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# ON QUADRATIC ISOPARAMETRIC TRANSITION ELEMENTS FOR A CRACK NORMAL TO A BIMATERIAL INTERFACE

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## SUMMARY

The quadratic isoparametric crack-tip elements proposed by R. E. Abdi and G. Valentin (*Computers and Structures* **33,** 241–248) are reconsidered and a simpler method for calculating the optimal position of the side nodes proposed. Quadratic isoparametric transition elements for an $r^{\lambda-1}$ $(0<\lambda<1)$ strain singularity are formulated. The effects of these transition elements on the accuracy of the calculated stress intensity factors are shown numerically for a crack normal to and terminating at a bimaterial interface. Finally, layered transition elements are formulated for this case and their effects studied numerically. 

KEY WORDS: transition finite elements; crack normal to bimaterial interface; variable power singularity

## 1. INTRODUCTION

In a homogeneous elastic medium, the $r^{-1/2}$ strain singularity is independent of the mechanical properties of the material, and the stress intensity factor (SIF) is the same for all stress components. Due to the complexity of the problems encountered in engineering applications, numerical calculations of SIFs are of great importance. Among various numerical methods, the finite element method is the most common and the most widely used. To date many special crack-tip elements have been suggested [1]. The most successful crack-tip elements are the isoparametric quarter-point elements proposed by Barsoum [2, 3] and Henshell and Shaw [4].

Lynn and Ingraffea [5] introduced the concept of utilizing transition elements (TEs) to improve the accuracy of SIFs for very small crack-tip elements. TEs have aroused some controversy [6–8]. Hussain *et al.* [9], extended Lynn and Ingraffea's concept to cubic elements. Horváth [10] studied *n*th-order isoparametric elements as TEs for an $r^{(1-m)/m}$ strain singularity; in the appendix, we show that these transition elements are not practical. Yavari *et al.* [11], reconsidered TEs,

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studying their strain singularity in particular. They formulated nine-noded TEs and compared them numerically to eight-noded TEs. They also studied the effects (on the calculated SIFs) of combinations of TEs and various isoparametric crack-tip elements. Finally, they correctly formulated the layered transition elements proposed by Murti and Valliappan [7], and concluded that using layered transition elements makes the calculated SIFs less mesh dependent.

The simplest model of a crack in an inhomogeneous medium is a crack in a bimaterial medium. As a special case, a crack terminating at and normal to the bimaterial interface is of great importance in the design of composite materials and welded joints. Zak and Williams [12] and Cook and Erdogan [13] showed that, in this model, the strain singularity is $r^{\lambda-1}$ $(0<\lambda<1)$. Here, $\lambda$ depends on the mechanical properties of both materials. For each stress component, a stress intensity factor is defined. However, the most important SIF is the one corresponding to the cleavage stress in the uncracked material. Many different methods for introducing an $r^{\lambda-1}$ singularity in finite elements have been proposed [14–18] to date. All these methods require that the existing finite element codes be modified.

Abdi and Valentin [19] generalized the idea of quarter-point elements for modelling an $r^{\lambda-1}$ strain singularity. They showed that this strain singularity can be modelled approximately by quadrilateral isoparametric elements. They obtained the optimal position of the side nodes adjacent to the crack tip for quadratic and cubic isoparametric elements. Wu [20] generalized this idea for collapsed triangular isoparametric elements and showed that they give better results. Lim and Kim [21] proposed a simpler method for calculating the optimal position of the side nodes of quadratic isoparametric crack-tip elements adjacent to the crack tip. Lim and Lee [22] introduced a two-point formula for calculating SIFs and compared it with the standard one-point formula.

This article proposes an even simpler method for finding the optimal positions of the side nodes of quadratic isoparametric crack-tip elements. Quadratic isoparametric transition elements are formulated. These are shown numerically to improve the accuracy of SIFs for large $L_T/L_Q$ ($L_T$ is the TE length and $L_Q$ is the crack-tip element length) and small $L_Q/a$ ratios. Finally, we formulate layered TEs and perform a numerical study of their effects on the calculated SIFs.

## 2. ISOPARAMETRIC CRACK-TIP ELEMENTS

A simple method for finding the optimal position of the side nodes of a quadratic isoparametric crack-tip element is offered in this section. Using this simpler method makes these crack-tip elements even more practical than other crack-tip elements. Figure 1(a) shows a crack normal to and terminating at a bimaterial interface. Eight crack-tip elements are shown. Four of them (A) are quarter-point elements, while for the other four (B) the positions of the mid-nodes depend on the singularity of the crack tip that lies on the interface. Here we propose a simple method to calculate the optimal position of mid-nodes ($\alpha$).

