On applications of generalized functions to beam bending problems

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Abstract

Using a mathematical approach, this paper seeks an efficient solution to the problem of beams bending under singular loading conditions and having various jump discontinuities. For two instances, the boundary-value problem that describes beam bending cannot be written in the space of classical functions. In the first instance, the beam is under singular loading conditions, such as point forces and moments, and in the second instance, the dependent variable(s) and its derivatives have jump discontinuities. In the most general case, we consider both instances. First, we study singular loading conditions and present a theorem by which the equivalent distributed force of a general class of singular loading conditions can be found. As a consequence of obtaining the equivalent distributed force of a distributed moment, we find a mathematical explanation for the corner condition in classical plate theory. While plate theory is not the focus of this paper, this explanation is interesting. Then beams with various jump discontinuities are considered. When beams have jump discontinuities the form of the governing differential equations changes. We find the governing differential equations in the space of generalized functions. It is shown that for Euler–Bernoulli beams with jump discontinuities the operator of the differential equation remains unchanged, only the force term changes so that delta function and its distributional derivatives appear within it. But for Timoshenko beams with jump discontinuities, in addition to changes in the force terms, the operator of one of the governing differential equations changes. We then propose a new method for solving these equations. This method which we term the auxiliary beam method, is to solve the governing differential equations not in the space of generalized functions but rather to solve them by means of solving equivalent boundary-value problems in the space of classical functions. The auxiliary beam method reduces the number of differential equations and at the same time obviates the need to solve these differential equations in the space of generalized functions which can be more difficult. © 2000 Elsevier Science Ltd. All rights reserved.

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Nomenclature

EI	flexural stiffness of an Euler–Bernoulli or Timoshenko beam
EI_1, EI_2	flexural stiffnesses of beam segments
GA'	shear stiffness of a Timoshenko beam
GA'_1, GA'_2	shear stiffnesses of Timoshenko beam segments
K_r	stiffness of a rotational spring
K_t	stiffness of a translational spring
K_S	shear correction factor
L	length of a beam
M	a concentrated moment
M_1, M_2	bending moments
M_{xy}	twisting moment in a classical plate
M_0	a concentrated bending moment or bending moment at a discontinuity point
M^n	moment of order n of a distributed force
M_0^n	point moment of order n
P	a concentrated force
Q'_x	equivalent shearing force distribution for twisting moment in a classical plate
V_0	shear force at a discontinuity point
V_1, V_2	shear forces
W	a component of beam deflection
a_i	real coefficients
m, m_0	distributed moments
m, n, r, s	integer numbers
q	a distributed force (force function)
q_P	distributed force equivalent to a concentrated force
q_M	distributed force equivalent to a concentrated moment
q_m	distributed force equivalent to a distributed moment
u_1, u_2, u_3	displacement components
x	longitudinal axis of a beam
x_0	position of a discontinuity point
x_0^-	$x_0 - \varepsilon$ for a very small ε
x_0^+	$x_0 + \varepsilon$ for a very small ε
x_1, x_2	specific points on a beam
w	deflection of an Euler–Bernoulli beam
$\bar{w}$	deflection of the auxiliary beam of an Euler–Bernoulli beam
w^T	deflection of a Timoshenko beam
$\bar{w}^T$	deflection of the auxiliary beam of a Timoshenko beam
w_h, w_p	components of beam deflection
Δ	strength of a jump discontinuity in deflection of an Euler–Bernoulli beam
Δ^T	strength of a jump discontinuity in deflection of a Timoshenko beam
Θ	strength of a jump discontinuity in rotation of an Euler–Bernoulli beam
Θ^T	strength of a jump discontinuity in rotation of a Timoshenko beam
Φ	rotation in a Timoshenko beam
$\bar{\Phi}$	rotation in an auxiliary Timoshenko beam
Φ_1, Φ_2	rotation of Timoshenko beam segments

Ω	ratio of shear and flexural stiffnesses of a Timoshenko beam
α, β	ratio of flexural and shear stiffnesses of two beam segments
$\varepsilon, \lambda, \tau$	parameters
D	space of test functions
D'	space of distributions
D'_R	space of right-sided distributions
$H(x-x_0)$	Heaviside's unit step function
$\mathscr{P}$	set of all polynomials
f, g	distributions
$\delta(x-x_0)$	Dirac delta function
$\delta^{(n)}(x-x_0)$	the nth distributional derivative of Dirac delta function
φ	a test function
$*$	convolution symbol
$\langle\,\rangle$	distribution symbol

1. Introduction

In beam deflection problems, one sometimes encounters discontinuous loading conditions. The classical method for solving these problems is to partition the beam into beam segments between any two successive discontinuity points. Solving the differential equation of each beam segment and applying boundary and continuity conditions then yields the beam deflection equation. The first attempt to simplify these problems by writing a single expression for the bending moment was published by Clebsch (1862). Later, Macaulay (1919) introduced the so-called Macaulay's bracket. This method is also referred to in the literature as the singularity function method. The advantage of this method is that it reduces an uncoupled system of ordinary second-order differential equations to a single ordinary second-order equation.

Later, the singularity function method was generalized to two-dimensional problems. Wittrick (1965) analyzed beams with lateral loads and circular plates with axisymmetric lateral loads. Mahing (1964) used the method for rectangular plates whose opposite sides are simply supported under a point load and for circular plates with axisymmetric loading. Conway (1980) and Selek and Conway (1983) generalized the singularity function method to two-dimensional problems governed by partial differential equations.

Arbabi (1991) generalized the singularity function method for a beam with an internal hinge and a beam with jump discontinuities in flexural stiffness. In these beam deflection problems he did not solve the problem as a boundary-value problem; instead he started the analysis from the bending moment expression.

Schwarz's distribution theory (Schwarz, 1966) provides a rigorous justification for a number of very common formal manipulations in the engineering literature. Certain types of distributions, in particular, the Dirac delta function and its derivatives, were used in engineering problems years before the development of distribution theory. The delta function dates back to the 19th century and the works of Hermite, Cauchy, Poisson, Kirchhoff, Helmholtz, Lord Kelvin, and Heaviside (Van der Pol and Bremmer, 1955, pp. 62–66). Dirac (1930) introduced this function in quantum mechanics and since then the function has been known as the Dirac delta function.

This article uses the distribution theory of Schwarz to analyze beams with various discontinuities.

Schwarz's theory is used first to obtain the equivalent force distributions of different singular loading conditions by presenting a theorem, and then to obtain the equivalent distributed force of a distributed moment. Then the corner condition phenomenon in classical plate theory is mathematically explained.

The paper continues with an investigation of an Euler–Bernoulli beam having internal jump discontinuities in slope, deflection, and flexural stiffness. In the most general case, it is assumed that at the point of hinge and shear-free connection there are translational and rotational springs. The governing differential equation of the beam is derived in the space of generalized functions. It is observed that the form of the operator of governing differential equation remains unchanged and only the force term changes. The auxiliary beam method is then introduced. Using this method, instead of solving a fourth-order differential equation in the space of generalized functions, another fourth-order differential equation is solved in the space of classical functions. Then three examples are solved in order to show the capability and efficiency of the method. As an alternative to the classical method, the equivalent distributed force of a concentrated force and a concentrated moment are obtained using the discontinuities they produce in the bending moment and shearing forces of an Euler–Bernoulli beam.

The system of differential equations of a Timoshenko beam with jump discontinuities in slope, deflection, flexural stiffness, and shear stiffness is obtained in the space of generalized functions. It is shown that, in addition to changes in force terms, the form of the operator of one of the governing differential equations changes. Following the same mode of presentation used for the case of an Euler–Bernoulli beam, the auxiliary beam method is introduced and an example solved to demonstrate the auxiliary beam method's capability and efficiency.

Basic definitions and theorems of Schwarz's distribution theory appear in Appendix A. Appendices B and C In some mathematical details are given.

2. Singular loading conditions

In this section the equivalent distributed force for a family of singular loading conditions is found using Schwarz's distribution theory. Appendix A discusses the basic definitions and some theorems of the distribution theory of Schwarz.

Definition 1. Suppose that $q(x)$ is a distributed force. The nth-order moment of $q(x)$ about a point x_0 is denoted by $M^n(x_0)$ and is defined as:

$$M^n(x_0) = \int_{-\infty}^{+\infty} (x - x_0)^n q(x)\, dx \tag{1}$$

Definition 2. Suppose that $q(x)$ is a distributed force in the small interval $(x_0 - \varepsilon,\, x_0 + \varepsilon)$. Assume that:

$$M^n(\varepsilon) = \int_{-\infty}^{+\infty} (x - x_0)^n q_\varepsilon(x)\, dx \neq 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} (x - x_0)^m q_\varepsilon(x)\, dx = 0, \quad m \neq n \tag{2}$$

Then

$$M_0^n = \lim_{\varepsilon \to 0} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} (x - x_0)^n q_\varepsilon(x)\, dx \tag{3}$$

is called a point moment of order n. Clearly, a concentrated force is a point moment of order zero and a concentrated moment is a point moment of order one.

A concentrated double moment is the limiting case of two opposite moments acting on two points apart. This loading condition was introduced by Timoshenko and Woinowsky-Krieger (1959), and shown by those authors to result in a deflection with a discontinuous slope at the point of the concentrated double moment. We show that the equivalent distributed force (loading function) for this loading condition can be expressed by:

$$q(x) = \frac{M^2}{2} \delta^{(2)}(x - x_0) \tag{4}$$

where M^2 is the value of the double moment and $\delta^{(2)}$ is the second distributional derivative of δ. This is not a common loading condition but it will be useful for us in subsequent sections. Here we want to obtain the equivalent distributed force of a point moment of order n using distribution theory.

Theorem. *The equivalent distributed force of a unit moment of order n applied at $x = x_0$ is:*

$$q_n(x) = \frac{(-1^n)}{n!} \delta^{(n)}(x - x_0) \tag{5}$$

where $\delta^{(n)}$ is the nth distributional derivative of the Dirac delta function. The proof is given in Appendix B.

Corollary 1. *The equivalent distributed force for an upward concentrated force of magnitude P is*

$$q_P(x) = p\delta(x - x_0) \tag{6}$$

This result was obtained in Timoshenko (1976) and Shames (1989), where a concentrated force is considered to be the limiting case of a load distributed over a very short portion of a beam. Another proof of this representation, using the discontinuity a concentrated force causes in the shearing forces of an Euler–Bernoulli beam, appears in Section 4.4.

Corollary 2. *The equivalent distributed force of a clockwise concentrated moment of magnitude M is:*

$$q_M(x) = M\delta^{(1)}(x - x_0) \tag{7}$$

The same result was obtained by Shames (1989) in which this loading is considered to be the limiting case of two concentrated forces M/ε, ε apart, when ε tends to zero. Another proof of this representation, using the discontinuity a concentrated moment introduces in the bending moment of an Euler–Bernoulli beam, appears in Section 4.4.

Corollary 3. *The equivalent distributed force of a concentrated double moment is given by* Eq. (4). *As* Timoshenko and Woinowsky-Krieger (1959) *mention, this loading results in a deflection with a discontinuous slope at the point of double moment. We see later that, in an Euler–Bernoulli beam with a jump discontinuity in slope, this forcing function appears.*

3. Equivalent distributed force for a distributed moment and a mathematical explanation for corner condition in classical plate theory

In this section the equivalent distributed force of a distributed moment is obtained and then a mathematical explanation for corner condition in classical plate theory is offered.

