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Riemannian and Euclidean material structures in anelasticity

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Abstract
In this paper, we discuss the mechanics of anelastic bodies with respect to a Riemannian and a Euclidean geometric structure on the material manifold. These two structures provide two equivalent sets of governing equations that correspond to the geometrical and classical approaches to non-linear anelasticity. This paper provides a parallelism between the two approaches and explains how to go from one to the other. We work in the setting of the multiplicative decomposition of deformation gradient seen as a non-holonomic change of frame in the material manifold. This allows one to define, in addition to the two geometric structures, a Weitzenböck connection on the material manifold. We use this connection to express natural uniformity in a geometrically meaningful way. The concept of uniformity is then extended to the Riemannian and Euclidean structures. Finally, we discuss the role of non-uniformity in the form of material forces that appear in the configurational form of the balance of linear momentum with respect to the two structures.

Keywords
Anelasticity, defects, configurational forces, material uniformity, residual stresses

1. Introduction

According to Eckart [1], anelasticity can be formulated starting from elasticity theory by relaxing the assumption that for a given body a global and time-independent relaxed state always exists. This means that an anelastic body is represented by a time-dependent non-Euclidean manifold, and hence, the theory of anelasticity can be reduced to the elasticity problem of mapping a Riemannian material manifold to the Euclidean ambient space. In a similar way, Epstein and Maugin [2] defined anelasticity as the "result of evolving distributions of inhomogeneity", where inhomogeneity is understood in the sense of Noll [3] and Wang [4]. Anelasticity is usually modeled through the multiplicative decomposition of deformation gradient, introduced by Bilby et al. [5], Kondo [6, 7], and Kröner [8]. For a short review, see Sadik and Yavari [9]. In a geometric approach to anelasticity, one formulates the balance laws in a Riemannian geometric structure defined on the material manifold: distances and strains are defined with

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respect to a non-Euclidean material metric, densities are defined with respect to the corresponding material volume form, and derivatives are taken using the associated Levi–Civita connection. Conversely, in the classical approach, everything is formulated in the standard Euclidean space. In this paper, we unify these two approaches. More specifically, using a geometric formalism, we will discuss the relations between the governing equations in the two frameworks. The differences between the two approaches are quite obscure when considered from the configurational perspective of Eshelby [10], Epstein and Maugin [2], and Gurtin [11], where the concept of uniformity, first discussed in the works of Noll [3], Wang [4], and Wang and Bloom [12], is involved. As a matter of fact, configurational forces arise as a consequence of non-uniformity of the material, but, depending on the setting in which the governing equations are written, these might show up as the effect of inhomogeneity, even in the case of uniform materials. It should be emphasized that in this paper we are not concerned with considering driving forces for the evolution of the distribution of inhomogeneities. These are related to some anelastic variables (as well as to the elastic deformations) via a specific flow rule that depends on the class of problems one is considering, such as dislocations or growth. As we are interested in investigating the geometric structures that the anelastic deformations induce on the material manifold, regardless of the underlying dynamics, in this paper, time evolutions of distributions of inhomogeneities will be considered as given.

Anelasticity is a general term that can refer to many phenomena. The present work does not make any assumption on the nature of the source of anelasticity; therefore, it applies to many different problems. Yielding of materials and plasticity is an example [13–16]. Plastic behavior is associated to the evolution of distribution of dislocations in a solid. In recent years, many workers have discussed anelasticity from a configurational point of view, involving the Eshelby stress and the concept of uniformity. Epstein and Maugin [2] provide a different perspective on the multiplicative decomposition of deformation gradient, with a focus on the inverse of plastic deformation—the uniformity map—representing the deformation of the reference crystal into a compatible reference configuration. In their theory, equivalence classes of uniformity maps are defined based on the symmetry group of the reference crystal. Evolution laws involving the inhomogeneity velocity gradient are obtained, together with thermodynamical restrictions involving Eshelby's tensor. We point out the work by Menzel and Steinmann [17], where different formats of the balance of linear momentum in the framework of the multiplicative decomposition of deformation gradient are presented. Menzel and Steinmann [17] defined different stress tensors with respect to different configurations, and used them to express the balance of linear momentum in several different forms, some of which involve the dislocation density tensor. Alhasadi et al. [18] discussed material forces and uniformity in the context of thermo-anelastic bodies. They used a geometric approach, although it is not clear whether they viewed the multiplicative decomposition of deformation gradient as a change of frame or as a local deformation tensor. However, their work has some similarities with this paper, e.g., the definition of modified quantities using the multiplication by a volume ratio, and the expression of the configurational forces in terms of the Mandel stress and some geometric objects defined on the material manifold.

This paper is organized as follows. In Section 2, we define metric tensors and connections on the material manifold, and introduce the Riemannian and the Euclidean structures with respect to which the balance laws of anelasticity will be written. This is followed by a discussion on the natural Weitzenböck derivative. In Section 3, we review some concepts related to the multiplicative decomposition of deformation gradient and define some measures of deformation with respect to both structures. Time evolutions of the moving frame are also discussed. In Section 4, we define stress tensors with respect to both the Riemannian and the Euclidean structures in the context of hyperelasticity. We also discuss uniformity with respect to the natural moving frame, and extend this concept to the Riemannian and Euclidean structures. In Section 5, we derive the balance of linear momentum for an anelastic body with respect to both the Riemannian and the Euclidean structures. We discuss the role of non-uniformity in the material forces that appear in the configurational form of the balance laws. A list of the notation used in this paper is given in Appendix A.

2. Geometric structures on the material manifold

In this section, we define two geometric structures on the material manifold of a solid body. The term "geometric material structure" is inspired by the work of Wang [4] and Wang and Bloom [12], and by it we mean a metric tensor with its associated volume form and the Levi–Civita connection. It should be emphasized that, for us, "structure" does not have the same meaning as in Epstein and Maugin [2], where it refers to a reduction to classes of anelastic deformations based on the symmetry group of the solid. The two geometric structures discussed here are: (i) the Riemannian structure, which provides information on the distances in the body in its natural configuration, and (ii) the Euclidean structure, inherited from the ambient space. Natural distances are provided, starting from a moving frame representing the local natural state of the body. The relation between the two structures allows one to define a distribution of local anelastic deformations. We also define a third connection, the Weitzenböck connection, which parallelizes the natural moving frame, and which contains information about the defect content of the anelastic deformation.

2.1. The Riemannian material structure

A body is represented by a 3-manifold $\mathcal{B}$ called the material manifold, which is embeddable in the Euclidean space $\mathcal{S}$. We indicate by $\{\boldsymbol{e}_\alpha\}_{\alpha=1,2,3}$, or simply $\{\boldsymbol{e}_\alpha\}$, a moving frame that represents the local natural state of the body. This is related to the constitutive behavior of the material and will be discussed in Section 4. It should be emphasized that this natural frame is not unique, as will be discussed in Remark 5. This moving frame is, in general, non-holonomic, meaning that it is not necessarily induced from any coordinate chart. Its associated moving co-frame field $\{\boldsymbol{\vartheta}^\alpha\}$ is such that $\langle \boldsymbol{\vartheta}^\alpha, \boldsymbol{e}_\beta \rangle = \delta^\alpha_\beta$,[1] or equivalently, $\boldsymbol{e}_\alpha \otimes \boldsymbol{\vartheta}^\alpha = \boldsymbol{I}$, where the summation convention for repeated indices is used. On $\mathcal{B}$ one defines the material metric $\boldsymbol{G}$ as the $\binom{0}{2}$-tensor that has components $\delta_{\alpha\beta}$ in the moving frame, viz.

$$\boldsymbol{G} = \delta_{\alpha\beta}\, \boldsymbol{\vartheta}^\alpha \otimes \boldsymbol{\vartheta}^\beta . \tag{1}$$

This means that the moving frame represents the state in which one observes the natural distances in the body. Note that the moving frame $\{\boldsymbol{e}_\alpha\}$ is orthonormal with respect to the material metric $\boldsymbol{G}$. The natural moving frame $\{\boldsymbol{e}_\alpha\}$ and co-frame $\{\boldsymbol{\vartheta}^\alpha\}$ can be written in terms of the local frame $\{\partial_A\}$ and co-frame $\{\mathrm{d}X^A\}$ induced by a generic coordinate chart $\{X^A\}$ as

$$\boldsymbol{e}_\alpha = (\mathsf{F}^{-1})^A{}_\alpha \partial_A , \qquad \boldsymbol{\vartheta}^\alpha = \mathsf{F}^\alpha{}_A \mathrm{d}X^A , \qquad \alpha = 1,2,3 . \tag{2}$$

Equation (2) represents a passive interpretation of the multiplicative decomposition of deformation gradient in the sense that the matrix $[\mathsf{F}^\alpha{}_A]$ is a change of frame and not a tensor. In this interpretation, anelasticity is modeled by endowing $\mathcal{B}$ with just a moving frame. In the chart $\{X^A\}$, using equation (2), $\boldsymbol{G}$ is given by

$$G_{AB} = \boldsymbol{G}(\partial_A, \partial_B) = \boldsymbol{G}\big(\mathsf{F}^\alpha{}_A \boldsymbol{e}_\alpha, \mathsf{F}^\beta{}_B \boldsymbol{e}_\beta\big) = \mathsf{F}^\alpha{}_A \mathsf{F}^\beta{}_B \boldsymbol{G}\big(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta\big) = \mathsf{F}^\alpha{}_A \mathsf{F}^\beta{}_B \delta_{\alpha\beta} . \tag{3}$$

The volume form $\boldsymbol{\mu}$ associated with $\boldsymbol{G}$ is defined as

$$\boldsymbol{\mu} = \sqrt{\det \boldsymbol{G}}\, \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 ,$$

with components $\sqrt{\det \boldsymbol{G}}\epsilon_{ABC}$, where ϵ_{ABC} indicates the permutation symbol.[2] The Riemannian mass density is denoted by ϱ_o. The total Riemannian volume $\mathcal{V}$ and the total mass $\mathcal{M}$ are therefore given by

$$\mathcal{V} = \int_{\mathcal{B}} \boldsymbol{\mu}, \qquad \mathcal{M} = \int_{\mathcal{B}} \varrho_o \boldsymbol{\mu} = \int_{\mathcal{B}} \mathbf{m},$$

where $\mathbf{m} = \varrho_o \boldsymbol{\mu}$ is the mass form. The Levi–Civita connection ∇ associated with $\boldsymbol{G}$ has the following coefficients:

$$\Gamma^A{}_{BC} = \frac{1}{2} G^{AD} (\partial_C G_{DB} + \partial_B G_{DC} - \partial_D G_{BC}) .$$

By construction, $\nabla \boldsymbol{G} = \boldsymbol{0}$. The torsion of ∇ vanishes, whereas its curvature is in general non-zero, meaning that, in general, it is not possible to isometrically embed $(\mathcal{B}, \boldsymbol{G})$ into the flat ambient manifold $(\mathcal{S}, \boldsymbol{g})$,

$\boldsymbol{g}$ being the standard Euclidean metric in the ambient space. Such an embedding is associated with the lowest energetic state; hence, when this cannot be achieved, residual stresses develop (see Section 4). We refer to the triplet $(\boldsymbol{G}, \boldsymbol{\mu}, \nabla)$ as the Riemannian structure.

2.2. The Euclidean material structure

The governing equations of anelasticity are often expressed with respect to an auxiliary reference Euclidean structure that is inherited from the ambient space $(\mathcal{S}, \boldsymbol{g})$, and does not provide any information about the anelastic frustration of the material. As mentioned earlier, the material manifold $\mathcal{B}$ is globally embeddable in the ambient space $\mathcal{S}$, which is endowed with the standard Euclidean metric $\boldsymbol{g}$. For this reason, $\mathcal{B}$ can in turn be endowed with a Euclidean metric $\bar{\boldsymbol{G}}$ inherited from $(\mathcal{S}, \boldsymbol{g})$. This can be done by considering an embedding $\psi : \mathcal{B} \to \mathcal{S}$ (i.e., a configuration of the body) and endowing $\mathcal{B}$ with the pulled-back geometry via ψ, viz. $\bar{\boldsymbol{G}} = \psi^* \boldsymbol{g}$. When $\mathcal{B}$ is defined as a subset of $\mathcal{S}$, one can simply take ψ to be the inclusion map, and hence, define $\bar{\boldsymbol{G}} = \boldsymbol{g}|_{\mathcal{B}}$. We fix Cartesian coordinates $\xi = \{\xi^{\bar{a}}\}$ and the corresponding frame $\{\partial_{\bar{a}}\}$ on $\mathcal{S}$ and take the global chart $\Xi = \{\Xi^{\bar{A}}\} = \xi \circ \psi$, so that on $\mathcal{B}$ a Cartesian moving frame $\{\bar{\partial}_{\bar{A}}\}$ is defined using Ξ as $\{\bar{\partial}_{\bar{A}}\} = (T\psi)^{\bar{a}}{}_{\bar{A}} \partial_{\bar{a}}$. Note that $\{\bar{\partial}_{\bar{A}}\}$ is orthonormal with respect to $\bar{\boldsymbol{G}}$, i.e., $\bar{G}_{\bar{A}\bar{B}} = \delta_{\bar{A}\bar{B}}$.

We indicate the associated volume form with $\bar{\boldsymbol{\mu}}$, defined as

$$\bar{\boldsymbol{\mu}} = \sqrt{\det \bar{\boldsymbol{G}}}\, \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 ,$$

with components $\sqrt{\det \bar{\boldsymbol{G}}}\epsilon_{ABC}$. The total Euclidean volume and the total mass are respectively given by

$$\bar{\mathcal{V}} = \int_{\mathcal{B}} \bar{\boldsymbol{\mu}}, \qquad \mathcal{M} = \int_{\mathcal{B}} \bar{\varrho}_o \bar{\boldsymbol{\mu}} = \int_{\mathcal{B}} \mathbf{m},$$

where $\bar{\varrho}_o$ denotes the Euclidean mass density, and $\mathbf{m} = \bar{\varrho}_o \bar{\boldsymbol{\mu}} = \varrho_o \boldsymbol{\mu}$ is the mass form defined in Section 2.1. Note that, unlike the volume forms, the mass form $\mathbf{m}$ is the same in both structures as we want the mass to be independent of the geometric structure. Since the two structures are defined on the same material manifold $\mathcal{B}$, the total mass $\mathcal{M}$ is the same with respect to both structures as well. We denote by $\bar{\nabla}$ the induced Levi–Civita connection with the following coefficients in a coordinate chart $\{X^A\}$:

$$\bar{\Gamma}^A{}_{BC} = \frac{1}{2} \bar{G}^{AD} (\partial_C \bar{G}_{DB} + \partial_B \bar{G}_{DC} - \partial_D \bar{G}_{BC}) . \tag{4}$$

The Christoffel symbols vanish with respect to the Cartesian chart $\{\Xi^{\bar{A}}\}$. Hence, they can be written in a generic chart $\{X^A\}$ as

$$\bar{\Gamma}^A{}_{BC} = \frac{\partial X^A}{\partial \Xi^{\bar{A}}} \frac{\partial^2 \Xi^{\bar{A}}}{\partial X^B \partial X^C} .$$

By construction, $\bar{\nabla} \bar{\boldsymbol{G}} = \mathbf{0}$. Both the torsion and the curvature tensors vanish. Note that $\bar{\nabla}$ is the Weitzenböck connection for $\{\bar{\partial}_{\bar{A}}\}$ (see Section 2.5) and, at the same time, the Levi–Civita connection for $\bar{\boldsymbol{G}}$. We refer to the triplet $(\bar{\boldsymbol{G}}, \bar{\boldsymbol{\mu}}, \bar{\nabla})$ as the Euclidean structure.

2.3. Change of material structure

A goal of this paper is to find the relation between the geometric framework of anelasticity (represented by the Riemannian structure) and the classical framework (represented by the Euclidean structure). The first step is to define objects that allow one to switch from one structure to the other. In particular, it is possible to switch from one metric to the other using the $\binom{1}{1}$-tensor $\boldsymbol{\Theta}$, defined as

$$\Theta^A{}_B = \bar{G}^{AD} G_{DB} , \qquad \left(\Theta^{-1}\right)^A{}_B = G^{AD} \bar{G}_{DB} , \tag{5}$$

so that $\boldsymbol{G} = \bar{\boldsymbol{G}}\boldsymbol{\Theta}$, and $\bar{\boldsymbol{G}} = \boldsymbol{G}\boldsymbol{\Theta}^{-1}$. For the inverse metric tensors, one has $\boldsymbol{G}^\sharp = \boldsymbol{\Theta}^{-1}\bar{\boldsymbol{G}}^\sharp$, and $\bar{\boldsymbol{G}}^\sharp = \boldsymbol{\Theta}\boldsymbol{G}^\sharp$. We call $\boldsymbol{\Theta}$ the change of metric and will show that the tensor $\boldsymbol{\Theta}$ represents a measure of the anelastic deformation, seen as a local transformation of the material manifold. As for the Riemannian and the Euclidean volume forms, they are related by the volume ratio J, defined as $\boldsymbol{\mu} = \mathsf{J}\bar{\boldsymbol{\mu}}$, where

$$\mathsf{J} = \sqrt{\frac{\det \boldsymbol{G}}{\det \bar{\boldsymbol{G}}}} = \sqrt{\det \boldsymbol{\Theta}}. \tag{6}$$

The Riemannian total volume can therefore be written as $\mathcal{V} = \int_{\mathcal{B}} \mathsf{J}\bar{\boldsymbol{\mu}}$. Having defined $\mathbf{m} = \bar{\varrho}_o\bar{\boldsymbol{\mu}} = \varrho_o\boldsymbol{\mu}$, the change of volume relates the two mass densities as $\mathsf{J}\varrho_o = \bar{\varrho}_o$. Area elements inherited from the Riemannian and the Euclidean structures are related by the area ratio. Let $\Omega \subset \mathcal{B}$ be an oriented surface, with the 2-forms $\boldsymbol{\eta}$ and $\bar{\boldsymbol{\eta}}$ being the area elements induced on Ω by $\boldsymbol{\mu}$ and $\bar{\boldsymbol{\mu}}$, respectively. Then, the scalar field J_Ω on Ω is defined as $\boldsymbol{\eta} = \mathsf{J}_\Omega\bar{\boldsymbol{\eta}}$. Indicating by $\boldsymbol{N}$ and $\bar{\boldsymbol{N}}$ the $\boldsymbol{G}$-normal and the $\bar{\boldsymbol{G}}$-normal unit vector fields, respectively, one can write the material analog of Nanson's formula for anelasticity, viz.

$$\mathsf{J}_\Omega \boldsymbol{N}^\flat = \mathsf{J}\bar{\boldsymbol{N}}^\flat, \tag{7}$$

where $(\,\cdot\,)^\flat$ indicates the lowering of indices of a vector using the corresponding metric, i.e., $N_A = G_{AB}N^B$, and $\bar{N}_A = \bar{G}_{AB}\bar{N}^B$. The following result defines the analog of the Piola transformation for anelastic deformations. We adopt the abuse of notation $\nabla_A U^B$ to indicate a tensor that acts on a vector $\boldsymbol{V}$ as $V^A\nabla_A U^B\partial_B = \nabla_{\boldsymbol{V}}\boldsymbol{U}$. This is trivially generalized to differential forms and tensors.