For an eight-noded isoparametric element we have

$$x=\sum_{i=1}^{8} N_i(\xi,\eta)x_i \tag{1a}$$

$$y=\sum_{i=1}^{8} N_i(\xi,\eta)y_i \tag{1b}$$

 

The eight-noded Serendipity element has the following shape functions in the normalized system:

$$N_i = [(1+\xi\xi_i)(1+\eta\eta_i) - (1-\xi^2)(1+\eta\eta_i) - (1-\eta^2)(1+\xi\xi_i)]\,\xi_i^2\eta_i^2/4$$
$$+ (1-\xi^2)(1+\eta\eta_i)(1-\xi_i^2)\eta_i^2/2 + (1-\eta^2)(1+\xi\xi_i)(1-\eta_i^2)\xi_i^2/2 \quad (2)$$

For an isoparametric element, displacements are interpolated similarly:

$$u = \sum_{i=1}^{8} N_i(\xi, \eta) u_i \quad (3a)$$

$$v = \sum_{i=1}^{8} N_i(\xi, \eta) v_i \quad (3b)$$

where $u_i$ and $v_i$ are nodal displacements. Now consider the edge 152 ($\eta = -1$). For this edge we have

$$N_1 = -\tfrac{1}{2}\xi(1-\xi) \quad (4a)$$

$$N_5 = (1-\xi^2) \quad (4b)$$

$$N_2 = \tfrac{1}{2}\xi(1+\xi) \quad (4c)$$

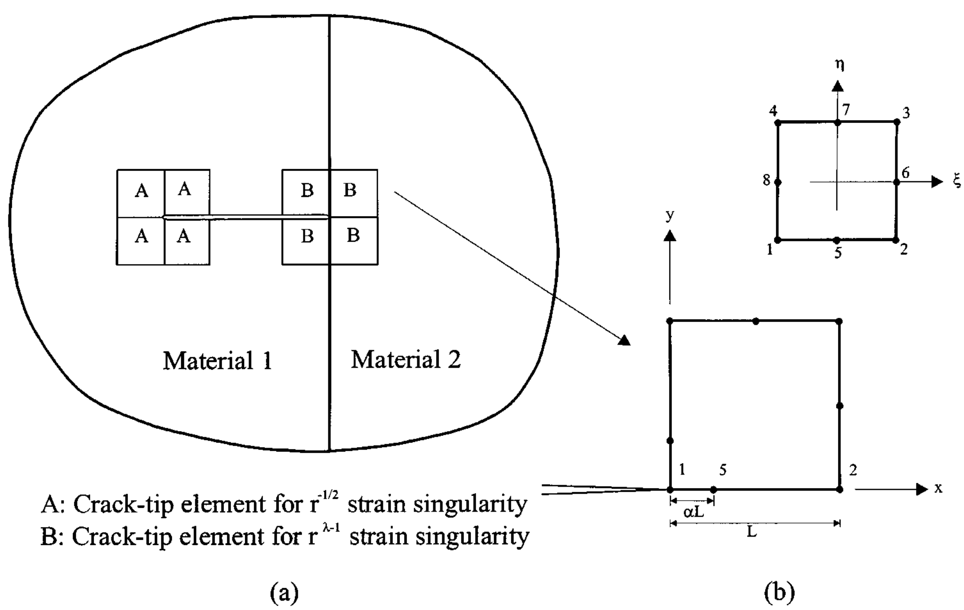


Figure 1. (a) A crack normal to and terminating at a bimaterial interface; (b) a square crack-tip element

 

Along this edge, $x$ may be expressed as

$$x = x_5 + \frac{x_2 - x_1}{2}\xi + \frac{x_1 + x_2 - 2x_5}{2}\xi^2 \tag{5}$$

Similarly,

$$u = A_0 + A_1\xi + A_2\xi^2 \tag{6}$$

From Figure 1(b) we know that

$$x_1 = 0, \quad x_5 = \alpha L, \quad x_2 = L \tag{7}$$

Hence,

$$x = \alpha L + \frac{L}{2}\xi + \frac{1 - 2\alpha}{2}L\xi^2 \tag{8}$$

Now let $X = x/L$. Abdi and Valentin [19] considered the following transformation and showed that when it is used, strains will have an $r^{\lambda - 1}$ singularity.

$$X = \left(\frac{1 + \xi}{2}\right)^{1/\lambda}, \quad X = \frac{x}{L} \tag{9}$$

