

Nonlinear and Linear Elastodynamic Transformation Cloaking

Arash Yavari & Ashkan Golgoon

Archive for Rational Mechanics and Analysis

ISSN 0003-9527

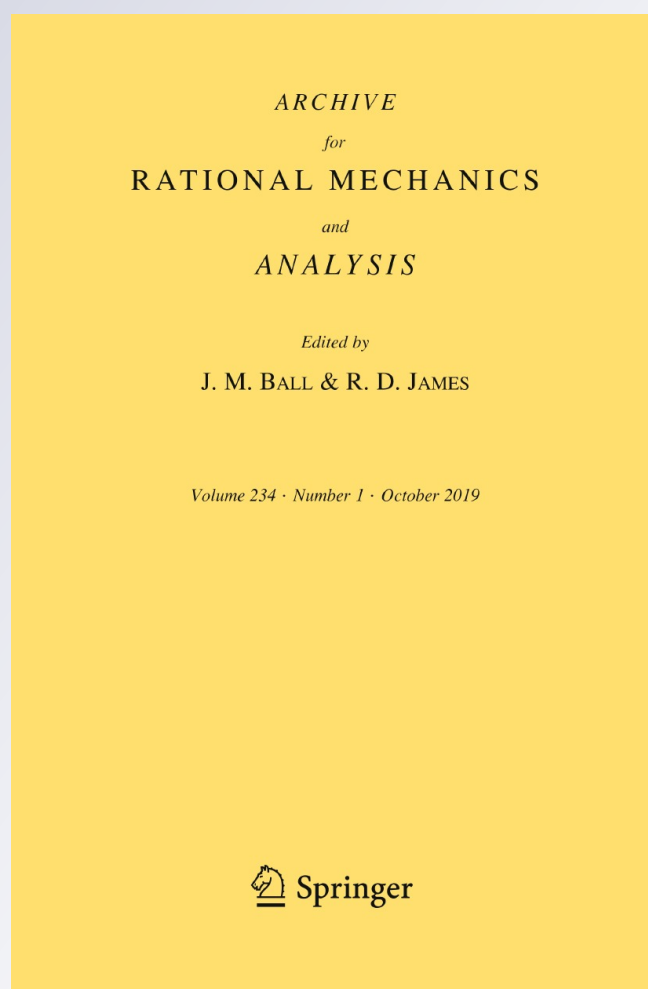
Volume 234

Number 1

Arch Rational Mech Anal (2019)

234:211-316


DOI 10.1007/s00205-019-01389-2



Your article is protected by copyright and all rights are held exclusively by Springer-Verlag GmbH Germany, part of Springer Nature. This e-offprint is for personal use only and shall not be self-archived in electronic repositories. If you wish to self-archive your article, please use the accepted manuscript version for posting on your own website. You may further deposit the accepted manuscript version in any repository, provided it is only made publicly available 12 months after official publication or later and provided acknowledgement is given to the original source of publication and a link is inserted to the published article on Springer's website. The link must be accompanied by the following text: "The final publication is available at link.springer.com".



Nonlinear and Linear Elastodynamic Transformation Cloaking

ARASH YAVARI  & ASHKAN GOLGOON

*Dedicated to Professor ZDENĚK P. BAŽANT on the occasion of his 80th birthday.
Communicated by M. ORTIZ*

Abstract

In this paper we formulate the problems of nonlinear and linear elastodynamic transformation cloaking in a geometric framework. In particular, it is noted that a cloaking transformation is neither a spatial nor a referential change of frame (coordinates); a cloaking transformation maps the boundary-value problem of an isotropic and homogeneous elastic body (virtual problem) to that of an anisotropic and inhomogeneous elastic body with a hole surrounded by a cloak that is to be designed (physical problem). The virtual body has a desired mechanical response while the physical body is designed to mimic the same response outside the cloak using a cloaking transformation. We show that nonlinear elastodynamic transformation cloaking is not possible while nonlinear elastostatic transformation cloaking may be possible for special deformations, e.g., radial deformations in a body with either a cylindrical or a spherical cavity. In the case of classical linear elastodynamics, in agreement with the previous observations in the literature, we show that the elastic constants in the cloak are not fully symmetric; they do not possess the minor symmetries. We prove that elastodynamic transformation cloaking is not possible regardless of the shape of the hole and the cloak. It is shown that the small-on-large theory, i.e., linearized elasticity with respect to a pre-stressed configuration, does not allow for transformation cloaking either. However, elastodynamic cloaking of a cylindrical hole is possible for in-plane deformations while it is not possible for anti-plane deformations. We next show that for a cavity of any shape elastodynamic transformation cloaking cannot be achieved for linear gradient elastic solids; similar to classical linear elasticity the balance of angular momentum is the obstruction to transformation cloaking. We finally prove that transformation cloaking is not possible for linear elastic generalized Cosserat solids in dimension two for any shape of the hole and the cloak. In particular, in dimension two transformation cloaking cannot be achieved in linear Cosserat elasticity. We show that transformation cloaking for a spherical cavity covered by a spherical cloak is not possible in the setting of linear elastic generalized Cosserat elasticity. We conjecture that this

result is true for a cavity of any shape. It should be emphasized that in this paper we do not consider the so-called metamaterials [70,72].

Contents

1. Introduction	212
2. Nonlinear Elastodynamics	222
2.1. Spatial Covariance of the Governing Equations of Nonlinear Elasticity	226
2.2. Material Covariance of the Governing Equations of Nonlinear Elasticity	227
3. Linearized Elastodynamics	232
3.1. Spatial Covariance of Linearized Elasticity	240
3.2. Material Covariance of Linearized Elasticity	242
4. A Mathematical Formulation of the Problem of Cloaking a Cavity in Nonlinear and Linearized Elastodynamics	247
4.1. Nonlinear Elastodynamic Transformation Cloaking	248
4.2. Nonlinear Elastostatic Cloaking	255
4.3. Linear Elastodynamic Transformation Cloaking	259
4.4. Small-on-Large Transformation Cloaking	267
5. Elastodynamic Transformation Cloaking in Solids with Microstructure	272
5.1. Elastodynamic Transformation Cloaking in Gradient Elastic Solids	273
5.2. Elastodynamic Transformation Cloaking in Generalized Cosserat Solids	284
Appendix A Riemannian Geometry	307
References	311

1. Introduction

Invisibility has been a dream for centuries. Making objects invisible to electromagnetic waves has been a subject of intense research in recent years. PENDRY et al. [89] and LEONHARDT [59] independently showed the possibility of electromagnetic cloaking. This was later experimentally verified by SCHURIG et al. [102], LIU et al. [62], and ERGIN et al. [25].¹ In many references, including [89], it is argued that the main idea of cloaking in electromagnetism is the invariance of Maxwell's equations under coordinate transformations (covariance). The covariance of Maxwell's equations has been known for a long time [90]. However, one should note that covariance of Maxwell's equations is not the direct underlying principle of transformation cloaking. In electromagnetic cloaking one maps one problem to another problem with some desirable response. For example, a domain with a hole surrounded by a cloak with unknown physical properties is mapped to a domain without a hole (or with a very small one) made of an isotropic and homogeneous material. Then one tries to find the transformed fields such that both problems satisfy Maxwell's equations [56]. This will then determine the physical properties of the cloak. In particular, the transformed quantities are not necessarily what one would expect under a coordinate transformation, i.e., the two problems are not related by push-forward or pull-back using the cloaking map.

¹ One should note the frequency limitation in the existing electromagnetic cloaking works; the existing works have been limited to microwaves frequencies.

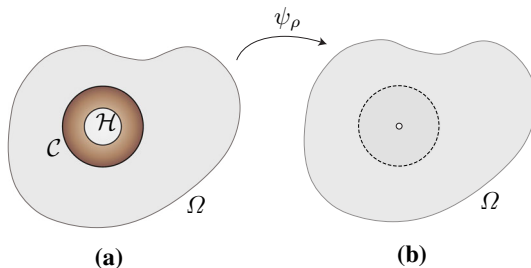


Fig. 1. Cloaking an object in a hole \mathcal{H} by a cloak \mathcal{C} . The system **b** has uniform physical properties. The cloaking transformation is the identity map outside the cloaking region \mathcal{C}

More specifically, the idea of cloaking for electromagnetism is as follows: suppose one is given a body (domain) Ω with a hole \mathcal{H} surrounded by a cloaking region \mathcal{C} (see Fig. 1). The hole can be of any shape and the one shown in Fig. 1 is assumed to be circular (spherical in 3D). Suppose the physical properties in $\Omega \setminus \mathcal{C}$ are uniform and isotropic. One is interested in designing the cloak \mathcal{C} such that an electromagnetic wave passing through Ω would not interact with \mathcal{H} . In other words, \mathcal{C} redirects the waves such that the boundary measurements are identical to those of another body with the same outer boundary as Ω without the hole and made of the same homogeneous (and isotropic) material. Let us consider a smooth mapping $\psi_\rho : \Omega \rightarrow \Omega$ such that $\psi_\rho|_{\Omega \setminus \mathcal{C}} = id$ (ψ_ρ restricted to the outside of the cloak is the identity map) and it shrinks the hole \mathcal{H} to a small circle (sphere) of radius $\rho > 0$ (see Fig. 1b). Note that this map is not unique, and hence, many possibilities for a cloak \mathcal{C} . One then transforms the physical fields such that the two problems satisfy Maxwell's equations. This usually makes the physical properties of the cloaking region \mathcal{C} both inhomogeneous and anisotropic. The perfect cloaking case (the limit $\rho \rightarrow 0$) may require singular physical properties.

Cloaking in the context of conductivity [40,41], electrical impedance tomography, and electromagnetism has been studied rigorously and is well understood [9,42–44]. Interestingly, the idea of cloaking has been explored in many other fields of science and engineering, e.g., acoustics [13,18,19,29–31,78,133], optics [60], thermodynamics (design of thermal cloaks) [46], diffusion [45], quantum mechanics [129], and elastodynamics [73]. The recent reviews [53,54] discuss these applications in some detail. The least understood among these applications is elastodynamics. In our opinion, the main problem is that none of the existing works in the literature has formulated the problem of elastodynamic cloaking properly. In particular, boundary and continuity conditions and the restrictions they impose on cloaking maps have not been discussed. In this paper, we formulate both the nonlinear and linearized cloaking problems in a mathematically precise form. One should note that transformation cloaking is an inherently geometric problem. This is explained in the case of optical cloaking and invisibility in [60]. We will see that this is the case for elastodynamic cloaking as well; geometry plays a critical role in a proper formulation of elastodynamic transformation cloaking.

The first ideas related to cloaking in elasticity go back to the 1930s and 1940s in the works of GURNEY [47] and REISSNER and MORDUCHOW [91] on reinforced holes

in elastic sheets in the framework of linear elasticity. The first systematic study of cloaking in linear elasticity is due to MANSFIELD [66] who introduced the concept of neutral holes. Mansfield considered a sheet (a plane problem) under a given (far-field) load. For the same far-field load applied to an uncut sheet one knows (or can calculate) the corresponding Airy stress function $\phi = \phi(x, y)$. He then put a hole(s) in the sheet and asked if the hole(s) can be reinforced such that the stress field outside the hole(s) is identical to that of the uncut sheet. In other words, the reinforcement hides the hole(s) from the stress field. MANSFIELD [66] showed that the boundary of a neutral hole is given by the equation $\phi(x, y) + ax + by + c = 0$, where a, b, c are constants. The reinforcement is a pre-stressed axially-loaded member (no bending stiffness) that may have a non-uniform cross sectional area. Design of such neutral holes depend on the external loading; unlike the cloaking problem in which a hole is to be hidden from arbitrary elastic disturbances, the shape of a neutral hole and the characteristics of its reinforcement explicitly depend on the stress field of the uncut sheet. In other words, a reinforced hole neutral under one far-field load may not be neutral under another one. In this paper we will present a nonlinear analogue of Mansfield's neutral holes for radial deformations in Section 4.2. Another problem related to cloaking is the idea of neutral inhomogeneities. A neutral inhomogeneity when inserted in an elastic medium that is under some far field loading would not perturb the stress field of the outside medium. This has been extensively studied for spherical [48, 49, 51], cylindrical [50], and ellipsoidal [5] inhomogeneities. It should be noted that the properties of the neutral inhomogeneities depend on the imposed loading.

The main difference between electromagnetic (and optical) cloaking and elastodynamic cloaking is that the governing equations of elasticity are written with respect to two frames and that they are tensor-valued. In elasticity, one writes the governing equations with respect to a reference and a current configuration; this leads to two-point tensors in the governing equations. If formulated properly, both nonlinear and linearized elasticity are spatially covariant, i.e., their governing equations are invariant under arbitrary time-dependent changes of frame (or coordinate transformations if viewed passively) [110, 126]. Invariance under referential changes of frame is more subtle as can be seen in the work of YAVARI et al. [127], who showed that the balance of energy is not invariant under an arbitrary time-dependent referential diffeomorphism. They obtained the transformed balance of energy that has some new terms corresponding to the velocity of the referential change of frame. MAZZUCATO and RACHELE [69] showed that the balance of linear momentum is invariant under any time-independent change in the reference configuration. In this paper, we will show that the results of YAVARI et al. [127] and MAZZUCATO and RACHELE [69] are consistent and the balance of energy and all the governing equations of nonlinear elasticity are invariant under arbitrary time-independent referential coordinate transformations. More recently, motivated by applications in seismology, AL-ATTAR and CRAWFORD [1] showed the invariance of the governing equations of elastodynamics under referential changes of frame that they called particle relabelling transformations, a term borrowed from fluid mechanics. They also correctly pointed out that the elastic constants with respect to any equilibrium configuration transform tensorially under a particle relabelling transformation.

The goal in elastodynamic cloaking is to make a hole (cavity) invisible to elastic waves. One idea is to cover the boundary of the cavity by a cloak that has inhomogeneous and anisotropic elastic properties, in general. The cloak will deflect the elastic waves resulting in elastic measurements away from the cavity (more specifically, outside the cloak) identical to those when the cavity is absent. Pulling back the homogeneous material properties using a cloaking transformation (that will be defined later), one may obtain the desired inhomogeneous and anisotropic mechanical properties of the cloak. To design an elastic cloak one would need to answer the following questions: (i) Are the governing equations of nonlinear (and linear) elasticity invariant under coordinate transformations? First, one must define what a coordinate transformation means in nonlinear elasticity. There are two types of transformations that have very different physical meanings; (ii) Can cloaking be achieved using a spatial or referential coordinate transformation? Or is a cloaking transformation more than a change of coordinates? Related to question i, note that any properly formulated physical field theory has to be covariant. This was Einstein's idea in the theory of general relativity. No coordinate system can be a distinguished one; nature does not discriminate between different observers and they all see the same physical laws. In nonlinear elasticity covariance is understood as the invariance of the governing equations under arbitrary time-dependent coordinate transformations in the ambient space [52, 68, 127]. Regarding question ii we will show that a cloaking transformation is neither a spatial nor a referential change of frame (coordinates); a cloaking transformation maps the boundary-value problem of an isotropic and homogeneous elastic body (virtual problem) to that of an anisotropic and inhomogeneous elastic body with a hole surrounded by a cloak that is to be designed (physical problem).

Traditionally, many workers in solid mechanics start from linear elasticity. This is appropriate for many practical applications, and linear elasticity has been quite successful in numerous engineering applications. The governing equations of linear elasticity are linear partial differential equations, and hence, superposition is applicable, one can use Green's functions, etc. However, there are many problems for which linearized elasticity is not appropriate. The first practical application of nonlinear elasticity was in the rubber industry in the 1940s and 1950s, which motivated Rivlin's seminal contributions [92–98]. In recent years, nonlinear elasticity has been revived motivated by the biomechanics applications in which biological tissues undergo large strains [37]. Unlike electromagnetism with only one configuration (ambient space), in nonlinear elasticity, there are two inherently different configurations: reference and current. Linear elasticity does not distinguish between these two configurations, and this has been a source of confusion in the recent literature of elastodynamic transformation cloaking. In the reference configuration the body is stress free and any measure of strain is defined with respect to this configuration.² Consequently, the stored energy of an elastic body explicitly depends on the reference configuration as well. In the classical formulation of nonlinear elasticity,

² We should mention that there are recent geometric developments using non-Euclidean reference configurations that allow for sources of residual stress [23, 33–36, 86, 99, 101, 106, 107, 116–119, 123].

it is well understood that coordinate transformations in the reference and current configurations are very different. Local referential transformations are related to material symmetries, while the global transformations of the ambient space (current configuration) are related to objectivity (or material frame indifference). This implies that any cloaking study, even when strains are small, should be formulated in the framework of nonlinear elasticity.

A classic example of an improper use of the governing equations of linear elasticity can be seen in almost all the existing discussions on the objectivity of linear elasticity. It has long been argued that linearized elasticity is not objective, i.e., its governing equations are not invariant under rigid body translations and rotations of the ambient space (see [110] for references). This is unnatural and accepting it, one, at least implicitly, is assuming that linear elasticity is a “special” field theory. This cannot be true and linear elasticity, like any other field theory, has to be objective (and, more generally, spatially covariant) if it is properly formulated. This problem was revisited independently by STEIGMANN [110] and YAVARI and OZAKIN [126]. These authors showed that if formulated and interpreted properly, linear elasticity is objective (covariant) as expected. In short, Navier’s equations are written with respect to one coordinate system and do not have the proper geometric structure to be used in studying the transformation properties of the balance of linear momentum in linear elasticity.

The work of Lodge [63,64]. Arthur S. Lodge showed that the static equilibrium solutions of certain anisotropic homogeneous linear elastic bodies can be mapped to those of isotropic homogeneous linear elastic bodies using affine transformations of position and displacement vectors. In other words, knowing an equilibrium solution for a homogeneous isotropic linear elastic body, new equilibrium solutions can be generated for certain anisotropic bodies. We should point out that in [63,64] and [85] the matrices of the coordinate and displacement transformations are inverses of each other. In particular, Lodge [63] in his first paper considered the following transformations for coordinates and displacements: $(x, y, z) \rightarrow (x', y', z') = (x, y, v^{-\frac{1}{2}}z)$ and $(u, v, w) \rightarrow (u', v', w') = (u, v, v^{\frac{1}{2}}z)$, for some $v > 0$. He showed that the governing equations of linear elasticity are invariant under these transformations. Note that these transformations map the equilibrium solution of an isotropic elastic body to that of another elastic body that is anisotropic. This should not be confused with the transformation of the governing equations of one given body under referential or spatial coordinate transformations. In other words, LODGE [64] (see also [57]) finds an equilibrium solution for an anisotropic body using that of another elastic body, which is isotropic. The position and displacement vectors are linearly related. However, the two problems are not related by a coordinate transformation. OLVER [84] showed that any planar anisotropic linear elastic solid is equivalent to an orthotropic solid through some linear transformations of coordinates and displacements that are independent.

LODGE [64]’s idea of mapping the boundary-value problem of an anisotropic linearly elastic body to that of an isotropic body can be summarized as follows. The position and the displacement vectors are transformed homogeneously as

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{u}' = \mathbf{A}^{-\mathbf{T}}\mathbf{u}. \quad (1.1)$$

We will refer to the transformation $(\mathbf{x}', \mathbf{u}') = (\mathbf{A}\mathbf{x}, \mathbf{A}^{-\top}\mathbf{u})$ as a *Lodge transformation*. Using this pair of linear transformations, strain is transformed as $\boldsymbol{\epsilon}' = \mathbf{A}^{-\top}\boldsymbol{\epsilon}\mathbf{A}^{-1}$. Assuming that under (1.1), $\boldsymbol{\sigma}' : \boldsymbol{\epsilon}' = \boldsymbol{\sigma} : \boldsymbol{\epsilon}$, stress is transformed as $\boldsymbol{\sigma}' = \mathbf{A}\boldsymbol{\sigma}\mathbf{A}^{\top}$. LODGE [64] assumed that body force transforms like a vector, i.e., $\mathbf{b}' = \mathbf{A}\mathbf{b}$. Implicitly, he assumed that mass density transforms like a scalar, i.e., it remains unchanged: $\rho' = \rho$. This is similar to the way mass density transforms under a change of spatial frame. Under these assumptions one can show that

$$\operatorname{div}' \boldsymbol{\sigma}' + \rho' \mathbf{b}' = \mathbf{A} (\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b}). \quad (1.2)$$

The inertial force is transformed as $\rho' \mathbf{a}' = \mathbf{A}^{-\top}(\rho \mathbf{a})$. Therefore, starting from $\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}$, one obtains the balance of linear momentum for the transformed body as

$$\operatorname{div}' \boldsymbol{\sigma}' + \rho' \mathbf{b}' = \mathbf{A}\mathbf{A}^{\top} \rho' \mathbf{a}. \quad (1.3)$$

Lodge then concluded that the balance of linear momentum is invariant under the transformation (1.1) when inertial forces are ignored. Under a Lodge transformation the elastic constants, and hence, the anisotropy type changes. Lodge finally calculated the matrix \mathbf{A} such that the transformed body is isotropic. Using this transformation, one can generate equilibrium solutions for certain anisotropic bodies having the corresponding solutions for an isotropic linearly elastic body. When inertial forces are not ignored the balance of linear momentum is invariant only when $\mathbf{A}\mathbf{A}^{\top} = \mathbf{I}$, i.e., \mathbf{A} is a rotation and can be interpreted as a coordinate transformation.

Remark 1. Fifty years later, not being aware of the work of Lodge, MILTON et al. [73] for a completely different purpose used the Lodge transformations (1.1) but with a position-dependent \mathbf{A} . They assumed a harmonic time dependence and ignored the body forces. Let us examine their Eq. (2.4) for a constant matrix \mathbf{A} . Their transformed wave equation in this special case reads

$$\operatorname{div}' \boldsymbol{\sigma}' = -\omega^2 \rho' \mathbf{u}', \quad (1.4)$$

where their matrix-valued mass density is defined as $\rho' = \frac{1}{\det \mathbf{A}} \rho \mathbf{A}\mathbf{A}^{\top}$. They justify a matrix-valued mass density arguing that it has been observed for composites. Two comments are in order here: (i) Note that the factor $\frac{1}{\det \mathbf{A}}$ appears when one uses the Piola identity as we will discuss in Section 4.1 and Section 4.3. When \mathbf{A} is a constant matrix, $\operatorname{div}' \boldsymbol{\sigma}'$ is $\frac{1}{\det \mathbf{A}}$ times that of Lodge's; (ii) If the body force term $\rho \mathbf{b}$ is kept, this matrix-valued mass density would not work unless one assumes that $\mathbf{b}' = \mathbf{A}^{-\top} \mathbf{b}$, which is different from Lodge's original transformation.

The work of Milton et al. [73] on elastodynamic cloaking. The first theoretical study of elastodynamic transformation cloaking is due to MILTON et al. [73]. They observed that the governing equations of linear elasticity are not invariant under coordinate transformations (or what they called “curvilinear transformations”), and hence, the cloaking of elastic waves cannot be achieved using coordinate transformations. One should note that Navier's equations are written with respect to one

coordinate system, and hence, do not have the proper geometric structure to be used in studying the transformation properties of the governing equations of linear elasticity under coordinate (or cloaking) transformations. MILTON et al. [73] start with the wave equation in the setting of linear elasticity, i.e., $\nabla \cdot \boldsymbol{\sigma} + \omega^2 \rho \mathbf{u} = \mathbf{0}$, where $\boldsymbol{\sigma} = \mathbf{C} \nabla \mathbf{u}$, and \mathbf{C} is the elasticity tensor, and consider mappings $\mathbf{x} \rightarrow \mathbf{x}'(\mathbf{x})$ and $\mathbf{u}(\mathbf{x}) \rightarrow \mathbf{u}'(\mathbf{x}')$. They point out that the two mappings can be chosen freely. Denoting the derivative of the map $\mathbf{x}'(\mathbf{x})$ by $\mathbf{A}(\mathbf{x})$, i.e., $A_{i'j} = \partial x_{i'}/\partial x_j$, they assume that instead of $\mathbf{u}' = \mathbf{A}(\mathbf{x})\mathbf{u}$ (assuming that the change of coordinates is a spatial coordinate transformation), displacement is transformed as $\mathbf{u}' = \mathbf{A}^{-T}(\mathbf{x})\mathbf{u}$. MILTON et al. [73] point out that these (Lodge-type) transformations preserve the symmetries of the elasticity tensor. They finally show that under these changes of variables the Cauchy stress and the wave equation transform as

$$\operatorname{div}' \boldsymbol{\sigma}' = -\omega^2 \rho' \mathbf{u}' + \mathbf{D}' \nabla' \mathbf{u}' \quad \text{and} \quad \boldsymbol{\sigma}' = \mathbf{C}' \nabla' \mathbf{u}' + \mathbf{S}' \mathbf{u}', \quad (1.5)$$

where ($J = \det \mathbf{A}$)

$$\begin{aligned} C'_{pqrs} &= \frac{1}{J} \frac{\partial x'^p}{\partial x^i} \frac{\partial x'^q}{\partial x^j} \frac{\partial x'^r}{\partial x^k} \frac{\partial x'^s}{\partial x^l} C_{ijkl}, \\ S'_{pqr} &= \frac{1}{J} \frac{\partial x'^p}{\partial x^i} \frac{\partial x'^q}{\partial x^j} \frac{\partial^2 x'^r}{\partial x^k \partial x^l} C_{ijkl} = S'_{qrp}, \end{aligned} \quad (1.6)$$

$D'_{pqr} = S'_{qrp}$, and the matrix-valued mass density is written as³

$$\rho'_{pq} = \frac{\rho}{J} \frac{\partial x'_p}{\partial x_i} \frac{\partial x'_q}{\partial x_i} + \frac{1}{J} \frac{\partial^2 x'_p}{\partial x_i \partial x_j} C_{ijkl} \frac{\partial^2 x'_q}{\partial x_k \partial x_l}. \quad (1.7)$$

Banerjee [4] discusses this in more detail and concludes that: “Therefore, unlike Maxwell’s equations the equations of elastodynamics change form under a coordinate transformation”. This conclusion is, unfortunately, incorrect; governing equations of linear elasticity are form-invariant under both spatial and referential coordinate transformations. However, as we will show cloaking transformations are not coordinate transformations. It is, nevertheless, correct that transformation cloaking is not possible in classical linear elastodynamics.