Lemma 1. *The divergence of a vector field $\boldsymbol{U}$ with respect to the Euclidean and Riemannian connections are related as*

$$\bar{\nabla}_A(\mathsf{J}U^A) = \mathsf{J}\nabla_A U^A. \tag{8}$$

Proof. Since the connection ∇ and volume form $\boldsymbol{\mu}$ are induced from the same metric $\boldsymbol{G}$, one has $(\nabla_A U^A)\boldsymbol{\mu} = \mathfrak{L}_{\boldsymbol{U}}\boldsymbol{\mu}$ for any vector field $\boldsymbol{U}$. Using Cartan's magic formula, one writes $\mathfrak{L}_{\boldsymbol{U}}\boldsymbol{\mu} = \iota_{\boldsymbol{U}}\mathrm{d}\boldsymbol{\mu} + \mathrm{d}(\iota_{\boldsymbol{U}}\boldsymbol{\mu}) = \mathrm{d}(\iota_{\boldsymbol{U}}\boldsymbol{\mu})$, where $\iota_{\boldsymbol{U}}\boldsymbol{\mu}$ indicates the interior product, i.e., $(\iota_U\mu)_{AB} = \mu_{ABC}U^C$, and d is the exterior derivative. Hence, one has $(\nabla_A U^A)\boldsymbol{\mu} = \mathrm{d}(\iota_{\boldsymbol{U}}\boldsymbol{\mu})$. Similarly, the Euclidean divergence $\bar{\nabla}_A U^A$ satisfies $(\bar{\nabla}_A U^A)\boldsymbol{\mu} = \mathrm{d}(\iota_{\boldsymbol{U}}\bar{\boldsymbol{\mu}})$, with $(\iota_U\bar{\mu})_{AB} = \bar{\mu}_{ABC}U^C$. Therefore, one can write

$$\mathsf{J}\nabla_A U^A\,\boldsymbol{\mu} = \mathsf{J}\mathrm{d}(\iota_{\boldsymbol{U}}\boldsymbol{\mu}) = \mathsf{J}\mathrm{d}(\mathsf{J}\iota_{\boldsymbol{U}}\bar{\boldsymbol{\mu}}) = \mathsf{J}\mathrm{d}(\iota_{\mathsf{J}\boldsymbol{U}}\bar{\boldsymbol{\mu}}) = \mathsf{J}\,\bar{\nabla}_A\big(\mathsf{J}U^A\big)\,\bar{\boldsymbol{\mu}} = \bar{\nabla}_A\big(\mathsf{J}U^A\big)\,\boldsymbol{\mu},$$

and hence equation (8). □

As for the change of connection, covariant derivatives with respect to the Riemannian and Euclidean connections are related by the $\binom{1}{2}$-tensor $\boldsymbol{H}$, defined as $\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{V}) = \nabla_{\boldsymbol{U}}\boldsymbol{V} - \bar{\nabla}_{\boldsymbol{U}}\boldsymbol{V}$, or in components, $H^A{}_{BC} = \Gamma^A{}_{BC} - \bar{\Gamma}^A{}_{BC}$. It is straightforward to show that $H^A{}_{BC}$ are the components of a tensor, and hence, $\boldsymbol{H}$ is well-defined. Moreover, $\boldsymbol{H}$ is symmetric in the two lower indices by virtue of the symmetry of both ∇ and $\bar{\nabla}$. Thus, given a tensor field $\boldsymbol{\mathcal{T}}$, one can write

$$\nabla_C\mathcal{T}^{A_1\ldots}{}_{B_1\ldots} - \bar{\nabla}_C\mathcal{T}^{A_1\ldots}{}_{B_1\ldots} = H^{A_1}{}_{CD}\mathcal{T}^{D\ldots}{}_{B_1\ldots} + \ldots - H^D{}_{CB_1}\mathcal{T}^{A_1\ldots}{}_{D\ldots} - \cdots \tag{9}$$

Using Lemma 1, given a vector field $\boldsymbol{U}$, one writes

$$\langle \mathrm{d}\mathsf{J}, \boldsymbol{U}\rangle = \bar{\nabla}_A\big(\mathsf{J}U^A\big) - \mathsf{J}\big(\bar{\nabla}_A U^A\big) = \mathsf{J}\big[\big(\nabla_A U^A\big) - \big(\bar{\nabla}_A U^A\big)\big] = \mathsf{J}H^B{}_{BA}U^A.$$

Therefore, the differential of the volume ratio $\mathrm{d}\mathsf{J}$, with components $\partial_A\mathsf{J}$, can be expressed as[3]

$$\partial_A\mathsf{J} = \mathsf{J}H^B{}_{BA}. \tag{10}$$

2.4. Anelastic deformations

As mentioned earlier, the introduction of the natural moving frame $\{\boldsymbol{e}_\alpha\}$ through the change of frame (equation (2)) provides a passive interpretation of the multiplicative decomposition of deformation gradient, as no transformation has yet been defined on the material manifold. However, the definition of a reference structure, such as the Euclidean structure, allows one to formulate anelasticity starting from a $\binom{1}{1}$-tensor field defined on the material manifold that relates the natural moving frame to the Euclidean moving frame. This tensor is called the local anelastic deformation that maps one structure to the other, i.e., an alternative interpretation of equation (2) for the multiplicative decomposition of deformation gradient.

Let $\{\bar{\partial}_{\bar{A}}\}$ be the orthonormal frame field with respect to $\bar{\boldsymbol{G}}$ induced from the chart Ξ defined previously. We define the $\binom{1}{1}$-tensor A as the linear mapping taking the natural moving frame to the Cartesian frame, viz.

$$\mathsf{A} : \boldsymbol{e}_1 \mapsto \bar{\partial}_1 \,, \boldsymbol{e}_2 \mapsto \bar{\partial}_2 \,, \boldsymbol{e}_3 \mapsto \bar{\partial}_3 \,, \tag{11}$$

and call it the local anelastic deformation. Equation (11) can be expressed compactly as $\mathsf{A} : \boldsymbol{e}_\alpha \mapsto \delta^{\bar{A}}_\alpha \, \bar{\partial}_{\bar{A}}$, where we use the Kronecker delta $\delta^{\bar{A}}_\alpha$ to keep the consistency of indices with their corresponding frames. In terms of co-frames, one writes $\mathsf{A}^\star : \mathrm{d}\bar{X}^{\bar{A}} \mapsto \delta^{\bar{A}}_\alpha \boldsymbol{\vartheta}^\alpha$, where $\mathsf{A}^\star$ is the dual of A.

Remark 1. If the body or one of its parts is allowed to fully relax, i.e., if it can be mapped to $\mathcal{S}$ through an isometric embedding pushing forward $\boldsymbol{G}$ to $\boldsymbol{g}$, then the $\boldsymbol{G}$-orthonormal natural moving frame $\{\boldsymbol{e}_\alpha\}$ will be mapped to a $\boldsymbol{g}$-orthonormal frame in the ambient space. In the material manifold, this relaxation is represented as $\mathsf{A} : \{\boldsymbol{e}_\alpha\} \mapsto \{\delta^{\bar{A}}_\alpha \bar{\partial}_{\bar{A}}\}$, where $\{\bar{\partial}_{\bar{A}}\}$ is a $\boldsymbol{g}$-orthonormal frame pulled back via some global map ψ, which can be taken to be the aforementioned isometry. For this reason, the local anelastic deformation can be seen as a local relaxation map. Vice versa, one can interpret the natural moving frame as the one that, if the body is fully relaxed, becomes orthonormal in the ambient space.

By virtue of equation (11), one can express the change of frame from $\{\partial_A\}$ to $\{\bar{\partial}_{\bar{A}}\}$ in the following two ways:

$$\bar{\partial}_{\bar{A}} = \frac{\partial X^A}{\partial \Xi^{\bar{A}}} \partial_A \qquad \text{and} \qquad \delta^{\bar{A}}_\alpha \, \bar{\partial}_{\bar{A}} = \mathsf{A}^A{}_B (\boldsymbol{e}_\alpha)^B \partial_A = \mathsf{A}^A{}_B (\mathsf{F}^{-1})^B{}_\alpha \partial_A \,.$$

Therefore, the components of A with respect to a frame $\{\partial_A\}$ are related to $[\mathsf{F}^\alpha{}_A]$ as

$$\mathsf{F}^\alpha{}_A = \delta^\alpha_{\bar{A}} \frac{\partial \Xi^{\bar{A}}}{\partial X^B} \mathsf{A}^B{}_A \,, \qquad \mathsf{A}^A{}_B = \frac{\partial X^A}{\partial \Xi^{\bar{A}}} \delta^{\bar{A}}_\alpha \mathsf{F}^\alpha{}_B \,. \tag{12}$$

When one works with Cartesian coordinates, i.e., when $\partial \Xi^{\bar{A}} / \partial X^A = \delta^{\bar{A}}_A$, and $\partial X^A / \partial \Xi^{\bar{A}} = \delta^A_{\bar{A}}$, one has $\mathsf{A}^A{}_B = \delta^A_\alpha \mathsf{F}^\alpha{}_B$, which means that the components of A are given by the matrix $[\mathsf{F}^\alpha{}_A]$.

Finally, from equations (3) and (12), one obtains

$$G_{AB} = \mathsf{F}^\alpha{}_A \mathsf{F}^\beta{}_B \delta_{\alpha\beta} = \delta^\alpha_{\bar{A}} \frac{\partial \Xi^{\bar{A}}}{\partial X^H} \mathsf{A}^H{}_A \delta^\beta_{\bar{B}} \frac{\partial \Xi^{\bar{B}}}{\partial X^K} \mathsf{A}^K{}_B \delta_{\alpha\beta} = \mathsf{A}^H{}_A \mathsf{A}^K{}_B \left(\delta_{\bar{A}\bar{B}} \frac{\partial \Xi^{\bar{A}}}{\partial X^H} \frac{\partial \Xi^{\bar{B}}}{\partial X^K} \right).$$

Note that, since in the frame $\{\bar{\partial}_{\bar{A}}\}$ the Euclidean metric $\bar{\boldsymbol{G}}$ has components $\delta_{\bar{A}\bar{B}}$, in a generic frame $\{\partial_A\}$ the metric $\bar{\boldsymbol{G}}$ is represented by

$$\bar{G}_{AB} = \delta_{\bar{A}\bar{B}} \frac{\partial \Xi^{\bar{A}}}{\partial X^H} \frac{\partial \Xi^{\bar{B}}}{\partial X^K} \,,$$

and therefore one obtains

$$G_{AB} = \mathsf{A}^H{}_A \bar{G}_{HK} \mathsf{A}^K{}_B \,, \qquad \Theta^A{}_B = \bar{G}^{AC} \mathsf{A}^H{}_C \bar{G}_{HK} \mathsf{A}^K{}_B \,. \tag{13}$$

This means that $\boldsymbol{G}$ and $\boldsymbol{\Theta}$ are two different representations of the right Cauchy–Green tensor for the local anelastic deformation A (see Section 3). Moreover, plugging equation (13) into equation (6), one obtains

$$\mathsf{J} = \sqrt{\det \boldsymbol{\Theta}} = \det \mathsf{A},$$

recovering the change of volume defined in classical plasticity.

2.5. The Weitzenböck connection

Given the natural moving frame $\{\boldsymbol{e}_\alpha\}$, one defines its Weitzenböck connection $\hat{\nabla}$ as the connection that makes $\{\boldsymbol{e}_\alpha\}$ parallel.[4] This means that one requires $\hat{\nabla}_{\boldsymbol{U}}\boldsymbol{e}_\alpha = \boldsymbol{0}$ for any vector $\boldsymbol{U}$, $\alpha = 1, 2, 3$. The coefficients of $\hat{\nabla}$ with respect to a generic coordinate chart $\{X^A\}$ are defined as $\hat{\Gamma}^A{}_{BC}\partial_A = \hat{\nabla}_{\partial_B}\partial_C$, and are calculated starting from

$$\hat{\nabla}_{\partial_B}\partial_C = \hat{\nabla}_{\partial_B}(\mathsf{F}^\gamma{}_C\boldsymbol{e}_\gamma) = \partial_B\mathsf{F}^\gamma{}_C\boldsymbol{e}_\gamma + \mathsf{F}^\gamma{}_C\hat{\nabla}_{\partial_B}\boldsymbol{e}_\gamma = \partial_B\mathsf{F}^\gamma{}_C\left(\mathsf{F}^{-1}\right)^A{}_\gamma\partial_A + \mathsf{F}^\gamma{}_C\hat{\nabla}_{\partial_B}\boldsymbol{e}_\gamma.$$

Since, by assumption, $\hat{\nabla}\boldsymbol{e}_\gamma = \boldsymbol{0}$, the last term vanishes, and one obtains

$$\hat{\Gamma}^A{}_{BC} = \left(\mathsf{F}^{-1}\right)^A{}_\alpha\partial_B\mathsf{F}^\alpha{}_C. \tag{14}$$

By construction, the Christoffel symbols with respect to the moving frame $\{\boldsymbol{e}_\alpha\}$ vanish.

Remark 2. To give an interpretation for the Weitzenböck derivative, we compute the components of $\hat{\nabla}\boldsymbol{U}$ for a given vector field $\boldsymbol{U}$. In general, components with respect to the moving frame will be indicated with Greek letters, while those in coordinate charts will be indicated with Latin letters. Note that a covariant derivative in a non-holonomic frame is calculated as $\hat{\nabla}_\beta U^\alpha = \partial_C U^\alpha(\mathsf{F}^{-1})^C{}_\beta + \hat{\Gamma}^\alpha{}_{\beta\gamma}U^\gamma$, where $\hat{\Gamma}^\alpha{}_{\beta\gamma} = 0$, following directly from $\hat{\nabla}_{\boldsymbol{e}_\beta}\boldsymbol{e}_\gamma = \boldsymbol{0}$. Therefore, one obtains

$$\hat{\nabla}_\beta U^\alpha = \partial_C U^\alpha(\mathsf{F}^{-1})^C{}_\beta, \qquad \hat{\nabla}_B U^A = \partial_B U^\alpha(\mathsf{F}^{-1})^A{}_\alpha.$$

This means that the Weitzenböck derivative of a vector can be calculated as the ordinary derivative of its components in the moving frame. This result can be extended to tensors as

$$\hat{\nabla}_\gamma \mathcal{T}^{\alpha_1\ldots}{}_{\beta_1\ldots} = \partial_C\mathcal{T}^{\alpha_1\ldots}{}_{\beta_1\ldots}(\mathsf{F}^{-1})^C{}_\gamma, \qquad \hat{\nabla}_C\mathcal{T}^{A_1\ldots}{}_{B_1\ldots} = \partial_C\mathcal{T}^{\alpha_1\ldots}{}_{\beta_1\ldots}(\mathsf{F}^{-1})^{A_1}{}_{\alpha_1}\ldots\mathsf{F}^{\beta_1}{}_{B_1}\ldots \tag{15}$$

Hence, a tensor field is uniform with respect to $\hat{\nabla}$ if and only if its components with respect to the natural moving frame are uniform. Therefore, $\hat{\nabla}$ can be seen as a natural connection for the body.