Since $1/\lambda$ is not always an integer, $X$ cannot be expressed by polynomials. Therefore, quadratic elements can only express this transformation approximately. Abdi and Valentin [19] used an optimization method for calculating the optimum value of $\alpha$. Here we propose a simpler method. Along the edge $\eta = -1$, we have

$$X = P(\xi) = \alpha + \tfrac{1}{2}\xi + (\tfrac{1}{2} - \alpha)\xi^2 \tag{10}$$

where $\alpha L$ is the position of the side node. Using the Taylor series expansion method, we obtain

$$X(\xi) = \left(\frac{1 + \xi}{2}\right)^{1/\lambda} = \frac{1}{2^{1/\lambda}} + \frac{1}{\lambda 2^{1/\lambda}}\xi + \frac{(1/\lambda) - 1}{\lambda 2^{(1/\lambda)+1}}\xi^2 + \cdots \tag{11}$$

Therefore,

$$P(\xi) \approx \frac{1}{2^{1/\lambda}}\left[1 + \frac{1}{\lambda}\xi + \frac{(1/\lambda) - 1}{2\lambda}\xi^2\right] \tag{12}$$

Comparing (10) and (12) yields

$$\alpha = \frac{1}{2^{1/\lambda}} \tag{13}$$

This $\alpha$ value can also be obtained using the least-squares method, with $\xi = -1, 0, 1$. Actually, in Lim and Kim's formula if one chooses $n = 3$ the results are the same as those of this investigation. Lim and Kim did not point out that their formula converges very rapidly and thus there is no need to use $n > 3$. In Figure 2, formula (13) is compared to Abdi and Valentin's results. As can be seen, the $\alpha$ values of both methods are very close. Obviously, our proposed method is simpler. It is seen from Figure 2 that for $0 < \lambda < 0{\cdot}5$, $0 < \alpha < 0{\cdot}25$ and that for $0{\cdot}5 < \lambda < 1$, $0{\cdot}25 < \alpha < 0{\cdot}5$.

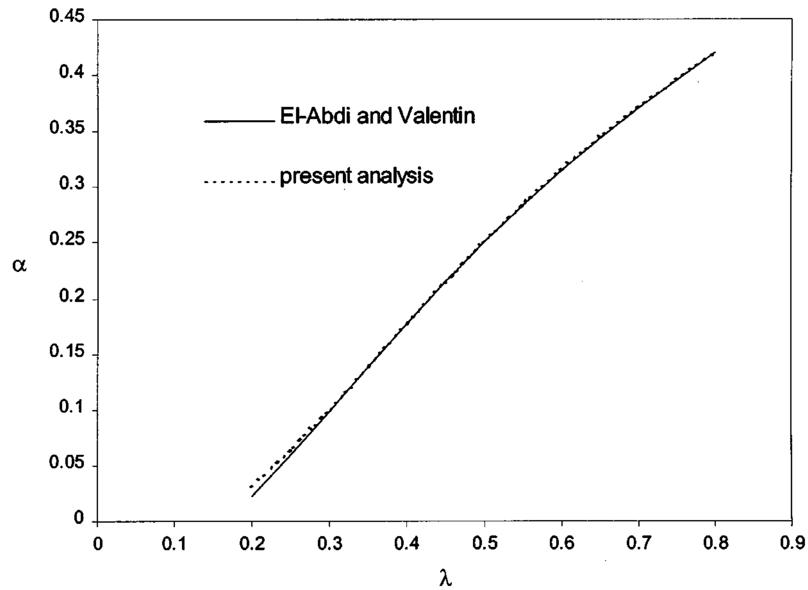


Figure 2. Comparison between $\alpha$ values calculated by El-Abdi and Valentin and the present analysis

In the next section the idea of Abdi and Valentin [19] is generalized for creating quadratic isoparametric transition elements for an $r^{\lambda-1}$ strain singularity problem.

## 3. QUADRATIC ISOPARAMETRIC TRANSITION ELEMENTS

In this section we find the optimal position for the mid-nodes—a position that allows the singularity point to remain outside the element. The transformation between $X$ and $\xi$ is modified so that if these elements are used in the second row of finite elements around the crack tip, their singularity points coincide with the crack tip. An assemblage of crack-tip and transition elements is shown in Figure 3. The following transformation between $X$ and $\xi$ is considered:

$$X + \frac{q}{2} = \left(\frac{c_0 + c_1\xi}{2}\right)^{1/\lambda}, \quad q = \frac{2L_Q}{L_T} \tag{14}$$

where $L_Q$ and $L_T$ are the lengths of the crack-tip and transition elements, respectively. We know that for $\xi = -1$, $X = 0$ and for $\xi = 1$, $X = 1$. Applying these two conditions yields:

$$c_0 = \left(1 + \frac{q}{2}\right)^{\lambda} + \left(\frac{q}{2}\right)^{\lambda}, \quad c_1 = \left(1 + \frac{q}{2}\right)^{\lambda} - \left(\frac{q}{2}\right)^{\lambda} \tag{15}$$