Several authors have looked at in-plane waves arguing that in this particular case the governing equations are invariant under coordinate transformations (however, it is not clear what type of coordinate transformations is being considered), and hence, transformation cloaking can be achieved. In particular, BRUN et al. [8] observed that for in-plane elastic waves, the balance of linear momentum is invariant under an arbitrary change of coordinates (they do not distinguish between referential and spatial transformations), and introduced a cylindrical cloak. They considered an

³ There is a typo in their transformed mass density. The second term does not have the correct physical dimension. The similar expression in [4] has the correct dimension with a factor $1/\omega^2$ but has the opposite sign.

annular cloak of inner and outer radii r_0 and r_1 , respectively, and used PENDRY et al. [89]'s coordinate transformation $(r, \theta) \rightarrow (r', \theta') = (r'(r), \theta)$, where

$$r'(r) = \begin{cases} r_0 + \frac{r_1 - r_0}{r_1} r, & r \leq r_1, \\ r, & r \geq r_1. \end{cases} \quad (1.8)$$

This (singular) transformation maps a disk of radius r_1 to an annulus of inner and outer radii r_0 and r_1 , respectively, and is the identity transformation outside the disk. These authors observed that assuming $\mathbf{u}' = \mathbf{u}$, Navier's equations are form-invariant, and mass density transforms as $\rho'(r) = \frac{r-r_0}{r} \left(\frac{r_1}{r_1-r_0}\right)^2 \rho(r)$. In the transformed coordinates, they obtained an elasticity tensor that does not possess the minor symmetries. Then their recourse is to argue that the transformed body is made of a Cosserat solid.⁴ We will show in this paper that the map (1.8), which has been borrowed from the literature of electromagnetic cloaking is not admissible for elastodynamic cloaking; the derivative of this map is not the identity on the outer boundary of the cloak, i.e., $f'(r_1) \neq 1$ (see Section 5.2). Instead of assuming that the cloak is made of a Cosserat solid without even discussing its elastic constants, we believe one should assume that both the physical and virtual bodies (that will be defined in Section 4) are made of Cosserat solids and see if a Cosserat cloak can be designed while all the balance laws are respected in both bodies. We will show that the minor symmetries of the elastic constants are not preserved under cloaking transformations (that will be defined in Section 4.1). Our conclusion is that classical linear elasticity is not flexible enough to allow for cloaking. A possible solution may be linearized elasticity with respect to a pre-stressed configuration, i.e., the small-on-large theory of GREEN et al. [38]. We will show that cloaking is not possible in this more general framework either. However, we will see that a cylindrical hole can be cloaked for in-plane deformations provided that the cloak is pre-stressed. This, however, is not possible for the anti-plane deformations. These observations would force one to start from some kind of a solid with microstructure. In addition to finding the classical elastic constants in the cloak, the non-classical elastic constants must be calculated as well. This has not been discussed in the literature to date. This is a non-trivial calculation that will be discussed in Section 5. In the literature, it has been implicitly assumed that the material outside the cloak is a classical linear elastic solid. We will see that this is not possible (Remark 14).

Starting from linear elasticity, NORRIS and SHUVALOV [80] tried to find the governing equations in a transformed domain using Lodge-type transformations. Similar to the work of MILTON et al. [73], they assumed that the displacement field does not transform the way it does under a spatial coordinate transformation. They refer to this as a linear gauge change. Unlike that of MILTON et al. [73], their displacement transformation is completely independent of the coordinate transformation. This is similar to the transformations that had earlier been used by OLVER [84]. They

⁴ Surprisingly, in none of the works that accept non-symmetric Cauchy stresses in the cloak is there any mention of the balance of angular momentum and the distribution of couple stresses. There is also no discussion on what the extra elastic constants of the Cosserat cloak should be.

discussed several possibilities for the displacement transformation and observed a loss of minor symmetries of the elastic constants in the cloak and assumed that it is made of a Cosserat solid. However, the balance of angular momentum and the calculation of the non-classical elastic constants of the Cosserat elastic cloak were not discussed.

OLSSON and WALL [83] studied time-harmonic cloaking of a finite rigid body that is fixed, i.e., cannot move, and is embedded in an elastic matrix. In the case of both a rigid circular disk and a rigid spherical ball they assumed that the matrix is an elastic medium with inextensible radial fibers. To the best of our knowledge, this is the first paper on elastic cloaking that actually discusses boundary conditions. In the physical and virtual bodies (that we will define in Section 4) in both 2D and 3D they assumed the following relation between displacement vectors: $\mathbf{U}/R = \tilde{\mathbf{U}}/\tilde{R}$. They further observed that the balance of linear momentum of the two configurations have the same form, and that the elastic constants retain their full symmetries. KHLOPOTIN et al. [55] investigated cloaking a finite rigid body embedded in an elastic medium. Their motivation was cloaking an object in a soft matrix from surface elastic waves. They assumed that the matrix is a micropolar solid and discussed boundary conditions. However, in this work there is no discussion on how the non-classical elastic constants of the cloak should be calculated. In particular, there is no mention of how the couple stiffness tensor and the couple stress tensor of the micropolar medium are transformed under a cloaking transformation. We believe that a proper formulation of linear elastodynamic cloaking should consider both the physical and virtual bodies to be made of generalized Cosserat solids and all the elastic constants of the cloak must be calculated. This will be discussed in Section 5.

PARNELL [87] and PARNELL et al. [88] (see also [79] for in-plane deformations) first considered anti-plane deformations of an isotropic linear elastic solid for which the displacement field in cylindrical coordinates has the form $(0, 0, W(R, \Theta))$. They concluded that the balance of linear momentum (wave equation) is form-invariant under coordinate transformations. The transformed mass density is identical to that of BRUN et al. [8]. Next, Parnell considered the wave equation in the small-on-large theory of GREEN et al. [38], which is simply linearization about a finitely-deformed (and stressed) configuration [81]. He considered a body with a small hole made of an incompressible neo-Hookean solid. Using a (static) applied internal pressure the hole is inflated. Parnell showed that the incremental wave equation with respect to this pre-stressed configuration has anisotropic shear moduli. However, they are different from those of a cloak. One should also note that inflating an initially small hole in a body the entire body would deform. In other words, the small-on-large elasticity of such a body cannot be identical to that of a stress-free and homogeneous linear elastic body outside any finite region. In Section 4.4 we will show that in the setting of small-on-large theory cloaking a cylindrical hole for anti-plane deformations is not possible. However, it is possible to cloak a cylindrical hole for in-plane deformations.

There have been other efforts in the literature on guiding elastic waves in structures. We should mention AMIRKHIZI et al. [2], who proposed the idea of redirecting stress waves by smoothly changing anisotropy of a structure. In particular, they experimentally and numerically showed that when the direction of a stress wave is

known and is fixed its propagation in a structure made of a transversely isotropic material with a varying axis of anisotropy can be guided. Of course, their construction is restricted and useful only for one specific direction of wave propagation. Using a pre-stressed neo-Hookean solid CHANG et al. [12] demonstrated the possibility of manipulating shear waves. SKLAN et al. [105] proposed a symmetrized elastic cloak by symmetrizing the non-symmetric elastic constants induced from a cloaking map. They used the symmetrized elastic constants in a cloaking region around a cylindrical hole in order to shield an object against elastic waves. In their numerical examples for certain frequencies they observed reduction of average displacements and elastic energy in the cloaking region. This approach based on brute-force symmetrization has clear limitations.

Contributions of this paper. In this paper, we investigate the problem of hiding a hole from elastic waves in both nonlinear and linear elastodynamics. It should be emphasized that we do not consider the so-called metamaterials [70,72]. We start by discussing the invariance of the governing equations of nonlinear and linearized (with respect to any finitely-deformed configuration) elasticity under both arbitrary time-dependent spatial changes of frame (coordinate transformations) and arbitrary time-independent referential changes of frame. We note, however, that cloaking cannot be achieved using either (or both) spatial or referential coordinate transformations. We define a cloaking map to be a mapping that transforms the boundary-value problem of an elastic body with a hole reinforced by a cloak (physical body) to that of a homogeneous and isotropic body with an infinitesimal hole (virtual body). The cloak needs to be designed while the loads and boundary conditions in the virtual body are not known a priori. We define a cloaking transformation to be a map between the boundary-value problems of an elastic body to be designed and a virtual elastic body that has some desired mechanical response. The main contributions of this work can be summarized as follows:

- We provide a geometric formulation of transformation cloaking in nonlinear elasticity. It is shown that nonlinear elastodynamic transformation cloaking is not possible (Proposition 2).
- It is shown that nonlinear elastostatic transformation cloaking may be possible for special deformations. This is somehow a nonlinear analogue of Mansfield's neutral holes. We provide one such example, namely radial deformations in an infinitely long solid cylinder with a cylindrical hole or a finite spherical ball with a spherical cavity.
- Classical linear elasticity is not flexible enough to allow for transformation cloaking (Proposition 3). More specifically, linear elastodynamic cloaking cannot be achieved because the elastic constants in the cloak lose their minor symmetries. This is true for a hole of any shape reinforced by a cloak with an arbitrary shape.
- Transformation cloaking is not possible even in the small-on-large theory, i.e., linearized elasticity with respect to a pre-stressed configuration (Proposition 4). This is true for a hole with any shape. We show that a cylindrical hole can be cloaked for in-plane deformations while it is not possible to cloak it for anti-plane deformations.

- Assuming that the virtual body is isotropic and centro-symmetric, elastodynamic transformation cloaking is not possible in the setting of gradient elasticity (Proposition 5). This result is independent of the shape of the hole.
- Elastodynamic transformation cloaking is not possible in the setting of generalized Cosserat elasticity in dimension two (Proposition 6). In particular, in dimension two transformation cloaking cannot be achieved in Cosserat elasticity (with rigid directors) either. This result is independent of the shapes of the hole and the cloak. Elastodynamic transformation cloaking is not possible for a spherical cavity using a spherical cloak in the setting of generalized Cosserat elasticity (Proposition 8). We conjecture that this result in dimension three is independent of the shapes of the cavity and the cloak (Conjecture 1).

This paper is structured as follows: in Section 2, we revisit nonlinear elasticity and discuss the invariance of its governing equations under both arbitrary time-dependent transformations of the ambient space (current configuration) and arbitrary time-independent transformations of the reference configuration. In Section 3, linearization of nonlinear elasticity is discussed in detail. Then spatial and referential covariance of the governing equations of linearized elasticity are investigated. The problems of cloaking for both nonlinear elastodynamics and elastostatics are formulated in Section 4.1 and Section 4.2, respectively. Transformation cloaking in classical linear elasticity is investigated in Section 4.3. In Section 4.4, transformation cloaking in the small-on-large theory is formulated. In Section 5, elastodynamic transformation cloaking in solids with microstructure is investigated. The impossibility of transformation cloaking in linearized gradient elasticity is discussed in Section 5.1. Elastodynamic transformation cloaking in generalized Cosserat solids is formulated and discussed in Section 5.2.

2. Nonlinear Elastodynamics

Kinematics. In nonlinear elasticity, motion is a time-dependent mapping between a reference configuration (or natural configuration) and the ambient space (see Fig. 2). Geometrically, we write this as $\varphi_t : \mathcal{B} \rightarrow \mathcal{S}$, where $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ are the material and the ambient space Riemannian manifolds, respectively [68]. Here, \mathbf{G} is the material metric (that allows one to measure distances in a natural stress-free configuration) and \mathbf{g} is the background metric of the ambient space. The Levi-Civita connections associated with the metrics \mathbf{G} and \mathbf{g} are denoted as $\nabla^{\mathbf{G}}$ and $\nabla^{\mathbf{g}}$, respectively. The corresponding Christoffel symbols of $\nabla^{\mathbf{G}}$ and $\nabla^{\mathbf{g}}$ in the local coordinate charts $\{X^A\}$ and $\{x^a\}$ are denoted by Γ^A_{BC} and γ^a_{bc} , respectively. These can be directly expressed in terms of the metric components as

$$\begin{aligned} \gamma^a_{bc} &= \frac{1}{2} g^{ak} (g_{kb,c} + g_{kc,b} - g_{bc,k}), \\ \Gamma^A_{BC} &= \frac{1}{2} G^{AK} (G_{KB,C} + G_{KC,B} - G_{BC,K}). \end{aligned} \tag{2.1}$$

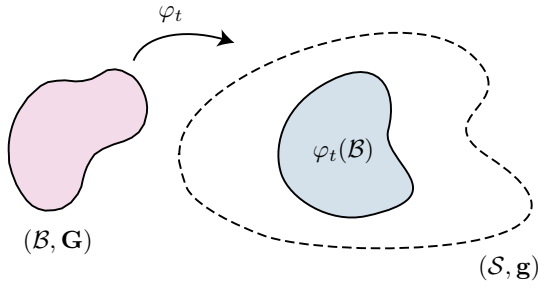


Fig. 2. Motion in nonlinear elasticity is a time-dependent mapping between two Riemannian manifolds

The deformation gradient \mathbf{F} is the tangent map of φ_t , which is defined as $\mathbf{F}(X, t) = T\varphi_t(X) : T_X\mathcal{B} \rightarrow T_{\varphi_t(X)}\mathcal{S}$. The transpose of \mathbf{F} is denoted by \mathbf{F}^\top , where

$$\begin{aligned} \mathbf{F}^\top(X, t) : T_{\varphi_t(X)}\mathcal{S} &\rightarrow T_X\mathcal{B}, \\ \langle\langle \mathbf{W}, \mathbf{F}^\top \mathbf{w} \rangle\rangle_{\mathbf{G}} &= \langle\langle \mathbf{F} \mathbf{W}, \mathbf{w} \rangle\rangle_{\mathbf{g}}, \quad \forall \mathbf{W} \in T_X\mathcal{B}, \mathbf{w} \in T_{\varphi_t(X)}\mathcal{S}. \end{aligned} \quad (2.2)$$

In components, $(\mathbf{F}^\top)^A{}_a = G^{AB} F^b{}_{B} g_{ab}$. The right Cauchy-Green deformation tensor is defined as $\mathbf{C} = \mathbf{F}^\top \mathbf{F} : T_X\mathcal{B} \rightarrow T_X\mathcal{B}$, which in components reads $C^A{}_B = F^a{}_L F^b{}_{B} g_{ab} G^{AL}$.⁵

The material velocity of the motion is the mapping $\mathbf{V} : \mathcal{B} \times \mathbb{R}^+ \rightarrow T\mathcal{S}$, where $\mathbf{V}(X, t) \in T_{\varphi_t(X)}\mathcal{S}$, and in components, $V^a(X, t) = \frac{\partial \varphi^a}{\partial t}(X, t)$. The spatial velocity is defined as $\mathbf{v} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \rightarrow T\mathcal{S}$ such that $\mathbf{v}_t(x) = \mathbf{V}_t \circ \varphi_t^{-1}(x) \in T_x\mathcal{S}$, where $x = \varphi_t(X)$. The convected velocity is defined as $\mathcal{V}_t = \varphi_t^*(\mathbf{v}_t) = T\varphi_t^{-1} \circ \mathbf{v}_t \circ \varphi_t = \mathbf{F}^{-1} \cdot \mathbf{V}$. The material acceleration is a mapping $\mathbf{A} : \mathcal{B} \times \mathbb{R}^+ \rightarrow T\mathcal{S}$ defined as $\mathbf{A}(X, t) := D_t^{\mathbf{g}} \mathbf{V}(X, t) = \nabla_{\mathbf{V}(X, t)}^{\mathbf{g}} \mathbf{V}(X, t) \in T_{\varphi_t(X)}\mathcal{S}$, where $D_t^{\mathbf{g}}$ denotes the covariant derivative along the curve $\varphi_t(X)$ in \mathcal{S} .⁶ In components, $A^a = \frac{\partial V^a}{\partial t} + \gamma^a{}_{bc} V^b V^c$. To motivate this definition note that the time derivative of the kinetic energy density is calculated as⁷

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \rho_0(X) \langle\langle \mathbf{V}(X, t), \mathbf{V}(X, t) \rangle\rangle_{\mathbf{g}} &= \rho_0(X) \langle\langle \mathbf{V}(X, t), D_t^{\mathbf{g}} \mathbf{V}(X, t) \rangle\rangle_{\mathbf{g}} \\ &= \rho_0(X) \langle\langle \mathbf{V}(X, t), \mathbf{A}(X, t) \rangle\rangle_{\mathbf{g}}. \end{aligned} \quad (2.3)$$

Therefore, $\mathbf{A}(X, t)$ is the covariant time derivative of the velocity vector field. The spatial acceleration is defined as $\mathbf{a} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \rightarrow T\mathcal{S}$ such that $\mathbf{a}_t(x) = \mathbf{A}_t \circ \varphi_t^{-1}(x) \in T_x\mathcal{S}$. In components, it reads $a^a = \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^b} v^b + \gamma^a{}_{bc} v^b v^c$. The spatial acceleration can also be expressed as the material time derivative of \mathbf{v} , i.e.,

⁵ Note that \mathbf{C}^b agrees with the pull-back of the ambient space metric by φ_t , i.e., $\mathbf{C}^b = \varphi_t^* \mathbf{g}$.

⁶ This definition is not to be confused with a Lodge transformation.

⁷ Note that if a connection ∇ is \mathbf{G} -compatible, then $\frac{d}{dt} \langle\langle \mathbf{X}, \mathbf{Y}(X, t) \rangle\rangle_{\mathbf{G}} = \langle\langle D_t \mathbf{X}, \mathbf{Y} \rangle\rangle_{\mathbf{G}} + \langle\langle \mathbf{X}, D_t \mathbf{Y} \rangle\rangle_{\mathbf{G}}$, where D_t is the covariant time derivative.

$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \nabla_{\mathbf{v}}^{\mathbf{g}} \mathbf{v}$. The convected acceleration is defined as [104]

$$\mathcal{A}_t = \varphi_t^*(\mathbf{a}_t) = \frac{\partial \mathcal{Y}_t}{\partial t} + \nabla_{\mathcal{Y}_t}^{\varphi_t^* \mathbf{g}} \mathcal{Y}_t = \frac{\partial \mathcal{Y}_t}{\partial t} + \nabla_{\mathcal{Y}_t}^{\mathbf{C}^{\flat}} \mathcal{Y}_t. \quad (2.4)$$

Balance laws. The balance of linear momentum in spatial and material forms reads

$$\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}, \quad \operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}, \quad (2.5)$$

where $\boldsymbol{\sigma}$ and \mathbf{P} are the Cauchy stress and the first Piola-Kirchhoff stress, respectively. ρ_0 , \mathbf{B} , and \mathbf{A} are the material mass density, material body force, and material acceleration, respectively, and ρ , \mathbf{b} , and \mathbf{a} are their corresponding spatial counterparts. Note that $\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}$ and $\operatorname{Div} \mathbf{P}$ have the following coordinate expressions:

$$\begin{aligned} \operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma} &= \sigma^{ab} |_{|b} \frac{\partial}{\partial x^a} = \left(\frac{\partial \sigma^{ab}}{\partial x^b} + \sigma^{ac} \gamma^b{}_{cb} + \sigma^{cb} \gamma^a{}_{cb} \right) \frac{\partial}{\partial x^a}, \\ \operatorname{Div} \mathbf{P} &= P^{aA} |_{|A} \frac{\partial}{\partial x^a} = \left(\frac{\partial P^{aA}}{\partial X^A} + P^{aB} \Gamma^A{}_{AB} + P^{cA} F^b{}_{A\gamma} \gamma^a{}_{bc} \right) \frac{\partial}{\partial x^a}. \end{aligned} \quad (2.6)$$

In coordinates, $J \sigma^{ab} = F^a{}_A P^{bA}$, where J is the Jacobian of deformation that relates the deformed and undeformed Riemannian volume elements as $dv(x, \mathbf{g}) = J dV(X, \mathbf{G})$, and

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}. \quad (2.7)$$

One can pull back the balance of linear momentum to the reference configuration, i.e., $\varphi_t^*(\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}) + \varphi_t^*(\rho \mathbf{b}) = \varphi_t^*(\rho \mathbf{a})$. This can be written as

$$\operatorname{div}_{\mathbf{C}^{\flat}} \boldsymbol{\Sigma} + \varrho \mathcal{B}_t = \varrho \mathcal{A}_t, \quad (2.8)$$

where $\boldsymbol{\Sigma} = \varphi_t^* \boldsymbol{\sigma}$ is the convected stress, $\mathcal{B}_t = \varphi_t^* \mathbf{b}$ is the convected body force, and $\varrho = \rho \circ \varphi_t$.

Identifying a material point with its position in the material manifold $X \in \mathcal{B}$, we have $x = \varphi_t(X)$. When the ambient space is Euclidean one defines the material displacement field as $\mathbf{U} = \varphi_t(X) - X$. The spatial displacement field is denoted by $\mathbf{u} = \mathbf{U} \circ \varphi_t^{-1}$.

The balance of angular momentum in local form reads $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$ or $\mathbf{FP}^* = \mathbf{PF}^*$, where \mathbf{P}^* and \mathbf{F}^* are duals of \mathbf{P} and \mathbf{F} , respectively, and are defined as

$$\begin{aligned} \mathbf{F} &= F^a{}_A \frac{\partial}{\partial x^a} \otimes dX^A, & \mathbf{F}^* &= F^a{}_A dX^A \otimes \frac{\partial}{\partial x^a}, \\ \mathbf{P} &= P^{aA} \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial X^A}, & \mathbf{P}^* &= P^{aA} \frac{\partial}{\partial X^A} \otimes \frac{\partial}{\partial x^a}. \end{aligned} \quad (2.9)$$

Note that $\mathbf{F}^* : T_{\varphi_t(X)}^* \mathcal{S} \rightarrow T_X^* \mathcal{B}$, where $T_{\varphi_t(X)}^* \mathcal{S}$ and $T_X^* \mathcal{B}$ denote the cotangent spaces of $T_{\varphi_t(X)} \mathcal{S}$ and $T_X \mathcal{B}$, respectively.

Conservation of mass implies that $\rho dv = \rho_0 dV$, or $\rho J = \rho_0$, where ρ_0 and ρ denote the material and spatial mass densities, respectively. In terms of Lie derivatives, conservation of mass can be written as $\mathbf{L}_{\mathbf{v}} \rho = 0$ [68].