It can be easily shown that the force function of a distributed moment $m(x)$ can be expressed as the convolution of m and $\delta^{(1)}$:
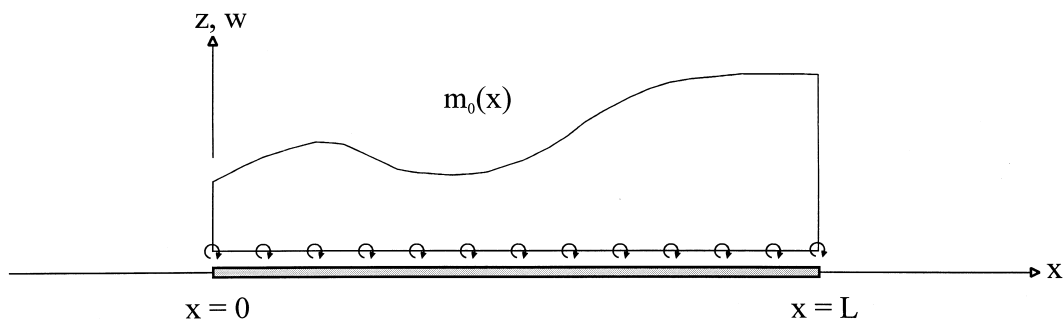
$$q(x) = \left(m * \delta^{(1)} \right)(x) \tag{8}$$

But it is known from distribution theory that for any function f:

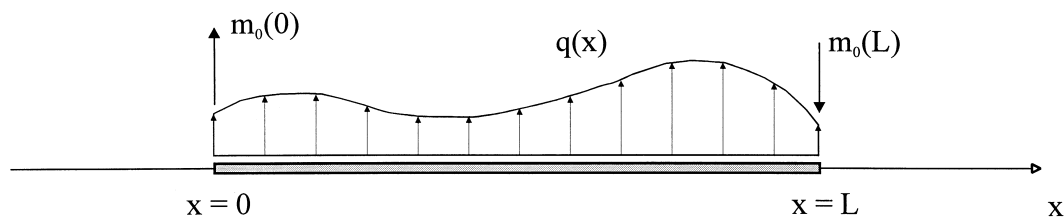$$\left(f * \delta^{(n)} \right)(x) = f^{(n)}(x) \tag{9}$$

Hence,

$$q(x) = m^{(1)}(x) \tag{10}$$

Therefore, for a distributed moment, the forcing function is the first distributional derivative of $m(x)$. Consider a beam with length L under a distributed moment $m_0(x)$ as shown in Fig. 1(a). The moment distribution function $m(x)$ can be written as:

Fig. 1. (a) A beam under a distributed moment. (b) The equivalent force system.

$$m(x) = \begin{cases} m_0(x) & 0 < x < L \\ 0 & L \leq x \text{ or } x \leq 0 \end{cases} \tag{11}$$

Hence,

$$m(x) = m_0(x)\big[H(x) - H(x - L)\big] \tag{12}$$

where H is Heaviside's function. Substituting Eq. (12) into (10) yields:

$$q(x) = m^{(1)}(x) = m_0'(x)\big[H(x) - H(x - L)\big] + m_0(x)\big[\delta(x) - \delta(x - L)\big]$$

$$= m_x'\big[H(x) - H(x - L)\big] + m_0(0)\delta(x) - m_0(L)\delta(x - L) \tag{13}$$

Clearly, the distributed moment is equivalent to a distributed force $m_0'(x)$ in $(0, L)$ and two concentrated forces $m_0(0)$ and $-m_0(L)$ at $x = 0$ and $x = L$, respectively, as shown in Fig. 1(b). Similarly, if $m(x)$ is a partially distributed moment in the interval (x_1, x_2), it is equivalent to a distributed force in this interval and two concentrated forces at $x = x_1$ and $x = x_2$.

According to Timoshenko and Woinowsky-Krieger (1959), Kelvin and Tait transformed the twisting moments along an edge of a classical plate to a system of shearing forces; for example, along an edge parallel to the y-axis, the equivalent distribution of shearing forces is (Timoshenko and Woinowsky-Krieger, 1959):

$$Q_x' = \left(\frac{\partial M_{xy}}{\partial y}\right)_{x=a} \tag{14}$$

where M_{xy} is the twisting moment. It is known, in the framework of classical theory, that polygonal plates loaded laterally will usually produce concentrated reactions at corner points, in addition to the distributed reactions along the edges. This phenomenon is known as the "corner condition," and was physically explained but not mathematically proven by Timoshenko and Woinowsky-Krieger (1959). It should be mentioned that this phenomenon does not appear in shear deformation plate theories.

According to our previous discussion, Q_x' consists of a system of distributed forces and two concentrated forces at the corner points. This is a mathematical explanation of the "corner condition" according to classical plate theory.

In the subsequent sections, differential equation of beams with various jump discontinuities are obtained and solution procedures are given.

4. Jump discontinuities in slope, deflection, and flexural stiffness of an Euler-Bernoulli beam

In this section the differential equation of an Euler–Bernoulli beam with jump discontinuities in slope, deflection, and flexural stiffness is obtained in the space of generalized functions. The classical method of solving this problem is to solve the problem on both sides of the discontinuities and then apply the boundary and continuity conditions. Here we solve the problem as a single beam using generalized functions. For the sake of simplicity, only one point of jump discontinuity is considered. The generalization to the case of an Euler–Bernoulli beam with n singular points is straightforward.

In the Euler–Bernoulli beam theory we have the following displacement field assumptions:

$$u_1(x, y, z) = -z\frac{\mathrm{d}w(x)}{\mathrm{d}x} \tag{15a}$$

$$u_2(x, y, z) = 0 \tag{15b}$$

$$u_3(x, y, z) = w(x) \tag{15c}$$

where u_1, u_2, and u_3 are displacement components along x, y, and z axes, respectively. The beam is along the x-axis and the loads are applied along the z-axis. Using Eq. (15) and the principle of virtual work, the governing equilibrium equation may be expressed as:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 w}{dx^2}\right) = q(x) \tag{16}$$

where EI is the flexural stiffness and q is a distributed force and is called the loading function. For the case of constant flexural stiffness, Eq. (16) can be simplified as:

$$\frac{dw^4}{dx^4} = \frac{q(x)}{EI} \tag{17}$$

When q is a piecewise continuous function, w and its first three derivatives are continuous and the fourth derivative is piecewise continuous. However, there are some loading conditions for which the loading function cannot be expressed as a classical function. A very general class of these loading conditions was studied in Section 2. However, sometimes displacement of the beam or its derivatives have discontinuities independent of the loading condition. This is the focus of this section.

The beam shown in Fig. 2 is of length L and has arbitrary boundary conditions at $x = 0, L$. The flexural stiffness of the beam is changed discontinuously at $x = x_0$. There are also jump discontinuities in slope and deflection at this point. In the most general case we have a combination of an internal hinge with a rotational spring and a shear-free connection with a translational spring. The spring constants of the rotational and translational springs are K_r and K_t, respectively. Now let

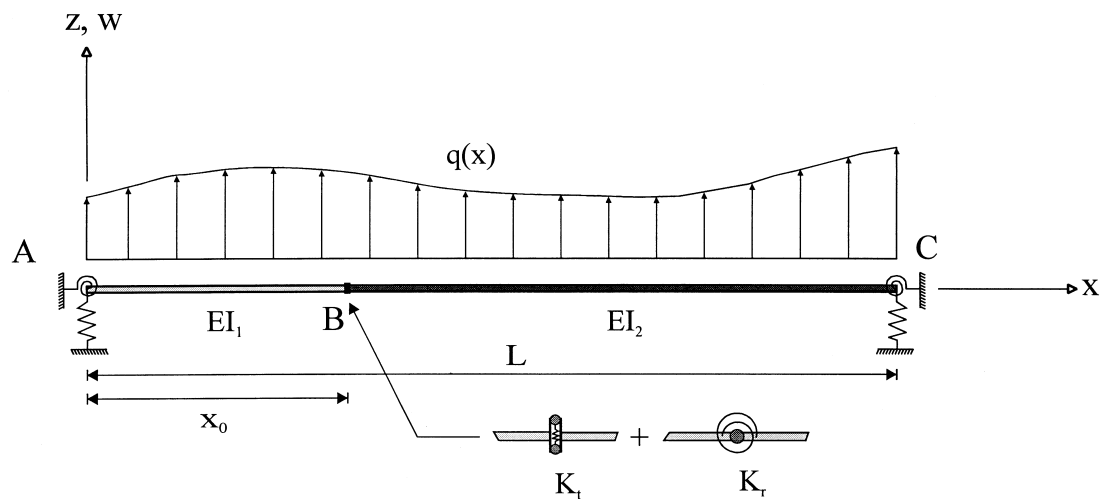$$w(x_0^+) - w(x_0^-) = \Delta \tag{18a}$$

Fig. 2. A beam with a jump discontinuity in slope, deflection, and flexural stiffness with arbitrary boundary conditions under a distributed force.

$$\frac{\mathrm{d}w(x_0^+)}{\mathrm{d}x} - \frac{\mathrm{d}w(x_0^-)}{\mathrm{d}x} = \Theta \tag{18b}$$

The beam is composed of two beam segments, AB and BC. Hence, using Heaviside's function:

$$w(x) = w_1(x) + [w_2(x) - w_1(x)]H(x - x_0) \tag{19}$$

where w is the deflection of the beam and w_1 and w_2 are the deflections of the beam segments, AB and BC, respectively. After some manipulations, which are given in Appendix C, the governing differential equation of the beam is found to be:

$$\frac{\bar{\mathrm{d}}^4 w}{\mathrm{d}x^4} = \frac{q(x)}{EI} + \frac{q(x)}{EI}\left(\frac{1}{\alpha} - 1\right)H(x - x_0) + \frac{K_t \Delta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta(x - x_0) + \frac{K_r \Theta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta^{(1)}(x - x_0)$$

$$+ \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0) \tag{20}$$

where a bar over the differentiation symbol implies distributional differentiation. As can be seen having jump discontinuities in slope and deflection is equivalent to having double and triple point moments $M^2 = 2\Theta$, $M^3 = 6\Delta$ at the point of jump discontinuities. Also, continuity conditions can be written as:

$$EI\frac{\mathrm{d}^2 w(x_0^-)}{\mathrm{d}x^2} = K_r\Theta, \qquad EI\frac{\mathrm{d}^3 w(x_0^-)}{\mathrm{d}x^3} = K_t\Delta \tag{21}$$

Applying the four boundary conditions at $x = 0$ and $x = L$ and the continuity conditions (21), we obtain the beam deflection w. As can be seen only the force term changes; the form of the operator of the differential equation is the same as that of Eq. (17).