Therefore, we have the following transformation between $X$ and $\xi$:

$$X = \left\{\frac{(1 + (q/2))^{\lambda} + (q/2)^{\lambda} + [(1 + (q/2))^{\lambda} - (q/2)^{\lambda}]\,\xi}{2}\right\} - \frac{q}{2} \tag{16}$$

 

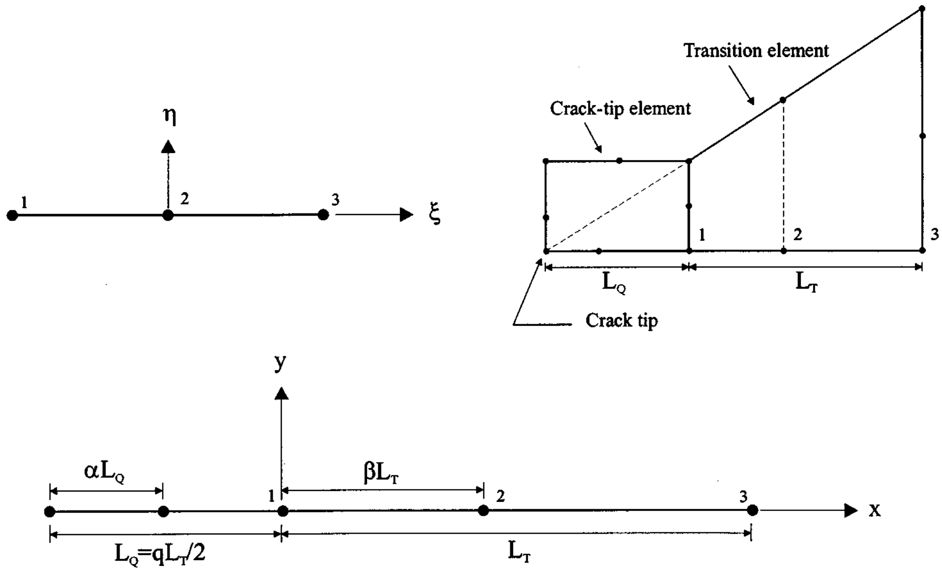


Figure 3. Assemblage of crack-tip and transition elements

From (14), we obtain

$$\xi = \frac{2}{c_1}\left(X + \frac{q}{2}\right)^{\lambda} - \frac{c_0}{c_1} \tag{17}$$

For a quadratic element

$$u = A_0 + A_1\xi + A_2\xi^2 \tag{18}$$

Substituting (17) into (18), we obtain

$$u = A_0 + A_1\left[\frac{2}{c_1}\left(X + \frac{q}{2}\right)^{\lambda} - \frac{c_0}{c_1}\right] + A_2\left[\frac{2}{c_1}\left(X + \frac{q}{2}\right)^{\lambda} - \frac{c_0}{c_1}\right]^2 \tag{19}$$

Therefore,

$$\varepsilon_x = \frac{1}{L}\frac{\partial u}{\partial X} = \frac{2\lambda}{Lc_1(X + (q/2))^{1-\lambda}}\left[\left(A_1 - \frac{2A_2c_0}{c_1}\right) + \frac{4A_2}{c_1}\left(X + \frac{q}{2}\right)^{\lambda}\right] \tag{20}$$

As can be seen at $X = -q/2$ (the crack tip), the strain has an $r^{\lambda-1}$ singularity. The transformation (16) can be approximated by a second-order polynomial using the Taylor series expansion method (or by the least-squares method with $\xi = -1, 0, 1$). For a quadratic element we know that

$$X = B_0 + B_1\xi + B_2\xi^2 \tag{21}$$

where

$$B_0 = \beta, \quad B_1 = \tfrac{1}{2}, \quad B_2 = \tfrac{1}{2} - \beta \tag{22}$$

 