Constitutive equations. In nonlinear elasticity, the energy function (per unit undeformed volume) of an inhomogeneous anisotropic hyperelastic material at a material point X is written in the following general form

$$W = \hat{W}(X, \mathbf{C}^b, \mathbf{G}, \xi_1, \dots, \xi_n), \quad (2.10)$$

where $\xi_i, i = 1, \dots, n$ are a collection of the so called *structural tensors* characterizing the material symmetry group at the point X . The inclusion of the structural tensors, along with \mathbf{C}^b in the energy function as shown in (2.10) constructs an isotropic function, i.e., it is invariant under the orthogonal group [61]. Therefore, (2.10) can be treated as the energy function of an isotropic material, and hence, the second Piola-Kirchhoff stress tensor is given as

$$\mathbf{S} = 2 \frac{\partial \hat{W}}{\partial \mathbf{C}^b}. \quad (2.11)$$

Alternatively, by the Doyle-Ericksen formula [22], the Cauchy, the first Piola-Kirchhoff, and the convected stress tensors are expressed as

$$\boldsymbol{\sigma} = \frac{2}{J} \frac{\partial \hat{W}}{\partial \mathbf{g}}, \quad \mathbf{P} = \mathbf{g}^\sharp \frac{\partial \hat{W}}{\partial \mathbf{F}}, \quad \boldsymbol{\Sigma} = \frac{2}{J} \frac{\partial \hat{W}}{\partial \mathbf{C}^b}, \quad (2.12)$$

where, with a slight abuse of notation, one may write

$$\begin{aligned} \hat{W}(x, \mathbf{G} \circ \varphi^{-1}, \mathbf{g}, \mathbf{F}, \xi_1 \circ \varphi^{-1}, \dots, \xi_n \circ \varphi^{-1}) &= \hat{W}(X, \mathbf{G}, \mathbf{g} \circ \varphi, \mathbf{F}, \xi_1, \dots, \xi_n) \\ &= \hat{W}(X, \mathbf{G}, \mathbf{C}^b, \xi_1, \dots, \xi_n). \end{aligned} \quad (2.13)$$

According to Hilbert's theorem, for any finite number of tensors, there exists a finite number of isotropic invariants forming a basis called *integrity basis* for the space of isotropic invariants of the collection of tensors.⁸ Thus, if $I_j, j = 1, \dots, m$, form an integrity basis for the set of tensors in (2.10), one has $W = W(X, I_1, \dots, I_m)$. Hence, using (2.11), one obtains

$$\mathbf{S} = \sum_{j=1}^m 2W_{I_j} \frac{\partial I_j}{\partial \mathbf{C}^b}, \quad W_{I_j} := \frac{\partial W}{\partial I_j}, \quad j = 1, \dots, m. \quad (2.14)$$

If the material is isotropic, i.e., $W = W(X, I_1, I_2, I_3)$, where $I_1 = \text{tr } \mathbf{C}$, $I_2 = \det \mathbf{C} \text{ tr } \mathbf{C}^{-1}$, and $I_3 = \det \mathbf{C}$ are the principal invariants of the right Cauchy-Green deformation tensor, it follows from (2.14) that

$$\mathbf{S} = 2 \left\{ W_{I_1} \mathbf{G}^\sharp + W_{I_2} (I_2 \mathbf{C}^{-1} - I_3 \mathbf{C}^{-2}) + W_{I_3} I_3 \mathbf{C}^{-1} \right\}. \quad (2.15)$$

⁸ See [109] for a detailed discussion on integrity basis for a finite set of tensors.

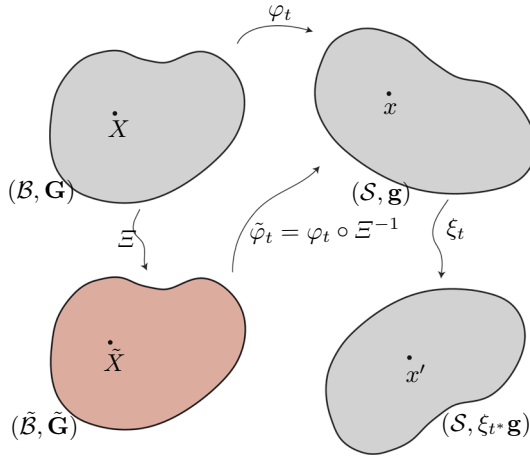


Fig. 3. Motion of a nonlinear elastic body and spatial and material changes of frame. ξ_t is a time-dependent spatial change of frame and \mathcal{E} is a time-independent material (referential) change of frame

2.1. Spatial Covariance of the Governing Equations of Nonlinear Elasticity

It turns out that in continuum mechanics (and even discrete systems) one can obtain all the balance laws using the energy balance and postulating its invariance under some groups of transformations. This idea was introduced by GREEN and RIVLIN [39] in the case of Euclidean ambient spaces and was later extended to manifolds by HUGHES and MARSDEN [52]. See also [68,103,121,124–127] for applications of covariance ideas in different continuous and discrete systems.

Consider an arbitrary time-dependent spatial diffeomorphism $\xi_t : S \rightarrow S$ (see Fig. 3). Denoting the fields in the transformed configuration by primes, we know that [68,127]

$$R' = R, \quad H' = H, \quad \rho'_0 = \rho_0, \quad \mathbf{T}' = \xi_{t*} \mathbf{T}, \quad \mathbf{V}' = \xi_{t*} \mathbf{V} + \mathbf{w} \circ \varphi_t, \quad (2.16)$$

where $\mathbf{w} = \frac{\partial}{\partial t} \xi_t \circ \varphi_t$ is the velocity of the change of frame and \mathbf{T} is the traction vector. Note that $\mathbf{A} = \nabla_{\mathbf{V}}^{\mathbf{g}} \mathbf{V}$, and hence

$$\begin{aligned} \mathbf{A}' &= \nabla_{\mathbf{V}'}^{\mathbf{g}'} \mathbf{V}' = \nabla_{\xi_{t*} \mathbf{V} + \mathbf{w} \circ \varphi_t}^{\xi_t * \mathbf{g}} (\xi_{t*} \mathbf{V} + \mathbf{w} \circ \varphi_t) \\ &= \xi_{t*} (\nabla_{\mathbf{V}}^{\mathbf{g}} \mathbf{V} + \nabla_{\mathbf{W}}^{\mathbf{g}} \mathbf{W} + \nabla_{\mathbf{V}}^{\mathbf{g}} \mathbf{W} + \nabla_{\mathbf{W}}^{\mathbf{g}} \mathbf{V}) \\ &= \xi_{t*} (\mathbf{A} + \nabla_{\mathbf{W}}^{\mathbf{g}} \mathbf{W} + 2 \nabla_{\mathbf{V}}^{\mathbf{g}} \mathbf{W} + [\mathbf{W}, \mathbf{V}]). \end{aligned} \quad (2.17)$$

It is assumed that body forces are transformed such that [68] $\mathbf{B}' - \mathbf{A}' = \xi_{t*} (\mathbf{B} - \mathbf{A})$. Similar transformations hold for the spatial quantities. It can be shown that [68, 127]

$$\begin{aligned} \operatorname{div}' \boldsymbol{\sigma}' + \rho' \mathbf{b}' - \rho' \mathbf{a}' &= \xi_{t*} (\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} - \rho \mathbf{a}), \\ \operatorname{Div}' \mathbf{P}' + \rho'_0 \mathbf{B}' - \rho'_0 \mathbf{A}' &= \xi_{t*} (\operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} - \rho_0 \mathbf{A}), \end{aligned} \quad (2.18)$$

i.e., the balance of linear momentum is spatially covariant provided that the Doyle-Ericksen formula (2.12)₁ is satisfied. This in turn restricts the body to be isotropic. A way out to have a covariant elasticity theory for anisotropic bodies is to include structural tensors in the energy function. Note that the balance of angular momentum (symmetry of the Cauchy stress) is covariant as well, i.e., $\boldsymbol{\sigma}'^T = \boldsymbol{\sigma}'$.

The main idea in covariant elasticity is that in the ambient space the physical laws (here, the balance of energy or the first law of thermodynamics) should be observer-independent.⁹ YAVARI et al. [127] investigated the possibility of the covariance of the energy balance under diffeomorphisms of the reference configuration. Their motivation was to see if there was any connection between material covariance and balance of the so-called configurational forces. It was observed that the energy balance is not invariant under material diffeomorphisms, in general. They obtained a transformation equation for the balance of energy. This is discussed next.

2.2. Material Covariance of the Governing Equations of Nonlinear Elasticity

Note that a spatial diffeomorphism is nothing but a change of observer. In other words, given an elastic body in (dynamic) equilibrium, a spatial diffeomorphism is simply representing the same configuration in another frame. For cloaking applications, one needs to know what the elastic properties of the cloak should be in order to make a given cavity invisible to elastic waves. This means that given a reference configuration, one should be looking at material (referential) diffeomorphisms. This is the motivation for the following discussion.

In this section, we discuss the transformation of the governing equations of nonlinear elasticity for an anisotropic and inhomogenous body under a time-independent material diffeomorphism. More specifically, consider a diffeomorphism $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$. We use the coordinate charts $\{X^A\}$, $\{\tilde{X}^{\tilde{A}}\}$, and $\{x^a\}$ for the reference configuration, the transformed reference configuration, and the ambient space, respectively, see Fig. 3. The deformation map $\varphi_t : \mathcal{B} \rightarrow \mathcal{S}$ is transformed under the material diffeomorphism to $\tilde{\varphi}_t : \tilde{\mathcal{B}} \rightarrow \mathcal{S}$, where $\tilde{\varphi}_t = \varphi_t \circ \mathcal{E}^{-1}$. Material velocity with respect to the new reference configuration reads

$$\tilde{\mathbf{V}}(\tilde{X}, t) = \frac{\partial}{\partial t} \tilde{\varphi}(\tilde{X}, t) = \frac{\partial}{\partial t} \varphi(\mathcal{E}^{-1}(\tilde{X}), t) = \mathbf{V}(\mathcal{E}^{-1}(\tilde{X}), t). \quad (2.19)$$

Thus, $\tilde{\mathbf{V}} = \mathbf{V} \circ \mathcal{E}^{-1}$. Material acceleration with respect to the new reference configuration reads

$$\begin{aligned} \tilde{\mathbf{A}}(\tilde{X}, t) &= \nabla_{\frac{\partial}{\partial t}}^{\mathbf{g}} \tilde{\mathbf{V}}(\tilde{X}, t) = \nabla_{\frac{\partial}{\partial t}}^{\mathbf{g}} \mathbf{V}(\mathcal{E}^{-1}(\tilde{X}), t) \\ &= \nabla_{\frac{\partial}{\partial t}}^{\mathbf{g}} \mathbf{V}_t \circ \mathcal{E}^{-1}(\tilde{X}) = \mathbf{A}_t \circ \mathcal{E}^{-1}(\tilde{X}). \end{aligned} \quad (2.20)$$

⁹ This is also known as the principle of material objectivity (see, e.g., [77]).

The deformation gradient $\mathbf{F} = T\varphi_t : T_X\mathcal{B} \rightarrow T_{\varphi_t(X)}\mathcal{S}$, under \mathcal{E} is transformed to $\tilde{\mathbf{F}} = T\tilde{\varphi}_t : T_{\tilde{X}}\mathcal{B} \rightarrow T_{\tilde{\varphi}_t(\tilde{X})}\mathcal{S} = T_{\varphi_t(X)}\mathcal{S}$, where

$$\tilde{\mathbf{F}} = T\tilde{\varphi}_t = T(\varphi_t \circ \mathcal{E}^{-1}) = T\varphi_t \circ T\mathcal{E}^{-1} = \mathbf{F} \circ \tilde{\mathbf{F}}^{-1}, \quad (2.21)$$

and $\tilde{\mathbf{F}} = T\mathcal{E}$. Moreover, note that $\tilde{\mathbf{G}} = \mathcal{E}_*\mathbf{G}$. In coordinates, $\tilde{G}_{\tilde{A}\tilde{B}} = (\tilde{F}^{-1})^A_{\tilde{A}} (\tilde{F}^{-1})^B_{\tilde{B}} G_{AB}$. The right Cauchy-Green deformation tensor is written as $\mathbf{C}^b = \varphi_t^*\mathbf{g}$. Hence, one notes that

$$\tilde{\mathbf{C}}^b = \tilde{\varphi}_t^*\mathbf{g} = (\varphi_t \circ \mathcal{E}^{-1})^*\mathbf{g} = \mathcal{E}_* \circ \varphi_t^*\mathbf{g} = \mathcal{E}_*(\varphi_t^*\mathbf{g}) = \mathcal{E}_*\mathbf{C}^b. \quad (2.22)$$

In coordinates, $\tilde{C}_{\tilde{A}\tilde{B}} = (\tilde{F}^{-1})^A_{\tilde{A}} (\tilde{F}^{-1})^B_{\tilde{B}} C_{AB}$.

Material symmetry. The material symmetry group \mathcal{G}_X associated with an elastic body made of a simple material¹⁰ with the response function \mathcal{R} ¹¹ at a point X with respect to the reference configuration $(\mathcal{B}, \mathbf{G})$ is defined as

$$\mathcal{R}(\mathbf{FK}) = \mathcal{R}(\mathbf{F}), \quad \forall \mathbf{K} \in \mathcal{G}_X, \quad (2.23)$$

for all deformation gradients \mathbf{F} , where $\mathbf{K} : T_X\mathcal{B} \rightarrow T_X\mathcal{B}$ is an invertible linear transformation. For a hyperelastic solid, objectivity requires that the energy function depend on the deformation through the right Cauchy-Green deformation tensor \mathbf{C}^b , i.e., $W = W(X, \mathbf{C}^b, \mathbf{G})$ at a material point X . Therefore, the material symmetry group \mathcal{G}_X for a hyperelastic solid is defined to be the subgroup of \mathbf{G} -orthogonal transformations $\text{Orth}(\mathbf{G})$ such that [24]¹²

$$W(X, \mathbf{Q}^{-*}\mathbf{C}^b\mathbf{Q}^{-1}, \mathbf{G}) = W(X, \mathbf{C}^b, \mathbf{G}), \quad \forall \mathbf{Q} \in \mathcal{G}_X \leq \text{Orth}(\mathbf{G}). \quad (2.24)$$

The symmetry group of the material relative to a transformed reference configuration $(\tilde{\mathcal{B}}, \tilde{\mathbf{G}})$ is denoted by $\tilde{\mathcal{G}}$. According to Noll's rule [14–16, 77], the relation between the two groups is

$$\tilde{\mathcal{G}} = \mathcal{E}_*\mathcal{G} = \tilde{\mathbf{F}}\mathcal{G}\tilde{\mathbf{F}}^{-1}. \quad (2.25)$$

In other words, in the sense of group theory, at each material point X , \mathcal{G} and $\tilde{\mathcal{G}}$ are conjugate subgroups of the general linear group, and hence, isomorphic ($\tilde{\mathcal{G}} \cong \mathcal{G}$). Note that if $\tilde{\mathbf{F}} \in \mathcal{G}$, then $\mathcal{G} = \tilde{\mathcal{G}}$. Also, the symmetry group is not affected by a change of reference configuration ($\tilde{\mathcal{G}} = \mathcal{G}$) if $\tilde{\mathbf{F}} = \alpha\mathbf{I}$ (pure dilatation) for some positive scalar α [81]. More generally, $\tilde{\mathcal{G}} = \mathcal{G}$ if and only if $\tilde{\mathbf{F}}$ belongs to the

¹⁰ The response of a *simple* material at any material point depends only on the first deformation gradient (and its evolution) at that point [77].

¹¹ Here we assume that \mathcal{R} is the energy function. Response function may be any measure of stress as well.

¹² Note that $\text{Orth}(\mathbf{G}) = \{\mathbf{Q} : T_X\mathcal{B} \rightarrow T_X\mathcal{B} \mid \mathbf{Q}^\top = \mathbf{Q}^{-1}\}$. We use the notation $\mathcal{G} \leq \mathcal{H}$ when \mathcal{G} is a subgroup of \mathcal{H} .

normalizer group of \mathcal{G} within the general linear group.¹³ It is straightforward to see that (2.25) is satisfied if and only if it holds for all the generators of the group \mathcal{G} . Therefore, if \mathcal{G} is finitely generated, the elements of the generating sets of \mathcal{G} and $\tilde{\mathcal{G}}$ denoted by $\{\mathbf{Q}_1, \dots, \mathbf{Q}_m\}$ and $\{\tilde{\mathbf{Q}}_1, \dots, \tilde{\mathbf{Q}}_m\}$, respectively, are related as $\tilde{\mathbf{Q}}_j = \tilde{\mathbf{F}}\mathbf{Q}_j\tilde{\mathbf{F}}^{-1}$, $j = 1, \dots, m$. The symmetry group can be characterized using a finite collection of structural tensors¹⁴ ξ_i of order μ_i , $i = 1, \dots, n$, as follows [7,61,65,69,130,131]

$$\mathbf{Q} \in \mathcal{G} \leq \text{Orth}(\mathbf{G}) \iff \langle \mathbf{Q} \rangle_{\mu_1} \xi_1 = \xi_1, \dots, \langle \mathbf{Q} \rangle_{\mu_n} \xi_n = \xi_n, \quad (2.26)$$

where the μ -th power Kronecker product $\langle \mathbf{Q} \rangle_{\mu}$ of a \mathbf{G} -orthogonal transformation \mathbf{Q} for any μ -th order tensor ξ is defined as¹⁵

$$(\langle \mathbf{Q} \rangle_{\mu} \xi)^{\bar{A}_1 \dots \bar{A}_{\mu}} = Q^{\bar{A}_1}_{A_1} \dots Q^{\bar{A}_{\mu}}_{A_{\mu}} \xi^{A_1 \dots A_{\mu}}. \quad (2.27)$$

Note that (2.26) suggests that the material symmetry group \mathcal{G} is the invariance group of the set of the structural tensors ξ_i , $i = 1, \dots, n$. Using (2.25) and (2.26), one obtains the following relation for the transformed structural tensors under the material diffeomorphism, which characterize the transformed symmetry group $\tilde{\mathcal{G}}$

$$\tilde{\mathbf{Q}} \in \tilde{\mathcal{G}} \leq \text{Orth}(\tilde{\mathbf{G}}) \iff \langle \tilde{\mathbf{Q}} \rangle_{\mu_1} \tilde{\xi}_1 = \tilde{\xi}_1, \dots, \langle \tilde{\mathbf{Q}} \rangle_{\mu_n} \tilde{\xi}_n = \tilde{\xi}_n, \quad (2.28)$$

where $\tilde{\mathbf{Q}} = \mathcal{E}_* \mathbf{Q} = \tilde{\mathbf{F}} \mathbf{Q} \tilde{\mathbf{F}}^{-1}$ and $\tilde{\xi}_i = \mathcal{E}_* \xi_i$, $i = 1, \dots, n$. Therefore, the type of the symmetry group of the material is preserved under a material change of frame \mathcal{E} .

Balance of energy. YAVARI et al. [127] showed that the balance of energy is not invariant under an arbitrary time-dependent material diffeomorphism. However, it can be shown that the balance of energy is always invariant under time-independent material diffeomorphisms. The strain energy function satisfies (2.24) if and only if it is represented as an isotropic function (invariant under the special orthogonal group) of the structural tensors and \mathbf{C}^b at a material point X (see (2.29)) [6,24,128]. Thus, we write the general form of the energy function (per unit undeformed volume) of an inhomogeneous anisotropic hyperelastic material with a set of structural tensors ξ_i , $i = 1, \dots, n$ (cf. (2.26)) characterizing the material symmetry group (at a referential point X) as

$$W = \hat{W}(X, \mathbf{C}^b, \mathbf{G}, \xi_1, \dots, \xi_n). \quad (2.29)$$

¹³ The normalizer group $\mathcal{N}_{\mathcal{G}}(\mathcal{Q})$ of a subgroup \mathcal{Q} of \mathcal{G} ($\mathcal{Q} \leq \mathcal{G}$) is defined as $\mathcal{N}_{\mathcal{G}}(\mathcal{Q}) = \{g_i \in \mathcal{G} : g_i \mathcal{Q} g_i^{-1} = \mathcal{Q}\}$.

¹⁴ Note that such a collection forms a basis for the space of tensors that are invariant under the action of \mathcal{G} .

¹⁵ Note that $\langle \mathbf{Q} \rangle_m (\mathbf{v}_1 \otimes \dots \otimes \mathbf{v}_m) = \mathbf{Q} \mathbf{v}_1 \otimes \dots \otimes \mathbf{Q} \mathbf{v}_m$, where $\mathbf{v}_i \in T_X \mathcal{B}$, $i = 1, \dots, m$, are arbitrary vectors.

Similarly, using (2.22) and (2.28), one obtains

$$\begin{aligned}\tilde{W} &= \tilde{W}(\tilde{X}, \tilde{\mathbf{C}}^b, \tilde{\mathbf{G}}, \tilde{\xi}_1, \dots, \tilde{\xi}_n) \\ &= \hat{W}(\mathcal{E}(X), \mathcal{E}_* \mathbf{C}^b, \mathcal{E}_* \mathbf{G}, \mathcal{E}_* \xi_1, \dots, \mathcal{E}_* \xi_n) = \mathcal{E}_* W.\end{aligned}\tag{2.30}$$

Using the theory of invariants [108, 109], the energy function can be represented in terms of a finite set of isotropic invariants. It can be shown that these invariants do not change under material diffeomorphisms, which in turn implies the material covariance of the energy function, i.e., (see also [65])

$$\tilde{W} = W \circ \mathcal{E}^{-1}.\tag{2.31}$$

Balance of linear momentum. The balance of linear momentum in terms of the first Piola-Kirchhoff stress reads $\text{Div } \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}$. We now examine the transformed first and second Piola-Kirchhoff stress tensors, denoted, respectively, by $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{S}}$, and the material acceleration $\tilde{\mathbf{A}}$ pertaining to the transformed configuration $(\tilde{\mathcal{B}}, \tilde{\mathbf{G}})$ with the deformation $\tilde{\varphi}_t = \varphi_t \circ \mathcal{E}^{-1}$. The second Piola-Kirchhoff stress tensor is written as $\mathbf{S} = 2 \frac{\partial \hat{W}}{\partial \mathbf{C}^b}$. Therefore, using (2.30), one obtains

$$\tilde{\mathbf{S}} = 2 \frac{\partial \tilde{W}}{\partial \tilde{\mathbf{C}}^b} = 2 \frac{\partial (\mathcal{E}_* \hat{W})}{\partial (\mathcal{E}_* \mathbf{C}^b)}.\tag{2.32}$$

Moreover, employing (2.31) and (2.32), one can write

$$\begin{aligned}(\mathcal{E}_* \mathbf{S})^{\tilde{A}\tilde{B}} &= \tilde{F}^{\tilde{A}}_{\tilde{A}} \tilde{F}^{\tilde{B}}_{\tilde{B}} S^{AB} \circ \mathcal{E}^{-1} = 2 \tilde{F}^{\tilde{A}}_{\tilde{A}} \tilde{F}^{\tilde{B}}_{\tilde{B}} \frac{\partial \hat{W}}{\partial C_{AB}} \circ \mathcal{E}^{-1} \\ &= 2 \tilde{F}^{\tilde{A}}_{\tilde{A}} \tilde{F}^{\tilde{B}}_{\tilde{B}} \frac{\partial \tilde{W}}{\partial \tilde{C}_{\tilde{D}\tilde{H}}} \frac{\partial \tilde{C}_{\tilde{D}\tilde{H}} \circ \mathcal{E}}{\partial C_{AB}} \\ &= 2 \tilde{F}^{\tilde{A}}_{\tilde{A}} \tilde{F}^{\tilde{B}}_{\tilde{B}} (\tilde{F}^{-1})^A_{\tilde{D}} (\tilde{F}^{-1})^B_{\tilde{H}} \frac{\partial \tilde{W}}{\partial \tilde{C}_{\tilde{D}\tilde{H}}} = 2 \frac{\partial \tilde{W}}{\partial \tilde{C}_{\tilde{A}\tilde{B}}} \\ &= \tilde{\mathcal{S}}^{\tilde{A}\tilde{B}}.\end{aligned}\tag{2.33}$$

Hence, it immediately follows that $\tilde{\mathbf{S}} = \mathcal{E}_* \mathbf{S}$. Under the diffeomorphism $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$, the two-point tensor \mathbf{P} is transformed to $\mathcal{E}_* \mathbf{P}$, where in components

$$\tilde{P}^{a\tilde{A}} = \frac{\partial \tilde{X}^{\tilde{A}}}{\partial X^a} P^{aA} = \tilde{F}^{\tilde{A}}_{\tilde{A}} P^{aA} \circ \mathcal{E}^{-1}.\tag{2.34}$$

We note that, using (2.21), one has $\tilde{\mathbf{P}} = \tilde{\mathbf{F}}\tilde{\mathbf{S}} = (\mathbf{F} \circ \tilde{\mathbf{F}}^{-1})\tilde{\mathbf{S}}$. Therefore, in components (cf. (2.34) and $\tilde{\mathbf{S}} = \mathcal{E}_* \mathbf{S}$), we have

$$\begin{aligned}\tilde{P}^{a\tilde{A}} &= F^a_{\tilde{A}} (\tilde{F}^{-1})^{\tilde{A}}_{\tilde{B}} \tilde{S}^{\tilde{B}\tilde{A}} = F^a_{\tilde{A}} (\tilde{F}^{-1})^{\tilde{A}}_{\tilde{B}} \tilde{F}^{\tilde{B}}_{\tilde{C}} \tilde{F}^{\tilde{A}}_{\tilde{B}} S^{CB} \circ \mathcal{E}^{-1} \\ &= F^a_{\tilde{A}} \tilde{F}^{\tilde{A}}_{\tilde{B}} S^{AB} \circ \mathcal{E}^{-1} = \tilde{F}^{\tilde{A}}_{\tilde{B}} P^{aB} \circ \mathcal{E}^{-1} = (\mathcal{E}_* P)^{a\tilde{A}},\end{aligned}\tag{2.35}$$

that is, $\tilde{\mathbf{P}} = \mathcal{E}_* \mathbf{P}$. The divergence term has components $(\text{Div } \mathbf{P})^a = \partial P^{aA} / \partial X^A + \Gamma^A_{AB} P^{aB} + (\gamma^a_{bc} \circ \varphi_t) F^b_A P^{cA}$. Thus, $(\widetilde{\text{Div}} \tilde{\mathbf{P}})^a = \partial \tilde{P}^{a\tilde{A}} / \partial \tilde{X}^{\tilde{A}} + \tilde{\Gamma}^{\tilde{A}}_{\tilde{A}\tilde{B}} \tilde{P}^{a\tilde{B}} + (\gamma^a_{bc} \circ \tilde{\varphi}_t) \tilde{F}^b_{\tilde{A}} \tilde{P}^{c\tilde{A}}$. Note that $\tilde{\mathbf{F}} = T \tilde{\varphi}_t = T(\varphi_t \circ \mathcal{E}^{-1}) = T\varphi_t \circ (T\mathcal{E})^{-1} = \mathbf{F} \circ \tilde{\mathbf{F}}^{-1}$. Thus, $\tilde{\mathbf{F}} \circ \tilde{\mathbf{P}} = \mathbf{F} \circ \mathbf{P} \circ \mathcal{E}^{-1}$. It can be shown that $\partial \tilde{P}^{a\tilde{A}} / \partial \tilde{X}^{\tilde{A}} = \partial P^{aA} / \partial X^A \circ \mathcal{E}^{-1}$. Using the transformation of connection coefficients, it is straightforward to show that $\tilde{\Gamma}^{\tilde{A}}_{\tilde{A}\tilde{B}} = (\tilde{F}^{-1})^B_{\tilde{B}} \Gamma^A_{AB} \circ \mathcal{E}^{-1}$. Therefore, $\widetilde{\text{Div}} \tilde{\mathbf{P}} = \text{Div } \mathbf{P} \circ \mathcal{E}^{-1}$. From conservation of mass $\rho_0 dV = \tilde{\rho}_0 d\tilde{V} = \tilde{\rho}_0 J_{\mathcal{E}} dV$, where

$$J_{\mathcal{E}} = \sqrt{\frac{\det \tilde{\mathbf{G}}}{\det \mathbf{G}}} \det \tilde{\mathbf{F}}. \quad (2.36)$$

Note that $\tilde{\mathbf{G}} = \mathcal{E}_* \mathbf{G}$, and $\det \tilde{\mathbf{G}} = \det \mathbf{G} (\det \tilde{\mathbf{F}})^{-2}$. Hence, $J_{\mathcal{E}} = 1$, and therefore, $\tilde{\rho}_0 = \rho_0 \circ \mathcal{E}^{-1}$. For the spatial mass density, $\tilde{\rho} = \rho$.