The torsion $\hat{\boldsymbol{T}}$ of the Weitzenböck connection has the following components with respect to a coordinate chart $\{X^A\}$:

$$\hat{T}^A{}_{BC} = \left(\mathsf{F}^{-1}\right)^A{}_\alpha(\partial_B\mathsf{F}^\alpha{}_C - \partial_C\mathsf{F}^\alpha{}_B), \tag{16}$$

which, in general, is non-vanishing. Using the symmetry of both connections ∇ and $\bar{\nabla}$, one can express $\hat{\boldsymbol{T}}$ as

$$\hat{T}^\alpha{}_{BC} = \nabla_B\boldsymbol{\vartheta}^\alpha{}_C - \nabla_C\boldsymbol{\vartheta}^\alpha{}_B = \bar{\nabla}_B\boldsymbol{\vartheta}^\alpha{}_C - \bar{\nabla}_C\boldsymbol{\vartheta}^\alpha{}_B. \tag{17}$$

Note that in the active approach, the coefficients of the Weitzenböck connection cannot be written as equation (14), taking Euclidean covariant derivatives of the local deformation A. The reason is that while $(\mathsf{A}^{-1})^A{}_D\,\bar{\nabla}_B\mathsf{A}^D{}_C$ is a tensor, $\hat{\Gamma}^A{}_{BC}$ is not. However, this can be done for the torsion tensor $\hat{\boldsymbol{T}}$, providing an alternative expression to equation (16). As a matter of fact, plugging equations (12) and (4) into equation (17), one obtains

$$\hat{T}^A{}_{BC} = \left(\mathsf{A}^{-1}\right)^A{}_D\left(\bar{\nabla}_B\mathsf{A}^D{}_C - \bar{\nabla}_C\mathsf{A}^D{}_B\right). \tag{18}$$

The same does not hold when using the Riemannian connection. The torsion tensor associated to the Weitzenböck connection expresses the local incompatibility of the anelastic deformation or, equivalently, the non-holonomicity of the natural moving frame $\{\boldsymbol{e}_\alpha\}$. To this extent, one defines the Burgers vector relative to the closed curve $\gamma : [0, 1] \to \mathcal{B}$ as the triplet of scalars

$$\mathsf{b}^\alpha[\gamma] = \oint_\gamma \boldsymbol{\vartheta}^\alpha = \int_0^1 (\mathsf{F}^\alpha{}_A \circ \gamma)\, \mathsf{T}^A \, \mathrm{d}s, \tag{19}$$

where $\mathbf{T}$ is the tangent vector to γ (see Appendix B). If the three scalars $\mathsf{b}^\alpha[\gamma]$ vanish for any γ, then the 1-forms $\boldsymbol{\vartheta}^\alpha$ are exact. This implies the existence of charts $\Xi : \mathcal{B} \supset \mathcal{U} \to \mathbb{R}^3$ such that $\mathrm{d}\Xi^\alpha = \boldsymbol{\vartheta}^\alpha$, or equivalently, $\boldsymbol{e}_\alpha = \partial/\partial\Xi^\alpha$, $\alpha = 1, 2, 3$. When $\mathcal{B}$ is simply connected, a closed 1-form is necessarily exact; therefore, compatibility is equivalent to $\mathrm{d}\boldsymbol{\vartheta}^\alpha = \mathbf{0}$. Conversely, note that

$$\hat{T}^\alpha{}_{BC} = \mathsf{F}^\alpha{}_A \hat{T}^A{}_{BC} = \partial_B \mathsf{F}^\alpha{}_C - \partial_C \mathsf{F}^\alpha{}_B = (\mathrm{d}\boldsymbol{\vartheta}^\alpha)_{BC}, \qquad \hat{T}^\alpha{}_{\beta\gamma}\partial_\alpha = [\boldsymbol{e}_\beta, \boldsymbol{e}_\gamma].$$

Thus, holonomicity can be expressed as $\hat{\boldsymbol{T}} = \mathbf{0}$. The curvature of a Weitzenböck connection, instead, vanishes by construction.

The next two lemmas are well-known results that establish the compatibility between the material metric $\boldsymbol{G}$ and the Weitzenböck connection $\hat{\nabla}$, and a relation between the presence of defects and residual stresses in a solid.

Lemma 2. *The Weitzenböck derivative of the material metric vanishes.*

Proof. By virtue of equation (15), one can write $\hat{\nabla}_C G_{AB} = \partial_C G_{\alpha\beta} \mathsf{F}^\alpha{}_A \mathsf{F}^\beta{}_B$. Hence, using equation (3), one obtains

$$\hat{\nabla}_C G_{AB} = \mathsf{F}^\alpha{}_A \mathsf{F}^\beta{}_B \partial_C \delta_{\alpha\beta} = 0. \qquad \square$$

Lemma 3. *If the Weitzenböck connection is torsion-free, then the curvature of the material Levi–Civita connection, i.e., the material Riemann curvature, vanishes.*

Proof. By virtue of the compatibility of $\hat{\nabla}$ with the material metric $\boldsymbol{G}$ established in Lemma 2, when $\hat{\boldsymbol{T}} = \mathbf{0}$, the Weitzenböck connection $\hat{\nabla}$ is also the Levi–Civita connection associated to $\boldsymbol{G}$, i.e., $\hat{\nabla} = \nabla$. Conversely, the curvature of $\hat{\nabla}$ vanishes by construction. $\square$

Note that the non-vanishing of the Riemannian curvature means that $(\mathcal{B}, \boldsymbol{G})$ is not isometrically embeddable in $(\mathcal{S}, \boldsymbol{g})$, and therefore a non-vanishing curvature is related to the presence of residual stresses in the body, at least from a local perspective (see Remark 6). Thus, Lemma 3 implies that when $\{\boldsymbol{e}_\alpha\}$ is holonomic, the body is stress-free. The converse does not hold. As a matter of fact, there exist incompatible anelastic deformations that leave the body stress-free, e.g., zero-stress distributions of dislocations [19–21], which are called contorted aleotropy by Noll [3].

The contorsion tensor $\boldsymbol{K}$ is defined relative to the Weitzenböck and Levi–Civita connections as $\boldsymbol{K}(\boldsymbol{U}, \boldsymbol{V}) = \hat{\nabla}_{\boldsymbol{U}}\boldsymbol{V} - \nabla_{\boldsymbol{U}}\boldsymbol{V}$, and in components it reads $K^A{}_{BC} = \hat{\Gamma}^A{}_{BC} - \Gamma^A{}_{BC}$. It is straightforward to show that $K^A{}_{BC}$ constitute a tensor, and that they are given by

$$K^A{}_{BC} = \frac{1}{2}\left(\hat{T}^A{}_{BC} - \hat{T}_{BC}{}^A - \hat{T}_{CB}{}^A\right), \tag{20}$$

where indices have been raised and lowered using the material metric $\boldsymbol{G}$. By virtue of the anti-symmetry of the torsion tensor, $\boldsymbol{K}$ satisfies the following two identities:

$$K^B{}_{BA} = \hat{T}^B{}_{BA}, \qquad K^B{}_{AB} = 0. \tag{21}$$

Finally, given a tensor field $\boldsymbol{\mathcal{T}}$, one can write

$$\hat{\nabla}_C \mathcal{T}^{A_1\ldots}{}_{B_1\ldots} - \nabla_C \mathcal{T}^{A_1\ldots}{}_{B_1\ldots} = K^{A_1}{}_{CD} \mathcal{T}^{D\ldots}{}_{B_1\ldots} + \ldots - K^D{}_{CB_1} \mathcal{T}^{A_1\ldots}{}_{D\ldots} - \cdots \tag{22}$$

3. Kinematics

Next we discuss kinematics of anelastic bodies. Looking at embeddings of the material manifold $\mathcal{B}$ in the ambient space $\mathcal{S}$, one can define measures of deformation with respect to both the Riemannian and the Euclidean structures defined in the previous section. By extending the derivatives defined in the previous section to two-point tensors (geometric objects with one leg in the material manifold and one leg in the ambient space), we will be able to obtain the Piola transformation with respect to both structures. Time evolutions of the moving frame and motions are also discussed.

3.1. Measures of deformation

Let $\varphi : \mathcal{B} \to \mathcal{S}$ be an embedding representing a configuration of the body with deformation gradient $\boldsymbol{F}$, defined as the tangent map $T\varphi$ and representing a two-point tensor with components $F^a{}_A = \partial x^a / \partial X^A$ with respect to the two charts $\{x^a\}$ and $\{X^A\}$ on $\mathcal{S}$ and $\mathcal{B}$, respectively. We denote by $\boldsymbol{C}^\flat = \varphi^* \boldsymbol{g}$ the pull-back of the ambient space metric using φ. This metric is flat by construction. In components, $C_{AB} = F^a{}_A F^b{}_B g_{ab}$. Starting from this object, which is independent of any material metric or connection defined on $\mathcal{B}$, one can define the right Cauchy–Green tensors $\boldsymbol{C}$ and $\bar{\boldsymbol{C}}$ referred to the Riemannian and Euclidean metric tensors as

$$C^A{}_B = G^{AD} C_{DB} = G^{AD} F^a{}_D g_{ab} F^b{}_B \,, \qquad \bar{C}^A{}_B = \bar{G}^{AD} C_{DB} = \bar{G}^{AD} F^a{}_D g_{ab} F^b{}_B \,.$$

We indicate with $\boldsymbol{F}^\mathsf{T}$ the adjoint of $\boldsymbol{F}$ with respect to $\boldsymbol{G}$ and, similarly, with $\bar{\boldsymbol{F}}^\mathsf{T}$ the adjoint of $\boldsymbol{F}$ with respect to $\bar{\boldsymbol{G}}$. In components,

$$\left(F^\mathsf{T}\right)^A{}_a = g_{ab} F^b{}_B G^{AB} = F_a{}^A \,, \qquad \left(\bar{F}^\mathsf{T}\right)^A{}_a = \bar{G}^{AB} g_{ab} F^b{}_B = \bar{F}_a{}^A \,. \tag{23}$$

Then one can write the right Cauchy–Green tensors as $\boldsymbol{C} = \boldsymbol{F}^\mathsf{T} \boldsymbol{F}$ and $\bar{\boldsymbol{C}} = \bar{\boldsymbol{F}}^\mathsf{T} \bar{\boldsymbol{F}}$. Recalling the change of metric tensor $\boldsymbol{\Theta}$ defined in equation (5), one has $\bar{\boldsymbol{C}} = \boldsymbol{\Theta} \boldsymbol{C}$. It is possible to extend the previous definitions formally to anelastic deformations and obtain equation (13), suggesting that the material metric and the change of metric represent the pulled-back metric and the right Cauchy–Green tensor for the local anelastic deformation A. Note that $\boldsymbol{C}$ is self-adjoint with respect to $\boldsymbol{G}$ and $\bar{\boldsymbol{C}}$ is self-adjoint with respect to $\bar{\boldsymbol{G}}$, while $\boldsymbol{\Theta}$ is self-adjoint with respect to both $\boldsymbol{G}$ and $\bar{\boldsymbol{G}}$.

3.2. Elastic deformations

Next we provide some insight on elastic measures of deformations in relation to the total measures previously defined with respect to the two structures. In the active approach involving the anelastic deformation A, one can define the elastic deformation as $\mathsf{E} = \boldsymbol{F}\mathsf{A}^{-1}$, which is equivalent to the classical multiplicative decomposition of the deformation gradient $\boldsymbol{F} = \mathsf{E}\mathsf{A}$. Therefore, one has

$$\{\boldsymbol{e}_\alpha\} \overset{\mathsf{A}}{\mapsto} \left\{\delta_\alpha^{\bar{A}} \bar{\partial}_{\bar{A}}\right\} \overset{\mathsf{E}}{\mapsto} \{\boldsymbol{F}\boldsymbol{e}_\alpha\} \,, \qquad \{\boldsymbol{F}^{-\star} \boldsymbol{\vartheta}^\alpha\} \overset{\mathsf{E}^\star}{\mapsto} \left\{\delta^\alpha{}_{\bar{A}}\, \mathrm{d}\bar{X}^{\bar{A}}\right\} \overset{\mathsf{A}^\star}{\mapsto} \{\boldsymbol{\vartheta}^\alpha\} \,,$$

where, of course, $\boldsymbol{F}^\star = \mathsf{A}^\star \mathsf{e}^\star$.

One defines the elastically pulled-back metric $\boldsymbol{\mathcal{C}}^\flat$ with components $\mathcal{C}_{AB} = \mathsf{E}^a{}_A g_{ab} \mathsf{E}^b{}_B$, which is related to $\boldsymbol{C}^\flat$ as $C_{AB} = \mathsf{A}^H{}_A \mathcal{C}_{HK} \mathsf{A}^K{}_B$, or simply $\boldsymbol{C}^\flat = \mathsf{A}^\star \boldsymbol{\mathcal{C}}^\flat \mathsf{A}$. This can also be written as

$$\delta_\alpha^{\bar{A}} \delta_\beta^{\bar{B}} \boldsymbol{\mathcal{C}}^\flat(\bar{\partial}_{\bar{A}}, \bar{\partial}_{\bar{B}}) = \boldsymbol{C}^\flat(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) \,. \tag{24}$$

The elastic right Cauchy–Green strain tensor $\bar{\boldsymbol{\mathcal{C}}}$ is defined as $\bar{\mathcal{C}}^A{}_B = \bar{G}^{AD} \mathcal{C}_{DB}$, for which the Euclidean metric is used to raise one of the two indices. From equation (13), one obtains $C^A{}_B = (\mathsf{A}^{-1})^A{}_H \bar{\mathcal{C}}^H{}_K \mathsf{A}^K{}_B$, or simply $\boldsymbol{C} = \mathsf{A} \bar{\boldsymbol{\mathcal{C}}} \mathsf{A}^{-1}$.

Finally, the Jacobian function associated to φ can be defined with respect to either $\boldsymbol{G}$ or $\bar{\boldsymbol{G}}$ as

$$J = \sqrt{\det \boldsymbol{C}} = \sqrt{\det \bar{\boldsymbol{\mathcal{C}}}} \,, \qquad \bar{J} = \sqrt{\det \bar{\boldsymbol{C}}} = \sqrt{\det \boldsymbol{\Theta}} \sqrt{\det \boldsymbol{C}} = \det \mathsf{A} \sqrt{\det \boldsymbol{C}} = \mathsf{J} J \,.$$

In the decomposition of the deformation gradient, J represents the change of volume due to anelastic deformations, J represents the change of volume due to elastic deformations, and $\bar{J}$ represents the total change of volume. Using Cartesian coordinates Ξ in the material manifold and Cartesian coordinates ξ in the ambient space, both $\bar{\boldsymbol{G}}$ and $\boldsymbol{g}$ are represented by identity matrices; thus, $\det \bar{\boldsymbol{C}} = (\det \boldsymbol{F})^2$. Hence, one obtains the classical relations $\bar{J} = \det \boldsymbol{F}$ and $J = \det \boldsymbol{F} / \det \mathbf{A} = \det \mathbf{E}$.

3.3. The Piola transformation in the two structures

We denote by $\gamma^a{}_{bc}$ the Christoffel symbols for the manifold $(\varphi(\mathcal{B}), \boldsymbol{G})$, and following Marsden and Hughes [22] we extend the Riemannian, Euclidean, and Weitzenböck derivations to two-point tensors as

$$\begin{aligned}
\nabla_C \mathcal{T}^A{}_B{}^a{}_b &= \partial_C \mathcal{T}^A{}_B{}^a{}_b + \Gamma^A{}_{CD}\mathcal{T}^D{}_B{}^a{}_b - \Gamma^D{}_{CB}\mathcal{T}^A{}_D{}^a{}_b + F^c{}_C\gamma^a{}_{cd}\mathcal{T}^A{}_B{}^d{}_b - F^c{}_C\gamma^d{}_{cb}\mathcal{T}^A{}_B{}^a{}_d\,,\\
\hat{\nabla}_C \mathcal{T}^A{}_B{}^a{}_b &= \partial_C \mathcal{T}^A{}_B{}^a{}_b + \hat{\Gamma}^A{}_{CD}\mathcal{T}^D{}_B{}^a{}_b - \hat{\Gamma}^D{}_{CB}\mathcal{T}^A{}_D{}^a{}_b + F^c{}_C\gamma^a{}_{cd}\mathcal{T}^A{}_B{}^d{}_b - F^c{}_C\gamma^d{}_{cb}\mathcal{T}^A{}_B{}^a{}_d\,,\\
\bar{\nabla}_C \mathcal{T}^A{}_B{}^a{}_b &= \partial_C \mathcal{T}^A{}_B{}^a{}_b + \bar{\Gamma}^A{}_{CD}\mathcal{T}^D{}_B{}^a{}_b - \bar{\Gamma}^D{}_{CB}\mathcal{T}^A{}_D{}^a{}_b + F^c{}_C\gamma^a{}_{cd}\mathcal{T}^A{}_B{}^d{}_b - F^c{}_C\gamma^d{}_{cb}\mathcal{T}^A{}_B{}^a{}_d\,.
\end{aligned} \tag{25}$$

One can extend equation (8) to two-point tensors, as

$$\mathsf{J}\nabla_B\left(\mathcal{T}^{a_1\cdots}{}_{b_1\ldots}{}^B\right) = \bar{\nabla}_B\left(\mathsf{J}\mathcal{T}^{a_1\cdots}{}_{b_1\ldots}{}^B\right), \tag{26}$$

which is valid only for two-point tensors with only one material upper index, e.g., the first Piola–Kirchhoff stress tensor. The Riemannian, Weitzenböck, and Euclidean derivatives of a two-point tensor are therefore related through the contorsion tensors $\boldsymbol{H}$ and $\boldsymbol{K}$, operating only on the material indices as equations (9) and (22).