Here $\beta L_T$ is the position of the side node. Using the Taylor series expansion method, we obtain

$$B_0 = -\frac{q}{2} + \frac{[(1+(q/2))^\lambda + (q/2)^\lambda]^{1/\lambda}}{2^{1/\lambda}} \tag{23a}$$

$$B_1 = \tfrac{1}{2} \tag{23b}$$

$$B_2 = \frac{1}{2} - \left(-\frac{q}{2} + \frac{[(1+(q/2))^\lambda + (q/2)^\lambda]^{1/\lambda}}{2^{1/\lambda}}\right) \tag{23c}$$

Comparing (22) and (23), yields

$$\beta(\lambda, q) = -\frac{q}{2} + \frac{[(1+(q/2))^\lambda + (q/2)^\lambda]^{1/\lambda}}{2^{1/\lambda}} \tag{24}$$

The variations of $\beta$ versus $\lambda$, for different $L_T/L_Q\,(=2/q)$ values, are shown in Figure 4. Clearly, for a specific $L_T/L_Q$, $\beta$ varies almost linearly with $\lambda$.

Layered transition elements may be defined for a crack normal to and terminating at a bimaterial interface in a manner similar to that used to define layered transition elements for a crack in a homogeneous medium [7, 11]. Assume that the crack-tip element and the $n-1$ other transition elements before the $n$th transition element constitute a fictitious crack-tip element whose length is denoted by $L'_Q$. For the $n$th layer of transition elements, $q_n$ is defined as

$$q_n = \frac{2L'_{Qn}}{L_T} = \frac{2\left(L_Q + \sum_{i=1}^{n-1} L_{Ti}\right)}{L_{Tn}} \tag{25}$$

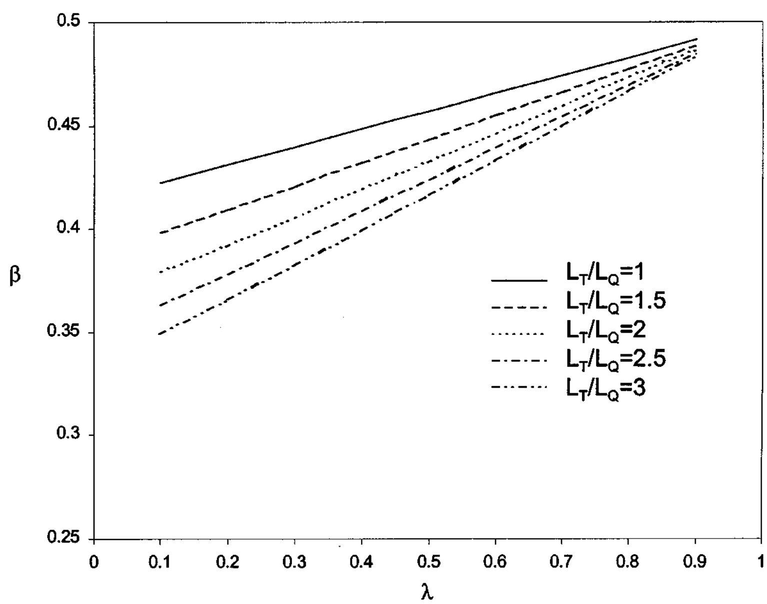


Figure 4. $\beta$ values for different $L_T/L_Q$ ratios

 

Therefore,

$$\beta_n(\lambda) = -\frac{q_n}{2} + \frac{[(1 + (q_n/2))^{\lambda} + (q_n/2)^{\lambda}]^{1/\lambda}}{2^{1/\lambda}} \qquad (26)$$

As can be seen for $q_n = 0$, equations (26) and (13) give the same result, as was expected.

## 4. NUMERICAL EXAMPLES

A plane strain bimaterial strip with a crack normal to and terminating at the interface is considered (Figure 5). A uniform pressure $P_0$ is applied at the crack faces. The finite element mesh is shown in Figure 6. When studying the effects of the TEs, $L = L_Q + L_T$ is assumed to be constant ($L = a/2$). Under this assumption we can change $L_Q/a$ and $L_T/L_Q$ simultaneously so as to study the effects of the TEs more easily. The bimaterial strip is an epoxy-aluminium combination and the cracked material is epoxy ($\mu_1 = \mu_{ep} = 1{\cdot}149 \times 10^6$ KPa and $v_{ep} = 0{\cdot}35$, $\mu_2 = \mu_{al} = 2{\cdot}652 \times 10^7$ KPa, $v_{al} = 0{\cdot}3$). For this cracked bimaterial strip, $m = \mu_2/\mu_1 = 23{\cdot}08$, $\lambda = 0{\cdot}6619$, and $\alpha = 0{\cdot}3509$ [13]. Stress intensity factors are calculated using one-point and two-point formulas [22]. Both eight-noded rectangular and eight-noded collapsed triangular crack-tip elements are used for comparison.