Body force is a vector field in the ambient space, i.e., $\mathbf{B}_X \in T_{\varphi_t(X)} \mathcal{S}$, and hence, $\tilde{\mathbf{B}} = \mathcal{E}_* \mathbf{B} = \mathbf{B} \circ \mathcal{E}^{-1}$. For a time-independent material diffeomorphism, $\tilde{\mathbf{V}} = \mathbf{V} \circ \mathcal{E}^{-1}$, and hence, acceleration transforms as $\tilde{\mathbf{A}} = \mathbf{A} \circ \mathcal{E}^{-1}$. Therefore, the balance of linear momentum is invariant under the diffeomorphism $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$, i.e.,

$$\begin{aligned} \widetilde{\text{Div}} \tilde{\mathbf{P}} + \tilde{\rho}_0 \tilde{\mathbf{B}} - \tilde{\rho}_0 \tilde{\mathbf{A}} &= \mathcal{E}_* (\text{Div } \mathbf{P} + \rho_0 \mathbf{B} - \rho_0 \mathbf{A}) \\ &= (\text{Div } \mathbf{P} + \rho_0 \mathbf{B} - \rho_0 \mathbf{A}) \circ \mathcal{E}^{-1}. \end{aligned} \quad (2.37)$$

This is identical to what MAZZUCATO and RACHELE [69] proved.

Balance of angular momentum. The balance of angular momentum in terms of the first Piola-Kirchhoff stress reads $P^{aA} F^b_A - P^{bA} F^a_A = 0$. It is straightforward to show that $\tilde{P}^{a\tilde{A}} \tilde{F}^b_{\tilde{A}} = P^{aA} F^b_A \circ \mathcal{E}^{-1}$. Therefore, $\tilde{P}^{a\tilde{A}} \tilde{F}^b_{\tilde{A}} - \tilde{P}^{b\tilde{A}} \tilde{F}^a_{\tilde{A}} = (P^{aA} F^b_A - P^{bA} F^a_A) \circ \mathcal{E}^{-1}$, i.e., the balance of angular momentum is invariant under material diffeomorphisms.

Conservation of mass. Conservation of mass in local form reads $\rho_0 - J\rho \circ \varphi = 0$, where the Jacobian is written as $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}$. Thus

$$\tilde{J} = \sqrt{\frac{\det \tilde{\mathbf{g}}}{\det \tilde{\mathbf{G}}}} \det \tilde{\mathbf{F}} = J \circ \mathcal{E}^{-1}. \quad (2.38)$$

Therefore, $\tilde{\rho}_0 - \tilde{J} \tilde{\rho} \circ \tilde{\varphi} = \mathcal{E}_* (\rho_0 - J\rho \circ \varphi) = (\rho_0 - J\rho \circ \varphi) \circ \mathcal{E}^{-1}$, i.e., conservation of mass is materially covariant.

We next observe that under the diffeomorphism $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$, the Cauchy stress tensor remains unchanged and is transformed as $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \mathcal{E}^{-1}$. To see this, note that (cf. (2.34) and (2.38))

$$\tilde{\boldsymbol{\sigma}} = \tilde{J}^{-1} \tilde{\mathbf{F}} \tilde{\mathbf{P}}^* = (J^{-1} \mathbf{F} \tilde{\mathbf{F}}^{-1} \tilde{\mathbf{F}} \mathbf{P}^*) \circ \mathcal{E}^{-1} = \boldsymbol{\sigma} \circ \mathcal{E}^{-1}. \quad (2.39)$$

3. Linearized Elastodynamics

Classical linear elasticity can be derived from nonlinear elasticity if one linearizes the governing equations of nonlinear elasticity with respect to a stress-free equilibrium configuration. More generally, nonlinear elasticity can be linearized with respect to any stressed and finitely-deformed (in either static or dynamic equilibrium) configuration. This is the so-called small-on-large theory of GREEN et al. [38]. In the language of geometric mechanics, MARSDEN and HUGHES [68] presented a geometric linearization of nonlinear elasticity. In particular, in their formulation elastic constants are properly defined in terms of two-point tensors. See also [126] for a discussion on covariance in linearized elasticity.

Variation of a map $\varphi_t : \mathcal{B} \rightarrow \mathcal{S}$ is a map $\Phi_t : \mathcal{B} \times I$, where $I = (-a, a)$ is some interval, such that $\Phi_t(X, 0) = \hat{\varphi}_t$, which we call the reference motion. Let us denote $\varphi_{t,\epsilon}(X) = \Phi_t(X, \epsilon)$, and hence, $\varphi_{t,0} = \hat{\varphi}_t$. The variation field is defined as

$$\delta\varphi_t(X) = \mathbf{U}(X, t) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \varphi_{t,\epsilon}(X). \tag{3.1}$$

The spatial variation (displacement) field is defined as $\mathbf{u} = \mathbf{U} \circ \hat{\varphi}_t^{-1}$ or $u^a(x, t) = U^a(X, t)$. Note that $\delta\varphi \in \Gamma(\hat{\varphi}^{-1}TS)$ (see the appendix for the definition of the induced bundle). The tangent space $T_{(X,\epsilon)}(\mathcal{B} \times I)$ of the product manifold $\mathcal{B} \times I$ at (X, ϵ) is identified with $T_X\mathcal{B} \otimes T_\epsilon I$.

Linearization of the material velocity and acceleration. Material velocity is defined as $\mathbf{V}_\epsilon(X, t) = \frac{\partial\varphi_{t,\epsilon}(X)}{\partial t}$. Note that $\mathbf{V}_\epsilon(X, t) \in T_{\varphi_{t,\epsilon}(X)}\mathcal{S}$, i.e., for different values of ϵ , velocity lies in different tangent spaces, and hence, a covariant derivative along the curve $\epsilon \mapsto \varphi_{t,\epsilon}(X)$ should be used to find the linearization of velocity [68]. Therefore

$$\begin{aligned} \delta\mathbf{V}(X, t) &= \nabla_{\frac{\partial}{\partial\epsilon}} \left. \frac{\partial\varphi_{t,\epsilon}(X)}{\partial t} \right|_{\epsilon=0} = \nabla_{\frac{\partial}{\partial t}} \left. \frac{\partial\varphi_{t,\epsilon}(X)}{\partial\epsilon} \right|_{\epsilon=0} \\ &= \nabla_t \delta\varphi_t(X) = D_t \mathbf{U}(X, t) = \nabla_{\mathbf{V}} \mathbf{U} =: \dot{\mathbf{U}}, \end{aligned} \tag{3.2}$$

i.e., variation of the velocity field is the covariant time derivative of the displacement field. In the above calculation in the second equality the symmetry lemma of Riemannian geometry [58] was used. In components, $(D_t \mathbf{U}(X, t))^a = \dot{U}^a = \frac{\partial U^a}{\partial t} + \gamma^a_{bc} \dot{V}^b U^c$. Note that $\frac{\partial U^a}{\partial t} = \frac{\partial u^a}{\partial t} + \frac{\partial u^a}{\partial x^b} \frac{\partial x^b}{\partial t}$. Therefore, $\delta\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} + \nabla_{\dot{\mathbf{v}}} \mathbf{u} =: \dot{\mathbf{u}}$.

Material acceleration is the covariant time derivative of velocity, i.e., $\mathbf{A} = D_t \mathbf{V} = \nabla_{\mathbf{V}} \mathbf{V}$, which in coordinates reads [68] $A^a = \frac{\partial V^a}{\partial t} + \gamma^a_{bc} V^b V^c$. Therefore

$$\begin{aligned} \delta\mathbf{A}(X, t) &= \nabla_{\frac{\partial}{\partial\epsilon}} \nabla_{\frac{\partial}{\partial t}} \left. \frac{\partial\varphi_{t,\epsilon}(X)}{\partial t} \right|_{\epsilon=0} \\ &= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial\epsilon}} \left. \frac{\partial\varphi_{t,\epsilon}(X)}{\partial t} \right|_{\epsilon=0} + \nabla_{[\frac{\partial}{\partial\epsilon}, \frac{\partial}{\partial t}]} \left. \frac{\partial\varphi_{t,\epsilon}(X)}{\partial t} \right|_{\epsilon=0} \\ &\quad + \mathcal{R}_{\mathbf{g}} \left(\frac{\partial}{\partial\epsilon}, \frac{\partial}{\partial t} \right) \left. \frac{\partial\varphi_{t,\epsilon}(X)}{\partial t} \right|_{\epsilon=0} \\ &= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial\epsilon}} \left. \frac{\partial\varphi_{t,\epsilon}(X)}{\partial\epsilon} \right|_{\epsilon=0} + \nabla_{[\mathbf{U}, \dot{\mathbf{V}}]} \dot{\mathbf{V}} + \mathcal{R}_{\mathbf{g}}(\mathbf{U}, \dot{\mathbf{V}}, \dot{\mathbf{V}}) \end{aligned}$$

$$\begin{aligned}
 &= D_t D_t \mathbf{U} + \nabla_{[\mathbf{U}, \dot{\mathbf{V}}]} \dot{\mathbf{V}} + \mathcal{R}_{\mathbf{g}}(\mathbf{U}, \dot{\mathbf{V}}, \dot{\mathbf{V}}) \\
 &= \nabla_{\dot{\mathbf{V}}} \nabla_{\dot{\mathbf{V}}} \mathbf{U} + \nabla_{[\mathbf{U}, \dot{\mathbf{V}}]} \dot{\mathbf{V}} + \mathcal{R}_{\mathbf{g}}(\mathbf{U}, \dot{\mathbf{V}}, \dot{\mathbf{V}}) \\
 &= \ddot{\mathbf{U}} + \nabla_{[\mathbf{U}, \dot{\mathbf{V}}]} \dot{\mathbf{V}} + \mathcal{R}_{\mathbf{g}}(\mathbf{U}, \dot{\mathbf{V}}, \dot{\mathbf{V}}), \tag{3.3}
 \end{aligned}$$

where $\mathcal{R}_{\mathbf{g}}$ is the curvature tensor of the metric \mathbf{g} and $\ddot{\mathbf{U}} = D_t D_t \mathbf{U}$ is the second covariant time derivative of the displacement field. Note that $\delta \mathbf{a}_t = \delta \mathbf{A}_t \circ \dot{\phi}_t^{-1} = \ddot{\mathbf{U}} \circ \dot{\phi}_t^{-1} + \nabla_{[\mathbf{u}, \dot{\mathbf{v}}]}^{\mathbf{g}} \dot{\mathbf{v}} + \mathcal{R}_{\mathbf{g}}(\mathbf{u}, \dot{\mathbf{v}}, \dot{\mathbf{v}})$.

Linearization of the right Cauchy-Green strain, the Jacobian, and the deformation gradient. The right Cauchy-Green strain of the motion $\varphi_{t,\epsilon}$ is defined as $\mathbf{C}_\epsilon^b = \varphi_{t,\epsilon}^* \mathbf{g} \circ \varphi_{t,\epsilon}$. Note that $\mathbf{C}_\epsilon^b \in \Gamma(\mathcal{B}, T^* \mathcal{B} \otimes T^* \mathcal{B})$ for all ϵ , where $\Gamma(\mathcal{B}, T^* \mathcal{B} \otimes T^* \mathcal{B})$ is the set of $\binom{0}{2}$ -tensors on \mathcal{B} . Linearization of \mathbf{C}^b is calculated as

$$\begin{aligned}
 \delta \mathbf{C}^b &= \frac{d}{d\epsilon} \mathbf{C}_\epsilon^b \Big|_{\epsilon=0} = \frac{d}{d\epsilon} \varphi_{t,\epsilon}^* \mathbf{g} \Big|_{\epsilon=0} = \dot{\phi}_t^* (\mathbf{L}_{\mathbf{u}} \mathbf{g}) \\
 &= \dot{\phi}_t^* (\mathcal{L}_{\mathbf{u}} \mathbf{g}) = \dot{\phi}_t^* (\nabla^{\mathbf{g}} \mathbf{u}^b + (\nabla^{\mathbf{g}} \mathbf{u}^b)^*) = 2\dot{\phi}_t^* \epsilon, \tag{3.4}
 \end{aligned}$$

where $\mathbf{L}_{\mathbf{u}}$ is Lie derivative with respect to the vector field \mathbf{u} and $\mathcal{L}_{\mathbf{u}}$ is the autonomous Lie derivative with respect to \mathbf{u} (see the appendix). The linearized strain ϵ has the components $2\epsilon_{ab} = u_{a|b} + u_{b|a}$.

The Jacobian for the perturbed motion is written as

$$J_\epsilon = \sqrt{\frac{\det \mathbf{g} \circ \varphi_{t,\epsilon}}{\det \mathbf{G}}} \det \mathbf{F}_\epsilon = \sqrt{\frac{\det \mathbf{C}_\epsilon^b}{\det \mathbf{G}}}. \tag{3.5}$$

Hence

$$\delta J = \frac{d}{d\epsilon} J_\epsilon \Big|_{\epsilon=0} = \frac{1}{2\sqrt{\det \mathbf{G} \det \dot{\mathbf{C}}^b}} \frac{d}{d\epsilon} \det \mathbf{C}_\epsilon^b \Big|_{\epsilon=0}. \tag{3.6}$$

Using Jacobi's formula we know that

$$\frac{d}{d\epsilon} \det \mathbf{C}_\epsilon^b \Big|_{\epsilon=0} = \det \dot{\mathbf{C}}^b \operatorname{tr} \left[(\dot{\mathbf{C}}^b)^{-1} 2\dot{\phi}_t^* \epsilon \right]. \tag{3.7}$$

Note that $(\dot{\mathbf{C}}^b)^{-1} = \dot{\phi}_t^* \mathbf{g}^\sharp$, and hence, the right-hand side of (3.7) is simplified to read $(\det \dot{\mathbf{C}}^b) \mathbf{g}^\sharp : \epsilon$. Therefore, $\delta J = \dot{J} \mathbf{g}^\sharp : \epsilon = \dot{J} \operatorname{div}_{\mathbf{g}} \mathbf{u}$.

Deformation gradient of the motion $\varphi_{t,\epsilon}$ is defined as $\mathbf{F}_{t,\epsilon} = \frac{\partial \varphi_{t,\epsilon}}{\partial X}$. Consider the vector fields $\left(\frac{\partial}{\partial X^A}, 0 \right)$ and $\left(0, \frac{\partial}{\partial \epsilon} \right)$ on $\mathcal{B} \times I$, and note that

$$\left[\left(\frac{\partial}{\partial X^A}, 0 \right), \left(0, \frac{\partial}{\partial \epsilon} \right) \right] = 0. \tag{3.8}$$

Thus, from (A.8) one can write

$$\nabla_{\left(0, \frac{\partial}{\partial \epsilon}\right)} \varphi_{t,\epsilon*} \left(\frac{\partial}{\partial X^A}, 0 \right) = \nabla_{\left(\frac{\partial}{\partial X^A}, 0\right)} \varphi_{t,\epsilon*} \left(0, \frac{\partial}{\partial \epsilon} \right), \tag{3.9}$$

or

$$\nabla_{\frac{\partial}{\partial \epsilon}} \frac{\partial \varphi_\epsilon^a}{\partial X^A} = \nabla_{\frac{\partial}{\partial X^A}} \frac{\partial \varphi_\epsilon^a}{\partial \epsilon}. \quad (3.10)$$

Therefore, $\delta F^a{}_A = \varphi^a|_A = U^a|_A = F^b{}_A U^a|_b$. Note that $U^a|_b = \frac{\partial U^b}{\partial x^b} + \gamma^a{}_{bc} U^c$.

Linearization of the first and the second Piola-Kirchhoff stresses. For a hyperelastic solid, given an energy function $W = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g} \circ \varphi, \zeta_1, \dots, \zeta_n)$, the first Piola-Kirchhoff stress is given as $\mathbf{P} = \mathbf{g}^\sharp \frac{\partial W}{\partial \mathbf{F}}$. In components, $P^{aA} = g^{ab} \frac{\partial W}{\partial F^b{}_A}$. For the perturbed motion, $W_\epsilon = W(X, \mathbf{F}_\epsilon, \mathbf{G}, \mathbf{g} \circ \varphi_\epsilon, \zeta_1, \dots, \zeta_n)$, and hence

$$\begin{aligned} \delta \mathbf{P} &= \mathbf{g}^\sharp \nabla_{\frac{\partial}{\partial \epsilon}} \left. \frac{\partial W_\epsilon}{\partial \mathbf{F}_\epsilon} \right|_{\epsilon=0} \\ &= \mathbf{g}^\sharp \left. \frac{\partial^2 W_\epsilon}{\partial \mathbf{F}_\epsilon \partial \mathbf{F}_\epsilon} : \nabla_{\frac{\partial}{\partial \epsilon}} \mathbf{F}_\epsilon \right|_{\epsilon=0} + \mathbf{g}^\sharp \left. \frac{\partial^2 W_\epsilon}{\partial \mathbf{g} \circ \varphi_\epsilon \partial \mathbf{F}_\epsilon} : \nabla_{\frac{\partial}{\partial \epsilon}} \mathbf{g} \circ \varphi_\epsilon \right|_{\epsilon=0} \\ &= \mathbf{g}^\sharp \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} : \nabla \mathbf{U}. \end{aligned} \quad (3.11)$$

In the above derivation, in the second equality the geometric ω -lemma [68], and in the last equality the metric compatibility of the Levi-Civita connection ($\nabla^{\mathbf{g}} \mathbf{g} = \mathbf{0}$) was used. Therefore, $\delta \mathbf{P} = \mathbf{A} : \nabla \mathbf{U}$, where \mathbf{A} is the first elasticity tensor [68] with components

$$\mathbf{A}^{aA}{}_b{}^B = \frac{\partial P^{aA}}{\partial F^b{}_B} = g^{ac} \frac{\partial^2 W}{\partial F^c{}_A \partial F^b{}_B}. \quad (3.12)$$

Thus, in components $\delta P^{aA} = \mathbf{A}^{aA}{}_b{}^B U^b|_B$.

The second Piola-Kirchhoff stress is defined as $S^{AB} = (F^{-1})^A{}_a P^{aB}$. Using the material frame-indifference (objectivity) the energy function is written as $W = \hat{W}(X, \mathbf{G}, \mathbf{C}^\flat, \zeta_1, \dots, \zeta_n)$. The second Piola-Kirchhoff stress is written as $\mathbf{S} = 2 \frac{\partial \hat{W}}{\partial \mathbf{C}^\flat}$. For the one-parameter family of motions $\varphi_{t,\epsilon}$, one has $\mathbf{S}_\epsilon = 2 \frac{\partial \hat{W}_\epsilon}{\partial \mathbf{C}^\flat_\epsilon}$, where $\hat{W}_\epsilon = \hat{W}(X, \mathbf{G}, \mathbf{C}^\flat_\epsilon, \zeta_1, \dots, \zeta_n)$. Therefore

$$\delta \mathbf{S} = 2 \frac{\partial^2 \hat{W}_\epsilon}{\partial \mathbf{C}^\flat_\epsilon \partial \mathbf{C}^\flat_\epsilon} : \frac{d}{d\epsilon} \mathbf{C}^\flat_\epsilon \Big|_{\epsilon=0} = 4 \frac{\partial^2 \hat{W}}{\partial \hat{\mathbf{C}}^\flat \partial \hat{\mathbf{C}}^\flat} : \hat{\varphi}^* \boldsymbol{\epsilon} = \mathbf{C} : \hat{\varphi}^* \boldsymbol{\epsilon}, \quad (3.13)$$

where \mathbf{C} is the second elasticity tensor with components

$$\mathbf{C}^{ABCD} = 4 \frac{\partial^2 \hat{W}}{\partial C_{AB} \partial C_{CD}}. \quad (3.14)$$

In components, $\delta S^{AB} = \mathbf{C}^{ABCD} (\hat{\varphi}^* \boldsymbol{\epsilon})_{CD}$. It is straightforward to show that the first and the second elasticity tensors are related as

$$\mathbf{A}_a{}^A{}_b{}^B = \mathbf{C}^{AMBN} \hat{F}^m{}_M \hat{F}^n{}_N g_{am} g_{bn} + \hat{S}^{AB} g_{ab}, \quad (3.15)$$

or

$$\mathbf{A}^{aAbB} = \mathbf{C}^{AMBN} \hat{F}^a{}_M \hat{F}^b{}_N + \hat{S}^{AB} g^{ab}. \quad (3.16)$$

Similarly, the spatial elasticity tensors are defined as $\mathbf{a} = \frac{1}{J} \varphi_* \mathbf{A}$, and $\mathbf{c} = \frac{1}{J} \varphi_* \mathbf{C}$. In components

$$\begin{aligned} \mathbf{a}^{ac}{}_b{}^d &= \frac{1}{J} F^c{}_A F^d{}_B \mathbf{A}^{AA}{}_b{}^B, \\ \mathbf{c}^{abcd} &= \frac{\partial \sigma^{ab}}{\partial g_{cd}} = \frac{1}{J} F^a{}_A F^b{}_B F^c{}_C F^d{}_D \mathbf{C}^{ABCD}. \end{aligned} \quad (3.17)$$

The relation analogous to (3.16) is given by $\mathbf{a}^{ac}{}_b{}^d = \sigma^{cd} \delta_b^a + \mathbf{c}^{aced} g_{eb}$. Note that the spatial elasticity tensors have the following symmetries $\mathbf{a}^{abcd} = \mathbf{a}^{bdac}$, and $\mathbf{c}^{abcd} = \mathbf{c}^{bacd} = \mathbf{c}^{abdc} = \mathbf{c}^{cdab}$.

