldsymbol{\mathcal{R}}$ on $(\mathcal{M}, \mathbf{G})$ was introduced in Appendix A. One can define in the same way the curvature tensor $\tilde{\boldsymbol{\mathcal{R}}}$ for $(\Omega, \tilde{\mathbf{G}})$. In components one has

$$\tilde{R}^A{}_{BCD} = \frac{\partial \tilde{\Gamma}^A{}_{DB}}{\partial X^C} - \frac{\partial \tilde{\Gamma}^A{}_{CB}}{\partial X^D} + \tilde{\Gamma}^A{}_{CE}\tilde{\Gamma}^E{}_{DB} - \tilde{\Gamma}^A{}_{DE}\tilde{\Gamma}^E{}_{CB}. \tag{B.7}$$

The curvature tensor has 16 components but, as mentioned earlier, only one is independent. Hence, it can be expressed by the scalar curvature R, or by the Gaussian curvature $g = \det(\tilde{G}^{AC} K_{CB})$. It can be shown that $R = 2g$ and $g \det \tilde{\mathbf{G}} = \frac{1}{2} R \det \tilde{\mathbf{G}} = R_{1212}$. The *Guass equation* relates the tangent part of $\boldsymbol{\mathcal{R}}$ to $\tilde{\boldsymbol{\mathcal{R}}}$ and $\mathbf{K}$. In components, it reads

$$R_{ABCD} = \tilde{R}_{ABCD} + K_{AD}K_{BC} - K_{AC}K_{BD}. \tag{B.8}$$

When the ambient space $(\mathcal{M}, \mathbf{G})$ is Euclidean the Gauss equation provides an expression for the curvature of the surface, viz. $\tilde{R}_{ABCD} = K_{AC}K_{BD} - K_{AD}K_{BC}$. The *Codazzi–Mainardi equations* relate $\boldsymbol{\mathcal{R}}$ to $\tilde{\nabla}\mathbf{K}$. In components, they are given by

$$N^H \mathcal{R}_{HBCD} = N_H \mathcal{R}^H{}_{BCD} = \tilde{\nabla}_D K_{BC} - \tilde{\nabla}_C K_{BD}. \tag{B.9}$$

When the ambient space $(\mathcal{M}, \mathbf{G})$ is Euclidean the Codazzi–Mainardi equations enforce the symmetry of the tensor $\tilde{\nabla}_A K_{BC}$ with respect to all permutations.

C Geometry of Foliations

Let $\mathcal{M}$ be an n-dimensional manifold. An $(n-1)$-foliation (or foliation of codimension 1) is an atlas of charts $(\mathcal{U}_a, \Xi_a)$ (a in some index set $\mathcal{I}$) such that $\Xi_a(\mathcal{U}_a) = V_a \times I_a$ with $V_a \subset \mathbb{R}^{n-1}$ and $I_a \subset \mathbb{R}$ open sets (see Fig. 20). For further details see Camacho and Neto (2013). The coordinate charts Ξ_a are called the foliation charts. When this property holds, one obtains a partition of $\mathcal{M}$ as a collection $\{\Omega_t\}_{t\in\mathbb{R}}$ of embedded submanifolds of dimension two — the leaves of the foliation. In particular, a Riemannian material manifold $(\mathcal{M}, \mathbf{G})$ is partitioned into $(n-1)$-dimensional Riemannian submanifolds $(\Omega_\tau, \tilde{\mathbf{G}}_\tau)$ with $\tau \in \mathbb{R}$. The metric $\tilde{\mathbf{G}}_\tau$ on the layer Ω_τ is inherited from $\mathbf{G}$, i.e., it is defined as

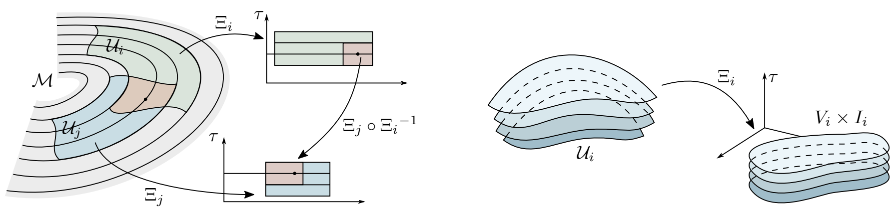

Fig. 20 Foliation charts. Left: sketch of two overlapping local charts in 2D. Right: sketch of a foliation chart in a 3-manifold

$$\tilde{\mathbf{G}}_\tau(\mathbf{Y}_\tau, \mathbf{Z}_\tau) = \mathbf{G}(\mathbf{Y}_\tau, \mathbf{Z}_\tau), \quad \forall \mathbf{Y}_\tau, \mathbf{Z}_\tau \in T\Omega_\tau, \tag{C.1}$$

and constitutes the first fundamental form of the layer.

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