In Figure 7, the effects (on the calculated SIFs) of the TEs with eight-noded square crack-tip elements are shown. It is seen that for small $L_Q/L$ the transition elements improve the accuracy; the one-point formula is preferable. In Figure 8, the effects of TEs with collapsed triangular crack-tip elements are shown. Again, for small $L_Q/L$ these elements are effective. In this case, the effects of TEs using one-point and two-point formulas are almost the same. Note that collapsed triangular crack-tip elements give better results, as was observed by Wu [20]. It should be noted also that $L_T/L_Q = (L_Q/L)^{-1} - 1$. Therefore, as $L_Q/L$ decreases,

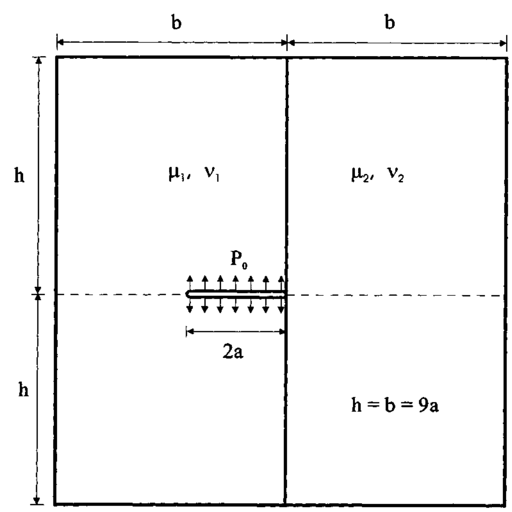


Figure 5. Cracked bimaterial strip under uniform pressure on the crack faces

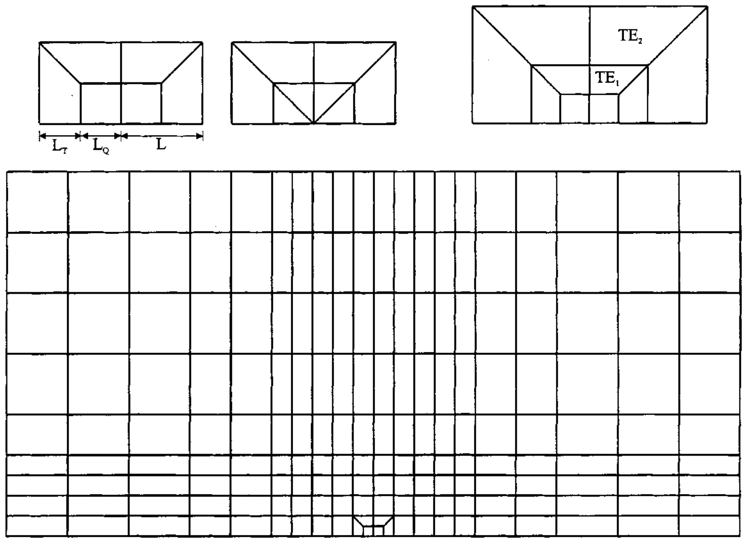


Figure 6. Finite element mesh and assemblage of crack-tip and transition elements

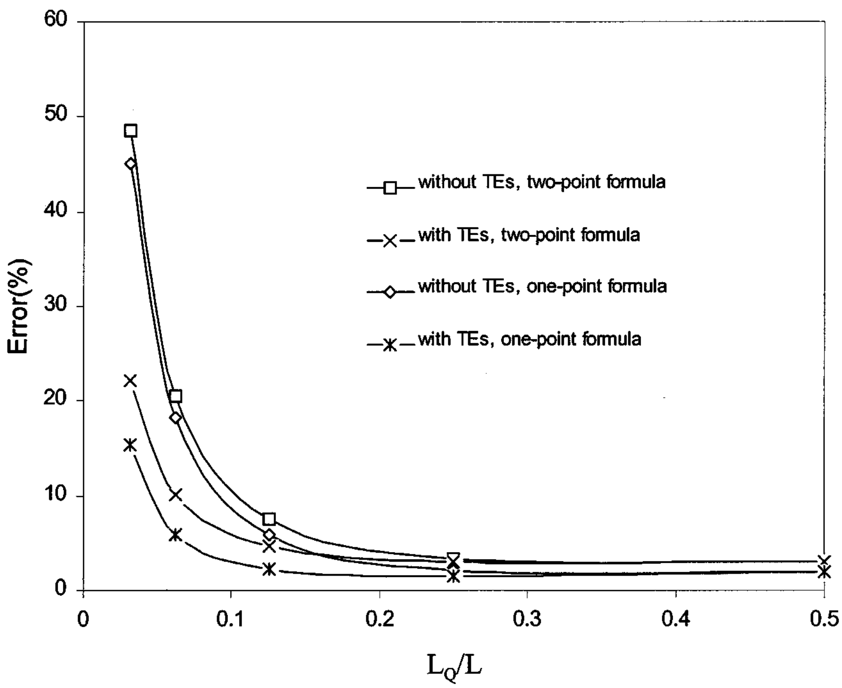


Figure 7. Percentage error of calculated $K$ values with eight-noded rectangular crack-tip elements