Linearization of conservation of mass. For the family of motions $\varphi_{t,\epsilon}$, conservation of mass is locally written as $\rho_0(X) = J_\epsilon \rho \circ \varphi_{t,\epsilon}(X)$. Taking derivatives of both sides with respect to ϵ and evaluating at $\epsilon = 0$, one obtains $0 = \delta J \rho \circ \dot{\varphi}_t(X) + \dot{J} \mathbf{d}\dot{\rho} \cdot \delta\varphi$, where $\mathbf{d}\dot{\rho}$ is the exterior derivative of the mass density of the reference motion (a 1-form). Thus, $\mathbf{d}\dot{\rho} \cdot \mathbf{u} + \dot{\rho} \mathbf{g}^\sharp : \boldsymbol{\epsilon} = 0$. Note that $\mathbf{g}^\sharp : \boldsymbol{\epsilon} = \mathbf{g}^\sharp : \nabla \mathbf{u}^b$. Hence, the linearized conservation of mass can be written as, $\mathbf{d}\dot{\rho} \cdot \mathbf{u} + \dot{\rho} \mathbf{g}^\sharp : \nabla \mathbf{g} \mathbf{u}^b = 0$. Therefore

$$\delta \rho = -\dot{\rho} \boldsymbol{\epsilon} : \mathbf{g}^\sharp = -\dot{\rho} \operatorname{div}_{\mathbf{g}} \mathbf{u}. \quad (3.18)$$

Linearization of the balance of linear momentum. The balance of linear momentum in terms of the first Piola-Kirchhoff stress reads $\operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}$. Note that the operator Div depends on both \mathbf{G} and \mathbf{g} and is defined through the Piola identity (see the appendix) as $\operatorname{Div} \mathbf{P} = J \operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}$. The balance of linear momentum in terms of the Cauchy stress reads $\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}$. Using the Doyle-Ericksen formula, the Cauchy stress is written as [22] $\boldsymbol{\sigma} = \frac{2}{J} \frac{\partial W}{\partial \mathbf{g}}$, where $W = W(X, \mathbf{F}, \mathbf{g} \circ \varphi, \mathbf{G}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_n)$ is the energy density per unit undeformed volume. The convected stress tensor $\boldsymbol{\Sigma} = \varphi_t^* \boldsymbol{\sigma}$ can be written as [104] $\boldsymbol{\Sigma} = \frac{2}{J} \frac{\partial \hat{W}}{\partial \mathbf{C}^b}$, where $W = \hat{W}(X, \mathbf{C}^b, \mathbf{G}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_n)$.

Linearization of the convected balance of linear momentum. For the one-parameter family of motions $\varphi_{t,\epsilon}$, one has

$$\operatorname{div}_{\mathbf{C}^b} \boldsymbol{\Sigma}_\epsilon + \varrho_\epsilon \mathcal{B}_\epsilon = \varrho_\epsilon \mathcal{A}_\epsilon. \quad (3.19)$$

Therefore, the linearized balance of linear momentum is written as

$$\frac{d}{d\epsilon} \left[\operatorname{div}_{\mathbf{C}^b} \boldsymbol{\Sigma}_\epsilon \right] \Big|_{\epsilon=0} + \frac{d}{d\epsilon} \left[\varrho_\epsilon \mathcal{B}_\epsilon \right] \Big|_{\epsilon=0} = \frac{d}{d\epsilon} \left[\varrho_\epsilon \mathcal{A}_\epsilon \right] \Big|_{\epsilon=0}. \quad (3.20)$$

Note that for $X \in \mathcal{B}$, all the terms in (3.19) lie in the same tangent space $T_X \mathcal{B}$, and hence, linearizing the balance of linear momentum is straightforward when using the convected stress. It is a simple calculation to show that the linearized convected stress reads

$$\delta \boldsymbol{\Sigma} = \frac{4}{J} \frac{\partial^2 \hat{W}}{\partial \mathbf{C}^b \partial \mathbf{C}^b} : \dot{\varphi}_t^* \boldsymbol{\epsilon} - (\boldsymbol{\epsilon} : \mathbf{g}^\sharp) \dot{\boldsymbol{\Sigma}}. \quad (3.21)$$

Note that $\delta\sigma = \dot{\varphi}_t^* \delta\Sigma$, and hence

$$\delta\sigma = \frac{4}{\dot{J}} \frac{\partial^2 W}{\partial \mathbf{g} \partial \mathbf{g}} : \boldsymbol{\epsilon} - (\boldsymbol{\epsilon} : \mathbf{g}^\sharp) \dot{\sigma}. \quad (3.22)$$

We define the following fourth-order elasticity tensor

$$\mathbb{C} := \frac{4}{\dot{J}} \frac{\partial^2 W}{\partial \mathbf{g} \partial \mathbf{g}}, \quad (3.23)$$

which in components reads $\mathbb{C}^{abcd} = \frac{4}{\dot{J}} \frac{\partial^2 W}{\partial g_{ab} \partial g_{cd}}$. Note that $\mathbb{C} = \dot{\varphi}_* \mathbf{C} / \dot{J}$. We now expand the linearization of each term in (3.20). For the body force term one can write

$$\begin{aligned} \frac{d}{d\epsilon} [\varrho_\epsilon \mathcal{B}_\epsilon] \Big|_{\epsilon=0} &= \frac{d}{d\epsilon} [(\rho \circ \varphi_{t,\epsilon}) \varphi_{t,\epsilon}^* \mathbf{b}] \Big|_{\epsilon=0} = \delta\rho \circ \dot{\varphi}_t \mathring{\mathcal{B}} + \dot{\varrho} \frac{d}{d\epsilon} [(\varphi_{t,\epsilon}^* \mathbf{b})] \Big|_{\epsilon=0} \\ &= -(\mathbf{g}^\sharp : \boldsymbol{\epsilon}) \circ \dot{\varphi}_t \dot{\varrho} \mathring{\mathcal{B}} + \dot{\varrho} \dot{\varphi}_t^* (\mathbf{L}_u \mathring{\mathbf{b}}). \end{aligned} \quad (3.24)$$

Note that $\partial \mathring{\mathbf{b}} / \partial \epsilon = \mathbf{0}$, and hence, $\mathbf{L}_u \mathring{\mathbf{b}} = \mathcal{L}_u \mathring{\mathbf{b}}$. Therefore, $\delta(\varrho \mathcal{B}) = -(\mathbf{g}^\sharp : \boldsymbol{\epsilon}) \circ \dot{\varphi}_t \dot{\varrho} \mathring{\mathcal{B}} + \dot{\varrho} \dot{\varphi}_t^* (\mathcal{L}_u \mathring{\mathbf{b}})$. Also note that $\mathcal{L}_u \mathring{\mathbf{b}} = \nabla_{\mathring{\mathbf{u}}} \mathring{\mathbf{b}} - \nabla_{\mathring{\mathbf{b}}} \mathring{\mathbf{u}}$. Hence

$$\delta(\varrho \mathcal{B}) = -(\mathbf{g}^\sharp : \boldsymbol{\epsilon}) \circ \dot{\varphi}_t \dot{\varrho} \mathring{\mathcal{B}} + \dot{\varrho} \dot{\varphi}_t^* (\nabla_{\mathring{\mathbf{u}}} \mathring{\mathbf{b}} - \nabla_{\mathring{\mathbf{b}}} \mathring{\mathbf{u}}). \quad (3.25)$$

The convected acceleration is linearized as follows

$$\begin{aligned} &\frac{d}{d\epsilon} [\mathcal{A}_\epsilon] \Big|_{\epsilon=0} \\ &= \varphi_{t,\epsilon}^* (\mathbf{L}_u \mathbf{a}_\epsilon) \Big|_{\epsilon=0} = \varphi_{t,\epsilon}^* (\mathcal{L}_u \mathbf{a}_\epsilon) \Big|_{\epsilon=0} = \varphi_{t,\epsilon}^* [\nabla_{\mathring{\mathbf{u}}} \mathring{\mathbf{a}}_\epsilon - \nabla_{\mathring{\mathbf{a}}_\epsilon} \mathring{\mathbf{u}}] \Big|_{\epsilon=0} \\ &= T \varphi_{t,\epsilon}^{-1} \circ [\nabla_{\mathbf{U}} \mathbf{A}_\epsilon - \nabla_{\mathbf{A}_\epsilon} \mathbf{U}] \Big|_{\epsilon=0} = T \varphi_{t,\epsilon}^{-1} \circ [D_\epsilon \mathbf{A}_\epsilon - \nabla_{\mathbf{A}_\epsilon} \mathbf{U}] \Big|_{\epsilon=0} \\ &= T \varphi_{t,\epsilon}^{-1} \circ [D_\epsilon D_t \mathbf{V}_\epsilon - \nabla_{\mathbf{A}_\epsilon} \mathbf{U}] \Big|_{\epsilon=0} \\ &= T \varphi_{t,\epsilon}^{-1} \circ \left[D_t D_\epsilon \frac{\partial \varphi_{t,\epsilon}}{\partial t} + \nabla_{[\mathbf{U}, \mathbf{V}_\epsilon]} \mathbf{V}_\epsilon + \mathcal{R}_{\mathbf{g}}(\mathbf{U}, \mathbf{V}_\epsilon, \mathbf{V}_\epsilon) - \nabla_{\mathbf{A}_\epsilon} \mathbf{U} \right] \Big|_{\epsilon=0} \\ &= \mathring{\mathbf{F}}_t^{-1} \cdot \left[D_t D_t \mathbf{U} + \nabla_{[\mathbf{U}, \mathring{\mathbf{V}}]} \mathring{\mathbf{V}} + \mathcal{R}_{\mathbf{g}}(\mathbf{U}, \mathring{\mathbf{V}}, \mathring{\mathbf{V}}) - \nabla_{\mathring{\mathbf{A}}} \mathbf{U} \right]. \end{aligned} \quad (3.26)$$

Therefore

$$\delta(\varrho \mathcal{A}) = -(\mathbf{g}^\sharp : \boldsymbol{\epsilon}) \circ \dot{\varphi}_t \dot{\varrho} \mathring{\mathcal{A}} + \dot{\varrho} \mathring{\mathbf{F}}_t^{-1} \cdot \left[\mathring{\mathbf{U}} + \nabla_{[\mathbf{U}, \mathring{\mathbf{V}}]} \mathring{\mathbf{V}} + \mathcal{R}_{\mathbf{g}}(\mathbf{U}, \mathring{\mathbf{V}}, \mathring{\mathbf{V}}) - \nabla_{\mathring{\mathbf{A}}} \mathbf{U} \right]. \quad (3.27)$$

To calculate $\delta(\operatorname{div}_{\mathbb{C}_\epsilon} \boldsymbol{\Sigma})$, SADIK and YAVARI [100] expanded $\operatorname{div}_{\mathbb{C}_\epsilon} \boldsymbol{\Sigma}_\epsilon$ in a coordinate chart and then took derivatives with respect to ϵ and finally evaluated

at $\epsilon = 0$. They also assumed a flat ambient space. Here, we do not assume a flat ambient space. Note that

$$\begin{aligned} \delta(\operatorname{div}_{\mathbf{C}^b} \boldsymbol{\Sigma}) &= \frac{d}{d\epsilon} \left[\operatorname{div}_{\mathbf{C}^b_\epsilon} \boldsymbol{\Sigma}_\epsilon \right] \Big|_{\epsilon=0} = \frac{d}{d\epsilon} \varphi_{t,\epsilon}^* \left[\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}_\epsilon \right] \Big|_{\epsilon=0} \\ &= \varphi_{t,\epsilon}^* \mathbf{L}_{\mathbf{u}} \left[\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}_\epsilon \right] \Big|_{\epsilon=0} = \varphi_{t,\epsilon}^* \mathcal{L}_{\mathbf{u}} \left[\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}_\epsilon \right] \Big|_{\epsilon=0}. \end{aligned} \quad (3.28)$$

We know that [68, 115]

$$\mathcal{L}_{\mathbf{u}} \nabla^{\mathbf{g}} = \nabla^{\mathbf{g}} \nabla^{\mathbf{g}} \mathbf{u} + \mathbf{u} \cdot \mathcal{R}_{\mathbf{g}}, \quad (3.29)$$

where in components, $(\mathcal{L}_{\mathbf{u}} \nabla^{\mathbf{g}})^a{}_{bc} = u^a{}_{|bc} + \mathcal{R}^a{}_{ebc} u^e$. We also know that [115]

$$\begin{aligned} \mathcal{L}_{\mathbf{u}} \sigma^{ab}{}_{|b} &= (\mathcal{L}_{\mathbf{u}} \sigma^{ab})_{|b} + (\mathcal{L}_{\mathbf{u}} \gamma^a{}_{bd}) \sigma^{bd} + (\mathcal{L}_{\mathbf{u}} \gamma^b{}_{bd}) \sigma^{ad} \\ &= (\mathcal{L}_{\mathbf{u}} \sigma^{ab})_{|b} + (u^a{}_{|bd} + \mathcal{R}^a{}_{cbd} u^c) \sigma^{bd} + (u^b{}_{|bd} + \mathcal{R}^b{}_{cbd} u^c) \sigma^{ad}. \end{aligned} \quad (3.30)$$

Note that $\mathcal{R}^a{}_{cbd} = -\mathcal{R}^a{}_{cdb}$, and hence, $\mathcal{R}^a{}_{cbd} \sigma^{bd} = 0$. Also, $\mathcal{R}^b{}_{cbd} = \operatorname{Ric}_{cd}$ is the Ricci curvature. Thus

$$\mathcal{L}_{\mathbf{u}} \sigma^{ab}{}_{|b} = (\mathcal{L}_{\mathbf{u}} \sigma^{ab})_{|b} + u^a{}_{|bd} \sigma^{bd} + \left[(u^b{}_{|b})_{,d} + u^c \operatorname{Ric}_{cd} \right] \sigma^{ad}. \quad (3.31)$$

Therefore

$$\mathcal{L}_{\mathbf{u}} (\operatorname{div}_{\mathbf{g}} \mathring{\boldsymbol{\sigma}}) = \operatorname{div}_{\mathbf{g}} (\mathcal{L}_{\mathbf{u}} \mathring{\boldsymbol{\sigma}}) + \nabla^{\mathbf{g}} \nabla^{\mathbf{g}} \mathbf{u} : \mathring{\boldsymbol{\sigma}} + \mathbf{d}(\boldsymbol{\epsilon} : \mathbf{g}^{\sharp}) \cdot \mathring{\boldsymbol{\sigma}} + \mathring{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u}. \quad (3.32)$$

Note that

$$\mathcal{L}_{\mathbf{u}} \mathring{\boldsymbol{\sigma}} = \mathcal{L}_{\mathbf{u}} \left(\frac{2}{j} \frac{\partial W}{\partial \mathbf{g}} \right) = -\frac{2}{j^2} \frac{\partial W}{\partial \mathbf{g}} \mathcal{L}_{\mathbf{u}} j + \frac{2}{j} \left(\frac{\partial^2 W}{\partial \mathbf{g} \partial \mathbf{g}} : \mathcal{L}_{\mathbf{u}} \mathbf{g} + \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{g}} : \mathcal{L}_{\mathbf{u}} \mathbf{F} \right). \quad (3.33)$$

However, $\mathcal{L}_{\mathbf{u}} \mathbf{F} = \mathbf{0}$, because for any vector $\mathbf{W}(X) \in T_X \mathcal{B}$, one can write

$$\begin{aligned} \mathcal{L}_{\mathbf{u}} \mathbf{F} \cdot \mathbf{W} &= \mathcal{L}_{\mathbf{u}} (\mathbf{F} \cdot \mathbf{W}) = \varphi_{t,*} \frac{d}{d\epsilon} \left[\varphi_{t,\epsilon}^* (\mathbf{F}_{t,\epsilon} \cdot \mathbf{W}) \right] \Big|_{\epsilon=0} \\ &= \varphi_{t,*} \frac{d}{d\epsilon} \left[\varphi_{t,\epsilon}^* (\varphi_{t,\epsilon*} \circ \mathbf{W}) \right] \Big|_{\epsilon=0} = \varphi_{t,*} \frac{d}{d\epsilon} [\mathbf{W}] \Big|_{\epsilon=0} = \mathbf{0}. \end{aligned} \quad (3.34)$$

Hence

$$\mathcal{L}_{\mathbf{u}} \mathring{\boldsymbol{\sigma}} = \frac{4}{j} \frac{\partial^2 W}{\partial \mathbf{g} \partial \mathbf{g}} : \boldsymbol{\epsilon} - (\boldsymbol{\epsilon} : \mathbf{g}^{\sharp}) \mathring{\boldsymbol{\sigma}}. \quad (3.35)$$

Therefore

$$\begin{aligned} \delta(\operatorname{div}_{\mathbf{C}^b} \boldsymbol{\Sigma}) &= \mathring{\varphi}_t^* \left[\nabla^{\mathbf{g}} \nabla^{\mathbf{g}} \mathbf{u} : \mathring{\boldsymbol{\sigma}} + \mathbf{d}(\boldsymbol{\epsilon} : \mathbf{g}^{\sharp}) \cdot \mathring{\boldsymbol{\sigma}} + \mathring{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u} \right. \\ &\quad \left. + \operatorname{div}_{\mathbf{g}} \left(\frac{4}{j} \frac{\partial^2 W}{\partial \mathbf{g} \partial \mathbf{g}} : \boldsymbol{\epsilon} - (\boldsymbol{\epsilon} : \mathbf{g}^{\sharp}) \mathring{\boldsymbol{\sigma}} \right) \right]. \end{aligned} \quad (3.36)$$

Or

$$\delta(\operatorname{div}_{\mathbb{C}} \boldsymbol{\Sigma}) = \dot{\varphi}_t^* \left[\nabla^{\mathbf{g}} \nabla^{\mathbf{g}} \mathbf{u} : \dot{\boldsymbol{\sigma}} - (\boldsymbol{\epsilon} : \mathbf{g}^\sharp) \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u} + \operatorname{div}_{\mathbf{g}} (\mathbb{C} : \boldsymbol{\epsilon}) \right]. \quad (3.37)$$

Now the push-forward of the balance of linear momentum by $\dot{\varphi}_t$ using (3.25), (3.27), and (3.37) reads

$$\begin{aligned} & \nabla^{\mathbf{g}} \nabla^{\mathbf{g}} \mathbf{u} : \dot{\boldsymbol{\sigma}} - (\boldsymbol{\epsilon} : \mathbf{g}^\sharp) \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u} + \operatorname{div}_{\mathbf{g}} (\mathbb{C} : \boldsymbol{\epsilon}) - \mathbf{g}^\sharp : \boldsymbol{\epsilon} \dot{\rho} \mathring{\mathbf{b}} \\ & + \dot{\rho} \left(\nabla_{\mathring{\mathbf{u}}}^{\mathbf{g}} \mathring{\mathbf{b}} - \nabla_{\mathring{\mathbf{b}}}^{\mathbf{g}} \mathbf{u} \right) = -\mathbf{g}^\sharp : \boldsymbol{\epsilon} \dot{\rho} \mathring{\mathbf{a}} \\ & + \dot{\rho} \left[\ddot{\mathbf{U}} \circ \dot{\varphi}_t^{-1} + \nabla_{[\mathring{\mathbf{u}}, \mathring{\mathbf{v}}]}^{\mathbf{g}} \mathring{\mathbf{v}} + \mathcal{R}_{\mathbf{g}}(\mathbf{u}, \mathring{\mathbf{v}}, \mathring{\mathbf{v}}) - \nabla_{\mathring{\mathbf{a}}}^{\mathbf{g}} \mathbf{u} \right]. \end{aligned} \quad (3.38)$$

We assume that the reference motion is “physical”, i.e., it satisfies all the balance laws. In particular, $\operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}} + \dot{\rho} \mathring{\mathbf{b}} = \dot{\rho} \mathring{\mathbf{a}}$. Thus, we have the following two identities:

$$\begin{aligned} & -\mathbf{g}^\sharp : \boldsymbol{\epsilon} \dot{\rho} \mathring{\mathbf{b}} + \mathbf{g}^\sharp : \boldsymbol{\epsilon} \dot{\rho} \mathring{\mathbf{a}} = (\mathbf{g}^\sharp : \boldsymbol{\epsilon}) \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}, \\ & -\dot{\rho} \nabla_{\mathring{\mathbf{b}}}^{\mathbf{g}} \mathbf{u} + \dot{\rho} \nabla_{\mathring{\mathbf{a}}}^{\mathbf{g}} \mathbf{u} = \nabla_{-\dot{\rho}(\mathring{\mathbf{b}}-\mathring{\mathbf{a}})}^{\mathbf{g}} \mathbf{u} = \nabla_{\operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}}^{\mathbf{g}} \mathbf{u} = \nabla^{\mathbf{g}} \mathbf{u} \cdot \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}. \end{aligned} \quad (3.39)$$

Therefore, the linearized balance of linear momentum is simplified to read

$$\begin{aligned} & \nabla^{\mathbf{g}} \nabla^{\mathbf{g}} \mathbf{u} : \dot{\boldsymbol{\sigma}} + \nabla^{\mathbf{g}} \mathbf{u} \cdot \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u} + \operatorname{div}_{\mathbf{g}} (\mathbb{C} : \boldsymbol{\epsilon}) + \dot{\rho} \nabla_{\mathring{\mathbf{u}}}^{\mathbf{g}} \mathring{\mathbf{b}} \\ & = \dot{\rho} \left[\ddot{\mathbf{U}} \circ \dot{\varphi}_t^{-1} + \nabla_{[\mathring{\mathbf{u}}, \mathring{\mathbf{v}}]}^{\mathbf{g}} \mathring{\mathbf{v}} + \mathcal{R}_{\mathbf{g}}(\mathbf{u}, \mathring{\mathbf{v}}, \mathring{\mathbf{v}}) \right]. \end{aligned} \quad (3.40)$$

Note that $\nabla^{\mathbf{g}} \nabla^{\mathbf{g}} \mathbf{u} : \dot{\boldsymbol{\sigma}} + \nabla^{\mathbf{g}} \mathbf{u} \cdot \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}} = \operatorname{div}_{\mathbf{g}} (\nabla^{\mathbf{g}} \mathbf{u} \cdot \dot{\boldsymbol{\sigma}})$. Hence, the linearized balance of linear momentum is written as¹⁶

$$\begin{aligned} & \operatorname{div}_{\mathbf{g}} (\mathbb{C} : \boldsymbol{\epsilon}) + \operatorname{div}_{\mathbf{g}} (\nabla^{\mathbf{g}} \mathbf{u} \cdot \dot{\boldsymbol{\sigma}}) + \dot{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u} + \dot{\rho} \nabla_{\mathring{\mathbf{u}}}^{\mathbf{g}} \mathring{\mathbf{b}} \\ & = \dot{\rho} \left[\ddot{\mathbf{U}} \circ \dot{\varphi}_t^{-1} + \nabla_{[\mathring{\mathbf{u}}, \mathring{\mathbf{v}}]}^{\mathbf{g}} \mathring{\mathbf{v}} + \mathcal{R}_{\mathbf{g}}(\mathbf{u}, \mathring{\mathbf{v}}, \mathring{\mathbf{v}}) \right]. \end{aligned} \quad (3.41)$$

Linearization of the spatial balance of linear momentum. For the motion $\varphi_{t,\epsilon}$, the balance of linear momentum reads $\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}_\epsilon + \rho_\epsilon \mathbf{b}_\epsilon = \rho_\epsilon \mathbf{a}_\epsilon$. Linearization of the body force term is calculated as $\delta(\rho \mathbf{b}) = \delta \rho \mathring{\mathbf{b}} + \dot{\rho} \nabla_{\frac{\partial}{\partial \epsilon}}^{\mathbf{g}} \mathbf{b}_\epsilon \Big|_{\epsilon=0} = -(\mathbf{g}^\sharp : \boldsymbol{\epsilon}) \dot{\rho} \mathring{\mathbf{b}} + \dot{\rho} \nabla_{\mathring{\mathbf{u}}}^{\mathbf{g}} \mathring{\mathbf{b}}$. Linearization of the inertial force term reads

$$\begin{aligned} \delta(\rho \mathbf{a}) &= \delta \rho \mathring{\mathbf{a}} + \dot{\rho} \nabla_{\frac{\partial}{\partial \epsilon}}^{\mathbf{g}} \mathbf{a}_\epsilon \Big|_{\epsilon=0} = -(\mathbf{g}^\sharp : \boldsymbol{\epsilon}) \dot{\rho} \mathring{\mathbf{a}} + \dot{\rho} \nabla_{\frac{\partial}{\partial \epsilon}}^{\mathbf{g}} \mathbf{a}_\epsilon \Big|_{\epsilon=0} \\ & - (\mathbf{g}^\sharp : \boldsymbol{\epsilon}) \dot{\rho} \mathring{\mathbf{a}} + \dot{\rho} \left[\ddot{\mathbf{U}} \circ \dot{\varphi}_t^{-1} + \nabla_{[\mathring{\mathbf{u}}, \mathring{\mathbf{v}}]}^{\mathbf{g}} \mathring{\mathbf{v}} + \mathcal{R}_{\mathbf{g}}(\mathbf{u}, \mathring{\mathbf{v}}, \mathring{\mathbf{v}}) \right]. \end{aligned} \quad (3.42)$$

¹⁶ For a flat ambient space this is identical to the corresponding equation in [100]. However, note that even for a flat ambient space this is not identical to what MARSDEN and HUGHES [68] obtained; they do not have the term $\operatorname{div}_{\mathbf{g}} (\nabla^{\mathbf{g}} \mathbf{u} \cdot \dot{\boldsymbol{\sigma}})$.