Two-point derivation can be applied to the deformation gradient $\boldsymbol{F}$, as in the case of compatibility equations. The deformation gradient must satisfy local compatibility equations, which are written as $\partial_A F^a{}_B = \partial_B F^a{}_A$.[5] Compatibility equations can also be written in terms of the two-point derivatives of equation (25) as

$$\nabla_A F^a{}_B = \nabla_B F^a{}_A\,, \qquad \bar{\nabla}_A F^a{}_B = \bar{\nabla}_B F^a{}_A\,, \qquad \hat{\nabla}_B F^a{}_A - \hat{\nabla}_A F^a{}_B = \hat{T}^D{}_{AB} F^a{}_D\,, \tag{27}$$

where the symmetry of the Levi–Civita connections in both the material manifold and in the ambient space, and the definition of torsion tensor $\hat{T}^D{}_{AB} = \hat{\Gamma}^D{}_{AB} - \hat{\Gamma}^D{}_{BA}$ are used. It should be emphasized that while the deformation gradient $\boldsymbol{F}$ is constructed in such a way as to be necessarily compatible, the anelastic deformation $\mathbf{A}$ can be either compatible or incompatible (see equation (18)). The differential $\mathrm{d}J = \partial_A J\, \mathrm{d}X^A$ can be written as

$$\partial_A J = \frac{1}{2} J \left(C^{-1}\right)^D{}_B \partial_A C^B{}_D = \frac{1}{2} J \left(C^{-1}\right)^D{}_B \nabla_A C^B{}_D = \frac{1}{2} J \left(C^{-1}\right)^{BD} \nabla_A C_{BD} = J F^B{}_b \nabla_A F^b{}_B\,, \tag{28}$$

where the second equality follows from a direct computation, the third from the compatibility of the Riemannian connection with the material metric $\boldsymbol{G}$, and the fourth from the compatibility of the ambient space connection with the spatial metric $\boldsymbol{g}$. Note that $F^B{}_b(\hat{\nabla}_A F^b{}_B - \nabla_A F^b{}_B) = K^B{}_{AB}$, which vanishes by virtue of equation (21). This allows one to write

$$\partial_A J = \frac{1}{2} J \left(C^{-1}\right)^D{}_B \hat{\nabla}_A C^B{}_D = \frac{1}{2} J \left(C^{-1}\right)^{BD} \hat{\nabla}_A C_{BD} = J F^B{}_b \hat{\nabla}_A F^b{}_B\,,$$

where the compatibility of the Weitzenböck connection with the material metric $\boldsymbol{G}$ established in Lemma 2 was used. We will see that this implies that the Jacobian is an isotropic and naturally uniform function. Using the Euclidean connection instead, one can write the differential $\mathrm{d}\bar{J}$ as

$$\partial_A \bar{J} = \frac{1}{2} \bar{J} \left(\bar{C}^{-1}\right)^D{}_B \partial_A \bar{C}^B{}_D = \frac{1}{2} \bar{J} \left(\bar{C}^{-1}\right)^D{}_B \bar{\nabla}_A \bar{C}^B{}_D = \frac{1}{2} J \left(C^{-1}\right)^{BD} \bar{\nabla}_A C_{BD} = \bar{J} F^B{}_b \bar{\nabla}_A F^b{}_B\,. \tag{29}$$

From equation (28), and using the compatibility of $\boldsymbol{F}$, one obtains

$$\nabla_B\left(JF^B{}_b\right) = \partial_B JF^B{}_b + J\nabla_B F^B{}_b = JF^C{}_c F^B{}_b \nabla_B F^c{}_C - JF^C{}_b F^B{}_c \nabla_B F^c{}_C .$$

The same holds in the Euclidean structure by equation (29), so that one has the identities

$$\nabla_B\left(JF^B{}_b\right) = 0, \qquad \bar{\nabla}_B\left(\bar{J}F^B{}_b\right) = 0,$$

which imply

$$\nabla_A\left(JF^A{}_a u^a\right) = J(\nabla_a u^a)\circ\varphi, \qquad \bar{\nabla}_A\left(\bar{J}F^A{}_a u^a\right) = \bar{J}(\nabla_a u^a)\circ\varphi. \tag{30}$$

Note that from the definition (equation (25)) of two-point derivatives, one has $\nabla u^a = \hat{\nabla} u^a = \bar{\nabla} u^a$, as u^a has no material index. Equation (30) allows one to define the Piola transformation of the spatial vector field $\boldsymbol{u}$ with respect to the Riemannian structure as $\boldsymbol{U} = J\varphi^*\boldsymbol{u}$, and the Piola transformation with respect to the Euclidean structure as $\bar{\boldsymbol{U}} = \bar{J}\varphi^*\boldsymbol{u}$. The two transformations are related as $\bar{\boldsymbol{U}} = \mathsf{J}\boldsymbol{U}$.

3.4. Rate of anelastic deformations

We first look at changes of the anelastic state of the body. A time evolution of the natural frame is a smooth map $t \mapsto \{\boldsymbol{e}_\alpha(t)\}$ or, equivalently, $t \mapsto \{\boldsymbol{\vartheta}^\alpha(t)\}$. We define the following $\binom{1}{1}$-tensor field:

$$\mathsf{L} = \boldsymbol{e}_\alpha \otimes \dot{\boldsymbol{\vartheta}}^\alpha = -\,\dot{\boldsymbol{e}}_\alpha \otimes \boldsymbol{\vartheta}^\alpha, \tag{31}$$

mapping a natural frame to the negative of its derivative, i.e., $\mathsf{L} : \boldsymbol{e}_\alpha \mapsto -\dot{\boldsymbol{e}}_\alpha$.[6] In components, one has $\mathsf{L}^A{}_B = (\mathsf{F}^{-1})^A{}_\alpha \dot{\mathsf{F}}^\alpha{}_B = -\,(\dot{\mathsf{F}}^{-1})^A{}_\alpha \mathsf{F}^\alpha{}_B$.[7] From the active perspective, recalling equation (12), equation (31) can be written as

$$\mathsf{L}^A{}_B = (\mathsf{A}^{-1})^A{}_C \frac{\partial X^C}{\partial \Xi^\alpha}\left(\frac{\partial \Xi^\alpha}{\partial X^D}\mathsf{A}^D{}_B\right)^{\cdot} = (\mathsf{A}^{-1})^A{}_C \dot{\mathsf{A}}^C_B,$$

as $\dot{\mathsf{A}}^A{}_C = (\partial X^A/\partial\Xi^\alpha)\dot{\mathsf{F}}^\alpha{}_B$ because the charts $\{\Xi^\alpha\}$ and $\{X^A\}$ do not depend on t. Hence, we have obtained $\mathsf{L} = \mathsf{A}^{-1}\dot{\mathsf{A}}$, and therefore L is called the rate of anelastic deformation. A time-dependent natural moving frame defines a time-dependent Riemannian structure, whereas the Euclidean structure is time-independent. The time-derivative of the material metric $\boldsymbol{G}(t)$ can be calculated by plugging equation (31) into equation (3), viz.

$$\dot{G}_{AB} = \left(\dot{\mathsf{F}}^\alpha{}_A \mathsf{F}^\beta{}_B + \mathsf{F}^\alpha{}_A \dot{\mathsf{F}}^\beta{}_B\right)\delta_{\alpha\beta} = G_{AD}\mathsf{L}^D{}_B + G_{BD}\mathsf{L}^D{}_A .$$

As for the volume ratio $\mathsf{J}(t)$ and Riemannian volume element $\boldsymbol{\mu}(t)$, using the chain rule, one obtains $(\det \boldsymbol{G})^{\cdot} = (\det \boldsymbol{G}) G^{AB}\dot{G}_{AB}$, allowing one to write

$$\dot{\mathsf{J}} = \frac{1}{2}\mathsf{J}G^{AB}\dot{G}_{AB}, \qquad \dot{\boldsymbol{\mu}} = \frac{1}{2}G^{AB}\dot{G}_{AB}\boldsymbol{\mu},$$

as $\bar{\boldsymbol{G}}$ is independent of t. Note that the quantity $\ell = \dot{\mathsf{J}}/\mathsf{J}$, which represents the variation of the Riemannian volume form, does not depend on the reference Euclidean structure, as $\delta\boldsymbol{\mu} = \ell\boldsymbol{\mu}$. Moreover, for the mass density, one has $\dot{\varrho}_o = -\,\ell\varrho_o$. From equation (31), one has $\ell = \frac{1}{2}G^{AB}\dot{G}_{AB} = \mathsf{L}^A{}_A = \mathrm{tr}\mathsf{L}$. Volume-preserving (or isochoric) variations of $\boldsymbol{G}$ are such that $\boldsymbol{\mu}_s$ is constant for every s, and therefore $\ell = \mathrm{tr}\mathsf{L} = 0$. Isochoric plastic flows are very important in plasticity [12, 23]. However, volumetric effects are key for problems involving geomaterials [16, 24].

Finally, we indicate by $\mathscr{C}$ the set of all embeddings $\mathcal{B} \to \mathcal{S}$. A motion is a smooth curve $\varphi : \mathbb{R}^+ \to \mathscr{C}$, $t \mapsto \varphi_t$. The velocity of a motion φ_t is a vector field $\boldsymbol{V} : \mathcal{B}\times\mathbb{R}^+ \to T\mathcal{S}$ defined as the tangent vector to the curves $t \mapsto \varphi_t(X)$. At time t, it can be expressed as a vector field $\boldsymbol{v}_t$ on $\varphi_t(\mathcal{B})$, given by $\boldsymbol{v}_t(x) = \boldsymbol{V}(\varphi_t^{-1}(x), t)$. The acceleration is a vector field $\boldsymbol{A} : \mathcal{B}\times\mathbb{R}^+ \to T\mathcal{S}$ defined as $\boldsymbol{A} = \nabla^{g}_{\boldsymbol{V}}\boldsymbol{V}$, inducing a vector field $\boldsymbol{a}_t$ on $\varphi_t(\mathcal{B})$ given by $\boldsymbol{a}_t(x) = \boldsymbol{A}(\varphi_t^{-1}(x), t)$. The deformation gradient $\boldsymbol{F}(X, t)$ is defined as the tangent map $T_X\varphi_t$.

Note that velocities and accelerations have been defined independently of any material structure, so the evolution of the natural moving frame is not involved in their definition.

4. Stress tensors in anelasticity

In this section, we define energy functions and stress tensors with respect to both the Riemannian and the Euclidean structures. Our goal is to compare the classical (Euclidean) formulation with the geometric (Riemannian) formulation of anelasticity. Working in the context of hyperelasticity, we start by assuming the existence of an energy function that depends only on the elastic part of the deformation, i.e., a function of distances in the deformed configuration as they are seen by an observer in the natural frame, and deriving other energy functions and related stress tensors from it. We discuss uniformity of the energy function with respect to the natural moving frame, and extend this concept to the Riemannian and Euclidean structures. Finally, we discuss stress tensors and uniformity in the case of isotropic hyperelastic materials.

4.1. Energy functions

We start by assuming the existence of an energy function that depends on distances in the deformed configuration as they are seen by an observer in the natural frame. In particular, indicating by $\mathsf{Pos}(3)$ the space of symmetric positive-definite 3×3 matrices, we assume the existence of a smooth function $\mathsf{W} : \mathcal{B} \times \mathsf{Pos}(3) \to \mathbb{R}$. The energy density per unit volume $W : \mathcal{B} \to \mathbb{R}$ is defined by evaluating the energy function W at $C_{\alpha\beta} = \boldsymbol{C}^{\flat}(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta)$,[8] representing the length and angles between natural frame vectors according to the pulled-back metric, i.e., $W(X) = \mathsf{W}(X, C_{\alpha\beta}(X))$. This is equivalent to assuming that the energy only depends on the elastic part of the deformation. This also means that at each point X the function W only depends on the values of $C_{\alpha\beta}$ at X. If this holds, the material is called simple. We also define the energy density per unit Euclidean volume, $\bar{W} = \mathsf{J}W$.

Remark 3. One assumes that at each point X the function W attains its minimum at $\delta_{\alpha\beta}$. This means that, at each point X, the function $\mathbb{W}_{\mathrm{X}}$ attains its minimum for $\boldsymbol{C}^{\flat} = \boldsymbol{G}$. Therefore, the stored elastic energy of a hyper-anelastic body is minimized by configurations that preserve the material metric $\boldsymbol{G}$, i.e., by maps $\psi : \mathcal{B} \to \mathcal{S}$ satisfying $\boldsymbol{G} = \psi^*\boldsymbol{g}$. In other words, the material metric is always encoded in the energy function because it is required to attain its minimum for $\boldsymbol{G}$-preserving configurations. However, we will see that $\boldsymbol{G}$ is not enough to fully characterize the constitutive equations of the material, as a natural moving frame is also needed.

We indicate by $F\mathcal{B}$ the bundle of frames, made of quadruplets $(X, \{\boldsymbol{v}_\beta\})$, where $\{\boldsymbol{v}_\beta\}$ indicates a triplet of linearly independent vectors in $T_X\mathcal{B}$. In a similar way, we indicate by $F^*\mathcal{B}$ the bundle of co-frames, made of quadruplets $(X, \{\boldsymbol{\lambda}^\beta\})$, where the $\{\boldsymbol{\lambda}^\beta\}$ are linearly independent 1-forms in $T^*_X\mathcal{B}$. Finally, we indicate by $P\mathcal{B}$ the tensor bundle of symmetric positive-definite $\binom{0}{2}$-tensors (e.g., metric tensors), made of all the pairs $(X, \boldsymbol{A})$. Then, the energy density for an anelastic body can be expressed by a smooth function $\mathbb{W} : F^*\mathcal{B} \times S\mathcal{B} \to \mathbb{R}$, such that

$$\mathbb{W}\big(X, \{\boldsymbol{\lambda}^\beta\}, \boldsymbol{A}\big) = \mathsf{W}(X, \boldsymbol{A}(\boldsymbol{v}_\alpha, \boldsymbol{v}_\beta))\,, \tag{32}$$

with $\langle \boldsymbol{\lambda}^\beta, \boldsymbol{v}_\beta \rangle = \delta^\alpha_\beta$. We also define the energy function

$$\bar{\mathbb{W}}\big(X, \{\boldsymbol{\lambda}^\beta\}, \boldsymbol{A}\big) = \mathsf{J}\big(X, \{\boldsymbol{\lambda}^\beta\}\big)\,\mathbb{W}\big(X, \{\boldsymbol{\lambda}^\beta\}, \boldsymbol{A}\big)\,, \tag{33}$$

where the function J is simply equation (6) evaluated at $\delta_{\alpha\beta}\boldsymbol{\lambda}^\alpha \otimes \boldsymbol{\lambda}^\beta$, in agreement with equation (1). The energy densities are then recovered by evaluating $\mathbb{W}$ and $\bar{\mathbb{W}}$ at the natural moving co-frame and at the pulled-back metric, viz.

$$W(X) = \mathbb{W}\Big(X, \{\boldsymbol{\vartheta}^\beta(X)\}, \boldsymbol{C}^\flat(X)\Big), \qquad \bar{W}(X) = \bar{\mathbb{W}}\Big(X, \{\boldsymbol{\vartheta}^\beta(X)\}, \boldsymbol{C}^\flat(X)\Big)\,. \tag{34}$$

Remark 4. In anelasticity, one may be given a material metric $\boldsymbol{G}$ representing the natural distances in the body without specifying a natural moving frame, although such a metric tensor can be induced by infinitely many moving frames. This is admissible for isotropic materials, whose elastic state depends only on $\boldsymbol{G}$, see Section 4.5. However, from equation (34), one observes that energy depends on the material co-frame $\{\boldsymbol{\vartheta}^\beta\}$, meaning that, in the case of anisotropic materials, different choices of natural frames inducing the same $\boldsymbol{G}$ might induce different anisotropy directions in the body. Thus, providing only a material metric, as suggested by Simo [23], is not sufficient for completely characterizing the mechanical response of an anelastic body.[9]

Finally, fixing a point $X \in \mathcal{B}$, we say that a matrix $[Q^\alpha{}_\beta]$ is a material symmetry at X if

$$\mathsf{W}\left(X, Q^\mu{}_\alpha C_{\mu\nu} Q^\nu{}_\beta\right) = \mathsf{W}(X, C_{\alpha\beta}), \tag{35}$$

or equivalently, if

$$\mathbb{W}\left(X, \left\{Q^\alpha{}_\beta \boldsymbol{\lambda}^\beta\right\}, \boldsymbol{C}^\flat\right) = \mathbb{W}\left(X, \{\boldsymbol{\lambda}^\alpha\}, \boldsymbol{C}^\sharp\right), \qquad \bar{\mathbb{W}}\left(X, \left\{Q^\alpha{}_\beta \boldsymbol{\lambda}^\beta\right\}, \boldsymbol{C}^\flat\right) = \bar{\mathbb{W}}\left(X, \{\boldsymbol{\lambda}^\alpha\}, \boldsymbol{C}^\flat\right). \tag{36}$$

The set of material symmetries at X forms the symmetry group and we denote it by $\mathcal{G}_X$. Note that

$$Q^\mu{}_\alpha C_{\mu\nu} Q^\nu{}_\beta = \left(\mathsf{F}^{-1}\right)^A{}_\alpha Q^M{}_A C_{MN} Q^N{}_B \left(\mathsf{F}^{-1}\right)^B{}_\beta, \qquad C_{\alpha\beta} = \left(\mathsf{F}^{-1}\right)^A{}_\alpha C_{AB} \left(\mathsf{F}^{-1}\right)^B{}_\beta.$$

Thus, we define the material symmetry group $\mathfrak{G}_X$ for $T_X\mathcal{B}$, made of $\binom{1}{1}$-tensors $\boldsymbol{Q} : T_X\mathcal{B} \to T_X\mathcal{B}$, such that

$$\mathbb{W}\left(X, \{\boldsymbol{\lambda}^\alpha\}, \boldsymbol{Q}^\star \boldsymbol{C}^\flat \boldsymbol{Q}\right) = \mathbb{W}\left(X, \{\boldsymbol{\lambda}^\alpha\}, \boldsymbol{C}^\flat\right) \qquad \text{or} \qquad \mathbb{W}\left(X, \{\boldsymbol{Q}^\star \boldsymbol{\lambda}^\alpha\}, \boldsymbol{C}^\flat\right) = \mathbb{W}\left(X, \{\boldsymbol{\lambda}^\alpha\}, \boldsymbol{C}^\flat\right).$$

This means that $\boldsymbol{Q} \in \mathfrak{G}_X$ if and only if $\boldsymbol{Q} = Q^\alpha{}_\beta \boldsymbol{e}_\alpha(X) \otimes \boldsymbol{\vartheta}^\beta(X)$ for some $[Q^\alpha{}_\beta] \in \mathscr{G}$. Hence, the material symmetry group $\mathfrak{G}_X$ for $T_X\mathcal{B}$ depends on the natural moving frame, as already observed in Wang and Bloom [12]. For a discussion on material symmetry and defects in solids see Golgoon and Yavari [25].