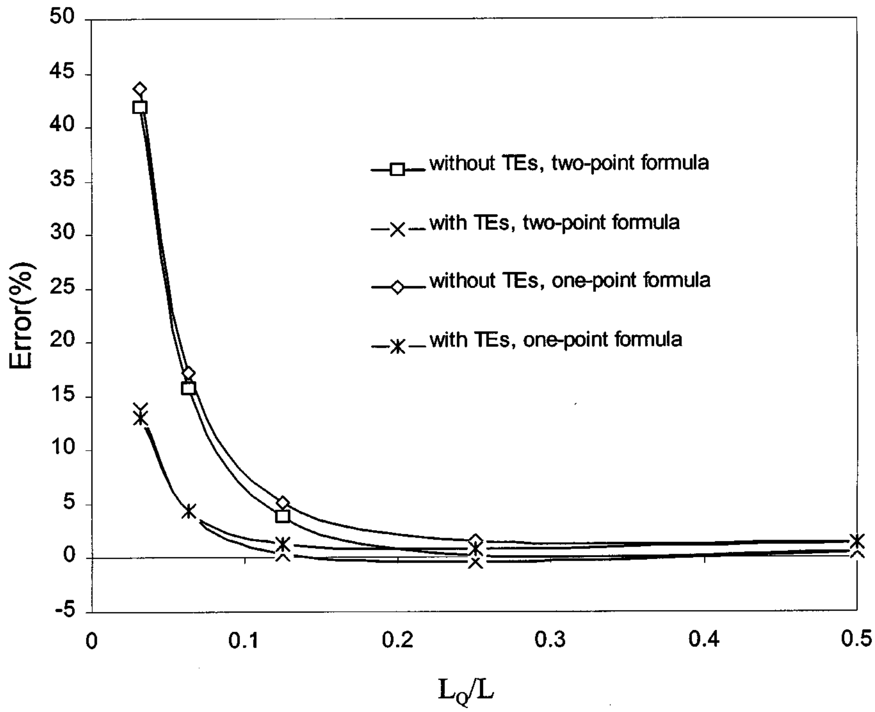


Figure 8. Percentage error of calculated $K$ values with eight-noded collapsed triangular crack-tip elements

$L_T/L_Q$ increases. As pointed out in Yavari *et al.* [11], both $L_Q/a$ and $L_T/L_Q$ control the effectiveness of the TEs.

Then the effects of layered TEs are studied. Here, two layers of TEs are considered, and $L = L_Q + L_{T1} + L_{T2}$ is assumed to be constant ($L = a/2$). The crack-tip elements are square and the one-point formula is used to calculate the SIFs. Three cases are compared: (1) without TEs; (2) with one layer of TEs; and (3) with two layers of TEs. As can be seen in Figure 9, the first layer of TEs has a marginal effect because $L_T/L_Q(=2)$ is small for this layer; hence if the elements in the next layer are able to model the strain singularity we can obtain a better calculated stress intensity factor. Clearly, using the second layer of TEs improves the accuracy for small crack-tip elements. This is exactly the same conclusion that we reached for the case of a crack in a homogeneous medium [11].

Around the crack tip and at a certain distance from the crack tip, strain singularity dominates (this is the singularity radius). If our finite elements are able to model this singularity and they are covering this singularity region we expect to obtain the best possible SIFs using this mesh; i.e., the results are better than the results obtained using standard elements.

When the crack-tip elements are small, they cannot cover the whole singularity domain. As the length of these elements decreases, the elements model a smaller and smaller part of the strain singularity, and hence their contribution to the accuracy of the SIFs decreases. Using the second layer of the (transition) elements improves the accuracy. In this case the small crack-tip elements have a marginal effect on the accuracy of the calculated SIFs, and actually transition elements improve the calculations.