Note that $\delta(\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}) = \nabla_{\frac{\partial}{\partial \boldsymbol{\epsilon}}}^{\mathbf{g}} (\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma} \boldsymbol{\epsilon}) \Big|_{\boldsymbol{\epsilon}=\mathbf{0}} = \nabla_{\mathbf{u}}^{\mathbf{g}} (\operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}})$. Knowing that the Levi-Civita connection is torsion-free, one can write $\nabla_{\mathbf{u}}^{\mathbf{g}} (\operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}) = \mathfrak{L}_{\mathbf{u}} (\operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}) + \nabla_{\operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}}^{\mathbf{g}} \mathbf{u} = \mathfrak{L}_{\mathbf{u}} (\operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}) + \nabla^{\mathbf{g}} \mathbf{u} \cdot \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}$. Hence, from (3.32), one has $\delta(\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}) = \operatorname{div}_{\mathbf{g}} (\mathfrak{L}_{\mathbf{u}} \dot{\boldsymbol{\sigma}}) + \operatorname{div}_{\mathbf{g}} (\nabla^{\mathbf{g}} \mathbf{u} \cdot \dot{\boldsymbol{\sigma}}) + \mathbf{d}(\boldsymbol{\epsilon} : \mathbf{g}^{\sharp}) \cdot \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u}$. Using (3.35), one obtains

$$\delta(\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}) = \operatorname{div}_{\mathbf{g}} (\nabla^{\mathbf{g}} \mathbf{u} \cdot \dot{\boldsymbol{\sigma}}) + \dot{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u} + \operatorname{div}_{\mathbf{g}} (\mathbb{C} : \boldsymbol{\epsilon}) - (\boldsymbol{\epsilon} : \mathbf{g}^{\sharp}) \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}. \quad (3.43)$$

Remark 2. Note that linearizing a convected vector and pushing it forward to the ambient space is not the same as linearizing the corresponding spatial vector using a covariant derivative. In particular, we note that

$$\begin{aligned} \delta(\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}) &= \dot{\varphi}_{t*} \delta(\operatorname{div}_{\mathbf{C}^{\flat}} \boldsymbol{\Sigma}) + \nabla^{\mathbf{g}} \mathbf{u} \cdot \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}, \\ \delta(\rho \mathbf{b}) &= \dot{\varphi}_{t*} \delta(\rho \mathcal{B}) + \dot{\rho} \nabla_{\mathbf{b}}^{\mathbf{g}} \mathbf{u}, \\ \delta(\rho \mathbf{a}) &= \dot{\varphi}_{t*} \delta(\rho \mathcal{A}) + \dot{\rho} \nabla_{\mathbf{a}}^{\mathbf{g}} \mathbf{u}. \end{aligned} \quad (3.44)$$

However, note that $\nabla^{\mathbf{g}} \mathbf{u} \cdot \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}} + \dot{\rho} \nabla_{\mathbf{b}}^{\mathbf{g}} \mathbf{u} - \dot{\rho} \nabla_{\mathbf{a}}^{\mathbf{g}} \mathbf{u} = \nabla_{\operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}} + \dot{\rho} \mathbf{b} - \dot{\rho} \mathbf{a}}^{\mathbf{g}} \mathbf{u} = \mathbf{0}$, i.e., the two approaches give the same linearized balance of linear momentum (3.41).

Linearization of the material balance of linear momentum. The balance of linear momentum in terms of the first Piola-Kirchhoff stress is linearized as follows. Linearized body and inertial forces are (note that $\rho_0 = \rho_0(X)$ has a vanishing variation) $\delta(\rho_0 \mathbf{B}) = \rho_0 \nabla_{\mathbf{U}}^{\mathbf{g}} \dot{\mathbf{B}}$, and $\delta(\rho_0 \mathbf{A}) = \rho_0 [\dot{\mathbf{U}} + \nabla_{[\mathbf{U}, \dot{\mathbf{V}}]}^{\mathbf{g}} \dot{\mathbf{V}} + \mathcal{R}_{\mathbf{g}}(\mathbf{U}, \dot{\mathbf{V}}, \dot{\mathbf{V}})]$. Note that $\delta(\operatorname{Div} \mathbf{P}) = \delta(J \operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}) = \delta J \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}} + \dot{J} \delta(\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}) = (\boldsymbol{\epsilon} : \mathbf{g}^{\sharp}) \operatorname{Div} \dot{\mathbf{P}} + \dot{J} \delta(\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma})$. From (3.43), we know that

$$\begin{aligned} \dot{J} \delta(\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}) &= \dot{J} \operatorname{div}_{\mathbf{g}} (\nabla^{\mathbf{g}} \mathbf{u} \cdot \dot{\boldsymbol{\sigma}}) + \dot{J} \dot{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u} \\ &\quad + \dot{J} \operatorname{div}_{\mathbf{g}} (\mathbb{C} : \boldsymbol{\epsilon}) - (\boldsymbol{\epsilon} : \mathbf{g}^{\sharp}) \dot{J} \operatorname{div}_{\mathbf{g}} \dot{\boldsymbol{\sigma}}. \end{aligned} \quad (3.45)$$

We next use the Piola identity and rewrite the divergence terms with respect to the reference configuration. Note that

$$\begin{aligned} \dot{J} \operatorname{div}_{\mathbf{g}} (\nabla^{\mathbf{g}} \mathbf{u} \cdot \dot{\boldsymbol{\sigma}}) &= \dot{J} \left(u^a {}_{|b} \dot{\sigma}^{bc} \right)_{|c} \frac{\partial}{\partial x^a} = \left[u^a {}_{|b} \dot{J} (\dot{F}^{-1})^B{}_c \dot{\sigma}^{bc} \right]_{|B} \frac{\partial}{\partial x^a} \\ &= \left(u^a {}_{|b} \dot{P}^{bB} \right)_{|B} \frac{\partial}{\partial x^a} = \operatorname{Div} (\nabla^{\mathbf{g}} \mathbf{U} \cdot \dot{\mathbf{P}}). \end{aligned} \quad (3.46)$$

The last term in (3.45) is simplified to read $-(\boldsymbol{\epsilon} : \mathbf{g}^{\sharp}) \operatorname{Div} \dot{\mathbf{P}}$. The second term on the right-hand side of (3.45) is simplified as $\dot{\mathbf{F}} \dot{\mathbf{P}}^{\star} \cdot \mathbf{Ric}_{\mathbf{g}} \circ \dot{\varphi} \cdot \mathbf{U}$. The second divergence term is simplified using the Piola identity as $\dot{J} \operatorname{div}_{\mathbf{g}} (\mathbb{C} : \boldsymbol{\epsilon}) = \dot{J} (\mathbb{C}^{abcd} u_{c|d})_{|b} \frac{\partial}{\partial x^a} = \left[\dot{J} (\dot{F}^{-1})^B{}_b \mathbb{C}^{abcd} u_{c|d} \right]_{|B} \frac{\partial}{\partial x^a}$. From (3.16), we know that $\dot{J} (\dot{F}^{-1})^B{}_b \mathbb{C}^{abcd} = F^a{}_M F^c{}_P F^d{}_Q \mathbb{C}^{BMPQ} = \mathbf{A}^{aBcQ} F^d{}_Q - \dot{P}^d B g^{ac}$. Thus

$$\begin{aligned} \left[\dot{J}(\dot{F}^{\circ-1})^B{}_b \mathbb{C}^{abcd} u_{c|d} \right]_{|B} &= \left(\mathbf{A}^{aBcQ} F^d{}_Q u_{c|d} \right)_{|B} - \left(\dot{P}^{dB} g^{ac} u_{c|d} \right)_{|B} \\ &= \left(\mathbf{A}^{aBcQ} u_{c|Q} \right)_{|B} - \left(\dot{P}^{dB} u^a{}_{|d} \right)_{|B}. \end{aligned} \quad (3.47)$$

Hence, $\dot{J} \operatorname{div}_{\mathbf{g}}(\mathbb{C} : \epsilon) = \operatorname{Div}(\mathbf{A} : \nabla \mathbf{U}) - \operatorname{Div}(\nabla \mathbf{U} \cdot \dot{\mathbf{P}})$. Therefore, $\delta(\operatorname{Div} \mathbf{P}) = \operatorname{Div}(\mathbf{A} : \nabla \mathbf{U}) + \dot{\mathbf{F}} \dot{\mathbf{P}}^* \cdot \mathbf{Ric}_{\mathbf{g}} \circ \dot{\varphi} \cdot \mathbf{U}$. In summary, the linearized material balance of linear momentum reads

$$\begin{aligned} \operatorname{Div}(\mathbf{A} : \nabla \mathbf{U}) + \dot{\mathbf{F}} \dot{\mathbf{P}}^* \cdot \mathbf{Ric}_{\mathbf{g}} \circ \dot{\varphi} \cdot \mathbf{U} + \rho_0 \nabla_{\mathbf{U}} \dot{\mathbf{B}} \\ = \rho_0 \left[\ddot{\mathbf{U}} + \nabla_{[\mathbf{U}, \dot{\mathbf{V}}]} \dot{\mathbf{V}} + \mathcal{R}_{\mathbf{g}}(\mathbf{U}, \dot{\mathbf{U}}, \dot{\mathbf{V}}) \right]. \end{aligned} \quad (3.48)$$

Linearization of the balance of angular momentum. In terms of the first Piola-Kirchhoff stress, the balance of angular momentum in coordinates reads $P^{aA} F^b{}_A = P^{bA} F^a{}_A$. Its linearization reads $\delta P^{aA} \dot{F}^b{}_A + \dot{P}^{aA} U^b{}_{|A} = \delta P^{bA} \dot{F}^a{}_A + \dot{P}^{bA} U^a{}_{|A}$. Or

$$\delta \mathbf{P} \cdot \dot{\mathbf{F}}^* + \dot{\mathbf{P}} \cdot (\nabla \mathbf{U})^* = \dot{\mathbf{F}} \cdot \delta \mathbf{P}^* + \nabla \mathbf{U} \cdot \dot{\mathbf{P}}^*. \quad (3.49)$$

In terms of the second Piola-Kirchhoff stress, $\delta \mathbf{S} = \mathbf{C} : \dot{\varphi}^* \epsilon$, the balance of angular momentum is equivalent to $\delta \mathbf{S}^* = \delta \mathbf{S}$, or $\mathbf{C}^{ABCD} = \mathbf{C}^{BACD}$. In terms of the first elasticity tensor, the balance of angular momentum reads

$$\mathbf{A}^{[aA}{}^M U^m{}_{|M} \dot{F}^{b]}{}_A + \dot{P}^{[aA} U^b]{}_{|A} = 0. \quad (3.50)$$

We next discuss the invariance of the governing equations of linear elasticity under both time-dependent spatial and time-independent referential changes of coordinates.

3.1. Spatial Covariance of Linearized Elasticity

Consider a spatial change of frame (or a coordinate transformation in the current configuration) $\xi_t : \mathcal{S} \rightarrow \mathcal{S}$. Under this change of frame, the one-parameter family of deformations $\varphi_{t,\epsilon}$ is transformed to $\varphi'_{t,\epsilon} = \xi_t \circ \varphi_{t,\epsilon} : \mathcal{B} \rightarrow \mathcal{S}$. We next study the effect of this change of frame on all the fields and governing equations of linear elasticity. The transformed variation field is defined as

$$\delta \varphi'_{t,\epsilon}(X) = \mathbf{U}'(X, t) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \xi_t \circ \varphi_{t,\epsilon}(X) = T \xi_t \cdot \delta \varphi_t = \dot{\mathbf{F}} \cdot \mathbf{U}(X, t). \quad (3.51)$$

Transformation of the linearized velocity, acceleration, and the deformation gradient. Linearization of the material velocity with respect to the new spatial frame is calculated as $\delta \mathbf{V}'(X, t) = \nabla_{\xi_*}^{\mathbf{g}'} \frac{\partial \varphi'_{t,\epsilon}(X)}{\partial t} \Big|_{\epsilon=0}$, where $\mathbf{g}' = \xi_* \mathbf{g}$. Note that

$$\begin{aligned} \frac{\partial \varphi'_{t,\epsilon}(X)}{\partial t} &= \frac{\partial}{\partial t} \xi_t \circ \varphi_{t,\epsilon}(X) = T \xi_t \cdot \frac{\partial \varphi_{t,\epsilon}(X)}{\partial t} + \frac{\partial \xi_t}{\partial t} \circ \varphi_{t,\epsilon}(X) \\ &= \xi_* \frac{\partial \varphi_{t,\epsilon}(X)}{\partial t} + \mathbf{w}_t \circ \varphi_{t,\epsilon}(X), \end{aligned} \quad (3.52)$$

where \mathbf{w}_t is the velocity of the change of frame. Thus

$$\begin{aligned} \delta \mathbf{V}'(X, t) &= \left(\nabla_{\xi_* \frac{\partial}{\partial \epsilon}} \xi_* \frac{\partial \varphi_{t, \epsilon}(X)}{\partial t} + \nabla_{\xi_* \frac{\partial}{\partial \epsilon}} \mathbf{w}_t \circ \varphi_{t, \epsilon}(X) \right) \Big|_{\epsilon=0} \\ &= \xi_* \left(\nabla_{\frac{\partial}{\partial \epsilon}} \frac{\partial \varphi_{t, \epsilon}(X)}{\partial t} + \nabla_{\mathbf{U}} \mathbf{W}_t \circ \varphi_{t, \epsilon}(X) \right) \Big|_{\epsilon=0}, \end{aligned} \quad (3.53)$$

where $\mathbf{W}_t = \xi_t^* \mathbf{w}_t$. Therefore, using (3.2), one obtains

$$\begin{aligned} \delta \mathbf{V}'(X, t) &= \xi_* (\delta \mathbf{V}(X, t) + \nabla_{\mathbf{U}} \mathbf{W}_t \circ \dot{\varphi}_t(X)) \\ &= \xi_* (D_t \mathbf{U}(X, t) + \nabla_{\mathbf{U}} \mathbf{W}_t \circ \dot{\varphi}_t(X)) \\ &= \xi_* (\dot{\mathbf{U}}(X, t) + \nabla_{\mathbf{U}} \mathbf{W}_t \circ \dot{\varphi}_t(X)). \end{aligned} \quad (3.54)$$

The transformed linearized acceleration is calculated as

$$\begin{aligned} \delta \mathbf{A}'(X, t) &= \nabla_{\dot{\mathbf{V}}'} \dot{\mathbf{V}}' = \nabla_{\xi_* \dot{\mathbf{V}} + \mathbf{w} \circ \varphi_t} (\xi_* \dot{\mathbf{V}} + \mathbf{w} \circ \varphi_t) \\ &= \xi_* \left\{ \nabla_{\dot{\mathbf{V}} + \mathbf{W} \circ \varphi_t} (\dot{\mathbf{V}} + \mathbf{W} \circ \varphi_t) \right\} \\ &= \xi_* \delta \mathbf{A}(X, t) + \xi_* \left(\nabla_{\dot{\mathbf{V}}} \mathbf{W} \circ \varphi_t + \nabla_{\mathbf{W} \circ \varphi_t} \dot{\mathbf{V}} + \nabla_{\mathbf{W} \circ \varphi_t} \mathbf{W} \circ \varphi_t \right). \end{aligned} \quad (3.55)$$

We assume that body force is transformed such that $\mathbf{A}'(X, t) - \mathbf{B}'(X, t) = \xi_t^* (\mathbf{A}(X, t) - \mathbf{B}(X, t))$ [68]. Therefore, $\delta \mathbf{A}(X, t) - \delta \mathbf{B}(X, t) = \xi_t^* (\delta \mathbf{A}(X, t) - \delta \mathbf{B}(X, t))$. Note that $\mathbf{F}'_{t, \epsilon} = T \varphi'_{t, \epsilon} = T(\xi_t \circ \varphi_{t, \epsilon}) = T \xi_t \cdot T \varphi_{t, \epsilon} = \xi_t^* T \varphi_{t, \epsilon}$. Therefore

$$\begin{aligned} \delta \mathbf{F}' &= \nabla_{\xi_* \frac{\partial}{\partial \epsilon}} \xi_t^* T \varphi_{t, \epsilon} \Big|_{\epsilon=0} = \xi_* \nabla_{\frac{\partial}{\partial \epsilon}} T \varphi_{t, \epsilon} \Big|_{\epsilon=0} = \xi_* \delta \mathbf{F} = \xi_* \nabla \mathbf{U} \\ &= \bar{\mathbf{F}} \cdot \nabla \mathbf{U}. \end{aligned} \quad (3.56)$$

Transformation of the linearized first and second Piola-Kirchhoff, and Cauchy stresses. The linearized first and second Piola-Kirchhoff stresses are written as $\delta \mathbf{P} = \mathbf{A} : \nabla^g \mathbf{U}$, and $\delta \mathbf{S} = \mathbf{C} : \dot{\varphi}^* \epsilon$. We know that both \mathbf{A} and \mathbf{C} are tensors and under $\xi_t : \mathcal{S} \rightarrow \mathcal{S}$ are transformed as

$$\mathbf{A}' = \xi_t^* \mathbf{A}, \quad \mathbf{A}'^{a' A}{}_{b' B} = \xi^{a'}_a (\xi^{-1})^b_{b'} \mathbf{A}^{a A}{}_{b B}, \quad \mathbf{C}' = \mathbf{C}. \quad (3.57)$$

Therefore

$$\delta \mathbf{P}' = \xi_t^* \mathbf{A} : \nabla^{\xi_t^* \mathbf{g}} \xi_t^* \mathbf{U} = \xi_t^* (\mathbf{A} : \nabla \mathbf{U}) = \xi_t^* \mathbf{P}, \quad \delta \mathbf{S}' = \delta \mathbf{S}. \quad (3.58)$$

In the case of Cauchy stress, $\delta \boldsymbol{\sigma} = \mathbb{C} : \epsilon - (\text{div}_{\mathbf{g}} \mathbf{u}) \hat{\boldsymbol{\sigma}}$, and hence

$$\begin{aligned} \delta \boldsymbol{\sigma}' &= \mathbb{C}' : \epsilon' - (\text{div}_{\mathbf{g}'} \mathbf{u}') \hat{\boldsymbol{\sigma}}' = \xi_t^* \mathbb{C} : \xi_t^* \epsilon - (\text{div}_{\xi_t^* \mathbf{g}} \xi_t^* \mathbf{u}) \xi_t^* \hat{\boldsymbol{\sigma}} \\ &= \xi_t^* [\mathbb{C} : \epsilon - (\text{div}_{\mathbf{g}} \mathbf{u}) \hat{\boldsymbol{\sigma}}] = \xi_t^* \delta \boldsymbol{\sigma}. \end{aligned} \quad (3.59)$$

Note that the elasticity tensor \mathbb{C} is transformed as

$$\mathbb{C}'^{a' b' c' d'} = \xi^{a'}_a \xi^{b'}_b \xi^{c'}_c \xi^{d'}_d \mathbb{C}^{abcd}. \quad (3.60)$$

Transformation of the linearized balance of linear momentum. The linearized spatial balance of linear momentum reads $\operatorname{div}_{\mathbf{g}}(\mathbb{C}:\boldsymbol{\epsilon}) + \operatorname{div}_{\mathbf{g}}(\nabla^{\mathbf{g}}\mathbf{u} \cdot \overset{\circ}{\boldsymbol{\sigma}}) + \overset{\circ}{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u} + \dot{\rho}(\delta\mathbf{b} - \delta\mathbf{a}) = \mathbf{0}$. Therefore, in the new spatial frame it reads

$$\begin{aligned} & \operatorname{div}_{\mathbf{g}'}(\mathbb{C}':\boldsymbol{\epsilon}') + \operatorname{div}_{\mathbf{g}'}\left(\nabla^{\mathbf{g}'}\mathbf{u}' \cdot \overset{\circ}{\boldsymbol{\sigma}'}\right) + \overset{\circ}{\boldsymbol{\sigma}'} \cdot \mathbf{Ric}'_{\mathbf{g}'} \cdot \mathbf{u}' + \dot{\rho}'(\delta\mathbf{b}' - \delta\mathbf{a}') \\ &= \operatorname{div}_{\mathbf{g}'}(\mathbb{C}':\boldsymbol{\epsilon}') + \operatorname{div}_{\xi_*\mathbf{g}}(\nabla^{\xi_*\mathbf{g}}\xi_*\mathbf{u} \cdot \xi_*\overset{\circ}{\boldsymbol{\sigma}}) + \xi_*\overset{\circ}{\boldsymbol{\sigma}} \cdot \xi_*\mathbf{Ric}_{\mathbf{g}} \cdot \xi_*\mathbf{u} \\ & \quad + (\dot{\rho} \circ \xi)\xi_*(\delta\mathbf{b} - \delta\mathbf{a}) \\ &= \xi_*\left[\operatorname{div}_{\mathbf{g}}(\mathbb{C}:\boldsymbol{\epsilon}) + \operatorname{div}_{\mathbf{g}}(\nabla^{\mathbf{g}}\mathbf{u} \cdot \overset{\circ}{\boldsymbol{\sigma}}) + \overset{\circ}{\boldsymbol{\sigma}} \cdot \mathbf{Ric}_{\mathbf{g}} \cdot \mathbf{u} + \dot{\rho}(\delta\mathbf{b} - \delta\mathbf{a})\right] = \mathbf{0}, \end{aligned} \quad (3.61)$$

i.e., the linearized balance of linear momentum is spatially covariant.

The linearized material balance of linear momentum reads $\operatorname{Div}(\mathbf{A}:\nabla_0^{\mathbf{g}}\mathbf{U}) + \mathring{\mathbf{F}}\mathring{\mathbf{P}}^* \cdot \mathbf{Ric}_{\mathbf{g}} \circ \dot{\boldsymbol{\varphi}} \cdot \mathbf{U} + \rho_0(\delta\mathbf{B} - \delta\mathbf{A}) = \mathbf{0}$. In the new spatial frame this reads

$$\begin{aligned} & \operatorname{Div}'(\mathbf{A}':\nabla\mathbf{U}') + \mathring{\mathbf{F}}'\mathring{\mathbf{P}}'^* \cdot \mathbf{Ric}'_{\mathbf{g}'} \circ \dot{\boldsymbol{\varphi}}' \cdot \mathbf{U}' + \rho'_0(\delta\mathbf{B}' - \delta\mathbf{A}') \\ &= \xi_*\left[\operatorname{Div}(\mathbf{A}:\nabla\mathbf{U}) + \mathring{\mathbf{F}}\mathring{\mathbf{P}}^* \cdot \mathbf{Ric}_{\mathbf{g}} \circ \dot{\boldsymbol{\varphi}} \cdot \mathbf{U} + \rho_0(\delta\mathbf{B} - \delta\mathbf{A})\right] = \mathbf{0}. \end{aligned} \quad (3.62)$$

The proofs of the spatial covariance of the balance of angular momentum and conservation of mass are straightforward.

3.2. Material Covariance of Linearized Elasticity

In this section, we find the transformations of the fields and governing equations of linearized elasticity under an arbitrary time-independent material diffeomorphism (change of coordinates in the reference configuration). Consider a referential change of frame (or a coordinate transformation in the reference configuration) $\mathcal{E}:\mathcal{B}\rightarrow\tilde{\mathcal{B}}$. Under this change of frame, the one-parameter family of deformations $\varphi_{t,\epsilon}$ is transformed to $\tilde{\varphi}_{t,\epsilon} = \varphi_{t,\epsilon} \circ \mathcal{E}^{-1}:\mathcal{B}\rightarrow\mathcal{S}$. The transformed variation field is defined as

$$\delta\tilde{\varphi}_t(\tilde{X}) = \tilde{\mathbf{U}}(\tilde{X}, t) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\varphi_{t,\epsilon}(\mathcal{E}^{-1}(\tilde{X})) = \delta\varphi_t \circ \mathcal{E}^{-1}(\tilde{X}) = \mathbf{U}(\mathcal{E}^{-1}(\tilde{X}), t). \quad (3.63)$$

Hence, $\delta\tilde{\varphi}_t = \delta\varphi_t \circ \mathcal{E}^{-1} = \mathbf{U} \circ \mathcal{E}^{-1}$.

Transformation of the linearized velocity, acceleration, and deformation gradient. Linearization of the material velocity with respect to the new reference configuration is calculated as

$$\delta\tilde{\mathbf{V}}(\tilde{X}, t) = \left.\nabla_{\frac{\partial}{\partial\epsilon}}\frac{\partial\tilde{\varphi}_{t,\epsilon}(\tilde{X})}{\partial t}\right|_{\epsilon=0} = \left.\nabla_{\frac{\partial}{\partial\epsilon}}\frac{\partial\varphi_{t,\epsilon}(\mathcal{E}^{-1}(\tilde{X}))}{\partial t}\right|_{\epsilon=0} = \delta\mathbf{V}(\mathcal{E}^{-1}(\tilde{X}), t). \quad (3.64)$$

Thus, $\delta\tilde{\mathbf{V}} = \delta\mathbf{V} \circ \mathcal{E}^{-1} = \dot{\mathbf{U}} \circ \mathcal{E}^{-1}$.