Remark 5. The natural moving frame is interpreted as an elastic observer, i.e., the frame where W sees deformations. However, such a moving frame is, in general, not uniquely defined. Given a material symmetry group $\mathscr{G}$, let us assume that two different moving frames $\{\boldsymbol{e}_\alpha\}$, $\{\tilde{\boldsymbol{e}}_\alpha\}$ are related by a change of frame $Q^\alpha{}_\beta \in \mathscr{G}$. Then, from equation (35), the energy functions built using the two moving frames are the same. Moreover, the two frames induce the same $\mathfrak{G}$ for the energy function. This means that there is a $\mathscr{G}$-ambiguity in the choice of the natural moving frame [12]. In other words, the natural moving frame is not uniquely defined; it is represented by an equivalence class that depends on the material symmetries of the body. This concept is discussed in Epstein and Maugin [2]. Note also that for solids $\mathscr{G} \subset \mathrm{SO}(3)$; hence, the material metric $\boldsymbol{G}$ is not affected by this ambiguity. Finally, note that any frame $\{\tilde{\boldsymbol{e}}_\alpha\} = \left\{A^\beta{}_\alpha \boldsymbol{e}_\beta\right\}$ gives the same Weitzenböck connection as $\{\boldsymbol{e}_\alpha\}$ does as long as the $A^\beta{}_\alpha$ are constant.[10] This means that if the symmetry group does not change from point to point, the natural derivative is not affected by this $\mathscr{G}$-ambiguity either.

4.2. Stress tensors

Next we look at derivatives of the energy functions $\mathbb{W}$ and $\bar{\mathbb{W}}$, keeping the base point fixed, i.e., letting the two arguments $\{\boldsymbol{\lambda}^\alpha\}$ and $\boldsymbol{A}$ of equations (32) and (33) change. All the derivatives are evaluated at a given pair of fields $(\{\boldsymbol{\vartheta}^\alpha\}, \boldsymbol{C}^\flat)$.[11] The following stress tensors are defined:

$$\boldsymbol{S} = 2\frac{\partial \mathbb{W}}{\partial \boldsymbol{C}^\flat}, \qquad \boldsymbol{Y}_\alpha = \frac{\partial \mathbb{W}}{\partial \boldsymbol{\vartheta}^\alpha}, \qquad \bar{\boldsymbol{S}} = 2\frac{\partial \bar{\mathbb{W}}}{\partial \boldsymbol{C}^\flat}, \qquad \bar{\boldsymbol{Y}}_\alpha = \frac{\partial \bar{\mathbb{W}}}{\partial \boldsymbol{\vartheta}^\alpha}. \tag{37}$$

The tensors $\boldsymbol{S}$ and $\bar{\boldsymbol{S}}$ are the second Piola–Kirchhoff stresses referred to the Riemannian and Euclidean structures, respectively, while the vectors $\boldsymbol{Y}_\alpha$ and $\bar{\boldsymbol{Y}}_\alpha$, $\alpha = 1, 2, 3$, represent material stresses that are dual to changes of the corresponding natural frame co-vector.[12] The symmetry of both $\boldsymbol{S}$ and $\bar{\boldsymbol{S}}$ is guaranteed by construction, and this in turn guarantees the satisfaction of the balance of angular momentum. Since the volume ratio between the two structures J does not depend on $\boldsymbol{C}^\flat$, one has

$$\bar{S}^{AB} = 2\frac{\partial\bar{\mathbb{W}}}{\partial C_{AB}} = 2\frac{\partial(\mathsf{J}\mathbb{W})}{\partial C_{AB}} = 2\mathsf{J}\frac{\partial\mathbb{W}}{\partial C_{AB}} = \mathsf{J}S^{AB}.$$

Moreover, recalling the definition (equation (6)) of J, and the relation between the material metric and the natural frame $G_{AB} = \mathsf{F}^\alpha{}_A\mathsf{F}^\beta{}_B\delta_{\alpha\beta}$ provided by equation (3), one can write

$$\begin{aligned}\bar{Y}_\alpha{}^A &= \frac{\partial\mathsf{J}}{\partial\mathsf{F}^\alpha{}_A}W + \mathsf{J}\frac{\partial\mathbb{W}}{\partial\mathsf{F}^\alpha{}_A}\\ &= \frac{1}{2}\mathsf{J}G^{MN}\left(\mathsf{F}^\nu{}_M\delta_{\alpha\nu}\delta^A{}_N + \mathsf{F}^\nu{}_M\delta_{\alpha\nu}\delta^A{}_N\right)W + \mathsf{J}Y_\alpha{}^A\\ &= \bar{W}\mathsf{F}^A{}_\alpha + \mathsf{J}Y_\alpha{}^A.\end{aligned}$$

Therefore, we have obtained the following relations between the objects that were defined in equation (37):

$$\bar{\boldsymbol{S}} = \mathsf{J}\boldsymbol{S}, \qquad \bar{\boldsymbol{Y}}_\alpha = \bar{W}\boldsymbol{e}_\alpha + \mathsf{J}\boldsymbol{Y}_\alpha. \tag{38}$$

Starting from $\boldsymbol{Y}_\alpha$ and $\bar{\boldsymbol{Y}}_\alpha$ the following material stress tensors of type $\binom{1}{1}$ are defined:

$$Z_A{}^B = \mathsf{F}^\alpha{}_A Y_\alpha{}^B, \qquad \bar{Z}_A{}^B = \mathsf{J}Z_A{}^B, \qquad E_A{}^B = \frac{1}{\mathsf{J}}\bar{E}_A{}^B, \qquad \bar{E}_A{}^B = \mathsf{F}^\alpha{}_A\bar{Y}_\alpha{}^B.$$

Their $\binom{2}{0}$ counterparts are defined using the material metric tensors $\boldsymbol{G}$ and $\bar{\boldsymbol{G}}$, viz.

$$\begin{aligned} Z^{AB} &= G^{AC}Z_C{}^B, \qquad & \bar{Z}^{AB} &= \bar{G}^{AC}\bar{Z}_C{}^B = \mathsf{J}\Theta^A{}_C Z^{CB},\\ E^{AB} &= G^{AC}E_C{}^B, \qquad & \bar{E}^{AB} &= \bar{G}^{AC}\bar{E}_C{}^B = \mathsf{J}\Theta^A{}_C E^{CB}.\end{aligned} \tag{39}$$

In component-free notation, they read

$$\begin{aligned}\boldsymbol{Z} &= \boldsymbol{G}^\sharp\left(\boldsymbol{\vartheta}^\beta\otimes\boldsymbol{Y}_\beta\right) = \delta^{\alpha\beta}\boldsymbol{e}_\alpha\otimes\boldsymbol{Y}_\beta, \quad \bar{\boldsymbol{Z}} = \mathsf{J}\bar{\boldsymbol{G}}^\sharp\left(\boldsymbol{\vartheta}^\beta\otimes\boldsymbol{Y}_\beta\right) = \mathsf{J}\delta^{\alpha\beta}(\boldsymbol{\Theta}\boldsymbol{e}_\alpha)\otimes\boldsymbol{Y}_\beta,\\ \boldsymbol{e} &= \frac{1}{\mathsf{J}}\boldsymbol{G}^\sharp\left(\boldsymbol{\vartheta}^\beta\otimes\bar{\boldsymbol{Y}}_\beta\right) = \frac{1}{\mathsf{J}}\delta^{\alpha\beta}\boldsymbol{e}_\alpha\otimes\bar{\boldsymbol{Y}}_\beta, \quad \bar{\boldsymbol{E}} = \bar{\boldsymbol{G}}^\sharp\left(\boldsymbol{\vartheta}^\beta\otimes\bar{\boldsymbol{Y}}_\beta\right) = \delta^{\alpha\beta}(\boldsymbol{\Theta}\boldsymbol{e}_\alpha)\otimes\bar{\boldsymbol{Y}}_\beta.\end{aligned} \tag{40}$$

The $\binom{2}{0}$-tensors $\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ are the negative of the Mandel stress referred to the Riemannian and Euclidean structures, respectively, while $\boldsymbol{E}$ and $\bar{\boldsymbol{E}}$ are the Eshelby or energy–momentum tensors. Since the energy function $\mathbb{W}$ is derived from the natural energy function W through equation (32), using the chain rule and equation (40), one obtains

$$\begin{aligned} Z^{AB} &= \mathsf{F}^A{}_\alpha\delta^{\alpha\beta}\frac{\partial\mathsf{W}}{\partial C_{\gamma\nu}}\frac{\partial\left(\mathsf{F}^M{}_\gamma\mathsf{F}^N{}_\nu C_{MN}\right)}{\partial\mathsf{F}^B{}_\beta}\\ &= \mathsf{F}^A{}_\alpha\delta^{\alpha\beta}\frac{1}{2}S^{\gamma\nu}\left(-\mathsf{F}^M{}_\gamma\mathsf{F}^N{}_\beta\mathsf{F}^B{}_\nu - \mathsf{F}^M{}_\beta\mathsf{F}^N{}_\nu\mathsf{F}^B{}_\gamma\right)C_{MN}\\ &= -\,G^{AN}S^{MB}C_{MN} = -\,C^A{}_D S^{DB},\end{aligned}$$

indicating that $\boldsymbol{Z}$ is the negative of the Mandel stress. In short, from equations (38) and (40), we have obtained the relation between Mandel, Eshelby, and the second Piola–Kirchhoff stresses as

$$\begin{aligned} &\boldsymbol{Z} = -\,\boldsymbol{C}\boldsymbol{S}, \quad \bar{\boldsymbol{Z}} = \mathsf{J}\boldsymbol{\Theta}Z = -\,\bar{\bar{\boldsymbol{C}}}\bar{\boldsymbol{S}},\\ &\boldsymbol{e} = W\boldsymbol{G}^\sharp + \boldsymbol{Z} = W\boldsymbol{G}^\sharp - \boldsymbol{C}\boldsymbol{S}, \quad \bar{\boldsymbol{E}} = \mathsf{J}\boldsymbol{\Theta}E = \bar{W}\bar{\boldsymbol{G}}^\sharp + \bar{\boldsymbol{Z}} = \bar{W}\bar{\boldsymbol{G}}^\sharp - \bar{\bar{\boldsymbol{C}}}\bar{\boldsymbol{S}}.\end{aligned} \tag{41}$$

Since the energy function is defined up to an additive constant, the Eshelby stress referred to either structure is defined up to an additive multiple of the metric tensor in the corresponding structure. Finally, we define the first Piola–Kirchhoff stress tensors $\boldsymbol{P}$ and $\bar{\boldsymbol{P}}$, and the Cauchy stress σ as

$$P^{aB} = F^a{}_A S^{AB}, \qquad \bar{P}^{aB} = F^a{}_A \bar{S}^{AB}, \qquad \sigma^{ab} = \frac{1}{J} F^a{}_A F^b{}_B S^{AB} = \frac{1}{\bar{J}} F^a{}_A F^b{}_B \bar{S}^{AB}.$$

Note that $\bar{\boldsymbol{P}} = \mathsf{J}\boldsymbol{P}$. Recalling equation (23), the first Piola–Kirchhoff stress tensors allow one to express the Eshelby tensors as

$$\boldsymbol{E} = W\boldsymbol{G}^\sharp - \boldsymbol{F}^\mathsf{T}\boldsymbol{P}, \qquad \bar{\boldsymbol{E}} = \bar{W}\bar{\boldsymbol{G}}^\sharp - \bar{\boldsymbol{F}}^\mathsf{T}\bar{\boldsymbol{P}}. \tag{42}$$

In components, one has $E^{AB} = WG^{AB} - F_a{}^A P^{aB}$, and $\bar{E}^{AB} = \bar{W}\bar{G}^{AB} - \bar{F}_a{}^A \bar{P}^{aB}$.

Remark 6. Recall that at each point the energy function attains its minimum for $\boldsymbol{C}^\flat = \boldsymbol{G}$. This means that in the absence of internal constraints, such as incompressibility, $\boldsymbol{C}^\flat = \boldsymbol{G}$ implies $\boldsymbol{S} = \boldsymbol{0}$. Conversely, under the assumption of energy functions admitting no multiple minima, $\boldsymbol{C}^\flat \neq \boldsymbol{G}$ implies the presence of residual stresses $\boldsymbol{S} \neq \boldsymbol{0}$. In terms of the Riemannian curvature $\boldsymbol{\mathcal{R}}$ associated to $\boldsymbol{G}$, if it is non-zero, then for each point X there exists no embedding of a neighborhood of X into $\mathcal{S}$ such that $\boldsymbol{C}^\flat = \boldsymbol{G}$ (i.e., a local isometric embedding). This means that for those energy functions that do not admit a number of minima one should expect residual stresses $\boldsymbol{S} \neq \boldsymbol{0}$. Recalling Lemma 3, a vanishing torsion for $\hat{\nabla}$ is sufficient to guarantee a vanishing curvature for ∇. However, a non-vanishing torsion is not enough to ensure the non-vanishing of the curvature, and therefore, the presence of residual stresses.

Let us consider a time evolution of the anelastic deformation, as in Section 3.4. The energy densities evolve in time as $W(t) = \mathbb{W}(\{\boldsymbol{\vartheta}^\alpha(t)\}, \boldsymbol{C}^\flat(t))$, and $\bar{W}(t) = \bar{\mathbb{W}}(\{\boldsymbol{\vartheta}^\alpha(t)\}, \boldsymbol{C}^\flat(t))$. Their rates are then calculated as

$$\dot{W} = Y_\alpha{}^B \dot{\mathsf{F}}^\alpha{}_B = Z_A{}^B \mathsf{L}^A{}_B = \frac{1}{\mathsf{J}} \bar{Z}_A{}^B \mathsf{L}^A{}_B, \qquad \dot{\bar{W}} = \bar{Y}_\alpha{}^B \dot{\mathsf{F}}^\alpha{}_B = \mathsf{J} E_A{}^B \mathsf{L}^A{}_B = \bar{E}_A{}^B \mathsf{L}^A{}_B,$$

where indices are lowered according to equation (39). In the case of volume-preserving anelastic deformations (recall Section 3.4), one has

$$\dot{\bar{W}} = \mathsf{J} Z_A{}^B \mathsf{L}^A{}_B = -\mathsf{J} C_{AD} S^{DB} \mathsf{L}^A{}_B = \mathsf{J}\dot{W}.$$

Let us next consider a motion of $\mathcal{B}$. One can define the functions $W(t) = \mathbb{W}(\{\boldsymbol{\vartheta}^\alpha(t)\}, \boldsymbol{C}^\flat(t))$ and $\bar{W}(t) = \bar{\mathbb{W}}(\{\boldsymbol{\vartheta}^\alpha(t)\}, \boldsymbol{C}^\flat(t))$, and write

$$\dot{W} = \frac{1}{2} S^{AB} \dot{C}_{AB} = P_a{}^B \dot{F}^a{}_B, \qquad \dot{\bar{W}} = \frac{1}{2} \bar{S}^{AB} \dot{C}_{AB} = \bar{P}_a{}^B \dot{F}^a{}_B.$$

4.3. Material uniformity

According to Noll [3] and Wang [4], the idea of material uniformity of a simple body is related to the existence of maps between tangent spaces at different points that leave the constitutive relation unchanged.[13] The concept of uniformity is related to the notion of configurational forces, that goes back to Griffith [26] and Eshelby [10, 27, 28]. This notion is important in developing evolution laws for the motion of defects, including dislocations, vacancies, interfaces, cavities, and cracks. Driving forces on these defects cause climb and glide of dislocations, diffusion of point defects, migration of interfaces, shape changes of cavities, and propagation of cracks, to mention a few examples.

A body is said to be materially uniform if, for any two points $X, Y \in \mathcal{B}$, there exists a linear map $\boldsymbol{\mathcal{I}} : T_X\mathcal{B} \to T_Y\mathcal{B}$ such that $\mathbb{W}_\mathbb{Y} = \mathbb{W}_\mathbb{X} \circ \boldsymbol{\mathcal{I}}$, or more precisely

$$\mathbb{W}(Y, \{\boldsymbol{\lambda}^\alpha\}, \boldsymbol{A}) = \mathbb{W}\big(X, \{\boldsymbol{\mathcal{I}}^\star \boldsymbol{\lambda}^\alpha\}, \boldsymbol{\mathcal{I}}^\star \boldsymbol{A} \boldsymbol{\mathcal{I}}\big),$$

where $\boldsymbol{\mathcal{I}}^\star : T_Y^*\mathcal{B} \to T_X^*\mathcal{B}$ is the dual of $\boldsymbol{\mathcal{I}}$. The map $\boldsymbol{\mathcal{I}}$ is called a material isomorphism; in general, it is not unique. The existence of such a map between pairs of points allows one to express the energy function as the same function for all material points. This can be done by defining a field of frames that are related to each other through the material isomorphisms $\mathcal{I}$. Such a moving frame is called a reference frame field.

We define a naturally uniform body as one for which the natural moving frame $\{\boldsymbol{e}_\alpha\}$ is a reference field, and therefore the linear map

$$\begin{array}{cccc} \mathcal{N}: & T_X\mathcal{B} & \to & T_Y\mathcal{B} \\ & V^\alpha \boldsymbol{e}_\alpha(X) & \mapsto & V^\alpha \boldsymbol{e}_\alpha(Y), \end{array} \tag{43}$$

is a material isomorphism.[14] This means that the function W on the space of symmetric positive-definite 3×3 matrices does not explicitly depend on any material point; therefore, $W(X) = \mathsf{W}(C_{\alpha\beta}(X))$. This implies that

$$\partial_A W = \frac{\partial \mathsf{W}}{\partial C_{\alpha\beta}} \frac{\partial C_{\alpha\beta}}{\partial X^A}. \tag{44}$$

Since, from Remark 2, the ordinary derivative of the components in the moving frame is the Weitzenböck derivative, one obtains the following identity for naturally uniform bodies:

$$\partial_A W = \frac{1}{2} S^{\alpha\beta} \partial_A C_{\alpha\beta} = \frac{1}{2} S^{BD} \hat{\nabla}_A C_{BD} = P_b{}^B \hat{\nabla}_A F^b{}_B, \tag{45}$$

where the last identity uses two-point derivatives (equation (25)) and the compatibility of the ambient metric $\boldsymbol{g}$. To this extent, the Weitzenböck connection can be seen as the natural connection.