 

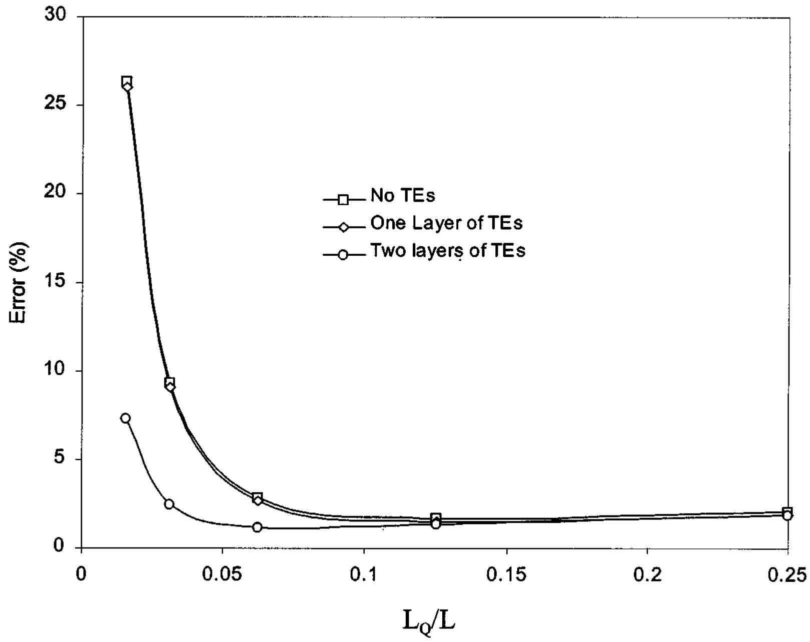


Figure 9. Percentage error of the calculated $K$ values for a cracked bimaterial strip

## 5. CONCLUSIONS

This article has proposed a simple method for calculating the optimal position of the side nodes of a quadratic isoparametric crack-tip element for modeling an $r^{\lambda-1}$ strain singularity.

A quadratic isoparametric transition element for an $r^{\lambda-1}$ singularity was formulated. It was shown that for small $L_Q/a$ and large $L_T/L_Q$, transition elements improve the accuracy of the calculated SIFs, just as they have been shown—in the case of $r^{-1/2}$ strain singularity—to improve the accuracy of SIFs for very small crack-tip elements. Layered transition elements were formulated for this strain singularity and it was observed that if the TEs in the first layer are small, a second layer of TEs can improve the accuracy of the calculation of the SIF, and so on.

It is to be noted that we considered a crack in the softer of the two materials. We did not report the results for the case of a crack in stiffer material because in that case it is very difficult to obtain good numerical results for stress intensity factors, although transition elements are effective. As can be seen in Lin and Mar [14], even using a hybrid finite element method does not give us good numerical results for stress intensity factors.

As was concluded for the $r^{-1/2}$ strain singularity [11], incorporating a layer of TEs produces results which always are better than or equal to those obtained when TEs are not used. Generally, around the crack tip and at a certain distance from the crack tip, strain singularity dominates (this is the singularity radius). Inside this region, finite elements should be able to model the singularity to obtain the best results with the same mesh. If crack-tip elements are large ($L_Q$ is equal to or greater than the singularity radius), using TEs has no improving effect. When $L_Q$ is small, with large $L_T/L_Q$ ratios ($L_Q + L_T$ is equal to or greater than the singularity radius), TEs can improve the accuracy. If $L_T/L_Q$ is small, the TEs in the first layer have a marginal effect, and a second layer of TEs can improve the accuracy, and so on. In general, using TEs yields results that are less dependent on the size of the crack-tip elements and more reliable than can be achieved without them.

## APPENDIX

Horváth [10] used an $n$th order isoparametric transition element for an $r^{(1-m)/m}$ ($n \geqslant m, m, n \in \mathbb{N}$) strain singularity problem and obtained the positions of the transition points. In order to use these elements in the second row of finite elements around the crack tip, we must have

$$\frac{1-m}{m} = \lambda - 1 \tag{27}$$

or

$$m = \frac{1}{\lambda} \tag{28}$$

Because $1/\lambda$ is not always an integer, we can use $m = [1/\lambda + 1]$, where $[\cdot]$ means the integer part. In the following table, the $m$ values for various $\lambda$ values are shown. It is clear that, for $0 < \lambda < 0{\cdot}5$, elements with high orders are needed, which is not practical. For $0{\cdot}5 < \lambda < 1$, $m$ is constant, which does not make any sense.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 |
| $m$ | 10 | 5 | 4 | 3 | 2 | 2 | 2 | 2 | 2 |

Thus, these transition elements are not practically useful.

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