4.1. A solution procedure

A common method of solving differential equations in the space of generalized functions is the Laplace transform method. Here, instead of using the Laplace transform method, we follow the method proposed by Kanwal (1983). The general solution can be written as:

$$w(x) = w_h(x) + w_p(x) \tag{22}$$

where w_h and w_p are the solutions of the following differential equations:

$$\frac{\mathrm{d}^4 w_h}{\mathrm{d}x^4} = \frac{q(x)}{EI} \tag{23a}$$

$$\frac{\bar{\mathrm{d}}^4 w_p}{\mathrm{d}x^4} = \frac{q(x)}{EI}\left(\frac{1}{\alpha} - 1\right)H(x - x_0) + \frac{K_t \Delta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta(x - x_0) + \frac{K_r \Theta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta^{(1)}(x - x_0)$$

$$+ \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0) \tag{23b}$$

For finding w_p, it is assumed that:

$$w_p(x) = W(x)H(x - x_0) \tag{24}$$

Hence:

$$\frac{\bar{d}^4 w_p}{dx^4} = \frac{d^4 W(x)}{dx^4}H(x - x_0) + \frac{d^3 W(x_0)}{dx^2}\delta(x - x_0) + \frac{d^2 W(x_0)}{dx^2}\delta^{(1)}(x - x_0) + \frac{dW(x_0)}{dx}\delta^{(2)}(x - x_0)$$

$$+ W(x_0)\delta^{(3)}(x - x_0) \tag{25}$$

Equating the coefficient of the generalized functions in Eqs. (23b) and (25), we obtain:

$$\frac{d^4 W(x)}{dx^4} = \frac{q(x)}{EI}\left(\frac{1}{\alpha} - 1\right) \tag{26a}$$

$$\frac{d^3 W(x_0)}{dx^3} = \frac{K_t\Delta}{EI}\left(\frac{1}{\alpha} - 1\right), \qquad \frac{d^2 W(x_0)}{dx^2} = \frac{K_r\Theta}{EI}\left(\frac{1}{\alpha} - 1\right)$$

$$\frac{dW(x_0)}{dx} = \Theta, \quad W(x_0) = \Delta \tag{26b}$$

Solving Eq. (26a) and applying the initial conditions (26b), we obtain:

$$W = W(x, \Delta, \Theta) \tag{27}$$

Solving Eq. (23a) for w_h, we have four integration constants. Applying the four boundary conditions at $x = 0, L$ for $w = w_h + w_p$ and the continuity conditions (21), we obtain the beam deflection. Obviously, this is not an efficient method and has no superiority over the classical method. A more efficient method is proposed here for calculating the beam deflection.

4.2. Auxiliary beam method

Suppose that w is the deflection of an Euler–Bernoulli beam with jump discontinuities in slope, deflection, and flexural stiffness at a point $x = x_0$. The deflection of the auxiliary beam is defined as follows:

$$\bar{w}(x) = w(x) - \Delta H(x - x_0) - \Theta(x - x_0)H(x - x_0) - \frac{K_r\Theta}{2EI}\left(\frac{1}{\alpha} - 1\right)(x - x_0)^2 H(x - x_0)$$

$$- \frac{K_t\Delta}{6EI}\left(\frac{1}{\alpha} - 1\right)(x - x_0)^3 H(x - x_0) \tag{28}$$

Clearly, $w(x)$ is a classical function. Substituting Eq. (28) into (20) yields:

$$\frac{d^4\bar{w}(x)}{dx^4} = \frac{q(x)}{EI} + \frac{q(x)}{EI}\left(\frac{1}{\alpha} - 1\right)H(x - x_0) \tag{29}$$

Also, from Eq. (28) we have:

$$\bar{w}(0) = w(0), \quad \bar{w}(L) = w(L) - \Delta - \Theta(L - x_0) - \frac{K_r\Theta}{2EI}\left(\frac{1}{\alpha} - 1\right)(L - x_0)^2 - \frac{K_t\Delta}{6EI}\left(\frac{1}{\alpha} - 1\right)(L - x_0)^3 \tag{30a}$$

$$\frac{d\bar{w}(0)}{dx} = \frac{dw(0)}{dx}, \quad \frac{d\bar{w}(L)}{dx} = \frac{dw(L)}{dx} - \Theta - \frac{K_r\Theta}{EI}\left(\frac{1}{\alpha} - 1\right)(L - x_0) - \frac{K_t\Delta}{2EI}\left(\frac{1}{\alpha} - 1\right)(L - x_0)^2 \tag{30b}$$

$$\frac{d^2\bar{w}(0)}{dx^2} = \frac{d^2w(0)}{dx^2}, \quad \frac{d^2\bar{w}(L)}{dx^2} = \frac{d^2w(L)}{dx^2} - \frac{K_r\Theta}{EI}\left(\frac{1}{\alpha} - 1\right) - \frac{K_t\Delta}{EI}\left(\frac{1}{a} - 1\right)(L - x_0) \tag{30c}$$

The continuity conditions for the auxiliary beam are:

$$\frac{d^2\bar{w}(x_0)}{dx^2} = \frac{d^2w(x_0^-)}{dx^2} = \frac{K_r\Theta}{EI}, \quad \frac{d^3\bar{w}(x_0)}{dx^3} = \frac{d^3w(x_0^-)}{dx^3} = \frac{K_t\Delta}{EI} \tag{31}$$

Therefore, instead of solving two differential equations for the two beam segments and applying eight boundary and continuity conditions, only one differential equation with six boundary and continuity equations is solved. To clarify the method, three examples are solved in the next section

4.3. Examples

Here three examples are solved to show the efficiency of the auxiliary beam method in analyzing Euler–Bernoulli beams with jump discontinuities.

Example 1. In this example a beam with an internal hinge under a uniform distributed force is considered. The beam shown in Fig. 3(a) is clamped at $x = 0$ and simply supported at $x = L$, and the flexural stiffness is constant. For this beam:

$$q(x) = -q_0, \quad \alpha = 1, \quad K_r = K_t = 0, \quad \Delta = 0 \tag{32}$$

From Eq. (29), the governing differential equation of the auxiliary beam is:

$$\frac{d^4\bar{w}}{dx^4} = -\frac{q_0}{EI} \tag{33}$$

Hence,

$$\bar{w}(x) = -\frac{q_0}{24EI}x^4 + a_0 + a_1x + a_2x^2 + a_3x^3 \tag{34}$$

We know that for this beam:

$$w(0) = \frac{dw(0)}{dx} = w(L) = \frac{d^2w(L)}{dx^2} = 0 \tag{35}$$

Hence from Eq. (30), we obtain:

$$\bar{w}(0) = \frac{d\bar{w}(0)}{dx} = 0, \quad \bar{w}(L) = (\lambda - 1)\Theta L, \quad \frac{d^2\bar{w}(L)}{dx^2} = 0 \tag{36a}$$

$$\frac{d^2\bar{w}(\lambda L)}{dx^2} = 0 \tag{36b}$$
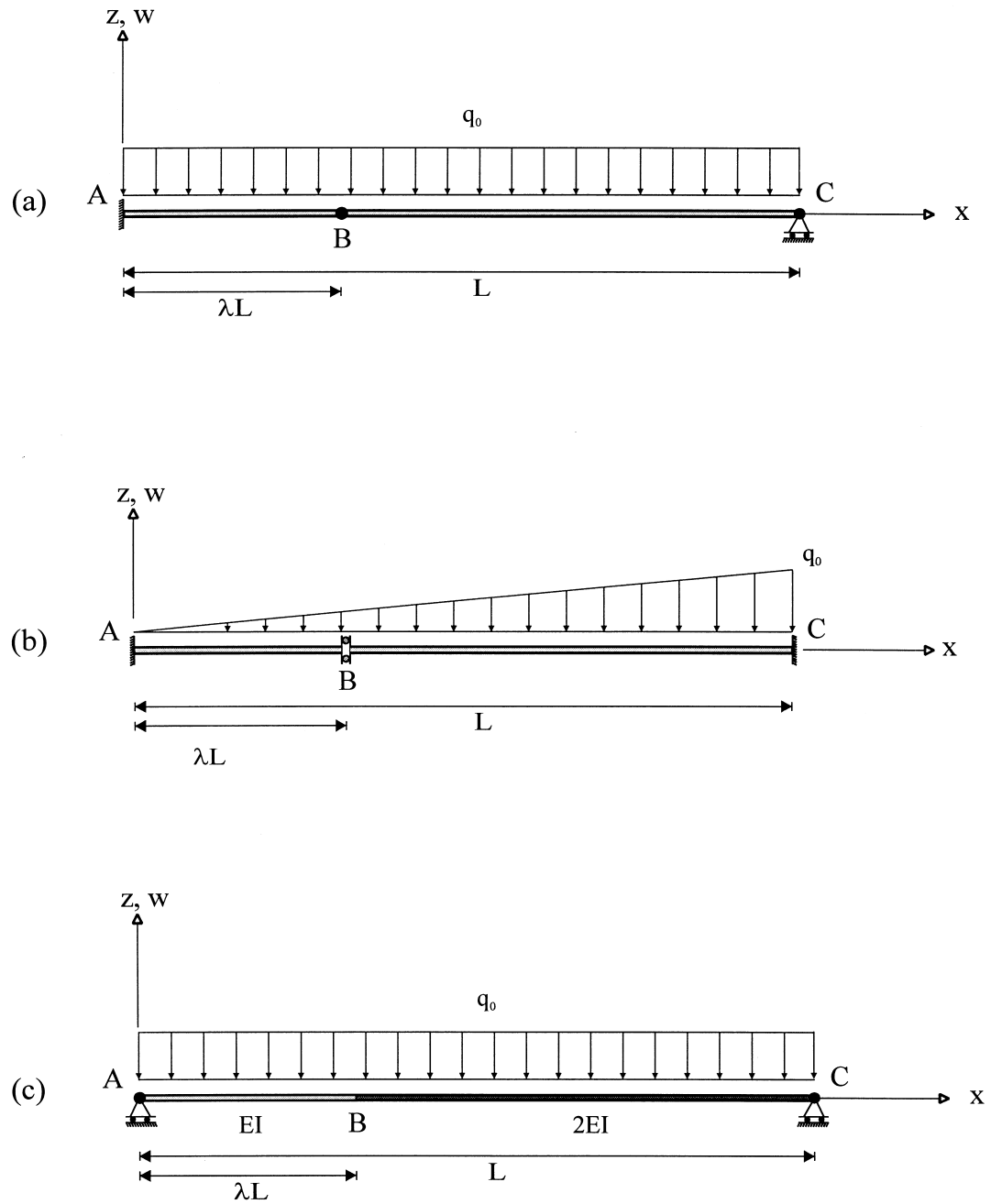
Thus, from Eqs. (34) and (36), we obtain:

Fig. 3. (a) A clamped, simply-supported beam with an internal hinge under a uniform distribute force. (b) A clamped–clamped beam with an internal shear-free connection under a linearly varying distributed force. (c) A simply-supported beam with a jump discontinuity in flexural stiffness under a uniform distributed force.

$$\bar{w}(x) = -\frac{q_0}{24EI}\left[x^4 - 2(\lambda + 1)Lx^3 + 6\lambda L^2 x^2\right] \tag{37a}$$

$$\Theta = \frac{(4\lambda - 1)q_0 L^3}{24(1 - \lambda)EI} \tag{37b}$$

Therefore, from Eq. (28) we have:

$$w(x) = -\frac{q_0}{24EI}\left[x^4 - 2(\lambda + 1)Lx^3 + 6\lambda L^2 x^2 - \frac{(4\lambda - 1)}{1 - \lambda}L^3(x - \lambda L)H(x - \lambda L)\right] \tag{38}$$

Example 2. The beam shown in Fig. 3(b) has a jump discontinuity in deflection at $x = \lambda L$. For this beam:

$$q(x) = -\frac{q_0}{L}x, \quad \alpha = 1, \quad K_r = K_t = 0, \quad \Theta = 0 \tag{39}$$

From Eq. (39), the governing differential equation of the auxiliary beam is:

$$\frac{d^4\bar{w}}{dx^4} = -\frac{q_0}{EIL}x \tag{40}$$

Thus:

$$\bar{w} = -\frac{q_0}{120EIL}x^5 + a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{41}$$