Linearization of the material acceleration with respect to the new reference configuration is calculated as

$$\delta\tilde{\mathbf{A}}(\tilde{X}, t) = \left.\nabla_{\frac{\partial}{\partial\epsilon}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\tilde{\varphi}_{t,\epsilon}(\tilde{X})}{\partial t}\right|_{\epsilon=0}$$

$$\begin{aligned}
 &= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial \epsilon}} \frac{\partial \varphi_{t,\epsilon}(\mathcal{E}^{-1}(\tilde{X}))}{\partial t} \Big|_{\epsilon=0} + \nabla_{[\frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial t}]} \frac{\partial \varphi_{t,\epsilon}(\mathcal{E}^{-1}(\tilde{X}))}{\partial t} \Big|_{\epsilon=0} \\
 &\quad + \mathcal{R}_{\mathbf{g}} \left(\frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial t} \right) \frac{\partial \varphi_{t,\epsilon}(\mathcal{E}^{-1}(\tilde{X}))}{\partial t} \Big|_{\epsilon=0} \\
 &= \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial \varphi_{t,\epsilon}(\mathcal{E}^{-1}(\tilde{X}))}{\partial \epsilon} \Big|_{\epsilon=0} + \nabla_{[\tilde{\mathbf{U}}, \tilde{\mathbf{V}}]} \tilde{\mathbf{V}} + \mathcal{R}_{\mathbf{g}}(\tilde{\mathbf{U}}, \tilde{\mathbf{V}}, \tilde{\mathbf{V}}) \\
 &= \delta \mathbf{A}(\mathcal{E}^{-1}(\tilde{X}), t), \tag{3.65}
 \end{aligned}$$

i.e., $\delta \tilde{\mathbf{A}} = \delta \mathbf{A} \circ \mathcal{E}^{-1}$.

Deformation gradient for the motion $\tilde{\varphi}_{t,\epsilon}$ is written as $\tilde{\mathbf{F}}_{t,\epsilon} = T \tilde{\varphi}_{t,\epsilon} = T(\varphi_{t,\epsilon} \circ \mathcal{E}^{-1}) = T \varphi_{t,\epsilon} \circ (T\mathcal{E})^{-1} = \mathcal{E}_{t*} T \varphi_{t,\epsilon}$. Therefore

$$\delta \tilde{\mathbf{F}} = \nabla_{\mathcal{E}_{t*} \frac{\partial}{\partial \epsilon}} \mathcal{E}_{t*} T \varphi_{t,\epsilon} \Big|_{\epsilon=0} = \mathcal{E}_{t*} \nabla_{\frac{\partial}{\partial \epsilon}} T \varphi_{t,\epsilon} \Big|_{\epsilon=0} = \mathcal{E}_{t*} \delta \mathbf{F} = \mathcal{E}_{t*} \nabla \mathbf{U}. \tag{3.66}$$

In components, $(\delta \tilde{F})^a_{\tilde{A}} = (\tilde{F}^{-1})^A_{\tilde{A}} U^a_{|A}$.

Transformation of the linearized first Piola-Kirchhoff stress and the first elasticity tensor. With respect to the new reference configuration (cf. (2.13))

$$\begin{aligned}
 \tilde{W}_\epsilon &= W(\tilde{X}, \tilde{\mathbf{F}}_\epsilon, \tilde{\mathbf{G}}, \mathbf{g} \circ \tilde{\varphi}_\epsilon, \tilde{\zeta}_1, \dots, \tilde{\zeta}_n) \\
 &= W(X, \mathcal{E}_{t*} \mathbf{F}_\epsilon, \mathcal{E}_{t*} \mathbf{G}, \mathbf{g} \circ \varphi_\epsilon \circ \mathcal{E}^{-1}, \mathcal{E}_{t*} \zeta_1, \dots, \mathcal{E}_{t*} \zeta_n) = \mathcal{E}_{t*} W_\epsilon.
 \end{aligned} \tag{3.67}$$

The above relation, in particular, implies that $\delta \tilde{W} = \mathcal{E}_{t*} \delta W$. Note that $\delta W = \frac{\partial W}{\partial \mathbf{F}} \cdot \delta \mathbf{F}$. Similarly

$$\delta \tilde{W} = \frac{\partial \tilde{W}}{\partial \tilde{\mathbf{F}}} \cdot \delta \tilde{\mathbf{F}} = \frac{\partial \tilde{W}}{\partial \tilde{\mathbf{F}}} \cdot \mathcal{E}_{t*} \delta \mathbf{F} = \mathcal{E}_{t*} \left(\mathcal{E}_{t*}^* \frac{\partial \tilde{W}}{\partial \tilde{\mathbf{F}}} \cdot \delta \mathbf{F} \right). \tag{3.68}$$

Hence

$$\mathcal{E}_{t*}^* \frac{\partial \tilde{W}}{\partial \tilde{\mathbf{F}}} \cdot \nabla \mathbf{U} = \frac{\partial W}{\partial \mathbf{F}} \cdot \nabla \mathbf{U}, \quad \forall \mathbf{U}. \tag{3.69}$$

This implies that

$$\frac{\partial \tilde{W}}{\partial \tilde{\mathbf{F}}} = \mathcal{E}_{t*}^* \frac{\partial W}{\partial \mathbf{F}}. \tag{3.70}$$

Therefore

$$\delta \tilde{\mathbf{P}} = \mathbf{g}^\sharp \nabla_{\frac{\partial}{\partial \epsilon}} \frac{\partial \tilde{W}_\epsilon}{\partial \tilde{\mathbf{F}}_\epsilon} \Big|_{\epsilon=0} = \mathbf{g}^\sharp \nabla_{\frac{\partial}{\partial \epsilon}} \mathcal{E}_{t*}^* \frac{\partial W_\epsilon}{\partial \mathbf{F}_\epsilon} \Big|_{\epsilon=0} = \mathcal{E}_{t*}^* \left(\mathbf{g}^\sharp \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} \cdot \nabla \mathbf{U} \right). \tag{3.71}$$

Hence, $\delta \tilde{\mathbf{P}} = \tilde{\mathbf{A}} \cdot \nabla \tilde{\mathbf{U}} = \mathcal{E}_{t*}^* \mathbf{A} \cdot \mathcal{E}_{t*}(\nabla \mathbf{U})$. In particular, the first elasticity tensor is transformed as

$$\tilde{\mathbf{A}}^{a\tilde{A}}_{\tilde{b}\tilde{B}} = \tilde{F}^{\tilde{A}}_{\tilde{A}} \tilde{F}^{\tilde{B}}_{\tilde{B}} \mathbf{A}^{aA}_{bB}. \tag{3.72}$$

Transformation of the second Piola-Kirchhoff stress and the second elasticity tensor. Note that the second elasticity tensor for hyperelastic solids (for which an energy function exists) is defined in components as $\mathbf{C}^{ABCD} = 4 \frac{\partial^2 W}{\partial C_{AB} \partial C_{CD}}$. Under a referential change of coordinates \mathbf{C} is transformed to $\tilde{\mathbf{C}} = \mathcal{E}_* \mathbf{C}$, which in components reads

$$\tilde{C}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = \tilde{F}^{\tilde{A}}_{\tilde{A}} \tilde{F}^{\tilde{B}}_{\tilde{B}} \tilde{F}^{\tilde{C}}_{\tilde{C}} \tilde{F}^{\tilde{D}}_{\tilde{D}} C^{ABCD} \circ \mathcal{E}^{-1}. \quad (3.73)$$

It then immediately follows that if \mathbf{C} possesses the minor ($\mathbf{C}^{ABCD} = \mathbf{C}^{BACD} = \mathbf{C}^{ABDC}$) and major ($\mathbf{C}^{ABCD} = \mathbf{C}^{CDAB}$) symmetries, so does $\tilde{\mathbf{C}}$, i.e., the major and minor symmetries of the elasticity tensor are preserved under a material diffeomorphism. The symmetry group $\text{Sym}_{\mathbf{G}}(\mathbf{T})$ of an m -order tensor field \mathbf{T} is the subgroup of \mathbf{G} -orthogonal transformations defined as

$$\langle \mathbf{Q} \rangle_m \mathbf{T} = \mathbf{T}, \quad \forall \mathbf{Q} \in \text{Sym}_{\mathbf{G}}(\mathbf{T}) \leq \text{Orth}(\mathbf{G}). \quad (3.74)$$

The space of elasticity tensors $\mathbb{E}la$ (consisting of all those fourth-order tensors satisfying the major and minor symmetries) may be endowed with an equivalence relation such that

$$\mathbf{C}_1 \sim \mathbf{C}_2 \iff \exists \mathbf{Q} \in \text{Orth}(\mathbf{G}) : \langle \mathbf{Q} \rangle_4 \mathbf{C}_2 = \mathbf{C}_1. \quad (3.75)$$

Equivalently¹⁷

$$\mathbf{C}_1 \sim \mathbf{C}_2 \iff \exists \mathbf{Q} \in \text{Orth}(\mathbf{G}) : \text{Sym}_{\mathbf{G}}(\mathbf{C}_1) = \mathbf{Q} \text{Sym}_{\mathbf{G}}(\mathbf{C}_2) \mathbf{Q}^T. \quad (3.76)$$

That is, two elasticity tensors are equivalent if and only if their symmetry groups are conjugate subgroups of $SO(3)$. Thus, by an abuse of notation, one may write $\mathbf{C}_1 \sim \mathbf{C}_2$ if and only if $\text{Sym}_{\mathbf{G}}(\mathbf{C}_1) \sim \text{Sym}_{\mathbf{G}}(\mathbf{C}_2)$, where use was made of the fact that conjugacy is also an equivalence relation. Under this equivalence relation, $\mathbb{E}la$ is divided into eight equivalence classes, known as *symmetry classes*, namely triclinic, monoclinic, trigonal, orthotropic, tetragonal, cubic, transversely isotropic, and isotropic (for proofs, see [11, 32]¹⁸). We note that under the material diffeomorphism \mathcal{E} the elasticity tensor is transformed such that $\tilde{\mathbf{C}} = \mathcal{E}_* \mathring{\mathbf{C}}$, where $\mathring{\mathbf{C}}$ is the linearized elasticity tensor with respect to the reference motion $\tilde{\phi}_t$. Therefore, from (3.73), it follows that $\tilde{\mathbf{C}} = \langle \tilde{\mathbf{F}} \rangle_4 (\mathring{\mathbf{C}} \circ \mathcal{E}^{-1})$. It is straightforward to verify that [65]

$$\mathbf{Q} \in \text{Orth}(\mathbf{G}) \iff \mathcal{E}_* \mathbf{Q} = \tilde{\mathbf{F}} \mathbf{Q} \tilde{\mathbf{F}}^{-1} \in \text{Orth}(\tilde{\mathbf{G}}), \quad (3.77)$$

where $\tilde{\mathbf{G}} = \mathcal{E}_* \mathbf{G}$. Using the properties of the Kronecker product (see, e.g., [132]), one concludes that

$$\text{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathbf{C}}) = \tilde{\mathbf{F}} \text{Sym}_{\mathbf{G}}(\mathring{\mathbf{C}}) \tilde{\mathbf{F}}^{-1}. \quad (3.78)$$

¹⁷ Note that (3.75) and (3.76) imply that two elasticity tensors are equivalent (represent the same type of material anisotropy) if and only if there exists an orthogonal transformation such that under its action the two elasticity tensors (or their symmetry groups) coincide.

¹⁸ Also, see [82] for calculation of the symmetry classes of an even-order tensor space.

In other words, the symmetry groups of $\tilde{\mathbf{C}}$ and $\mathring{\mathbf{C}}$ are conjugate, and hence, isomorphic.¹⁹ It then follows from (3.76) and (3.78) that

$$\begin{aligned} \mathring{\mathbf{C}}_1 \sim \mathring{\mathbf{C}}_2 &\iff \exists \mathbf{Q} \in \text{Orth}(\mathbf{G}) : \text{Sym}_{\mathbf{G}}(\mathring{\mathbf{C}}_1) = \mathbf{Q} \text{Sym}_{\mathbf{G}}(\mathring{\mathbf{C}}_2) \mathbf{Q}^T \\ &\iff \exists \tilde{\mathbf{Q}} \in \text{Orth}(\tilde{\mathbf{G}}) : \text{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathbf{C}}_1) = \tilde{\mathbf{Q}} \text{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathbf{C}}_2) \tilde{\mathbf{Q}}^T \quad (3.79) \\ &\iff \tilde{\mathbf{C}}_1 \sim \tilde{\mathbf{C}}_2, \end{aligned}$$

where $\tilde{\mathbf{Q}} = \tilde{\mathcal{E}}_* \mathbf{Q}$. Therefore, $\mathring{\mathbf{C}}_1 \sim \mathring{\mathbf{C}}_2 \iff \tilde{\mathbf{C}}_1 \sim \tilde{\mathbf{C}}_2$, i.e., there is a one-to-one correspondence between the symmetry classes of the elasticity tensors in the initial and transformed reference configurations. It is important to notice that the symmetry of a tensor explicitly depends on the metric. In particular, a tensor, e.g., the elasticity tensor, may belong to a different type of symmetry class with respect to a different metric under consideration. Note that if $\tilde{\mathbf{F}}^{-*} \mathbf{G} \tilde{\mathbf{F}}^{\tilde{\mathbf{F}}}^{-1} = \mathbf{G}$, i.e., if $\tilde{\mathbf{F}}$ is \mathbf{G} -orthogonal ($\in \text{Orth}(\mathbf{G})$), then the symmetry class is trivially unaffected under a change of reference configuration. MAZZUCATO and RACHELE [69] characterized the orbits of different symmetry classes of the elasticity tensor (i.e., $\psi^* \mathbf{C}$, where ψ is a diffeomorphism in the material manifold) with respect to the Euclidean metric under a change of reference configuration that fixes the boundary to the first-order. In particular, they found that the orbit of isotropic materials consists of some orthotropic and some transversely isotropic materials, in addition to isotropic materials. As they describe the material symmetry groups of \mathbf{C} and $\tilde{\mathbf{C}}$ with respect to the Euclidean metric for both configurations, they are implicitly characterizing the symmetry group of the transformed elasticity tensor $\tilde{\mathbf{C}}$ with respect to the untransformed metric \mathbf{G} , i.e., $\text{Sym}_{\mathbf{G}}(\tilde{\mathbf{C}})$.

Next, using (3.78) and (3.79), we show that symmetry classes are preserved under a change of reference configuration. In other words, the elasticity tensor $\mathring{\mathbf{C}}$ belongs to a given symmetry class if and only if $\tilde{\mathbf{C}}$ belongs to the same symmetry class in the transformed reference configuration. For instance, let $\mathring{\mathbf{C}}$ belong to the transversely isotropic symmetry class in the initial reference configuration at a material point X with the unit vector $\mathbf{N} \in T_X \mathcal{B}$ identifying the material preferred direction. It follows that the symmetry group of $\mathring{\mathbf{C}}$ is given by

$$\text{Sym}_{\mathbf{G}}(\mathring{\mathbf{C}}) = \left\{ \mathbf{Q} \in \text{Orth}(\mathbf{G}) : \mathbf{Q}\mathbf{N} = \pm\mathbf{N} \right\}. \quad (3.80)$$

From (3.78)

$$\text{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathbf{C}}) = \left\{ \tilde{\mathbf{Q}} \in \text{Orth}(\tilde{\mathbf{G}}) : \tilde{\mathbf{Q}} = \tilde{\mathbf{F}} \mathbf{Q} \tilde{\mathbf{F}}^{\tilde{\mathbf{F}}}^{-1}, \mathbf{Q} \in \text{Sym}_{\mathbf{G}}(\mathring{\mathbf{C}}) \right\}. \quad (3.81)$$

¹⁹ The isomorphism is trivially given by the conjugacy relations as follows. Let $\phi : \text{Sym}_{\mathbf{G}}(\mathring{\mathbf{C}}) \rightarrow \text{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathbf{C}})$ and let $\mathbf{H} \in \text{Sym}_{\mathbf{G}}(\mathring{\mathbf{C}})$. Then, $\phi(\mathbf{H}) = \tilde{\mathbf{F}} \mathbf{H} \tilde{\mathbf{F}}^{\tilde{\mathbf{F}}}^{-1} \in \text{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathbf{C}})$. It is straightforward to see that $\phi(\mathbf{H}_1 \mathbf{H}_2) = \phi(\mathbf{H}_1) \phi(\mathbf{H}_2)$ for $\mathbf{H}_1, \mathbf{H}_2 \in \text{Sym}_{\mathbf{G}}(\mathring{\mathbf{C}})$. Also, it is straightforward to see that ϕ is one-to-one, and thus, an isomorphism.

The material preferred direction is transformed to $\tilde{\mathbf{N}} = \tilde{\mathbf{F}}\mathbf{N}$. Clearly, employing (3.80) and (3.81), $\forall \tilde{\mathbf{Q}} \in \text{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathbf{C}})$ one has $\tilde{\mathbf{Q}}\tilde{\mathbf{N}} = \pm\tilde{\mathbf{N}}$. Moreover, if $\tilde{\mathbf{Q}} \in \text{Orth}(\tilde{\mathbf{G}})$ such that $\tilde{\mathbf{Q}}\tilde{\mathbf{N}} = \pm\tilde{\mathbf{N}}$, then from (3.77), there exists $\mathbf{Q} = \tilde{\mathbf{F}}^{-1}\tilde{\mathbf{Q}}\tilde{\mathbf{F}} \in \text{Orth}(\tilde{\mathbf{G}})$, for which one concludes that $\mathbf{Q}\mathbf{N} = \pm\mathbf{N}$, i.e., $\mathbf{Q} \in \text{Sym}_{\mathbf{G}}(\mathbf{C})$. Thus

$$\text{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathbf{C}}) = \left\{ \tilde{\mathbf{Q}} \in \text{Orth}(\tilde{\mathbf{G}}) : \tilde{\mathbf{Q}}\tilde{\mathbf{N}} = \pm\tilde{\mathbf{N}} \right\}, \quad (3.82)$$

and hence, $\tilde{\mathbf{C}}$ belongs to the transversely isotropic symmetry class in the transformed reference configuration. The generalization of this proof to the orthotropic case is immediate. One only needs to consider three \mathbf{G} -orthonormal vectors $\mathbf{N}_1, \mathbf{N}_2,$ and \mathbf{N}_3 specifying the orthotropic axes and define the symmetry group in the orthotropic case as $\mathbf{Q} \in \text{Orth}(\mathbf{G})$ such that $\mathbf{Q}\mathbf{N}_i = \pm\mathbf{N}_i, i = 1, 2, 3$. One can similarly prove that other symmetry classes are preserved under material diffeomorphisms as well. For the isotropic case, it immediately follows from (3.77) and (3.78) that $\text{Sym}_{\mathbf{G}}(\mathbf{C}) = \text{Orth}(\mathbf{G}) \iff \text{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathbf{C}}) = \text{Orth}(\tilde{\mathbf{G}})$. Therefore, we have proved the following result:

Proposition 1. *The elasticity tensor \mathbf{C} in the reference configuration $(\mathcal{B}, \mathbf{G})$ with respect to the reference motion $\tilde{\varphi} : \mathcal{B} \rightarrow \mathcal{S}$ belongs to a given symmetry class if and only if $\tilde{\mathbf{C}}$, which is the elasticity tensor in the transformed reference configuration $(\tilde{\mathcal{B}}, \mathcal{E}_*\mathbf{G})$ with respect to the reference motion $\tilde{\tilde{\varphi}} : \tilde{\mathcal{B}} \rightarrow \mathcal{S}$, belongs to the same symmetry class.*

Using (3.13), one can write

$$\delta\tilde{\mathbf{S}} = 2 \frac{\partial^2 \tilde{W}_\epsilon}{\partial \tilde{\mathbf{C}}_\epsilon^a \partial \tilde{\mathbf{C}}_\epsilon^b} : \frac{d}{d\epsilon} \tilde{\mathbf{C}}_\epsilon^b \Big|_{\epsilon=0} = \mathcal{E}_* \left(4 \frac{\partial^2 \hat{W}}{\partial \mathbf{C}^b \partial \mathbf{C}^b} : \tilde{\varphi}^* \epsilon \right) = \tilde{\mathbf{C}} : \tilde{\varphi}^* \epsilon, \quad (3.83)$$

where $\tilde{\mathbf{C}} = \mathcal{E}_*\mathbf{C}$ and $\tilde{\varphi}^* \epsilon = \mathcal{E}_* \circ \tilde{\varphi}^* \epsilon = \mathcal{E}_*(\tilde{\varphi}^* \epsilon)$, and thus, $\delta\tilde{\mathbf{S}} = \mathcal{E}_*\mathbf{C} : \mathcal{E}_*(\tilde{\varphi}^* \epsilon)$.

Transformation of the linearized balance of linear and angular momenta. The linearized spatial balance of linear momentum with respect to the new reference configuration reads

$$\begin{aligned} \text{div}_{\tilde{\mathbf{g}}}(\tilde{\mathbf{C}} : \tilde{\boldsymbol{\epsilon}}) + \text{div}_{\tilde{\mathbf{g}}}(\nabla^{\tilde{\mathbf{g}}} \tilde{\mathbf{u}} \cdot \tilde{\boldsymbol{\sigma}}) + \tilde{\boldsymbol{\sigma}} \cdot \tilde{\mathbf{Ric}}_{\tilde{\mathbf{g}}} \cdot \tilde{\mathbf{u}} + (\tilde{\rho} \nabla_{\tilde{\mathbf{u}}}^{\tilde{\mathbf{g}}} \tilde{\mathbf{b}}) \circ \mathcal{E}^{-1} \\ = \tilde{\rho} \left[\tilde{\mathbf{U}} \circ \tilde{\varphi}_t^{-1} + \nabla_{[\mathbf{u}, \hat{\mathbf{v}}]}^{\tilde{\mathbf{g}}} \hat{\mathbf{v}} + \mathcal{R}_{\tilde{\mathbf{g}}}(\mathbf{u}, \hat{\mathbf{v}}, \hat{\mathbf{v}}) \right] \circ \mathcal{E}^{-1}. \end{aligned} \quad (3.84)$$

Note that

$$\tilde{\mathbf{C}} = \frac{1}{\tilde{\rho}} \tilde{\varphi}_* \tilde{\mathbf{C}} = \frac{1}{\tilde{\rho}} (\tilde{\varphi} \circ \mathcal{E}^{-1})_* \mathcal{E}_*\mathbf{C} = \frac{1}{\tilde{\rho}} \tilde{\varphi}_* \mathbf{C} = \mathbf{C}. \quad (3.85)$$

Knowing that all the spatial tensors transform like a scalar, i.e., $\tilde{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon} \circ \mathcal{E}^{-1}$, $\tilde{\mathbf{g}} = \mathbf{g} \circ \mathcal{E}^{-1}$, $\tilde{\mathbf{Ric}}_{\tilde{\mathbf{g}}} = \mathbf{Ric}_{\mathbf{g}} \circ \mathcal{E}^{-1}$, $\tilde{\mathbf{u}} = \mathbf{u} \circ \mathcal{E}^{-1}$, and $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \mathcal{E}^{-1}$, one concludes that the linearized spatial balance of linear momentum is materially covariant.

Similarly, the linearized material balance of linear momentum transforms like a scalar, i.e.,

$$\begin{aligned} & \widetilde{\text{Div}}(\widetilde{\mathbf{A}} : \nabla \widetilde{\mathbf{U}}) + \widetilde{\mathbf{F}}\widetilde{\mathbf{P}}^* \cdot \widetilde{\mathbf{Ric}}_{\mathbf{g}} \circ \widetilde{\varphi} \cdot \widetilde{\mathbf{U}} + \widetilde{\rho}_0 \nabla_{\widetilde{\mathbf{U}}} \widetilde{\mathbf{B}} \\ & - \widetilde{\rho}_0 \left[\ddot{\widetilde{\mathbf{U}}} + \nabla_{[\widetilde{\mathbf{U}}, \widetilde{\mathbf{V}}]} \widetilde{\mathbf{V}} + \widetilde{\mathcal{R}}_{\mathbf{g}}(\widetilde{\mathbf{U}}, \dot{\widetilde{\mathbf{U}}}, \dot{\widetilde{\mathbf{V}}}) \right] = \left\{ \text{Div}(\mathbf{A} : \nabla \mathbf{U}) + \mathbf{F}\mathbf{P}^* \cdot \mathbf{Ric}_{\mathbf{g}} \circ \varphi \cdot \mathbf{U} \right. \\ & \left. + \rho_0 \nabla_{\mathbf{U}} \mathbf{B} - \rho_0 \left[\ddot{\mathbf{U}} + \nabla_{[\mathbf{U}, \mathbf{V}]} \mathbf{V} + \mathcal{R}_{\mathbf{g}}(\mathbf{U}, \dot{\mathbf{U}}, \dot{\mathbf{V}}) \right] \right\} \circ \mathcal{E}^{-1} = \mathbf{0}. \end{aligned} \quad (3.86)$$

The balance of angular momentum transforms like a scalar as well.