If, for two given points $X, Y \in \mathcal{B}$, the material isomorphism $\mathcal{I}$ is different from the natural isomorphism $\mathcal{N}$ defined in equation (43), then there exist two maps, $\boldsymbol{D}_X : T_X\mathcal{B} \to T_X\mathcal{B}$ and $\boldsymbol{D}_Y : T_Y\mathcal{B} \to T_Y\mathcal{B}$, such that $\mathcal{I} = \boldsymbol{D}_Y^{-1} \mathcal{N} \boldsymbol{D}_X$. In this, way a material isomorphism can be expressed in terms of the natural isomorphism and local deformations at the two points. In particular, defining the moving frame $\{\boldsymbol{d}_\alpha\}$ such that $\{\boldsymbol{e}_\alpha\} = \{\boldsymbol{D}\boldsymbol{d}_\alpha\}$, one can write

$$\begin{array}{cccc} \mathcal{I}: & T_X\mathcal{B} & \to & T_Y\mathcal{B} \\ & V^\alpha \boldsymbol{d}_\alpha(X) & \mapsto & V^\alpha \boldsymbol{d}_\alpha(Y), \end{array}$$

so that $\{\boldsymbol{d}_\alpha\}$ is a reference frame for $\mathcal{B}$. This means that the energy can be expressed as a function $\check{\mathsf{W}}$ on the space of 3×3 matrices such that it does not depend on any material point explicitly, i.e., $\mathsf{W}(X, \boldsymbol{C}(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta)(X)) = \check{\mathsf{W}}(\boldsymbol{C}(\boldsymbol{d}_\alpha, \boldsymbol{d}_\beta)(X))$. Finally, note that if the natural frame field evolves, then the natural isomorphism is subject to some evolution law $\mathcal{N} = \mathcal{N}(s)$, where s is some parameter. We assume that the general reference frame field $\{\boldsymbol{d}_\alpha\}$ follows the evolution of the natural frame field, being subject to the same anelastic deformation. This means that we assume that $\boldsymbol{D}$ is constant; hence, $\{\boldsymbol{e}_\alpha(s)\} = \{\boldsymbol{D}\boldsymbol{d}_\alpha(s)\}$. Therefore, the evolution law for the material isomorphism is $\mathcal{I}(s) = \boldsymbol{D}_Y^{-1} \mathcal{N}(s) \boldsymbol{D}_X$. In this way, it is possible to express non-natural uniformities in an evolving anelastic structure.

4.4. Configurational forces

As mentioned earlier, the concept of a non-uniform energy function is related to configurational forces. Eshelby studied inhomogeneities by considering the explicit dependence of the elastic energy density on position in the reference configuration and defined the force on a defect as the generalized force corresponding to the position of the defect in the reference configuration that is thought of as a generalized displacement. For instance, the configurational force acting on a crack (the crack tip is the defect) is related to the celebrated J-integral [29, 30]. However, uniformity is not a univocal concept. Imagine a body with a holonomic natural moving frame and a uniform energy function. If it undergoes anelastic deformations, it will still look uniform to an internal observer attached to the natural moving frame. We called this natural uniformity. Nonetheless, for an external observer it will no longer appear uniform as the natural structure has been deformed.

In relation to equations (44) and (45), we define the natural non-uniformity as the 1-form $\hat{\boldsymbol{M}}$ with components

$$\hat{M}_A = \left.\frac{\partial \mathsf{W}}{\partial X^A}\right|_{\text{explicit}} = \partial_A W - P_c{}^B \hat{\nabla}_A F^c{}_B\,, \tag{46}$$

where by “explicit” we mean the derivative of W with respect to X^A when the arguments $C_{\alpha\beta}$ are fixed. Note that, in the general case, natural uniformity does not imply uniformity in the sense of the Levi–Civita connection ∇. As a matter of fact, recalling equation (22), one has

$$P_c{}^B \nabla_A F^c{}_B - P_c{}^B \hat{\nabla}_A F^c{}_B = \frac{1}{2}\left(S^{BD}\nabla_A C_{BD} - S^{BD}\hat{\nabla}_A C_{BD}\right) = S^{BD}K^H{}_{AB}C_{HD}\,, \tag{47}$$

which, in general, does not vanish. Therefore, we define the Riemannian non-uniformity as the 1-form $\boldsymbol{M}$ with components

$$M_A = \partial_A W - P_c{}^B \nabla_A F^c{}_B\,, \tag{48}$$

representing a measure of the Riemannian non-uniformity of the body. We set $M^A = G^{AB}M_B$. In analogy with the natural uniformity, when $M_A = 0$ the body is said to be Riemannian uniform. By virtue of equation (47), the two non-uniformities $\hat{\boldsymbol{M}}$ and $\boldsymbol{M}$ are related as

$$M_A - \hat{M}_A = - S^{BD}K^H{}_{AB}C_{HD} = Z^{BD}K_{BAD}\,, \tag{49}$$

where use was made of equation (41), and $K_{BAD} = - K_{DAB}$ from equation (20). In particular, when the body is naturally uniform, one has

$$M_A^{\text{uni}} = Z^{BD}K_{BAD}\,. \tag{50}$$

Non-uniformity can be defined with respect to the Euclidean structure as well. It is sufficient to take the Cartesian orthonormal frame $\{\bar{\partial}_{\bar{A}}\}$ considered so far as the reference frame. Note that, unlike uniformity with respect to $\hat{\nabla}$ and with respect to ∇, the Euclidean connection $\bar{\nabla}$ is the Levi–Civita connection for $\bar{\boldsymbol{G}}$ and the Weitzenböck connection for $\{\bar{\partial}_{\bar{A}}\}$, so one needs to define only one uniformity. The Euclidean non-uniformity $\bar{\boldsymbol{M}}$ is defined as

$$\bar{M}_A = \partial_A \bar{W} - \bar{P}_c{}^B\, \bar{\nabla}_A F^c{}_B\,,$$

while we write $\bar{M}^A = \bar{G}^{AB}\bar{M}_B$. It should be emphasized that the Euclidean uniformity is not a physically meaningful object, as it is built with respect to the Euclidean structure, which does not contain any information about the anelastic frustration of the material. As a matter of fact, while it makes sense to expect uniformity in the natural frame $\{\boldsymbol{e}_\alpha\}$, and in many cases with respect to the Riemannian structure as well (e.g., for isotropic bodies, see Section 4.5), there is no reason to expect uniformity in a Euclidean frame, unless the anelastic deformation is compatible. Nevertheless, the definition of $\bar{\boldsymbol{M}}$ will turn out to be useful later in the paper. Using equation (10), one can write

$$\begin{aligned}\bar{M}_A &= \partial_A \mathsf{J}\, W + \mathsf{J}\partial_A W - \mathsf{J}P_c{}^B\nabla_A F^c{}_B - \mathsf{J}P_c{}^B H^D{}_{AB}F^c{}_D \\ &= \mathsf{J}M_A + \bar{W}H^D{}_{AD} - \bar{G}_{CD}F_c{}^C\bar{P}^{cB}H^D{}_{AB} \\ &= \mathsf{J}M_A + (\bar{W}\bar{G}^{BC} - F_c{}^C\bar{P}^{cB})\bar{G}_{CD}H^D{}_{AB}\,,\end{aligned}$$

having recovered the Eshelby tensor referred to the Euclidean structure written in equation (42). Therefore, we have obtained the relation between non-uniformities in the two structures as:

$$\bar{M}_A = \mathsf{J}M_A + H^D{}_{BA}\bar{E}_D{}^B\,, \qquad \bar{M}^A = \mathsf{J}\Theta^A{}_B M^B + \bar{G}^{AD}H^D{}_{BD}\bar{E}_D{}^B\,, \tag{51}$$

implying that, in general, material and Euclidean uniformities are different, as one can have $M_A = 0$ but $\bar{M}_A \neq 0$, and vice versa. In particular, when a body is naturally uniform, by virtue of equation (50), one has

$$\bar{M}_A^{\mathrm{uni}} = \mathsf{J} Z_B{}^D K^B{}_{AD} + \mathsf{J} E_B{}^D H^B{}_{AD} \,. \tag{52}$$

Hence, we have shown that for a naturally uniform anelastic body, i.e., when $\hat{\boldsymbol{M}} = \mathbf{0}$, in general, the Riemannian and Euclidean non-uniformities do not vanish.

4.5. Stress and uniformity in isotropic bodies

A body is isotropic if the symmetry group $\mathscr{G}$ defined in Section 4.1 is the entire SO(3), meaning that it is made of matrices $[Q^\alpha{}_\beta]$ such that $(Q^{-1})^\alpha{}_\beta = \delta^{\alpha\mu} Q^\nu{}_\mu \delta_{\nu\beta}$. It is straightforward to see that this is equivalent to assuming that the group $\mathfrak{G}$ is made of $\binom{1}{1}$-tensors $\boldsymbol{Q}$ that are $\boldsymbol{G}$-orthogonal, i.e., $(Q^{-1})^A{}_B = G^{AM} Q^N{}_M G_{MB}$. Note that two moving co-frames $\{\boldsymbol{\lambda}^\alpha\}$, $\{\tilde{\boldsymbol{\lambda}}^\alpha\}$ that are related as $\{\tilde{\boldsymbol{\lambda}}^\alpha\} = \{Q^\alpha{}_\beta \boldsymbol{\lambda}^\alpha\}$ belong to the same equivalence class, and from equation (36), the energy functions $\mathbb{W}$ and $\bar{\mathbb{W}}$ are constant on each equivalence class. Moreover, two moving co-frames $\{\boldsymbol{\lambda}^\alpha\}$, $\{\tilde{\boldsymbol{\lambda}}^\alpha\}$ are in the same equivalence class if and only if they induce the same metric, i.e., $\delta_{\alpha\beta} \tilde{\boldsymbol{\lambda}}^\alpha \otimes \tilde{\boldsymbol{\lambda}}^\beta = \delta_{\alpha\beta} \boldsymbol{\lambda}^\alpha \otimes \boldsymbol{\lambda}^\beta$, as the orthogonality condition $\delta_{\alpha\beta} = \delta_{\alpha\beta} Q^\alpha{}_\mu Q^\beta{}_\nu$ is equivalent to $\delta_{\alpha\beta} Q^\alpha{}_\mu Q^\beta{}_\nu \boldsymbol{\lambda}^\mu \otimes \boldsymbol{\lambda}^\nu = \delta_{\alpha\beta} \boldsymbol{\lambda}^\alpha \otimes \boldsymbol{\lambda}^\beta$. Therefore, each class of orthogonally related co-frames $\{\boldsymbol{\lambda}^\alpha\}$ is represented by a unique metric. Since $\mathbb{W}$ is constant on each equivalence class, one can define two energy functions $\mathbb{W}^{\mathrm{iso}}, \bar{\mathbb{W}}^{\mathrm{iso}} : P\mathcal{B} \times P\mathcal{B} \mapsto \mathbb{R}$ as follows:

$$\mathbb{W}^{\mathrm{iso}}(\delta_{\alpha\beta} \boldsymbol{\lambda}^\alpha \otimes \boldsymbol{\lambda}^\beta, \boldsymbol{C}^\flat) = \mathbb{W}(\{\boldsymbol{\lambda}^\alpha\}, \boldsymbol{C}^\flat) \,, \qquad \bar{\mathbb{W}}^{\mathrm{iso}}(\delta_{\alpha\beta} \boldsymbol{\lambda}^\alpha \otimes \boldsymbol{\lambda}^\beta, \boldsymbol{C}^\flat) = \bar{\mathbb{W}}(\{\boldsymbol{\lambda}^\alpha\}, \boldsymbol{C}^\flat) \,.$$

One recovers the energy densities W and $\bar{W}$ by evaluating $\mathbb{W}^{\mathrm{iso}}$ and $\bar{\mathbb{W}}^{\mathrm{iso}}$ at the material metric $\boldsymbol{G}$ and at the pulled-back spatial metric $\boldsymbol{C}^\flat$, viz.[15]

$$W(X) = \mathbb{W}^{\mathrm{iso}}(X, \boldsymbol{G}(X), \boldsymbol{C}^\flat(X)) \,, \qquad \bar{W}(X) = \bar{\mathbb{W}}^{\mathrm{iso}}(X, \boldsymbol{G}(X), \boldsymbol{C}^\flat(X)) \,.$$

It is straightforward to check that, under the isotropy assumption, the material stress tensors are simply given by

$$\boldsymbol{Z} = 2 \frac{\partial \mathbb{W}^{\mathrm{iso}}}{\partial \boldsymbol{G}} \,, \qquad \bar{\boldsymbol{Z}} = \mathsf{J} \boldsymbol{\Theta} Z \,, \qquad \mathsf{J} \boldsymbol{e} = 2 \frac{\partial \bar{\mathbb{W}}^{\mathrm{iso}}}{\partial \boldsymbol{G}} \,, \qquad \bar{\boldsymbol{E}} = \mathsf{J} \boldsymbol{\Theta} \boldsymbol{E} . \tag{53}$$

Note that, as pointed out by Epstein and Maugin [31], in the isotropic case, both $\boldsymbol{Z}$ and $\boldsymbol{e}$ are symmetric $\binom{2}{0}$-tensors. Moreover, in the isotropic case, it is possible to write the energy density as a function of the right Cauchy–Green tensor $\boldsymbol{C}$ (see Section 3). One can show this by defining a function $\mathcal{W}(\boldsymbol{G}, \boldsymbol{C}) = \mathbb{W}^{\mathrm{iso}}(\boldsymbol{G}, \boldsymbol{G}\boldsymbol{C})$ and taking its derivative with respect to $\boldsymbol{G}$. Recalling equation (41), one has $\boldsymbol{Z} = - \boldsymbol{C}\boldsymbol{S}$, and hence

$$\frac{\partial \mathcal{W}}{\partial G_{AB}} = \frac{\partial \mathbb{W}^{\mathrm{iso}}}{\partial G_{AB}} + C^A{}_D \frac{\partial \mathbb{W}^{\mathrm{iso}}}{\partial C_{DB}} = Z^{AB} + C^A{}_D S^{DB} = - C^A{}_D S^{DB} + C^A{}_D S^{DB} = 0 \,.$$

Therefore, $\mathcal{W}$ is a function of only $\boldsymbol{C}$, and in particular, of its three invariants. The same holds for the energy density per unit Euclidean volume.

Let us consider an evolution $\boldsymbol{G}_s$ of the material metric $\boldsymbol{G}$, such that $\boldsymbol{G}_0 = \boldsymbol{G}$. Then, one can define the functions $W_s(X) = \check{\mathcal{W}}(\boldsymbol{G}_s(X), \boldsymbol{C}^\flat(X))$ and $\bar{W}_s(X) = \tilde{\mathcal{W}}(\boldsymbol{G}_s(X), \boldsymbol{C}^\flat(X))$, expressing the evolution of the energy densities referred to the material volume and the Euclidean volume, and calculate their variations as

$$\delta W = \frac{1}{2} Z^{AB} \delta G_{AB} \,, \qquad \delta \bar{W} = \frac{1}{2} \mathsf{J} E^{AB} \delta G_{AB} \,,$$

where $\delta = \frac{\mathrm{d}}{\mathrm{d}s}\big|_0$. In the case of volume-preserving material variations, one obtains

$$\delta \bar{W} = \frac{1}{2} \mathsf{J} Z^{AB} \delta G_{AB} = - \frac{1}{2} \mathsf{J} C^A{}_D S^{DB} \delta G_{AB} = \mathsf{J} \delta W \,.$$

Finally, the following result establishes the equivalence of uniformity with respect to the connections $\hat{\nabla}$ and ∇ for isotropic bodies.

Lemma 4. *For an isotropic body, natural uniformity is equivalent to Riemannian uniformity. Moreover,* $M_A = \hat{M}_A$.

Proof. From equations (47) and (49), $S^{BD}\nabla_A C_{BD} - S^{BD}\hat{\nabla}_A C_{BD} = -2Z^{BD}K_{BAD}$, with $K_{BAD} = -K_{DAB}$ from equation (20). Therefore, using the symmetry of $\boldsymbol{Z}$ for isotropic bodies following from equation (53), one obtains $S^{BD}\nabla_A C_{BD} = S^{BD}\hat{\nabla}_A C_{BD}$, and $M_A = \hat{M}_A$. Hence, for uniform isotropic bodies one has $\partial_A W = P_c{}^B \nabla_A F^c{}_B$. □

5. The balance of linear momentum

In this section, we use the Lagrange–d'Alembert principle [32] and write the balance of linear momentum for an anelastic body with respect to both the Riemannian and the Euclidean structures. The balance of linear momentum, in either the standard or configurational form, is obtained by taking variations about a generic motion φ_t, while the balance of angular momentum is automatically satisfied when one assumes that W is a function of $\boldsymbol{C}^\flat$. We should emphasize that in anelasticity, in addition to the classical degrees of freedom represented by the configuration mapping φ_t, one must consider some set of variables $\mathcal{Y}$, which are related to the natural moving co-frame $\{\boldsymbol{\vartheta}^\alpha\}$ via a flow rule [33], and which represent such quantities as the density of defects or the temperature field. In this paper, we are not concerned with considering material evolutions for $\mathcal{Y}$. These strictly depend on the class of problems one is considering (dislocations, growth, thermal expansion), and will be the subject of a future communication. Note also that, as mentioned earlier, we work in the context of hyperelasticity, and therefore no dissipation is associated to standard motions, albeit we do not exclude dissipation phenomena associated to anelastic evolutions, whose effect would show up in the equations corresponding to variations of $\mathcal{Y}$.