For this beam:

$$w(0) = \frac{dw(0)}{dx} = w(L) = \frac{dw(L)}{dx} = 0 \tag{42a}$$

$$\frac{d^2 w(x_0^-)}{dx^2} = \frac{d^3 w(x_0^-)}{dx^3} = 0 \tag{42b}$$

Thus, using Eq. (30) we have:

$$\bar{w}(0) = \frac{d\bar{w}(0)}{dx} = 0, \quad \bar{w}(L) = w(L) - \Delta = -\Delta, \quad \frac{d\bar{w}(L)}{dx} = \frac{dw(L)}{dx} = 0 \tag{43a}$$

$$\frac{d^3\bar{w}(xL)}{dx^3} = 0 \tag{43b}$$

From Eqs. (41) and (43), we obtain:

$$\bar{w}(x) = -\frac{q_0 x^2}{24EIL}\left[2x^3 - 20\lambda^2 L^2 x - 5L^3\left(1 - 6\lambda^2\right)\right] \tag{44a}$$

$$\Delta = \frac{q_0 L^4\left(10\lambda^2 - 3\right)}{24EI} \tag{44b}$$

Hence, from Eq. (28) we have:

$$w(x) = -\frac{q_0 x^2}{24EIL}\left[2x^3 - 20\lambda^2 L^2 x - 5L^3\left(1 - 6\lambda^2\right)\right] + \frac{q_0 L^4\left(10\lambda^2 - 3\right)}{24EI}H(x - \lambda L) \tag{45}$$

Example 3. A simply-supported beam under a uniform distributed force with a jump discontinuity in flexural stiffness at $x = \lambda L$ is shown in Fig. 3(c). For this beam:

$$\Delta = \Theta = 0, \quad \alpha = 2, \quad q(x) = -q_0 \tag{46}$$

and the deflection of the auxiliary beam is defined as:

$$\bar{w}(x) = w(x) - \frac{M_0}{2EI}\left(\frac{1}{\alpha} - 1\right)(x - \lambda L)^2 H(x - \lambda L) - \frac{V_0}{6EI}\left(\frac{1}{\alpha} - 1\right)(x - \lambda L)^3 H(x - \lambda L) \tag{47}$$

The governing differential equation of the auxiliary beam may be written as:

$$\frac{d^4\bar{w}}{dx^4} = -\frac{q_0}{EI} + \frac{q_0}{2EI}H(x - \lambda L) \tag{48}$$

The boundary and continuity conditions are:

$$\bar{w}(0) = 0, \quad \bar{w}(L) = \frac{M_0}{4EI}(1 - \lambda)^2 L^2 \tag{49a}$$

$$\frac{d^2\bar{w}(0)}{dx^2} = 0, \quad \frac{d^2\bar{w}(L)}{dx^2} = \frac{M_0}{2EI} + \frac{V_0 L}{2EI}(1 - \lambda) \tag{49b}$$

$$\frac{d^2\bar{w}(\lambda L)}{dx^2} = \frac{M_0}{EI}, \quad \frac{d^3\bar{w}(\lambda L)}{dx^3} = \frac{V_0}{EI} \tag{49c}$$

Combining the boundary and continuity equations yields:

$$\bar{w}(0)\frac{d^2\bar{w}(0)}{dx^2} = 0 \tag{50a}$$

$$-\bar{w}(L) + \frac{(1 - \lambda)^2 L^2}{4}\frac{d^2\bar{w}(\lambda L)}{dx^2} = 0 \tag{50b}$$

$$-\frac{d^2\bar{w}(L)}{dx^2} + \frac{1}{2}\frac{d^2\bar{w}(\lambda L)}{dx^2} + \frac{L(1 - \lambda)}{2}\frac{d^3\bar{w}(\lambda L)}{dx^3} = 0 \tag{50c}$$

From Eq. (48) we obtain:

$$\bar{w}(x) = -\frac{q_0 x^4}{24EI} + \frac{q_0(x - \lambda L)^4}{48EI}H(x - \lambda L) + a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{51}$$

Applying the boundary and continuity conditions (50), we obtain:

$$\bar{w}(x) = -\frac{q_0}{48EI}\left\{2x^4 - 4Lx^3 + L^3\left[(1-\lambda)^4 - 6\lambda(1-\lambda)^3 + 2\right]x - (x-\lambda L)^4 H(x-\lambda L)\right\} \tag{52}$$

From Eq. (49c), we obtain:

$$M_0 = \frac{\lambda(1-\lambda)}{2}q_0L^2, \qquad V_0 = \left(\frac{1}{2} - \lambda\right)q_0L \tag{53}$$

Therefore, from Eq. (47) we obtain:

$$\begin{aligned} w(x) = & -\frac{q_0}{48EI}\left\{2x^4 - 4Lx^3 + L^3\left[(1-\lambda)^4 - 6\lambda(1-\lambda)^3 + 2\right]x + \left[-(x-\lambda L)^4 \right.\right. \\ & \left.\left. + 6L^2\lambda(1-\lambda)(x-\lambda L) + 4L\left(\frac{1}{2} - \lambda\right)(x-\lambda L)^3\right]H(x-\lambda L)\right\} \end{aligned} \tag{54}$$

Assuming $\lambda = 1$ in Eq. (54) gives us the deflection of a simply-supported beam with a constant flexural stiffness EI under a uniform distributed force:

$$w(x) = -\frac{q_0}{24EI}x^4 + \frac{q_0L}{12EI}x^3 - \frac{q_0L^3}{24EI}x \tag{55}$$

The advantage of using Macaulay's bracket is that we have only one expression for the bending moment or loading function. If there are n point loads (either force or moment), and one uses the singularity function method instead of solving $n + 1$ differential equations and applying $4(n + 1)$ boundary and continuity conditions, then only one differential equation with four boundary conditions need be solved.

In the case of n jump discontinuities, if one uses the auxiliary beam method presented in this article instead of solving $n + 1$ differential equations and applying $4(n + 1)$ boundary and continuity conditions, then only one differential equation with four boundary conditions and $2n$ continuity conditions need be solved. In most practical problems, we do not have all three kinds of discontinuities at the same point; for example, if a beam has n internal hinges, the number of continuity equations is reduced to n.

In Section 4 for finding the governing differential equation of an Euler–Bernoulli beam with jump discontinuities, the beam was partitioned to continuous beam segments. The next section uses the same idea to find the equivalent distributed forces for point forces and point moments.

4.4. Equivalent force function for concentrated force and moment: a nonclassical approach

This section offers a nonclassical proof for the representations (6) and (7). We use the fact that a concentrated force and a concentrated moment introduce respective jump discontinuities into the shearing force (the third derivative of the beam deflection) and the bending moment (the second derivative of the beam deflection) of an Euler–Bernoulli beam. As was mentioned in Section 2, the classical proof of these representations is based on considering these singular loading conditions as a distributed force over a very short length of the beam. Fig. 4 shows a beam with a concentrated force P_0 and a concentrated moment M_0 applied at $x = x_0$. The beam AC may be assumed to be composed of two beam segments, AB and BC. The deflections of the two beam segments AB and BC are denoted by w_1 and w_2, respectively. There is no loading for $0 < x < x_0$ and $x_0 < x < L$; hence, we have:

$$\frac{d^4w_1}{dx^4} = 0; \quad x \in (0, x_0) \tag{56a}$$

$$\frac{\mathrm{d}^4 w_2}{\mathrm{d}x^4} = 0; \quad x \in (x_0, L) \tag{56b}$$

The deflection of the beam w can be written as:

$$w(x) = w_1(x) + \left[w_2(x) - w_1(x)\right] H(x - x_0) \tag{57}$$

We know that the magnitudes and the first derivatives of w_1 and w_2 are equal at $x = x_0$; hence:

$$\frac{\bar{\mathrm{d}}^2 w(x)}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^2} + \left[\frac{\mathrm{d}^2 w_2(x)}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^2}\right] H(x - x_0) \tag{58}$$

From Fig. 4 we can write:

$$V_2(x_0^+) - V_1(x_0^-) = P_0 \tag{59a}$$

$$M_2(x_0^+) - M_1(x_0^-) = M_0 \tag{59b}$$

Thus,

$$\frac{\mathrm{d}^3 w_2(x_0^+)}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 w_1(x_0^-)}{\mathrm{d}x^3} = \frac{P_0}{EI} \tag{60a}$$

$$\frac{\mathrm{d}^2 w_2(x_0^+)}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1(x_0^-)}{\mathrm{d}x^2} = \frac{M_0}{EI} \tag{60b}$$
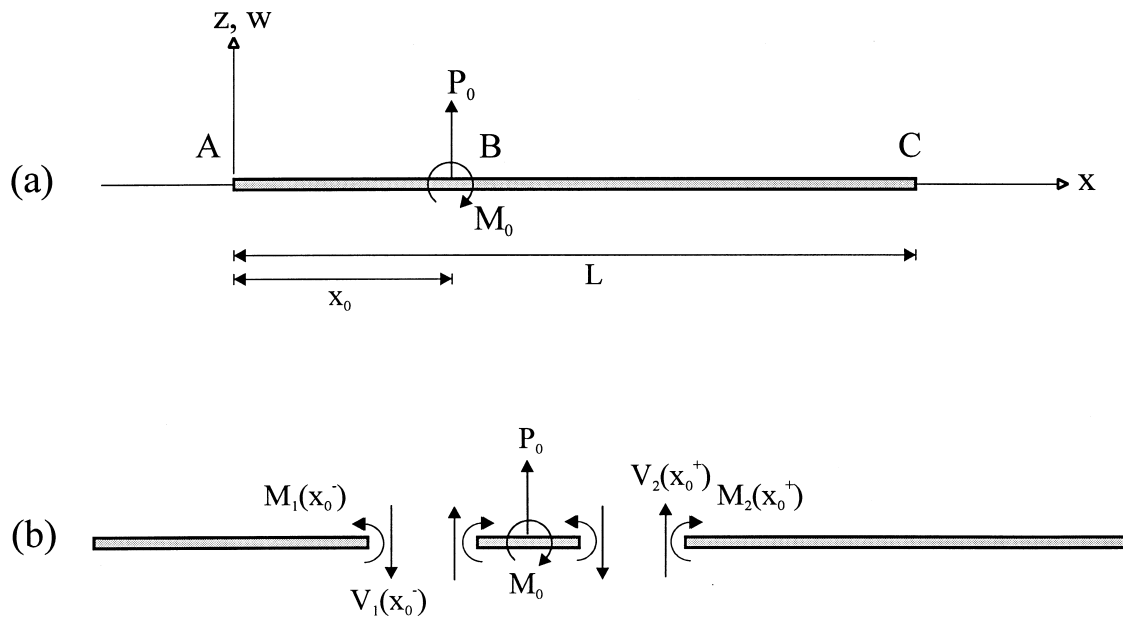

Fig. 4. (a) A beam under a concentrated force and a concentrated moment. (b) Moment and shear discontinuity at the point of the action of concentrated loads.

Differentiating both sides of Eq. (58) with respect to x yields:

$$\frac{\bar{\mathrm{d}}^3 w(x)}{\mathrm{d}x^3} = \frac{\mathrm{d}^3 w_1(x)}{\mathrm{d}x^3} + \left[\frac{\mathrm{d}^3 w_2(x)}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 w_1(x)}{\mathrm{d}x^3}\right] H(x - x_0) + \frac{M_0}{EI}\delta(x - x_0) \tag{61}$$

Also

$$\frac{\bar{\mathrm{d}}^4 w(x)}{\mathrm{d}x^4} = \frac{\mathrm{d}^4 w_1(x)}{\mathrm{d}x^4} + \left[\frac{\mathrm{d}^4 w_2(x)}{\mathrm{d}x^4} - \frac{\mathrm{d}^4 w_1(x)}{\mathrm{d}x^4}\right] H(x - x_0) + \frac{P_0}{EI}\delta(x - x_0) + \frac{M_0}{EI}\delta^{(1)}(x - x_0) \tag{62}$$

From Eqs. (56) and (62), we obtain:

$$\frac{\bar{\mathrm{d}}^4 w(x)}{\mathrm{d}x^4} = \frac{P_0\delta(x - x_0) + M_0\delta^{(1)}(x - x_0)}{EI} \tag{63}$$

Therefore, the equivalent force function is:

$$q(x) = q_\mathrm{P}(x) + q_\mathrm{M}(x) = P_0\delta(x - x_0) + M_0\delta^{(1)}(x - x_0). \tag{64}$$

5. Timoshenko beam with jump discontinuities

In this section we derive the system of governing differential equations of a Timoshenko beam with jump discontinuities in slope, deflection, flexural stiffness, and shear stiffness. For the sake of simplicity, only one point of jump discontinuity is considered; the generalization to the case of a Timoshenko beam with n discontinuity points is then straightforward. Timoshenko beam theory (Timoshenko, 1921) is the simplest shear deformation beam theory, and it is based on the following displacement field:

$$u_1(x, y, z) = z\Phi(x) \tag{65a}$$

$$u_2(x, y, z) = 0 \tag{65b}$$

$$u_3(x, y, z) = w^\mathrm{T}(x) \tag{65c}$$

where u_1, u_2, and u_3 are displacement components along the x, y, and z axes, respectively, and Φ is rotation about the y-axis. The superscript T in w^T denotes the deflection of the Timoshenko beam. The governing system of differential equations can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right) - GA'\left(\Phi + \frac{\mathrm{d}w}{\mathrm{d}x}\right) = 0 \tag{66a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[GA'\left(\Phi + \frac{\mathrm{d}w}{\mathrm{d}x}\right)\right] + q(x) = 0 \tag{66b}$$

where G is the shear modulus, $A' = K_\mathrm{s}A$ is the shear equivalent area of the beam cross-section, and K_s is the shear correction factor. Now let:

$$\Omega = \frac{GA'}{EI} \tag{67}$$

Substituting Eq. (67) into (66) and assuming constant flexural and shear stiffnesses yields:

$$\Omega \frac{dw^{\mathrm{T}}}{dx} - \frac{d^2\Phi}{dx^2} + \Omega\Phi = 0 \tag{68a}$$

$$\frac{d\Phi}{dx} + \frac{d^2 w^{\mathrm{T}}}{dx^2} + \frac{q(x)}{GA'} = 0 \tag{68b}$$

Now consider a Timoshenko beam of length L with internal jump discontinuities in slope, deflection, shear stiffness, and flexural stiffness at $x = x_0$. Using an expression similar to the one used in the case of an Euler–Bernoulli beam, we can write:

$$w^{\mathrm{T}}(x) = w_1^{\mathrm{T}}(x) + \left[w_2^{\mathrm{T}}(x) - w_1^{\mathrm{T}}(x)\right]H(x - x_0) \tag{69a}$$

$$\Phi(x) = \Phi_1(x) + \left[\Phi_2(x) - \Phi_1(x)\right]H(x - x_0) \tag{69b}$$

Deflection and rotation have jump discontinuities at $x = x_0$; i.e:

$$w_2^{\mathrm{T}}(x_0) - w_1^{\mathrm{T}}(x_0) = \Delta^{\mathrm{T}}, \qquad \Phi_2(x_0) - \Phi_1(x_0) = \Theta^{\mathrm{T}} \tag{70}$$

It is known that:

$$M_1(x_0) = EI_1 \frac{d\Phi_1(x_0)}{dx}, \qquad M_2(x_0) = EI_2 \frac{d\Phi_2(x_0)}{dx} \tag{71a}$$

$$V_1(x_0) = GA_1'\left[\Phi_1(x_0) + \frac{dw_1(x_0)}{dx}\right], \qquad V_2(x_0) = GA_2'\left[\Phi_2(x_0) + \frac{dw_2(x_0)}{dx}\right] \tag{71b}$$

Also, equilibrium for an infinitesimal element including the discontinuity point implies that:

$$M_1(x_0) = M_2(x_0) = K_{\mathrm{r}}\Theta^{\mathrm{T}} \tag{72a}$$

$$V_1(x_0) = V_2(x_0) = K_{\mathrm{t}}\Delta^{\mathrm{T}} \tag{72b}$$

where K_{t} and K_{r} are the stiffnesses of the translational and rotational springs at $x = x_0$. Comparing Eqs. (71) and (72), we obtain:

$$\frac{d\Phi_2(x_0)}{dx} - \frac{d\Phi_1(x_0)}{dx} = \frac{K_{\mathrm{r}}\Theta^{\mathrm{T}}}{EI}\left(\frac{1}{\alpha} - 1\right) \tag{73a}$$

$$\frac{dw_2(x_0)}{dx} - \frac{dw_1(x_0)}{dx} = \frac{K_{\mathrm{t}}\Delta^{\mathrm{T}}}{GA'}\left(\frac{1}{\beta} - 1\right) - \Theta^{\mathrm{T}} \tag{73b}$$

where

$$EI_1 = EI, \quad EI_2 = \alpha EI \tag{74a}$$

$$GA_1' = GA', \quad GA_2' = \beta GA' \tag{74b}$$

Differentiating Eqs. (69a) and (69b), we obtain:

$$\frac{\bar{\mathrm{d}}w^{\mathrm{T}}}{\mathrm{d}x} = \frac{\mathrm{d}w_1^{\mathrm{T}}}{\mathrm{d}x} + \left(\frac{\mathrm{d}w_2^{\mathrm{T}}}{\mathrm{d}x} - \frac{\mathrm{d}w_1^{\mathrm{T}}}{\mathrm{d}x}\right)H(x-x_0) + \Delta^{\mathrm{T}}\delta(x-x_0) \tag{75a}$$

$$\frac{\bar{\mathrm{d}}^2 w^{\mathrm{T}}}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 w_1^{\mathrm{T}}}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}^2 w_2^{\mathrm{T}}}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1^{\mathrm{T}}}{\mathrm{d}x^2}\right)H(x-x_0) + \left[\frac{K_{\mathrm{t}}\Delta^{\mathrm{T}}}{GA'}\left(\frac{1}{\beta}-1\right) - \Theta^{\mathrm{T}}\right]\delta(x-x_0) + \Delta^{\mathrm{T}}\delta^{(1)}(x-x_0) \tag{75b}$$

and

$$\frac{\bar{\mathrm{d}}\Phi}{\mathrm{d}x} = \frac{\mathrm{d}\Phi_1}{\mathrm{d}x} + \left(\frac{\mathrm{d}\Phi_2}{\mathrm{d}x} - \frac{\mathrm{d}\Phi_1}{\mathrm{d}x}\right)H(x-x_0) + \Theta^{\mathrm{T}}\delta(x-x_0) \tag{76a}$$

$$\frac{\bar{\mathrm{d}}^2 \Phi}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 \Phi_1}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}^2 \Phi_2}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 \Phi_1}{\mathrm{d}x^2}\right)H(x-x_0) + \frac{K_{\mathrm{r}}\Theta^{\mathrm{T}}}{EI}\left(\frac{1}{\alpha}-1\right)\delta(x-x_0) + \Theta^{\mathrm{T}}\delta^{(1)}(x-x_0) \tag{76b}$$

The deflection and rotation of each beam segment have continuous derivatives and hence they are governed by Eq. (68); thus:

$$\Omega_1 \frac{\mathrm{d}w_1^{\mathrm{T}}}{\mathrm{d}x} - \frac{\mathrm{d}^2 \Phi_1}{\mathrm{d}x^2} + \Omega_1 \Phi_1 = 0 \tag{77a}$$

$$\frac{\mathrm{d}\Phi_1}{\mathrm{d}x} + \frac{\mathrm{d}^2 w_1^{\mathrm{T}}}{\mathrm{d}x^2} + \frac{q(x)}{GA'} = 0 \tag{77b}$$

and

$$\Omega_2 \frac{\mathrm{d}w_2^{\mathrm{T}}}{\mathrm{d}x} - \frac{\mathrm{d}^2 \Phi_2}{\mathrm{d}x^2} + \Omega_2 \Phi_2 = 0 \tag{78a}$$

$$\frac{\mathrm{d}\Phi_2}{\mathrm{d}x} + \frac{\mathrm{d}^2 w_2^{\mathrm{T}}}{\mathrm{d}x^2} + \frac{q(x)}{\beta GA'} = 0 \tag{78b}$$

Now let $\Omega_1 = \Omega$ and $\Omega_2 = (\beta/\alpha)\Omega$. From Eqs. (75)–(78), we obtain the governing system of equilibrium equations for the beam:

$$\Omega \frac{\bar{\mathrm{d}}w^{\mathrm{T}}}{\mathrm{d}x} - \frac{\bar{\mathrm{d}}^2 \Phi}{\mathrm{d}x^2} + \Omega\Phi + \left(\frac{\beta}{\alpha}-1\right)\Omega\frac{\mathrm{d}w_2}{\mathrm{d}x}H(x-x_0) + \left(\frac{\beta}{\alpha}-1\right)\Omega\Phi_2 H(x-x_0)$$

$$= \left[\Omega\Delta^{\mathrm{T}} - \frac{K_{\mathrm{r}}\Theta^{\mathrm{T}}}{EI}\left(\frac{1}{\alpha}-1\right)\right]\delta(x-x_0) - \Theta^{\mathrm{T}}\delta^{(1)}(x-x_0) \tag{79a}$$

$$\frac{\bar{\mathrm{d}}^2 w^{\mathrm{T}}}{\mathrm{d}x^2} + \frac{\bar{\mathrm{d}}\Phi}{\mathrm{d}x} + \frac{q(x)}{GA'} + \frac{q(x)}{GA'}\left(\frac{1}{\beta} - 1\right)H(x - x_0) = \frac{K_{\mathrm{t}}\Delta^{\mathrm{T}}}{GA'}\left(\frac{1}{\beta} - 1\right)\delta(x - x_0) + \Delta^{\mathrm{T}}\delta^{(1)}(x - x_0) \qquad (79b)$$

Multiplying Eq. (69b) by $H(x - x_0)$ yields:

$$\Phi(x)H(x - x_0) = \Phi_1(x)H(x - x_0) + \left[\Phi_2(x) - \Phi_1(x)\right]H(x - x_0)H(x - x_0)$$

$$= \Phi_1(x)H(x - x_0) + \left[\Phi_2(x) - \Phi_1(x)\right]H(x - x_0)$$

$$= \Phi_2(x)H(x - x_0) \qquad (80)$$

Also multiplying Eq. (75a) by $H(x - x_0)$, we obtain:

$$\frac{\bar{\mathrm{d}}w^{\mathrm{T}}}{\mathrm{d}x}H(x - x_0) = \frac{\bar{\mathrm{d}}w_1^{\mathrm{T}}}{\mathrm{d}x}H(x - x_0) + \left(\frac{\bar{\mathrm{d}}w_2^{\mathrm{T}}}{\mathrm{d}x} - \frac{\bar{\mathrm{d}}w_1^{\mathrm{T}}}{\mathrm{d}x}\right)H(x - x_0) + \Delta^{\mathrm{T}}\delta(x - x_0)H(x - x_0)$$

$$= \frac{\mathrm{d}w_2^{\mathrm{T}}}{\mathrm{d}x}H(x - x_0) + \frac{\Delta^{\mathrm{T}}}{2}\delta(x - x_0) \qquad (81)$$