5.1. The Lagrange–D'Alembert principle

Using an action principle such as the Lagrange–d'Alembert principle, the inertial forces are taken into account. In particular, we take variations about a generic motion φ_t during a time interval $[t_1, t_2]$. Let us consider a one-parameter family of motions $\varphi_{t,\epsilon} : \mathcal{B} \times [t_1, t_2] \times \mathbb{R} \to \mathcal{S}$, such that for $\epsilon = 0$ one recovers φ_t, and such that all the trajectories agree at the end points, viz.

$$\varphi_{t,0} = \varphi_t, \forall t; \qquad \varphi_{t_1,\epsilon} = \varphi_{t_1}, \quad \varphi_{t_2,\epsilon} = \varphi_{t_2}, \forall \epsilon. \tag{54}$$

We denote by $\boldsymbol{U}_\epsilon(t)$ the vector field tangent to the curves $\epsilon \mapsto \varphi_{t,\epsilon}(X)$ for X and t fixed. Clearly, from equation (54) one has $\boldsymbol{U}_\epsilon(t_1) = \boldsymbol{0}$, and $\boldsymbol{U}_\epsilon(t_2) = \boldsymbol{0}$. We denote by $\boldsymbol{V}_\epsilon(t)$ the vector field tangent to the curves $t \mapsto \varphi_{t,\epsilon}(X)$ for X and ϵ fixed. Note that $\boldsymbol{V}_0(t) = \boldsymbol{V}(t)$ from equation (54), recovering at $\epsilon = 0$ the velocity field for φ_t defined in Section 3. Let us consider the following one-parameter families of kinetic energies and elastic energy densities

$$K_\epsilon(t) = \frac{1}{2}\varrho_o \|\boldsymbol{V}_\epsilon(t)\|^2_{\boldsymbol{G}}, \qquad W_\epsilon(t) = \mathbb{W}\left(\{\boldsymbol{\vartheta}^\alpha\}, \boldsymbol{C}^\flat_\epsilon(t)\right), \tag{55}$$

and define the action as

$$\mathcal{A}(\epsilon) = \int_{t_1}^{t_2} \left[\int_{\mathcal{B}} (K_\epsilon(t) - W_\epsilon(t))\boldsymbol{\mu}\right] \mathrm{d}t.$$

We denote by δ the derivative with respect to ϵ evaluated at $\epsilon = 0$, i.e., $\delta = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\big|_0$. The Lagrange–d'Alembert principle reads

$$\delta\mathcal{A} + \int_{t_1}^{t_2} \left[\int_{\mathcal{B}} \varrho_o \langle \boldsymbol{b}, \boldsymbol{u}\rangle \boldsymbol{\mu} + \int_{\partial\mathcal{B}} \langle \boldsymbol{T}, \boldsymbol{u}\rangle \boldsymbol{\eta}\right] \mathrm{d}t = 0, \tag{56}$$

where the 1-forms $\boldsymbol{b}$ and $\boldsymbol{t}$ represent, respectively, the body and contact forces, and $\boldsymbol{\eta}$ is the area form induced on $\partial\mathcal{B}$ by $\boldsymbol{\mu}$.

5.2. The balance of linear momentum in terms of the two structures

From the calculations in Appendix C, the Lagrange–d'Alembert principle for an anelastic body in terms of the Riemannian structure is written as

$$\int_{t_1}^{t_2}\left[\int_{\mathcal{B}}(-\varrho_o a^a u_a - \varrho_o b^a u_a - P^{aB}\nabla_B u_a)\boldsymbol{\mu} + \int_{\partial\mathcal{B}} T_a u^a\,\boldsymbol{\eta}\right]\mathrm{d}t = 0\,. \tag{57}$$

Applying the divergence theorem to equation (57), one writes

$$\int_{t_1}^{t_2}\left[\int_{\mathcal{B}}(-\rho_o a^a - \varrho_o b^a + \nabla_B P^{aB})u_a\,\boldsymbol{\mu} + \int_{\partial\mathcal{B}}(T^a - P^{aB}N_B)u_a\,\boldsymbol{\eta}\right]\mathrm{d}t = 0\,.$$

Using the arbitrariness of $\boldsymbol{u}$, one obtains the Euler–Lagrange equations expressed in terms of the first Piola–Kirchhoff stress $\boldsymbol{P}$, viz.

$$\nabla_B P^{aB} + \varrho_o b^a = \varrho_o a^a \quad \text{on } \mathcal{B}, \qquad P^{aB}N_B = T^a \quad \text{on } \partial\mathcal{B}. \tag{58}$$

Equation (58) is the balance of linear momentum in local form written with respect to the Riemannian structure, see Marsden and Hughes [22]. Note that information about the anelastic deformation is contained in the connection ∇, in the body forces (through the mass density), and in the $\boldsymbol{G}$-normal vector $\boldsymbol{N}$. Of course, anelasticity is also hidden in the constitutive relations (equation (32)), encoded in the stress tensor $\boldsymbol{P}$. Equation (57) can be written in terms of the Euclidean structure as

$$\int_{t_1}^{t_2}\left[\int_{\mathcal{B}}\left(-\mathsf{J}\varrho_o a^a u_a - \mathsf{J}\varrho_o b^a u_a - \bar{P}^{aB}\nabla_B u_a\right)\bar{\boldsymbol{\mu}} + \int_{\partial\mathcal{B}}\bar{T}^a u_a\,\bar{\boldsymbol{\eta}}\right]\mathrm{d}t = 0\,,$$

with $\bar{T}_a$ indicating traction with respect to the Euclidean metric, i.e., $T_a = \mathsf{J}_\partial \bar{T}_a$, where J_∂ is the area ratio between the two structures (see Appendix C). Therefore, using the divergence theorem, one writes

$$\int_{t_1}^{t_2}\left[\int_{\mathcal{B}}\left(-\mathsf{J}\varrho_o a^a - \mathsf{J}\varrho_o b^a + \bar{\nabla}_B\bar{P}^{aB}\right)u_a\,\bar{\boldsymbol{\mu}} + \int_{\partial\mathcal{B}}\left(T^a - \bar{P}^{aB}\bar{N}_B\right)u_a\,\bar{\boldsymbol{\eta}}\right]\mathrm{d}t = 0\,.$$

This gives the Euclidean or classical form of the balance of linear momentum as

$$\bar{\nabla}_B\bar{P}^{aB} + \bar{\varrho}_o b^a = \bar{\varrho}_o a^a \quad \text{on } \mathcal{B}, \qquad \bar{P}^{aB}\bar{N}_B = \bar{T}^a \quad \text{on } \partial\mathcal{B}, \tag{59}$$

where $\bar{\varrho}_o = \mathsf{J}\varrho_o$ denotes the mass density referred to the Euclidean structure. In this case, the information about the anelastic deformation is entirely carried by the constitutive relation that determines $\bar{\boldsymbol{P}}$. Note that equation (59) can also be obtained from equation (58), deriving the bulk part from the generalization (equation (26)) of Lemma 8 and the boundary conditions from the analog of Nanson's formula (equation (7)). In summary, the local forms of the balance of linear momentum in the Riemannian and Euclidean structures read

$$\begin{cases} \nabla_B P^{aB} + \varrho_o b^a = \varrho_o a^a \text{on } \mathcal{B}, \\ P^{aB}N_B = T^a \qquad\quad \text{on } \partial\mathcal{B}, \end{cases} \Leftrightarrow \begin{cases} \bar{\nabla}_B\bar{P}^{aB} + \bar{\varrho}_o b^a = \bar{\varrho}_o a^a \text{on } \mathcal{B}, \\ \bar{P}^{aB}\bar{N}_B = \bar{T}^a \qquad\quad \text{on } \partial\mathcal{B}, \end{cases}$$

$$\left(\begin{array}{c} \bar{P}^{aB} = \mathsf{J}P^{aB}\,, \bar{\varrho}_o = \mathsf{J}\varrho_o\,, \bar{T}^a = \mathsf{J}_{\partial\mathcal{B}}T^a\,, \\ \nabla = \nabla^{\boldsymbol{G}}\,,\ \bar{\nabla} = \nabla^{\bar{\boldsymbol{G}}}\,, \mathsf{J}_{\partial\mathcal{B}}N_A = \mathsf{J}\bar{N}_A\,. \end{array}\right)$$

Finally, as is well-known, the balance of linear momentum can be written in the ambient space in terms of the Cauchy stress $\boldsymbol{\sigma}$ as $\nabla_b\sigma^{ab} + \varrho b^a = \varrho a^a$. Note that, using the two-point derivative notation (equation (25)), for a spatial tensor the three symbols ∇, $\hat{\nabla}$, and $\bar{\nabla}$ are equivalent. Thus, there is only one form for the balance of linear momentum when it is written in terms of the Cauchy stress.[16]

5.3. The configurational balance of linear momentum with respect to the two structures

Next we look at the balance of linear momentum from a configurational point of view [10, 11]. Recalling equation (42), one has $E^{AB} = WG^{AB} - F_a{}^A P^{aB}$, and therefore

$$\nabla_B E^{AB} = \partial_B W\, G^{AB} - \nabla_B F_a{}^A P^{aB} - F_a{}^A \nabla_B P^{aB} ,$$

where the compatibility of $\boldsymbol{G}$ and ∇ was used. From equation (48), one has

$$\begin{aligned} \nabla_B E^{AB} &= M^A + P_c{}^C \nabla_B F^c{}_C G^{AB} - \nabla_B F^a{}_C P_a{}^B G^{AC} - F_a{}^A \nabla_B P^{aB} \\ &= M^A + P_a{}^C (\nabla_B F^a{}_C - \nabla_C F^a{}_B) G^{AB} - F_a{}^A \nabla_B P^{aB} . \end{aligned}$$

Therefore, using compatibility of the total deformation gradient (equation (27)), one obtains

$$\nabla_B E^{AB} = M^A - F_a{}^A \nabla_B P^{aB} .$$

Thus, defining the total configurational force referred to the Riemannian structure as

$$B^A = - M^A - \varrho_o F_a{}^A b^a + \varrho_o F_a{}^A a^a , \tag{60}$$

one writes the balance of linear momentum (equation (58)) in terms of the Eshelby stress, viz.

$$\nabla_B E^{AB} + B^A = 0 \quad \text{on } \mathcal{B}, \qquad E^{AB} N_B = W N^A - F_a{}^A T^a \quad \text{on } \partial\mathcal{B} . \tag{61}$$

Equation (61) is the geometric version of the configurational form of the balance of linear momentum. Following the same approach, one can refer everything to the Euclidean structure. By substituting equation (42) into equation (59), one obtains the total configurational force referred to the Euclidean structure as:

$$\bar{B}^A = - \bar{M}^A - \bar{\varrho}_o \bar{F}_a{}^A b^a + \bar{\varrho}_o \bar{F}_a{}^A a^a . \tag{62}$$

The classical form of the configurational format of the balance of linear momentum reads

$$\bar{\nabla}_B \bar{E}^{AB} + \bar{B}^A = 0 \text{ on } \mathcal{B}, \qquad \bar{E}^{AB} \bar{N}_B = \bar{W} \bar{N}^A - \bar{F}_a{}^A \bar{T}^a \text{ on } \partial\mathcal{B} . \tag{63}$$

From equations (60) and (62), and using equation (51), one obtains the relation between the two configurational forces as

$$\bar{B}^A = \mathsf{J} \Theta^A{}_D B^D - \bar{G}^{AC} \bar{E}_B{}^D H^B{}_{CD} .$$

In summary, the two forms of the configurational balance of linear momentum read

$$\left\{ \begin{array}{l} \nabla_B E^{AB} + B^a = 0 \text{ on } \mathcal{B}, \\ E^{AB} N_B = W N^A - F_a{}^A T^a \text{ on } \partial\mathcal{B}, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} \bar{\nabla}_B \bar{E}^{AB} + \bar{B}^a = 0 & \text{on } \mathcal{B}, \\ \bar{E}^{AB} \bar{N}_B = \bar{W} \bar{N}^A - \bar{F}_a{}^A \bar{T}^a & \text{on } \partial\mathcal{B}, \end{array} \right.$$

$$\left(\begin{array}{c} \bar{E}^{AB} = \mathsf{J} \Theta^A{}_D E^{DB} , \bar{T}^a = \mathsf{J}_{\partial\mathcal{B}} T^a , \\ \nabla = \nabla^{\boldsymbol{G}} ,\ \bar{\nabla} = \nabla^{\bar{\boldsymbol{G}}} , \mathsf{J}_{\partial\mathcal{B}} N_A = \mathsf{J} \bar{N}_A , \\ \bar{B}^A = \Theta^A{}_D B^D - \bar{G}^{AC} \bar{E}_B{}^D H^B{}_{CD} . \end{array} \right)$$

In the case of no inertial and no body forces, the configurational force is entirely given by the non-uniformity, as the bulk parts of equations (61) and (63) are simplified to read

$$\nabla_B E^{AB} = M^A , \qquad \bar{\nabla}_B \bar{E}^{AB} = \bar{M}^A . \tag{64}$$

As mentioned earlier, in general, both the Riemannian and the Euclidean non-uniformities are different from the non-uniformity defined with respect to the natural reference frame. Therefore, it is worth pointing out that the non-uniformities contributing to the configurational forces in both equations (61) and (63) are not the natural uniformity given by the explicit dependence of W on the point X. As a matter of fact, as shown in equations (50) and (52), for an anelastic uniform body, one has $\boldsymbol{M} \neq \mathbf{0}$ and $\bar{\boldsymbol{M}} \neq \mathbf{0}$. In this case, the configurational balance (equation (64)) reads

$$\nabla_B E^{AB} = Z_B{}^D K^B{}_{AD}, \qquad \bar{\nabla}_B \bar{E}^{AB} = \bar{Z}_B{}^D K^B{}_{AD} + \bar{E}_B{}^D H^B{}_{AD}.$$

In particular, the Euclidean structure carries no information about the anelastic state of the body. Conversely, being built on the natural metric, the Riemannian structure is affected by anelastic deformations; hence, uniformity in the Riemannian sense is related to that in the natural sense. Moreover, for isotropic bodies, the Riemannian non-uniformity is identical to the natural non-uniformity.

6. Conclusions

In this paper, we formulated the mechanics of anelastic bodies with respect to two different material structures, the Euclidean and the Riemannian structures. A material structure is defined as a triplet made of a metric tensor on the material manifold with its corresponding volume form, and the Levi–Civita connection. In particular, the Riemannian structure is built using the material metric induced from the natural moving frame, which provides information about the distances in the body in its natural configuration, and therefore, the anelastic frustration of the material, and corresponds to the geometric approach to anelasticity. The Euclidean structure, however, represents the classical formalism of non-linear anelasticity. The multiplicative decomposition of deformation gradient was approached in a geometric framework consistent with the interpretation that views the anelastic part of the deformation gradient as a non-holonomic change of frame in the material manifold.

In the setting of hyper-anelasticity, we defined stress tensors with respect to both the Riemannian and the Euclidean structures, and derived the balance of linear momentum for an anelastic body with respect to the two structures. These two sets of governing equations are very similar and differ only in the use of the corresponding derivative and of the volume ratio of the two structures. We discussed uniformity with respect to general moving frames and, in particular, with respect to the natural moving frame. This natural uniformity is expressed via the Weitzenböck connection, which parallelizes the non-holonomic natural frame. We extended the concept of uniformity to the Riemannian and Euclidean structures, and discussed the role of non-uniformity in the form of the material forces that appear in the configurational form of the balance of linear momentum. This was derived with respect to both the Riemannian and the Euclidean structures, and it was observed that in anelasticity, even for uniform bodies, a non-uniformity term appears in the configurational balance of linear momentum, whether it is expressed in the classical Euclidean format or in the geometric Riemannian format. Hyperelastic isotropic bodies are exceptional in the sense that, for them, uniformity in the natural sense is equivalent to uniformity in the Riemannian sense.

Extending the present theory to material variations will be the subject of a future communication. Material variations model evolutions of the set of variables that describe the specific nature of a particular anelastic process, e.g., the density of defects, and that trigger the evolution of the natural moving co-frame (or local anelastic deformation) via a flow rule. It would be interesting to see how the present theory applies to that setting and investigate material forces dual to these changes. A further extension of the present work would be to take into account dissipation phenomena associated to anelastic evolutions.