Hence,

$$\frac{\mathrm{d}w_2^{\mathrm{T}}}{\mathrm{d}x} = \frac{\bar{\mathrm{d}}w^{\mathrm{T}}}{\mathrm{d}x}H(x - x_0) - \frac{\Delta^{\mathrm{T}}}{2}\delta(x - x_0) \qquad (82)$$

Substituting Eqs. (80) and (82) we find the governing differential equations of the Timoshenko beam:

$$\Omega\left[1 + \left(\frac{\beta}{\alpha} - 1\right)H(x - x_0)\right]\frac{\bar{\mathrm{d}}w^{\mathrm{T}}}{\mathrm{d}x} - \frac{\bar{\mathrm{d}}^2\Phi}{\mathrm{d}x^2} + \Omega\left[1 + \left(\frac{\beta}{\alpha} - 1\right)H(x - x_0)\right]\Phi$$

$$= \left\{\Omega\left[1 + \frac{1}{2}\left(\frac{\beta}{\alpha} - 1\right)\right]\Delta^{\mathrm{T}} - \frac{K_{\mathrm{r}}\Theta^{\mathrm{T}}}{EI}\left(\frac{1}{\alpha} - 1\right)\right\}\delta(x - x_0) - \Theta^{\mathrm{T}}\delta^{(1)}(x - x_0) \qquad (83a)$$

$$\frac{\bar{\mathrm{d}}^2 w^{\mathrm{T}}}{\mathrm{d}x^2} + \frac{\bar{\mathrm{d}}\Phi}{\mathrm{d}x} + \frac{q(x)}{GA'} + \frac{q(x)}{GA'}\left(\frac{1}{\beta} - 1\right)H(x - x_0) = \frac{K_{\mathrm{t}}\Delta^{\mathrm{T}}}{GA'}\left(\frac{1}{\beta} - 1\right)\delta(x - x_0) + \Delta^{\mathrm{T}}\delta^{(1)}(x - x_0) \qquad (83b)$$

Eq. (83) is the system of governing differential equations for a Timoshenko beam with one point of jump discontinuities in the space of generalized functions. As can be seen for the first differential equation the operator of the differential equation has changed. For the special case when $\Omega_1 = \Omega_2$ ($\alpha = \beta$) the governing differential equations have a simpler form:

$$\Omega\frac{\bar{\mathrm{d}}w^{\mathrm{T}}}{\mathrm{d}x} - \frac{\bar{\mathrm{d}}^2\Phi}{\mathrm{d}x^2} + \Omega\Phi = \left[\Omega\Delta^{\mathrm{T}} - \frac{K_{\mathrm{r}}\Theta^{\mathrm{T}}}{EI}\left(\frac{1}{\alpha} - 1\right)\right]\delta(x - x_0) - \Theta^{\mathrm{T}}\delta^{(1)}(x - x_0) \qquad (84a)$$

$$\frac{\bar{\mathrm{d}}^2 w^{\mathrm{T}}}{\mathrm{d}x^2} + \frac{\bar{\mathrm{d}}\Phi}{\mathrm{d}x} + \frac{q(x)}{GA'} + \frac{q(x)}{GA'}\left(\frac{1}{\beta} - 1\right)H(x - x_0) = \frac{K_{\mathrm{t}}\Delta^{\mathrm{T}}}{GA'}\left(\frac{1}{\beta} - 1\right)\delta(x - x_0) + \Delta^{\mathrm{T}}\delta^{(1)}(x - x_0) \qquad (84b)$$

As was done in the case of an Euler–Bernoulli beam, an auxiliary beam method is introduced to solve

the governing equilibrium equations (84). For the more general case when $\alpha \neq \beta$ it is easier to solve the differential equations directly rather than referring to the auxiliary beam.

5.1. Auxiliary beam method

In this section an auxiliary beam is defined for a Timoshenko beam with internal jump discontinuities. The deflection and rotation of the auxiliary beam are defined as:

$$\bar{w}^{\mathrm{T}}(x) = w^{\mathrm{T}}(x) - \Delta^{\mathrm{T}} H(x - x_0) - \left[\frac{K_t \Delta^{\mathrm{T}}}{GA'} \left(\frac{1}{\beta} - 1 \right) - \Theta^{\mathrm{T}} \right] (x - x_0) H(x - x_0) \tag{85a}$$

$$\bar{\Phi}(x) = \Phi(x) - \Theta^{\mathrm{T}} H(x - x_0) - \left[\frac{K_r \Theta^{\mathrm{T}}}{EI} \left(\frac{1}{\alpha} - 1 \right) \right] (x - x_0) H(x - x_0) \tag{85b}$$

Substituting Eq. (85) into (84) yields:

$$\Omega \frac{\mathrm{d}\bar{w}^{\mathrm{T}}}{\mathrm{d}x} - \frac{\mathrm{d}^2 \bar{\Phi}}{\mathrm{d}x^2} + \Omega \bar{\Phi} = -\Omega \left[\frac{K_T \Delta^{\mathrm{T}}}{GA'} \left(\frac{1}{\beta} - 1 \right) + \frac{K_r \Theta^{\mathrm{T}}}{EI} \left(\frac{1}{\alpha} - 1 \right) (x - x_0) \right] H(x - x_0) \tag{86a}$$

$$\frac{\mathrm{d}^2 \bar{w}^{\mathrm{T}}}{\mathrm{d}x^2} + \frac{\mathrm{d}\bar{\Phi}}{\mathrm{d}x} + \frac{q(x)}{GA'} + \frac{q(x)}{GA'} \left(\frac{1}{\beta} - 1 \right) H(x - x_0) = \frac{-K_r \Theta^{\mathrm{T}}}{EI} \left(\frac{1}{\alpha} - 1 \right) H(x - x_0) \tag{86b}$$

The boundary conditions for the auxiliary beam can be obtained using the following relations:

$$\bar{w}^{\mathrm{T}}(0) = w^{\mathrm{T}}(0), \qquad \bar{w}^{\mathrm{T}}(L) = w^{\mathrm{T}}(L) - \Delta^{\mathrm{T}} - \left[\frac{K_t \Delta^{\mathrm{T}}}{GA'} \left(\frac{1}{\beta} - 1 \right) - \Theta^{\mathrm{T}} \right] (L - x_0) \tag{87a}$$

$$\frac{\mathrm{d}\bar{w}^{\mathrm{T}}(0)}{\mathrm{d}x} = \frac{\mathrm{d}w^{\mathrm{T}}(0)}{\mathrm{d}x}, \qquad \frac{\mathrm{d}\bar{w}^{\mathrm{T}}(L)}{\mathrm{d}x} = \frac{\mathrm{d}w^{\mathrm{T}}(L)}{\mathrm{d}x} - \left[\frac{K_t \Delta^{\mathrm{T}}}{GA'} \left(\frac{1}{\beta} - 1 \right) - \Theta^{\mathrm{T}} \right] \tag{87b}$$

and

$$\bar{\Phi}(0) = \Phi(0), \qquad \bar{\Phi}(L) = \Phi(L) - \Theta^{\mathrm{T}} - \left[\frac{K_r \Theta^{\mathrm{T}}}{EI} \left(\frac{1}{\alpha} - 1 \right) - \Theta^{\mathrm{T}} \right] (L - x_0) \tag{88a}$$

$$\frac{\mathrm{d}\bar{\Phi}(0)}{\mathrm{d}x} = \frac{\mathrm{d}\Phi(0)}{\mathrm{d}x}, \qquad \frac{\mathrm{d}\bar{\Phi}(L)}{\mathrm{d}x} = \frac{\mathrm{d}\Phi(L)}{\mathrm{d}x} - \left[\frac{K_r \Theta^{\mathrm{T}}}{EI} \left(\frac{1}{\alpha} - 1 \right) \right] \tag{88b}$$

The continuity conditions may be expressed as:

$$\frac{\mathrm{d}\bar{\Phi}(x_0)}{\mathrm{d}x} = \frac{K_r \Theta^{\mathrm{T}}}{EI}, \qquad \frac{\mathrm{d}\bar{w}(x_0)}{\mathrm{d}x} + \bar{\Phi}(x_0) = \frac{K_t \Delta^{\mathrm{T}}}{GA'} \tag{89}$$

In the next section an example clarifies the method.

5.2. Example

Suppose that the beam in Example 1 is a Timoshenko beam; thus, from Eqs. (86) and (32), we have:

$$\Omega\frac{d\bar{w}}{dx} - \frac{d^2\bar{\Phi}}{dx^2} + \Omega\bar{\Phi} = 0 \tag{90a}$$

$$\frac{d^2\bar{w}}{dx^2} + \frac{d\bar{\Phi}}{dx} - \frac{q_0}{GA'} = 0 \tag{90b}$$

The boundary and continuity conditions for this beam may be written as:

$$\bar{w}^{\mathrm{T}}(0) = 0, \quad \bar{w}^{\mathrm{T}}(L) = \Theta^{\mathrm{T}}L(1-\lambda), \quad \bar{\Phi}(0) = 0, \quad \frac{d\bar{\Phi}(L)}{dx} = 0 \tag{91a}$$

$$\frac{d\bar{\Phi}(\lambda L)}{dx} = 0 \tag{91b}$$

By solving this system of differential equations and applying the boundary and continuity conditions (91), we obtain:

$$\bar{\Phi}(x) = \frac{q_0}{12EI}\left[2x^3 - 3(1+\lambda)Lx^2 + 6\lambda L^2 x\right] \tag{92a}$$

$$\bar{w}^{\mathrm{T}}(x) = -\frac{q_0}{24EI}\left\{x^4 - 2(1+\lambda)Lx^3 + 6\lambda L^2 x^2 - \frac{12}{\Omega}\left[x^2 - (1+\lambda)Lx\right]\right\} \tag{92b}$$

$$\Theta^{\mathrm{T}} = -\frac{L^3 q_0}{24(1-\lambda)EI}\left(-1 + 4\lambda + \frac{12\lambda}{\Omega}\right) \tag{92c}$$

From Eq. (85) we obtain:

$$w^{\mathrm{T}}(x) = \bar{w}^{\mathrm{T}}(x) - \Theta^{\mathrm{T}}(x - \lambda L)H(x - \lambda L) \tag{93a}$$

$$\Phi(x) = \bar{\Phi}(x) + \Theta^{\mathrm{T}}H(x - \lambda L) \tag{93b}$$

Hence

$$w^{\mathrm{T}}(x) = -\frac{q_0}{24EI}\left\{x^4 - 2(1+\lambda)Lx^3 + 6\lambda L^2 x^2 - \frac{4\lambda - 1}{1-\lambda}L^3(x - \lambda L)H(x - \lambda L)\right. $$
$$\left. - \frac{12}{\Omega}\left[x^2 - (1+\lambda)Lx - \frac{\lambda L^3}{1-\lambda}(x - \lambda L)H(x - \lambda L)\right]\right\} \tag{94a}$$

$$\Phi(x) = \frac{q_0}{24EI}\left[4x^3 - 6(1+\lambda)Lx^2 + 12\lambda L^2 x - \frac{L^3}{1-\lambda}\left(-1 + 4\lambda + \frac{12\lambda}{\Omega}\right)H(x - \lambda L)\right] \tag{94b}$$

As shear stiffness tends to infinity the effect of shear deformation diminishes until it disappears entirely. From Eqs. (37b) and (92c) it is seen that:

$$\lim_{\Omega \to \infty} \Theta^{\mathrm{T}} = -\Theta \tag{95}$$

Therefore, in this limit case, the two theories describe the slope discontinuity due to the hinge, at the hinge point, as exactly the same except for the sign, which is always different. Also from Eqs. (38) and (94a) we obtain

$$\lim_{\Omega \to \infty} w^{\mathrm{T}}(x) = w(x) \tag{96}$$

Wang (1995) obtained relationships between the bending solutions of the Euler–Bernoulli beam theory and the Timoshenko beam theory to find response quantities of a Timoshenko beam using the Euler–Bernoulli beam solutions. Reddy et al. (1997a, 1997b), generalized the method for the third-order beam theory. Wang et al. (1998), presented relationships between Euler–Bernoulli and Timoshenko nonuniform beams which are applicable for beams with a discontinuity in flexural stiffness. However, here we have found that, in the case of a Timoshenko beam with jump discontinuities, it is easier to solve the governing equilibrium equations directly than to refer to the Euler–Bernoulli beam solutions. Also, the Timoshenko beam element given by Reddy et al. (1997a) and Reddy et al. (1997b) can be used to solve the class of Timoshenko beam problems considered in this paper.

6. Conclusions

In this paper we present some applications of the distribution theory of Schwarz to problems in beam bending. In the most general case, the equivalent distributed force of a point moment of order n is represented by the nth distributional derivative of the Dirac delta function. The equivalent distributed force for a distributed moment is shown to be a system comprised of a distributed force and two concentrated forces. Using this result, we offer a mathematical explanation of the corner condition in Kirchhoff's plate theory.

The governing differential equation of an Euler–Bernoulli beam with jump discontinuities in slope, deflection, and flexural stiffness is derived in the space of generalized functions. We find that the operator of the governing differential equation has the same form that the classical one does. However, the force term changes and the delta function and its first two distributional derivatives appear in the new force term. The auxiliary beam method is introduced to solve this problem. When the auxiliary beam method is used, instead of solving the governing differential equation for the beam in the space of generalized functions, one can solve another differential equation for the auxiliary beam in the space of classical functions. Examples demonstrate the efficiency of the auxiliary beam method.

The equivalent distributed force for a concentrated force and a concentrated moment is obtained with another method, using the discontinuities these forces introduce in an Euler–Bernoulli beam.

The governing system of differential equations of a Timoshenko beam with jump discontinuities in slope, deflection, flexural stiffness, and shear stiffness is obtained in the space of generalized functions. It is shown that the operator of one of the governing differential equations changes so that for both equations the delta function and its first distributional derivative appear in the new force terms. As was done in the case of an Euler–Bernoulli beam, the auxiliary beam method is introduced and an example solved to show its capability.

The beam-bending problem with jump discontinuities is already being solved practically for design challenges. The contribution of this paper is that it offers a deepened understanding of the mathematical description of this class of problems. In conclusion, this paper's use of the theory of generalized functions both deepens the understanding of beam-bending problems with discontinuities and gives the corresponding boundary-value problems a more compact form.

Acknowledgements

Appendix A. Schwarz's distribution theory

This appendix gives some definitions and operations in the Schwarz theory of distributions that are used in this paper. We restrict our discussion to distributions with a one-dimensional independent real variable. For more details, the reader may refer to Zemanian (1965), Kanwal (1983), Stakgold (1969) and Lighthill (1958).

The Heaviside function $H(x - x_0)$ is defined as:

$$H(x - x_0) = \begin{cases} 0 & x < x_0 \\ \dfrac{1}{2} & x = x_0 \\ 1 & x > x_0 \end{cases} \tag{A1}$$

It has a jump discontinuity at $x = x_0$. Its value at $x = x_0$ is usually taken to be 1/2. Clearly,

$$H(x_0 - x) = 1 - H(x - x_0) \tag{A2}$$

The Heaviside function is very useful in the study of functions with jump discontinuities. For example, let $F(x)$ be a function that is continuous everywhere except for the point $x = x_0$, at which it has a jump discontinuity:

$$F(x) = \begin{cases} F_1(x), & x < x_0 \\ F_2(x), & x > x_0 \end{cases} \tag{A3}$$

Then the function can be written as:

$$F(x) = F_1(x)H(x_0 - x) + F_2(x)H(x - x_0)$$

$$= F_1(x) + \left[F_2(x) - F_1(x)\right]H(x - x_0) \tag{A4}$$

Test functions are real-valued functions $\varphi(x)$ with the following two properties: (1) φ is infinitely smooth; (2) φ is zero outside a finite interval; i.e., φ has compact support. The space of the test functions is denoted by D.

A distribution is a continuous linear functional on the space D of test functions. The space of all distributions is denoted by D'; D' is itself a linear space and is called the dual space of D and is a larger space than D. The space D' forms a generalization of the class of locally integrable functions because it contains functions that are not locally integrable. Here the terms "distribution" and "generalized function" are used interchangeably.

A locally integrable function is integrable in the Lebesgue sense over every finite interval. Every locally integrable function $f(x)$ generates a distribution by means of the formula:

$$\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(x)\varphi(x)\, dx \tag{A5}$$

This is called a regular distribution. All other distributions are called singular distributions.

Two distributions, f and g, in D' are said to be equal if:

$$\langle f, \varphi \rangle = \langle g, \varphi \rangle \tag{A6}$$

for every test function $\varphi(x)$ in D.

The Dirac delta function is a singular generalized function defined as:

$$\langle \delta(x - x_0), \varphi(x) \rangle = \varphi(x_0) \tag{A7}$$

The nth derivative $f^{(n)}(x)$ of any generalized function $f(x)$ is given by:

$$\langle f^{(n)}(x), \varphi(x) \rangle = \langle f(x), (-1)^n \varphi^{(n)}(x) \rangle, \quad \varphi \in D \tag{A8}$$

The nth distributional derivative of the delta function is therefore defined as:

$$\langle \delta^{(n)}(x - x_0), \varphi(x) \rangle = \langle \delta(x - x_0), (-1)^n \varphi^{(n)}(x) \rangle = (-1)^n \varphi^{(n)}(x_0) \tag{A9}$$

Corollary.

$$\langle H^{(1)}(x - x_0), \varphi(x) \rangle = \langle H(x - x_0), -\varphi^{(1)}(x) \rangle = -\int_{x_0}^{+\infty} \varphi^{(1)}(x)\, dx$$

$$= \varphi(x_0) = \langle \delta(x - x_0), \varphi(x) \rangle \tag{A10}$$

Hence:

$$\frac{\bar{d}}{dx} H(x - x_0) = \delta(x - x_0) \tag{A11}$$

Theorem 1. *If $f(x)$ is a classical function and*

$$f(x) + a_0 \delta(x - x_0) + \cdots + a_n \delta^{(n)}(x - x_0) = 0 \tag{A12}$$

on the whole axis, $-\infty < x < +\infty$, then $f(x) = 0$ and $a_0 = \cdots = a_n = 0$.

Theorem 2. *Let a function $f(x)$ be n times continuously differentiable; then*:

$$f(x)\delta^{(n)}(x - x_0) = (-1)^n f^{(n)}(x_0)\delta(x - x_0) + (-1)^{n-1} n f^{(n-1)}(x_0)\delta^{(1)}(x - x_0)$$

$$+ (-1)^{n-2} \frac{n(n-1)}{2!} f^{(n-2)}(x_0)\delta^{(2)}(x - x_0) + \cdots + f(x_0)\delta^{(n)}(x - x_0) \tag{A13}$$

Corollary.

$$[f(x)H(x - x_0)]^{(n)} = f^{(n)}(x)H(x - x_0) + f^{(n-1)}(x_0)\delta(x - x_0) + f^{(n-2)}(x_0)\delta^{(1)}(x - x_0) + \cdots$$

$$+ f(x_0)\delta^{(n-1)}(x - x_0) \tag{A14}$$

The space of distributions D'_R having their supports bounded on the left is called the space of right-sided distributions, $D'_R \subset D'$ (proper subspace).

   *The Convolution of two right-sided distributions f and g, f * g is defined as:*

$$\langle h, \varphi \rangle = \langle f * g, \varphi \rangle = \langle f(x), \langle g(\tau), \varphi(x + \tau) \rangle \rangle$$

$$= \langle g * f, \varphi \rangle \tag{A15}$$

Theorem 3. *The convolution of the nth derivative of the delta function with any distribution yields the nth derivative of that distribution, i.e.,*

$$\delta^{(n)} * f = f^{(n)} \tag{A16}$$

Proof.

$$\langle \delta^{(n)} * f, \varphi \rangle = \langle f * \delta^{(n)}, \varphi \rangle = \langle f(x), \langle \delta^{(n)}(\tau), \varphi(x + \tau) \rangle \rangle$$

$$= \langle f(x), (-1)^n \varphi^{(n)}(x) \rangle = \langle f^{(n)}(x), \varphi(x) \rangle$$

Appendix B. Equivalent distributed force for generalized point moments

   In this appendix we present a proof for the theorem presented in Section 2.

Theorem. *The equivalent distributed force of a unit moment of order n applied at $x = x_0$ is:*

$$q_n(x) = \frac{(-1)^n}{n!} \delta^{(n)}(x - x_0) \tag{B1}$$

where $\delta^{(n)}$ is the nth distributional derivative of the Dirac delta function.

Proof. By definition

$$\int_{-\infty}^{+\infty} (x - x_0)^n q_n(x)\, dx = 1 \tag{B2a}$$

$$\int_{-\infty}^{+\infty} (x - x_0)^m q_n(x)\, dx = 0, \quad m \neq n \tag{B2b}$$

Using integration by parts, Eq. (B2a) may be written as:

$$\left[\frac{(x-x_0)^{n+1}}{n+1}q_n(x)\right]_{-\infty}^{+\infty} - \frac{1}{n+1}\int_{-\infty}^{+\infty}(x-x_0)^{n+1}q_n'(x)\,\mathrm{d}x = 1 \tag{B3}$$

Since $q_n(x)$ has a compact support, from Eq. (B3) we obtain:

$$\int_{-\infty}^{+\infty}(x-x_0)^{n+1}q_n'(x)\,\mathrm{d}x = -(n+1) = (-1)^1\frac{(n+1)!}{n!}$$

$$= \frac{(-1)^n}{n!}(-1)^{n+1}\frac{\mathrm{d}}{\mathrm{d}x^{n+1}}\left[(x-x_0)^{n+1}\right]_{x=x_0} \tag{B4}$$

Applying integration by parts for Eq. (B4) yields:

$$\int_{-\infty}^{+\infty}(x-x_0)^{n+2}q_n''(x)\,\mathrm{d}x = (n+2)(n+1) = (-1)^2\frac{(n+2)!}{n!}$$

$$= \frac{(-1)^n}{n!}(-1)^{n+2}\frac{\mathrm{d}}{\mathrm{d}x^{n+2}}\left[(x-x_0)^{n+2}\right]_{x=x_0} \tag{B5}$$

By mathematical induction we get:

$$\int_{-\infty}^{+\infty}(x-x_0)^{n+k}q_n^{(k)}(x) = \frac{(-1)^n}{n!}(-1)^{n+k}(n+k)! = \frac{(-1)^n}{n!}(-1)^{n+k}\left[\frac{\mathrm{d}}{\mathrm{d}x^{n+k}}(x-x_0)^{n+k}\right]_{x=x_0} \tag{B6}$$