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Notes

1. $\langle\cdot,\cdot\rangle$ is the natural pairing of 1-forms and vectors.
2. The permutation symbol is 1 for even permutations of (123), -1 for odd permutations, and 0 when an index is repeated.
3. Alhasadi et al. [18] obtain an expression similar to equation (10) but with the term $\Gamma^B{}_{BA} - \hat{\Gamma}^B{}_{BA}$ instead of $H^B{}_{BA} = \Gamma^B{}_{BA} - \bar{\Gamma}^B{}_{BA}$ (for the meaning of $\hat{\Gamma}$, see Section 2.5). This is because they define the volume ratio as the determinant of the change of frame, which depends on the coordinate functions with respect to which the natural frame is expressed, i.e., ∂_A in equation (2). This means that it is not a well-defined quantity. As an example, for unimodular anelastic deformations, i.e., when the volume ratio is one, they obtain $\Gamma^B{}_{BA} = \hat{\Gamma}^B{}_{BA}$, which disagrees with our theory. A counterexample to their result is constituted by

$$[\mathsf{F}^\alpha{}_A](X) = \begin{bmatrix} 1 & f(X) \\ 0 & 1 \end{bmatrix},$$

representing a unimodular anelastic deformation, but for which one has $[\Gamma^B{}_{BA}] = [0 \quad f'(X)]$, and $[\hat{\Gamma}^B{}_{BA}] = [0 \;\; 0]$. Another way to look at this inconsistency is to view the result of Alhasadi et al. [18] as an active approach (see Section 2.4). In this case, the definition of the volume ratio is well-defined. However, what is incorrect is the expression of the coefficients of the Weitzenböck connection, as will be explained in Section 2.5.
4. The Weitzenböck connection is called the material connection by Noll [3] and Wang [4]. See also Youssef and Sid-Ahmed [34] and Yavari and Goriely [21].
5. Global compatibility involves extra equations for every generator of the first homology group. These were investigated by Yavari [35] in both linear and non-linear elasticity.
6. The minus sign is because everything is defined with a focus on the co-frame field.
7. With $(\dot{\mathsf{F}}^{-1})^A{}_\alpha$, we imply the time-derivative of the inverse, i.e., $((\mathsf{F}^{-1})^{\cdot})^A{}_\alpha$, and not the inverse of the time-derivative, i.e., $((\dot{\mathsf{F}})^{-1})^A{}_\alpha$. Note that these two objects are, in general, different.
8. Recalling equation (24), from the classical active point of view one has $C_{\alpha\beta} = \boldsymbol{C}^\flat(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) = \delta^{\bar{A}}_\alpha \delta^{\bar{B}}_\beta C^\flat(\bar{\partial}_{\bar{A}}, \bar{\partial}_{\bar{B}})$, where $\{\bar{\partial}_{\bar{A}}\}$ is a Cartesian frame.
9. Simo [23] obtains this result as a consequence of "invariance under rigid-body motions superposed onto the intermediate configuration". This is equivalent to invariance under rotations of the natural moving frame. As we explain later in the paper, this invariance requires that any proper rotation be a material symmetry for W. In other words, Simo [23] enforces isotropy. See also Section 4.5.
10. A natural frame $\{\tilde{e}_\alpha\} = \{A^\beta{}_\alpha \boldsymbol{e}_\beta\}$ is induced by a change of frame similar to equation (2) with $\tilde{\mathsf{F}}^\alpha{}_C = (A^{-1})^\alpha{}_\gamma \mathsf{F}^\gamma{}_C$. Therefore, from equation (14), one obtains

$$\tilde{\hat{\Gamma}}^A{}_{BC} = \left(\mathsf{F}^{-1}\right)^A{}_\beta A^\beta{}_\alpha\, \partial_B \left(\left(A^{-1}\right)^\alpha{}_\gamma \mathsf{F}^\gamma{}_C\right) = \hat{\Gamma}^A{}_{BC} - \left(\mathsf{F}^{-1}\right)^A{}_\alpha\, \partial_B A^\alpha{}_\beta \left(A^{-1}\right)^\beta{}_\gamma \mathsf{F}^\gamma{}_C\,.$$

Hence, if the symmetry group changes continuously from point to point, then by changing the natural frame via material symmetry one obtains a different Weitzenböck connection. However, the material metric $\boldsymbol{G}$ would not be affected as, albeit non-uniform, the symmetry group is still everywhere a subgroup of the orthogonal group. The effect of this "change of frame according to the non-uniform material symmetry" is therefore similar to that of a superposed distribution of stress-free dislocations. For this reason, we believe that the notion of teleparallelism for non-uniform bodies is not as insightful as it is for the uniform case.
11. For the sake of clarity and simplicity, instead of indicating differentiation with generic $\boldsymbol{\lambda}^\beta$ and $\boldsymbol{A}$, we will use the fields $\boldsymbol{\vartheta}^\beta$ and $\boldsymbol{C}^\flat$ at which these derivatives are evaluated, i.e., we set

$$\frac{\partial}{\partial \boldsymbol{\vartheta}^\beta} = \frac{\partial}{\partial \boldsymbol{\lambda}^\alpha}\Big|_{(\boldsymbol{\vartheta}^\beta,\, \boldsymbol{C}^\flat)} \qquad \text{and} \qquad \frac{\partial}{\partial \boldsymbol{C}^\flat} = \frac{\partial}{\partial \boldsymbol{A}}\Big|_{(\boldsymbol{\vartheta}^\beta,\, \boldsymbol{C}^\flat)}\,.$$

12. To our best knowledge, no interpretation of the material stresses $\boldsymbol{Y}_\alpha$ and $\bar{\boldsymbol{Y}}_\alpha$ from a geometric perspective of the type discussed in Kanso et al. [36] has been provided in the literature.
13. We work with a $\boldsymbol{C}^\flat$-dependent energy function, whereas Wang [4] assumes a constitutive model for the stress in terms of the deformation gradient $\boldsymbol{F}$.
14. This map is equivalent to the parallel transport induced by the Weitzenböck connection $\hat{\nabla}$. Conversely, Wang [4] defines a material connection as one whose induced parallel transports are always material isomorphisms.
15. We point out the work of Mariano [37], in which the Eshelby tensor is associated to variations of the material metric. However, we emphasize that this association is valid only for isotropic bodies.
16. In the work of Menzel and Steinmann [17], different stress tensors are defined with respect to different configurations, and are used to express the balance of linear momentum in several different formats, including the configurational one. Some of these expressions involve the dislocation density tensor. In our framework, this is justified from Remark 2, stating that

differentiating components with respect to the moving frame is equivalent to using the Weitzenböck derivative, and therefore, the tensors $\hat{\boldsymbol{T}}$ and $\boldsymbol{K}$ might show up.

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Appendix

A Notation

$\boldsymbol{A}$, $\boldsymbol{a}$	acceleration
A	anelastic deformation
$\mathcal{A}$	action
$\boldsymbol{B}$, $\bar{\boldsymbol{B}}$	configurational forces
$\mathcal{B}$	material manifold
$\boldsymbol{b}$	body force
b^α	Burgers vector
$\boldsymbol{C}$, $\bar{\boldsymbol{C}}$, $\boldsymbol{C}^\flat$	right CG tensors
$\bar{\boldsymbol{C}}$, $\bar{\boldsymbol{C}}^\flat$	elastic right CG tensors
$\boldsymbol{E}$, $\bar{\boldsymbol{E}}$	Eshelby stress
E	elastic deformation
$\{\boldsymbol{e}_\alpha\}$	natural co-frame
$\boldsymbol{F}$	deformation gradient
$F\mathcal{B}$, $F^*\mathcal{B}$	tensor bundles
$\mathsf{F}^\alpha{}_A$	change of frame
$\boldsymbol{G}$, $\bar{\boldsymbol{G}}$, $\boldsymbol{g}$	metric tensors
$\boldsymbol{H}$	change of connection
$\mathcal{I}$	material isomorphisms
$\mathcal{J}$, J, $\bar{J}$,	volume ratios
K	kinetic energy
$\boldsymbol{K}$	change of connection
L	rate of anelastic deformation
$\hat{\boldsymbol{M}}$, $\boldsymbol{M}$, $\bar{\boldsymbol{M}}$	non-uniformity forms
$\boldsymbol{N}$, $\bar{\boldsymbol{N}}$	normal vectors
$\mathcal{N}$	material isomorphism
$\boldsymbol{P}$, $\bar{\boldsymbol{P}}$	first PK stress
$P\mathcal{B}$	tensor bundle
$\boldsymbol{\mathcal{R}}$	curvature tensor for ∇
$\boldsymbol{S}$, $\bar{\boldsymbol{S}}$	second PK stress
$\mathcal{S}$	ambient space
$\hat{\boldsymbol{T}}$	torsion tensor for $\hat{\nabla}$
$\boldsymbol{T}$, $\bar{\boldsymbol{T}}$	tractions
$T\mathcal{B}$, $T\mathcal{S}$	tangent spaces
$\boldsymbol{u}$	spatial variation

$\boldsymbol{V}$, $\boldsymbol{v}$	velocity
W, $\bar{W}$	energy densities
W, $\mathbb{W}$, $\bar{\mathbb{W}}$	energy functions
$\{\boldsymbol{Y}_\alpha\}$, $\{\bar{\boldsymbol{Y}}_\alpha\}$	vector stress
$\boldsymbol{Z}$, $\bar{\boldsymbol{Z}}$	negative Mandel stress
$\boldsymbol{\eta}$, $\bar{\boldsymbol{\eta}}$	area forms
$\boldsymbol{\Theta}$	change of metric
$\{\boldsymbol{\vartheta}^\alpha\}$	natural co-frame
$\boldsymbol{\mu}$, $\bar{\boldsymbol{\mu}}$	volume forms
ϱ_o, $\bar{\varrho}_o$, ϱ	mass densities
$\boldsymbol{\sigma}$	Cauchy stress
$\{\partial_A\}$	generic frame
$\{\partial_{\bar{A}}\}$	Cartesian frame
∇, $\Gamma^A{}_{BC}$	Riemannian connection
$\bar{\nabla}$, $\bar{\Gamma}^A{}_{BC}$	Euclidean connection
$\hat{\nabla}$, $\hat{\Gamma}^A{}_{BC}$	Weitzenböck connection

B The Burgers vector in the geometric and classical approaches

From a geometric point of view, the Burgers vector defined in equation (19) is not a vector but a triplet of scalars. It is a quantity associated to a curve, so it does not belong to any particular tangent space at any point. The classical definition of the Burgers vector is formally different, and uses the fact that in a flat space it is possible to integrate a vector field using components with respect to a Cartesian chart. It is defined using the active notation for the multiplicative decomposition of deformation gradient, which in our framework reads

$$b^{\bar{a}}[\psi(\gamma)] = \oint_{\psi(\gamma)} [\psi_*(\mathbf{AT})]^{\bar{a}}\,\mathrm{d}s, \tag{65}$$

where $\mathbf{T}$ is the velocity vector of γ, ψ is the global embedding defining the Cartesian chart $\Xi = \xi \circ \psi$ on $\mathcal{B}$, and the components are with respect to Ξ. Using the change of variables theorem, one can write equation (65) as

$$b^{\bar{a}}[\psi(\gamma)] = \oint_\gamma \delta^{\bar{a}}_{\bar{A}}(\mathbf{AT})^{\bar{A}}\,\mathrm{d}s.$$

Note that using a Cartesian chart one can write $\mathsf{A}^{\bar{A}}{}_{\bar{B}} = \delta^{\bar{A}}_\alpha \mathsf{F}^\alpha{}_{\bar{B}}$. Therefore, one has

$$\oint_\gamma \delta^{\bar{a}}_{\bar{A}}\mathsf{A}^{\bar{A}}{}_{\bar{B}}\mathsf{T}^{\bar{B}}\,\mathrm{d}s = \oint_\gamma \delta^{\bar{a}}_{\bar{A}}\delta^{\bar{A}}_\alpha\mathsf{F}^\alpha{}_{\bar{B}}\mathsf{T}^{\bar{B}}\,\mathrm{d}s = \oint_\gamma \delta^{\bar{a}}_\alpha\mathsf{F}^\alpha{}_{\bar{A}}\mathsf{T}^{\bar{A}}\,\mathrm{d}s = \delta^{\bar{a}}_\alpha\oint_\gamma \vartheta^\alpha = \delta^{\bar{a}}_\alpha \mathsf{b}^\alpha[\gamma],$$

and hence, the geometric definition of the Burgers vector coincides with the classical one.

C The Lagrange–D'Alembert principle and anelasticity

We start with the calculation of the first variation of the kinetic energy density $K_\epsilon(t)$. Let us fix $X \in \mathcal{B}$, and consider the surface Ω_X spanned by $\varphi_{t,\epsilon}(X)$ for $t \in [t_1, t_2]$ and $\epsilon \in [-\epsilon_o, \epsilon_o]$, $\epsilon_o > 0$. Note that Ω_0 is injectively immersed in $\mathcal{S}$, and that the pair (t, ϵ) is a global coordinate system for Ω_X, with the basis vectors $\boldsymbol{V}_\epsilon(t)$ and $\boldsymbol{U}_\epsilon(t)$. Therefore, one has $\partial_\epsilon K = \langle \mathrm{d}K, \boldsymbol{U}_\epsilon\rangle$, and hence, evaluating at $\epsilon = 0$, one obtains $\delta K(t)$ as

$$\delta K = \langle \mathrm{d}K, \boldsymbol{U}(t)\rangle.$$

Therefore, recalling the definition of the kinetic energy density (equation (55)), one writes

$$\delta K = \frac{1}{2}\varrho_o\left\langle \mathrm{d}\left(\|\boldsymbol{V}\|^2_{\boldsymbol{G}}\right), \boldsymbol{U}\right\rangle = \varrho_o\langle\!\langle \boldsymbol{V}, \nabla_{\boldsymbol{U}}\boldsymbol{V}\rangle\!\rangle_{\boldsymbol{g}},$$

as $\partial_a(g_{bc}V^bV^c)U^a = \nabla_a(g_{bc}V^bV^c)U^a = 2g_{bc}V^bU^a\nabla_aV^c$. Moreover, the holonomicity of $\boldsymbol{U}$ and $\boldsymbol{V}$ on Ω_X implies $\nabla_{\boldsymbol{V}}\boldsymbol{U} = \nabla_{\boldsymbol{U}}\boldsymbol{V}$; hence,

$$\delta K = \rho_o\langle\!\langle \boldsymbol{V}, \nabla_{\boldsymbol{V}}\boldsymbol{U}\rangle\!\rangle_{\boldsymbol{g}}.$$

Therefore, as $\nabla_{\boldsymbol{V}}\boldsymbol{V} = \boldsymbol{A}$ by definition, using the product rule one obtains

$$\delta K(t) = \varrho_o\nabla_{\boldsymbol{V}}\left(\langle\!\langle \boldsymbol{V}, \boldsymbol{U}\rangle\!\rangle_{\boldsymbol{g}}\right) - \varrho_o\langle\!\langle \boldsymbol{A}, \boldsymbol{U}\rangle\!\rangle_{\boldsymbol{g}}.$$

Note that the term $\nabla_{\boldsymbol{V}}(\langle\!\langle \boldsymbol{V}, \boldsymbol{U}\rangle\!\rangle_{\boldsymbol{g}})$ will be omitted; once integrated over time it vanishes because of equation (54), i.e.,

$$\int_{t_1}^{t_2}\left[\int_{\mathcal{B}}\nabla_{\boldsymbol{V}}(\langle\!\langle \boldsymbol{V}, \boldsymbol{U}\rangle\!\rangle_{\boldsymbol{G}})\boldsymbol{\mu}\right]\mathrm{d}t = \int_{\mathcal{B}}\langle\!\langle \boldsymbol{V}, \boldsymbol{U}\rangle\!\rangle_{\boldsymbol{G}}\boldsymbol{\mu}\Big|_{t_1}^{t_2} = 0.$$

Hence, using the spatial representation, one has

$$\delta K(t) = -\varrho_o\langle\!\langle \boldsymbol{a}, \boldsymbol{u}\rangle\!\rangle_{\boldsymbol{G}}.$$

As for the elastic energy, first note that at each time t, the configuration $\varphi_{t,\epsilon}$ can be written as $\zeta_{t,\epsilon}\circ\varphi_t$ with $\zeta_{t,\epsilon}:\varphi_t(\mathcal{B})\to\mathcal{S}$ generated by the field $\boldsymbol{u}_{t,\epsilon}$, in turn related to $\boldsymbol{U}_\epsilon(t)$ as $\boldsymbol{u}_{t,\epsilon}(x) = \boldsymbol{U}_\epsilon(\varphi_t^{-1}(x), t)$. As a matter of fact, for fixed X and t, the fields $\boldsymbol{u}_{t,\epsilon}(\varphi_t(X))$ and $\boldsymbol{U}_\epsilon(X, t)$ are tangent to the same curve $\epsilon\mapsto\varphi_{t,\epsilon}(X)$. From equation (54), the map $\zeta_{t,\epsilon}$ is such that $\zeta_{t,0} = \mathrm{id}_{\varphi_t(\mathcal{B})}$ for any t, $\zeta_{t_1,\epsilon} = \mathrm{id}_{\varphi_{t_1}(\mathcal{B})}$ and $\zeta_{t_2,\epsilon} = \mathrm{id}_{\varphi_{t_2}(\mathcal{B})}$ for any ϵ. Thus, for the pulled-back metric, one has

$$\partial_\epsilon\boldsymbol{C}_\epsilon(t) = \partial_\epsilon\left(\varphi_t^*\left(\zeta_{t,\epsilon}^*\boldsymbol{g}\right)\right) = \varphi_t^*\left(\partial_\epsilon\left(\zeta_{t,\epsilon}^*\boldsymbol{g}\right)\right).$$

Note that $\zeta_{t,\epsilon}^*\boldsymbol{g}$ is defined on $\varphi_t(\mathcal{B})$, and so the expression $\partial_\epsilon(\zeta_{t,\epsilon}^*\boldsymbol{g})$ is meaningful. The $\boldsymbol{C}$-variation is simplified to read

$$\delta\boldsymbol{C}(t) = \partial_\epsilon\boldsymbol{C}_\epsilon(t)|_{\epsilon=0} = \varphi_t^*(\mathfrak{L}_{\boldsymbol{u}}\boldsymbol{g}),$$

where $\mathfrak{L}$ denotes the Lie derivative. Note that $(\mathfrak{L}_u g)_{ab} = \nabla_a u_b + \nabla_b u_a$, and hence, the variation of the pulled-back metric reads $\delta C_{AB} = F^a{}_AF^b{}_B(\nabla_a u_b + \nabla_b u_a)$, where $u_a = g_{ac}u^c$. Hence, one obtains

$$\delta W(t) = \frac{1}{2}S^{AB}\delta C_{AB} = S^{AB}\delta C_{AB}F^a{}_AF^b{}_B\nabla_a u_b = P^{aB}\nabla_B u_a.$$

In components, the Lagrange–d'Alembert principle (equation (56)) is written as

$$\int_{t_1}^{t_2}\left[\int_{\mathcal{B}}\left(-\varrho_o a^a u_a - \varrho_o b^a u_a - P^{aB}\nabla_B u_a\right)\boldsymbol{\mu} + \int_{\partial\mathcal{B}}T_a u^a\boldsymbol{\eta}\right]\mathrm{d}t = 0. \tag{68}$$

Material mass density and the area and volume forms are time-dependent, as the anelastic deformations affect them. Equivalently, the Lagrange–d'Alembert principle (equation (56)) can be written in terms of the Euclidean structure as

$$\int_{t_1}^{t_2}\left[\int_{\mathcal{B}}\left(-\mathrm{J}\varrho_o a^a u_a - \mathrm{J}\varrho_o b^a u_a - \bar{P}^{aB}\nabla_B u_a\right)\bar{\boldsymbol{\mu}} + \int_{\partial\mathcal{B}}\bar{T}^a u_a\bar{\boldsymbol{\eta}}\right]\mathrm{d}t = 0, \tag{67}$$

with $\bar{T}_a$ indicating tractions with respect to the Euclidean metric, i.e., $T_a = \mathrm{J}_\partial\bar{T}_a$. Equations (66) and (67) can be used to derive the local form of the balance of linear momentum.