Similarly, from Eq. (B2b) we have:

$$\int_{-\infty}^{+\infty}(x-x_0)^{m+k}q_n^{(k)}(x)\,\mathrm{d}x = 0 = \frac{(-1)^n}{n!}(-1)^{n+k}\left[\frac{\mathrm{d}^{n+k}}{x^{n+k}}(x-x_0)^{m+k}\right]_{x=x_0},\quad m\neq n \tag{B7}$$

From Eqs. (B6) and (B7) it may be concluded that:

$$\int_{-\infty}^{+\infty}(x-x_0)^r q_n^{(s)}\,\mathrm{d}x = \frac{(-1)^n}{n!}\int_{-\infty}^{\infty}(x-x_0)^r\delta^{(s+n)}(x-x_0)\,\mathrm{d}x \quad \text{or}$$

$$\langle q_n^{(s)}, (x-x_0)^r\rangle = \frac{(-1)^n}{n!}\langle\delta^{(s+n)}(x-x_0), (x-x_0)^r\rangle \tag{B8}$$

Denote the set of all polynomials by $\mathscr{P}$. Considering Eq. (B8) and the fact that regular distributions are linear, we conclude that for any polynomial $P_\mathrm{m}(x)$ in $\mathscr{P}$:

$$\langle q_n^{(s)}(x), P_\mathrm{m}(x)\rangle = \frac{(-1)^n}{n!}\langle\delta^{(s+n)}(x-x_0), P_\mathrm{m}(x)\rangle \tag{B9}$$

According to the Weierstrass approximation theorem (Rudin, 1964; Stoll, 1997), polynomials are dense in the space of all continuous functions; hence one may approximate any test function uniformly by a polynomial. Therefore, Eq. (B9) implies that:

$$\langle q_n^{(s)}(x), \varphi(x)\rangle = \frac{(-1)^n}{n!}\langle\delta^{(s+n)}(x-x_0), \varphi(x)\rangle \quad \forall\varphi \in D \tag{B10}$$

Thus,

$$q_n^{(s)}(x) = \frac{(-1)^n}{n!}\delta^{(s+n)}(x-x_0) \tag{B11}$$

Therefore, from Eqs. (B2a) and (B11) we conclude that:

$$q_n(x) = \frac{(-1)^n}{n!} \delta^{(n)}(x - x_0) \tag{B12}$$

B.1. The Weierstrass approximation theorem

If f is a continuous real-valued function on the interval $[a, b]$, then for given $\varepsilon > 0$, there exists a polynomial P such that:

$$|f(x) - P(x)| < \varepsilon \quad \forall x \in [a, b] \tag{B13}$$

An equivalent version of the theorem may be stated as: If f is a continuous real-valued function on $[a, b]$, then there exists a sequence $\{P_n\}$ of polynomials such that:

$$f(x) = \lim_{n \to \infty} P_n(x) \quad \text{uniformly on } [a, b] \tag{B14}$$

The reader can refer to Stoll (1997) for more details and the proof.

Appendix C. Governing differential equation of an Euler–Bernoulli beam with various jump discontinuities

This appendix gives the mathematical details of obtaining the governing differential equation of an Euler–bernoulli beam with jump discontinuities.

The beam shown in Fig. 2 is composed of two beam segments AB and BC. By using Heaviside's function:

$$w(x) = x_1(x) + [w_2(x) - w_1(x)]H(x - x_0) \tag{C1}$$

where w is the deflection of the beam and w_1 and w_2 are the deflections of the beam segments, AB and BC, respectively.

The governing differential equations of beam segments can be written as

$$\frac{\mathrm{d}^4 w_1}{\mathrm{d}x^4} = \frac{q(x)}{EI_1} \qquad 0 \le x < x_0 \tag{C2a}$$

$$\frac{\mathrm{d}^4 w_2}{\mathrm{d}x^4} = \frac{q(x)}{EI_2} \qquad x_0 < x \le L \tag{C2b}$$

Differentiating both sides of Eq. (C1), we obtain:

$$\frac{\bar{\mathrm{d}} w}{\mathrm{d}x} = \frac{\mathrm{d}w_1}{\mathrm{d}x} + \left(\frac{\mathrm{d}w_2}{\mathrm{d}x} - \frac{\mathrm{d}w_1}{\mathrm{d}x}\right) H(x - x_0) + (w_2 - w_1)\delta(x - x_0)$$

$$= \frac{\mathrm{d}w_1}{\mathrm{d}x} + \left(\frac{\mathrm{d}w_2}{\mathrm{d}x} - \frac{\mathrm{d}w_1}{\mathrm{d}x}\right) H(x - x_0) + (w_2 - w_1)_{x=x_0}\delta(x - x_0)$$

$$= \frac{\mathrm{d}w_1}{\mathrm{d}x} + \left(\frac{\mathrm{d}w_2}{\mathrm{d}x} - \frac{\mathrm{d}w_1}{\mathrm{d}x}\right)H(x - x_0) + \Delta\delta(x - x_0) \tag{C3}$$

and

$$\frac{\bar{\mathrm{d}}^2 w}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} + \left[\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2}\right]H(x - x_0) + \left[\frac{\mathrm{d}w_2}{\mathrm{d}x} - \frac{\mathrm{d}w_1}{\mathrm{d}x}\right]_{x=x_0}\delta(x - x_0) + \Delta\delta^{(1)}(x - x_0)$$

$$= \frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} + \left[\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2}\right]H(x - x_0) + \Theta\delta(x - x_0) + \Delta\delta^{(1)}(x - x_0) \tag{C4}$$

where a bar over the differentiation symbol means distributional differentiation. We know that:

$$M_1(x_0) = \left[EI_1\frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2}\right]_{x=x_0}, \qquad M_2(x_0) = \left[EI_2\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2}\right]_{x=x_0} \tag{C5}$$

Therefore,

$$\frac{\mathrm{d}^2 w_1(x_0)}{\mathrm{d}x^2} = \frac{M_1(x_0)}{EI_1}, \qquad \frac{\mathrm{d}^2 w_2(x_0)}{\mathrm{d}x^2} = \frac{M_2(x_0)}{EI_2} \tag{C6}$$

Assuming $I_1 = I$, $I_2 = \alpha I$, and also considering $M_1(x_0) = M_2(x_0) = K_r\Theta$, we have:

$$\frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} = \frac{K_r\Theta}{EI}, \qquad \frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2} = \frac{K_r\Theta}{\alpha EI} \tag{C7}$$

Therefore, differentiating both sides of Eq. (C4) yields:

$$\frac{\bar{\mathrm{d}}^3 w}{\mathrm{d}x^3} = \frac{\mathrm{d}^3 w_1}{\mathrm{d}x^3} + \left[\frac{\mathrm{d}^3 w_2}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 w_1}{\mathrm{d}x^3}\right]H(x - x_0) + \left[\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2}\right]_{x=x_0}\delta(x - x_0) + \Theta\delta^{(1)}(x - x_0)$$

$$+ \Delta\delta^{(2)}(x - x_0)$$

$$= \frac{\mathrm{d}^3 w_1}{\mathrm{d}x^3} + \left[\frac{\mathrm{d}^3 w_2}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 w_1}{\mathrm{d}x^3}\right]H(x - x_0) + \frac{K_r\Theta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta(x - x_0) + \Theta\delta^{(1)}(x - x_0) + \Delta\delta^{(2)}(x - x_0) \tag{C8}$$

We also know that:

$$\left(EI_1\frac{\mathrm{d}^3 w_1}{\mathrm{d}x^3}\right)_{x=x_0} = \left(EI_2\frac{\mathrm{d}^3 w_2}{\mathrm{d}x^3}\right)_{x=x_0} = V(x_0) = K_t\Delta \tag{C9}$$

Therefore, differentiating both sides of Eq. (C8), we obtain:

$$\frac{\bar{\mathrm{d}}^4 w}{\mathrm{d}x^4} = \frac{\mathrm{d}^4 w_1}{\mathrm{d}x^4} + \left[\frac{\mathrm{d}^4 w_2}{\mathrm{d}x^4} - \frac{\mathrm{d}^4 w_1}{\mathrm{d}x^4}\right]H(x - x_0) + \left[\frac{\mathrm{d}^3 w_2}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 w_1}{\mathrm{d}x^3}\right]_{x=x_0}\delta(x - x_0) + \frac{K_r\Theta}{EI}\left(\frac{1}{\alpha}\right.$$

$$\left. - 1\right)\delta^{(1)}(x - x_0) + \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0) \tag{C10}$$

Hence,

$$\frac{\bar{d}^4 w}{dx^4} = \frac{d^4 w_1}{dx^4} + \left[\frac{d^4 w_2}{dx^4} - \frac{d^4 w_1}{dx^4}\right]H(x - x_0) + \frac{K_t \Delta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta(x - x_0)$$

$$+ \frac{K_r \Theta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta^{(1)}(x - x_0) + \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0) \tag{C11}$$

Therefore, considering Eqs. (C2a) and (C2b), the governing equilibrium equation of the beam can be written as:

$$\frac{\bar{d}^4 w}{dx^4} = \frac{q(x)}{EI} + \frac{q(x)}{EI}\left(\frac{1}{\alpha} - 1\right)H(x - x_0) + \frac{K_t \Delta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta(x - x_0) + \frac{K_r \Theta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta^{(1)}(x - x_0)$$

$$+ \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0) \tag{C12}$$

Eq. (C12) is the governing differential equation of an Euler–Bernoulli beam with one point of jump discontinuity in the space of generalized functions. The continuity conditions can be expressed as:

$$EI\frac{d^2 w(x_0^-)}{dx^2} = K_r \Theta, \qquad EI\frac{d^3(w_0^-)}{dx^3} = K_t \Delta \tag{C13}$$

Applying the four boundary conditions at $x = 0$ and $x = L$, and the continuity conditions in Eq. (C13), gives us the beam deflection w.

For a beam, if $\Theta = 0$ and/or $\Delta = 0$, the governing equilibrium equation of the beam must be modified. Consider the case of a beam with only a change in flexural stiffness. Other cases are treated similarly. Now let:

$$M_1(x_0) = M_2(x_0) = M_0 \tag{C14a}$$

$$V_1(x_0) = V_2(x_0) = V_0 \tag{C14b}$$

Therefore,

$$\left(\frac{d^2 w_2}{dx^2} - \frac{d^2 w_1}{dx^2}\right)_{x=x_0} = \frac{M_0}{EI}\left(\frac{1}{\alpha} - 1\right) \tag{C15a}$$

$$\left(\frac{d^3 w_2}{dx^3} - \frac{d^3 w_1}{dx^3}\right)_{x=x_0} = \frac{V_0}{EI}\left(\frac{1}{\alpha} - 1\right) \tag{C15b}$$

Hence,

$$\frac{\bar{d}^4 w}{dx^4} = \frac{q(x)}{EI} + \frac{q(x)}{EI}\left(\frac{1}{\alpha} - 1\right)H(x - x_0) + \frac{V_0}{EI}\left(\frac{1}{\alpha} - 1\right)\delta(x - x_0) + \frac{M_0}{EI}\left(\frac{1}{\alpha} - 1\right)\delta^{(1)}(x - x_0) \tag{C16}$$

In this case, continuity conditions can be written as:

$$EI\frac{d^2 w(x_0^-)}{dx^2} = M_0, \qquad EI\frac{d^3 w(x_0^-)}{dx^3} = V_0 \tag{C17}$$

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