Wave equation and its spatial and material covariance. Let us assume that $\mathbf{u}(x, t) = \bar{\mathbf{u}}(x)e^{-i\omega t}$, where ω is the frequency. For a Euclidean ambient space, the wave equation is written as $\text{div}_{\mathbf{g}}(\mathbb{C} : \bar{\boldsymbol{\epsilon}}) + \text{div}_{\mathbf{g}}(\nabla^{\mathbf{g}} \bar{\mathbf{u}} \cdot \bar{\boldsymbol{\sigma}}) + \dot{\rho} \nabla_{\bar{\mathbf{u}}}^{\mathbf{g}} \bar{\mathbf{b}} = \dot{\rho} \left[-\omega^2 \bar{\mathbf{u}} + \nabla_{[\bar{\mathbf{u}}, \dot{\bar{\mathbf{V}}}]^{\mathbf{g}}} \dot{\bar{\mathbf{V}}}] \right]$, where $2\bar{\boldsymbol{\epsilon}} = \nabla^{\mathbf{g}} \bar{\mathbf{u}}^{\flat} + (\nabla^{\mathbf{g}} \bar{\mathbf{u}}^{\flat})^*$. Similarly, in material form $\mathbf{U}(X, t) = \bar{\mathbf{U}}(X)e^{-i\omega t}$, and the wave equation in material form reads

$$\text{Div}(\mathbf{A} : \nabla \bar{\mathbf{U}}) + \rho_0 \nabla_{\bar{\mathbf{U}}} \bar{\mathbf{B}} = \rho_0 \left[-\omega^2 \bar{\mathbf{U}} + \nabla_{[\bar{\mathbf{U}}, \dot{\bar{\mathbf{V}}}]^{\mathbf{g}}} \dot{\bar{\mathbf{V}}} \right]. \quad (3.87)$$

If the reference motion is static, i.e., $\dot{\bar{\mathbf{V}}} = \mathbf{0}$ and body force is ignored, the spatial and material wave equations are simplified to read

$$\begin{aligned} & \text{div}_{\mathbf{g}}(\mathbb{C} : \nabla^{\mathbf{g}} \bar{\mathbf{u}}^{\flat}) + \text{div}_{\mathbf{g}}(\nabla^{\mathbf{g}} \bar{\mathbf{u}} \cdot \bar{\boldsymbol{\sigma}}) + \dot{\rho} \omega^2 \bar{\mathbf{u}} = \mathbf{0}, \\ & \text{Div}(\mathbf{A} : \nabla \bar{\mathbf{U}}) + \rho_0 \omega^2 \bar{\mathbf{U}} = \mathbf{0}. \end{aligned} \quad (3.88)$$

Note that as a consequence of the spatial and material covariance of the balance of linear momentum and its linearization, the wave equation is form-invariant under arbitrary spatial coordinate transformations and arbitrary time-independent referential coordinate transformations.

4. A Mathematical Formulation of the Problem of Cloaking a Cavity in Nonlinear and Linearized Elastodynamics

Suppose an object is to be hidden from elastic waves. This object lies in a cavity inside an elastic body. The hole needs to be reinforced by a cloak, which is a layer of specially designed inhomogeneous and anisotropic material that will deflect away any incoming elastic waves from the cloaked object. For an observer away from the hole the elastic waves are passing through the body as if there was no hole. Let us assume that the body is homogeneous and is made of an isotropic material. The idea of elastodynamic transformation cloaking is to first map the stress-free body in its reference configuration to a corresponding homogeneous and isotropic body in its stress-free reference configuration. We assume that the homogeneous and isotropic transformed body (virtual body) has a very small hole of radius ϵ ($\epsilon \rightarrow 0$). Consider a map that shrinks the hole to a very small hole and is the identity outside the cloak. Note that there are many such mappings. Next, the important requirement is that the

physical body with the hole and the homogeneous and isotropic virtual body must have identical current configurations outside the cloak. In other words, to an elastic wave the two bodies are identical outside the cloaking region. Inside the cloak we will impose certain requirements. The last step is to check if these requirements are enough to specify the elastic properties of the cloak, its mass density, and the external loads and the boundary conditions in the virtual body, or if they result in an overdetermined system with no solution. One should also check if all the balance laws are satisfied in both the virtual and physical bodies. To formulate this problem the understanding of the transformation properties of the governing equations of nonlinear and linearized elasticity that we established in the previous two sections is crucial.

4.1. Nonlinear Elastodynamic Transformation Cloaking

Let us consider a body \mathcal{B} with a hole \mathcal{H} (see Fig. 4). An object is placed inside \mathcal{H} and needs to be hidden from elastic waves. The hole is reinforced by a cloak \mathcal{C} , which we can assume is an annulus. The elastic properties and mass density of \mathcal{C} are, in general, inhomogeneous and anisotropic. For the sake of simplicity and without loss of generality, let us assume that in $\mathcal{B} \setminus \mathcal{C}$ the body is homogeneous and isotropic. This means that the mass density ρ_0 is a constant and the body has an energy function $W = W(I_1, I_2, I_3)$, where I_i ($i = 1, 2, 3$) are the principal invariants of the left (or right) Cauchy-Green strain. The motion of \mathcal{B} is represented by a map $\varphi_t : \mathcal{B} \rightarrow \mathcal{S}$ in Fig. 4. A spatial diffeomorphism leaves the governing equations form invariant. However, spatial diffeomorphisms (spatial coordinate transformations) would not be useful in designing a cloak because a change of spatial coordinates corresponds to the same body as seen through a warped lens — but the material is the same. The cloaking transformation is assumed to be a time-independent map $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ such that the annulus \mathcal{C} is transformed to a disk with a very small hole (or a spherical ball with a very small hole in 3D). The mapping \mathcal{E} is assumed to be the identity in $\mathcal{B} \setminus \mathcal{C}$. It is also assumed that the virtual body has the uniform and isotropic mechanical properties of $\mathcal{B} \setminus \mathcal{C}$.

One may be tempted to think that the cloak can be designed using a referential change of frame (coordinate transformation). We have shown that the governing equations of both nonlinear and linearized elasticity are spatially and referentially covariant. In other words, the governing equations of the same body are invariant under arbitrary time-dependent coordinate transformations in the current configuration and arbitrary time-independent coordinate transformations in the reference configuration. This, however, is not useful for cloaking applications, because a change of the material frame does not lead to a new elastic body. In other words, the two bodies $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{E}(\mathcal{B}), \mathcal{E}_* \mathbf{G})$ are isometric and essentially the same elastic body with the same mechanical response. One instead needs two different elastic bodies that cannot be distinguished by an elastic wave in their current configurations outside the cloak. Motion of the virtual (homogeneous and isotropic) body is represented by a map $\tilde{\varphi}_t$ as is schematically shown in Fig. 4. The spatial configurations of the two bodies with their corresponding deformations φ_t and $\tilde{\varphi}_t$ are required to be identical outside the cloaking region, i.e., in $\mathcal{B} \setminus \mathcal{C}$. In particular,

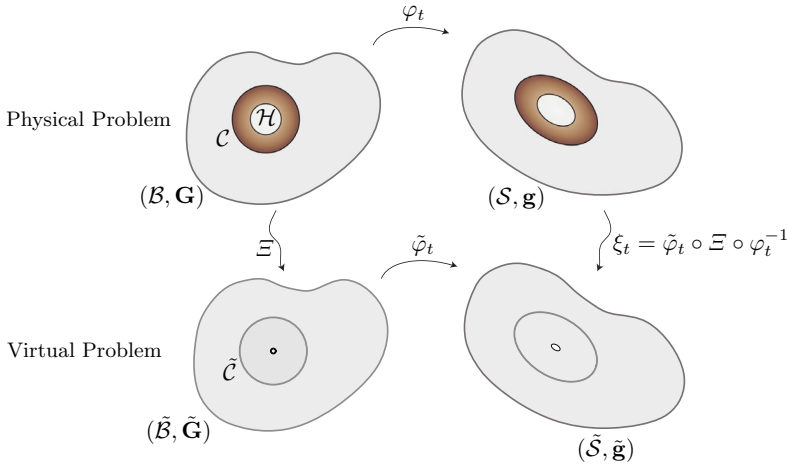


Fig. 4. A cloaking transformation \mathcal{E} transforms a body with a hole \mathcal{H} to another body with an infinitesimal hole that is homogeneous and isotropic. The cloaking transformation is defined to be the identity map outside the cloak \mathcal{C} . Note that \mathcal{E} is not a referential change of coordinates and ξ is not a spatial change of coordinates

this implies that any elastic measurements made in the current configurations of the two bodies outside the cloak are identical, and hence, the two bodies cannot be distinguished by an observer located in $\mathcal{B} \setminus \mathcal{C}$. Moreover, in the virtual body $\tilde{\mathcal{B}}$, the influence of the hole is negligible as \mathcal{H} in \mathcal{B} is mapped to an infinitesimal hole in $\tilde{\mathcal{B}}$. It is important to notice that our approach does not impose any restrictions on the size of the hole, and thus, that of the concealment as illustrated in Fig. 4.

Let us consider a body (physical body) that has a material manifold $(\mathcal{B}, \mathbf{G})$ and is in a time-dependent current configuration $\varphi_t(\mathcal{B})$. Suppose that the stress-free reference configuration of the body is in a one-to-one correspondence with that of another body $\tilde{\mathcal{B}}$ (virtual body) in its reference configuration $(\tilde{\mathcal{B}}, \tilde{\mathbf{G}})$ (see Fig. 4). Let us denote the bijection between the two bodies by $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$, which is a diffeomorphism. We assume that the two stress-free bodies are embedded in the Euclidean space, and hence, \mathbf{G} and $\tilde{\mathbf{G}}$ are their corresponding induced Euclidean metrics. This immediately implies that \mathcal{E} is not a simple change of material frame (referential coordinate transformation) as $\tilde{\mathbf{G}} \neq \mathcal{E}_* \mathbf{G}$, in general. The boundary-value problems related to the motions $\varphi_t : \mathcal{B} \rightarrow \mathcal{S}$ and $\tilde{\varphi}_t : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{S}}$ are called the physical and virtual problems, respectively. The deformation gradients corresponding to the physical and virtual problems are denoted by $\mathbf{F} = T\varphi_t$ and $\tilde{\mathbf{F}} = T\tilde{\varphi}_t$, respectively. The energy function of the physical problem in $\mathcal{B} \setminus \mathcal{C}$ is known, while it is not known a priori in \mathcal{C} . The energy function of the virtual body is known everywhere in $\tilde{\mathcal{B}}$. The applied loads (body force and traction boundary conditions) and the essential boundary conditions are given for the physical problem, but are not, however, known a priori for the virtual problem.

Shifters in Euclidean ambient space. We assume that the reference configurations of both the physical and virtual bodies are embedded in the Euclidean space. To relate vector fields in the physical problem to those in the virtual problem we would need to use shifters. We first consider the physical body $\mathcal{B} \subset \mathcal{S} = \mathbb{R}^n$ ($n = 2$ or 3).

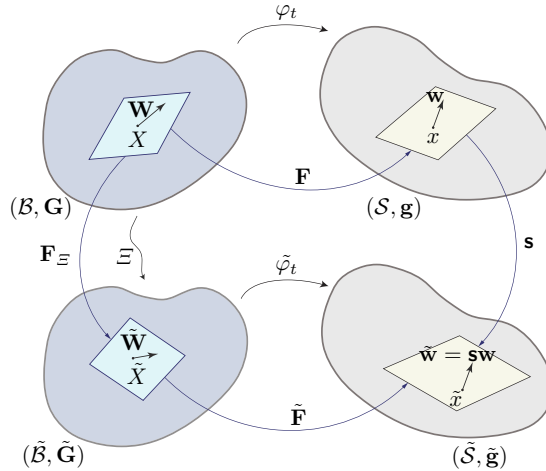


Fig. 5. The mapping \mathbf{s} is the shifter map, which parallel transports vectors in $T_x\mathcal{S}$ to vectors in $T_{\tilde{x}}\tilde{\mathcal{S}}$

The map $\mathbf{s} : T\mathcal{S} \rightarrow T\tilde{\mathcal{S}}$, $\mathbf{s}(x, \mathbf{w}) = (\tilde{x}, \mathbf{w})$ is called the shifter map (see Fig. 5). The restriction of \mathbf{s} to $x \in \mathcal{S}$ is denoted by $\mathbf{s}_x = \mathbf{s}(x) : T_x\mathcal{S} \rightarrow T_{\tilde{x}}\tilde{\mathcal{S}}$, and shifts \mathbf{w} based at $x \in \mathcal{S}$ to \mathbf{w} based at $\tilde{x} \in \tilde{\mathcal{S}}$ (shifter map in the reference configuration is defined similarly). For $\tilde{\mathcal{S}}$ and \mathcal{S} we choose two global colinear Cartesian coordinates $\{\tilde{z}^i\}$ and $\{z^i\}$ for the virtual and physical deformed configurations, respectively. We also use curvilinear coordinates $\{\tilde{x}^a\}$ and $\{x^a\}$ for these configurations. Note that $\tilde{s}^i_i = \delta^i_i$. One can show that [68]

$$\mathbf{s}^{\tilde{a}}_a(x) = \frac{\partial \tilde{x}^{\tilde{a}}}{\partial \tilde{z}^i}(\tilde{x}) \frac{\partial z^i}{\partial x^a}(x) \delta^i_i. \tag{4.1}$$

Note that \mathbf{s} preserves inner products, and hence, $\mathbf{s}^T = \mathbf{s}^{-1}$. In components, $(\mathbf{s}^T)^a_{\tilde{a}} = g^{ab} \mathbf{s}^{\tilde{b}}_b \tilde{g}^{\tilde{a}}_{\tilde{b}}$. Note also that

$$\mathbf{s}^{\tilde{a}}_{a|\tilde{b}} = \frac{\partial \mathbf{s}^{\tilde{a}}_a}{\partial \tilde{x}^{\tilde{b}}} + \tilde{\gamma}^{\tilde{a}}_{\tilde{b}\tilde{c}} \mathbf{s}^{\tilde{c}}_a - \frac{\partial x^b}{\partial \tilde{x}^{\tilde{b}}} \gamma^c_{ab} \mathbf{s}^{\tilde{a}}_c, \tag{4.2}$$

where $\gamma^a_{bc} = \frac{\partial x^a}{\partial z^k} \frac{\partial^2 z^k}{\partial x^b \partial x^c}$ and $\tilde{\gamma}^{\tilde{a}}_{\tilde{b}\tilde{c}} = \frac{\partial \tilde{x}^{\tilde{a}}}{\partial \tilde{z}^k} \frac{\partial^2 \tilde{z}^k}{\partial \tilde{x}^{\tilde{b}} \partial \tilde{x}^{\tilde{c}}}$. It is easily shown that $\mathbf{s}^{\tilde{a}}_{a|\tilde{b}} = 0$, i.e., the shifter is covariantly constant.

Example 1. Consider cylindrical coordinates (r, θ, z) and $(\tilde{r}, \tilde{\theta}, \tilde{z})$ at $x \in \mathbb{R}^3$ and $\tilde{x} \in \mathbb{R}^3$, respectively. The shifter map has the following matrix representation with respect to these coordinates:

$$\mathbf{s} = \begin{bmatrix} \cos(\tilde{\theta} - \theta) & r \sin(\tilde{\theta} - \theta) & 0 \\ -\sin(\tilde{\theta} - \theta)/\tilde{r} & r \cos(\tilde{\theta} - \theta)/\tilde{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.3}$$

Example 2. Consider spherical coordinates (r, θ, ϕ) and $(\tilde{r}, \tilde{\theta}, \tilde{\phi})$ at $x \in \mathbb{R}^3$ and $\tilde{x} \in \mathbb{R}^3$, respectively. The shifter map has the following matrix representation with

respect to these coordinates:

$$\mathbf{S} = \begin{bmatrix} \cos(\tilde{\phi} - \phi) \sin \tilde{\theta} \sin \theta + \cos \tilde{\theta} \cos \theta & r[\cos(\tilde{\phi} - \phi) \sin \tilde{\theta} \cos \theta - \cos \tilde{\theta} \sin \theta] & r \sin(\tilde{\phi} - \phi) \sin \tilde{\theta} \sin \theta \\ [\cos(\tilde{\phi} - \phi) \cos \tilde{\theta} \sin \theta - \sin \tilde{\theta} \cos \theta]/\tilde{r} & r[\cos(\tilde{\phi} - \phi) \cos \tilde{\theta} \cos \theta + \sin \tilde{\theta} \sin \theta]/\tilde{r} & r \sin(\tilde{\phi} - \phi) \cos \tilde{\theta} \sin \theta/\tilde{r} \\ -\sin(\tilde{\phi} - \phi) \sin \theta/\tilde{r} \sin \tilde{\theta} & -r \sin(\tilde{\phi} - \phi) \cos \theta/\tilde{r} \sin \tilde{\theta} & r \cos(\tilde{\phi} - \phi) \sin \theta/\tilde{r} \sin \tilde{\theta} \end{bmatrix}. \quad (4.4)$$

Balance of linear momentum in the physical and virtual bodies. The balance of linear momentum for the physical body reads: $\text{Div } \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}$. We use the Piola identity and write the divergence term with respect to the reference configuration of the virtual body. The Piola identity [68] tells us that for a vector field \mathbf{W} on \mathcal{B} , one can write (see the appendix)

$$\text{Div } \mathbf{W} = J_{\mathcal{E}} \widetilde{\text{Div}} \tilde{\mathbf{W}}, \quad \tilde{\mathbf{W}} = J_{\mathcal{E}}^{-1} \tilde{\mathbf{F}} \mathbf{W}. \quad (4.5)$$

In components, this reads

$$W^A|_A = J_{\mathcal{E}} \left[J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_A W^A \right]_{|\tilde{A}}. \quad (4.6)$$

$\tilde{\mathbf{W}}$ is called the Piola transform of \mathbf{W} . We use the local coordinate charts $\{X^A\}$ and $\{\tilde{X}^{\tilde{A}}\}$ for \mathcal{B} and $\tilde{\mathcal{B}}$, respectively. Let us start with $\text{Div } \mathbf{P}$ and rewrite it with respect to the reference configuration of the virtual body. In coordinates and using the Piola identity one can write²⁰

$$P^{aA}|_A = J_{\mathcal{E}} \left(J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_A P^{aA} \right)_{|\tilde{A}} = J_{\mathcal{E}} (\tilde{P}^{a\tilde{A}}|_{\tilde{A}}) \circ \mathcal{E}, \quad (4.8)$$

where

$$\tilde{P}^{a\tilde{A}} \circ \mathcal{E} = J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_A P^{aA}, \quad (4.9)$$

and $J_{\mathcal{E}}$ is the Jacobian of the *cloaking transformation* \mathcal{E} , which is written as

$$J_{\mathcal{E}} = \sqrt{\frac{\det \tilde{\mathbf{G}} \circ \mathcal{E}}{\det \mathbf{G}}} \det \tilde{\mathbf{F}}. \quad (4.10)$$

²⁰ Note that the Piola identity can be written as

$$\left[J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_A \right]_{|\tilde{A}} = 0. \quad (4.7)$$

Thus

$$\left[J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_A P^{aA} \right]_{|\tilde{A}} = \left[J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_A \right]_{|\tilde{A}} P^{aA} + J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_A P^{aA}|_{\tilde{A}}.$$

Therefore, using (4.7), one can write

$$J_{\mathcal{E}} \left[J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_A P^{aA} \right]_{|\tilde{A}} = \tilde{F}^{\tilde{A}}{}_A P^{aA}|_{\tilde{A}} = P^{aA}|_A.$$

However, this is a vector in $T_x\mathcal{S}$. The corresponding vector in $T_{\tilde{x}}\tilde{\mathcal{S}}$ is defined using the shifter $\mathbf{s}(x)$ as

$$\tilde{\mathbf{s}}^{\tilde{a}}_a \tilde{P}^{a\tilde{A}} \circ \mathcal{E} = J_{\mathcal{E}}^{-1} \mathbf{s}^{\tilde{a}}_a P^{aA} \tilde{F}^{\tilde{A}}_{\tilde{A}A}. \quad (4.11)$$

Note that $\tilde{\mathbf{s}}^{\tilde{a}}_a P^{aA}|_A = (\tilde{\mathbf{s}}^{\tilde{a}}_a P^{aA})|_A$. Therefore

$$\mathbf{s} \circ \varphi \operatorname{Div} \mathbf{P} = J_{\mathcal{E}}(\tilde{\operatorname{Div}} \tilde{\mathbf{P}}) \circ \mathcal{E}, \quad (4.12)$$

where

$$\tilde{\mathbf{P}} = J_{\mathcal{E}}^{-1} \mathbf{s} \circ \varphi \mathbf{P} \tilde{\mathbf{F}}^{\tilde{A}}_{\tilde{A}A}, \text{ or } \tilde{P}^{\tilde{a}\tilde{A}} \circ \mathcal{E} = J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}_{\tilde{A}A} \tilde{\mathbf{s}}^{\tilde{a}}_a P^{aA}. \quad (4.13)$$

Equivalently, the transformed second Piola-Kirchhoff stress reads

$$\tilde{\mathbf{S}} \circ \mathcal{E} = J_{\mathcal{E}}^{-1} \tilde{\mathbf{F}}^{-1} \circ \tilde{\varphi} \mathbf{s} \circ \varphi \mathbf{F} \mathbf{S} \tilde{\mathbf{F}}^{\tilde{A}}_{\tilde{A}A}. \quad (4.14)$$

Note that force on an infinitesimal area dA is calculated as

$$t^a dA = P^{aA} N_A dA = P^{aA} J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}_{\tilde{A}A} \tilde{N}_{\tilde{A}} d\tilde{A} = \tilde{P}^{a\tilde{A}} \tilde{N}_{\tilde{A}} d\tilde{A} = \tilde{t}^a d\tilde{A}, \quad (4.15)$$

and hence, $\tilde{\mathbf{s}}^{\tilde{a}}_a t^a dA = \tilde{t}^a d\tilde{A}$, i.e., this force is the same in the physical and the virtual bodies if one assumes that the first Piola-Kirchhoff stress in the virtual body is given by (4.13). Equivalently, if one assumes that forces on the corresponding infinitesimal areas in the two bodies are equal, i.e., are related by the shifter map, then the relation between the stresses is given in (4.13).

Remark 3. Instead of (4.11), i.e., assuming that $\tilde{P}^{\tilde{a}\tilde{A}} = \tilde{\mathbf{s}}^{\tilde{a}}_a \tilde{P}^{a\tilde{A}} \circ \mathcal{E}$ one may assume that

$$\tilde{P}^{\tilde{a}\tilde{A}} = \mathbf{w}^{\tilde{a}}_a \tilde{P}^{a\tilde{A}} \circ \mathcal{E}, \quad (4.16)$$

where $\mathbf{w}^{\tilde{a}}_a$ are the components of some two-point tensor \mathbf{w} . Hence

$$\begin{aligned} \tilde{P}^{\tilde{a}\tilde{A}}|_{\tilde{A}} &= \left(J_{\mathcal{E}}^{-1} \mathbf{w}^{\tilde{a}}_a P^{aA} \tilde{F}^{\tilde{A}}_{\tilde{A}A} \right)|_{\tilde{A}} \\ &= J_{\mathcal{E}}^{-1} \left(\mathbf{w}^{\tilde{a}}_a P^{aA} \right)|_{\tilde{A}} \tilde{F}^{\tilde{A}}_{\tilde{A}A} + \mathbf{w}^{\tilde{a}}_a P^{aA} \left(J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}_{\tilde{A}A} \right)|_{\tilde{A}} \\ &= J_{\mathcal{E}}^{-1} \left(\mathbf{w}^{\tilde{a}}_a P^{aA} \right)|_A = J_{\mathcal{E}}^{-1} \mathbf{w}^{\tilde{a}}_a P^{aA}|_A + J_{\mathcal{E}}^{-1} P^{aA} \mathbf{w}^{\tilde{a}}_a|_A, \end{aligned} \quad (4.17)$$

where use was made of the identity $(J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}_{\tilde{A}A})|_{\tilde{A}} = 0$. In order to preserve the form of the balance of linear momentum under a cloaking map one needs to impose $J_{\mathcal{E}}^{-1} P^{aA} \mathbf{w}^{\tilde{a}}_a|_A = J_{\mathcal{E}}^{-1} P^{aA} \mathbf{w}^{\tilde{a}}_{ab} F^b_A = 0$, which implies that $\mathbf{w}^{\tilde{a}}_{ab} = 0$, i.e., \mathbf{w} must be a covariantly constant two-point tensor defined on the Euclidean ambient spaces \mathcal{S} and $\tilde{\mathcal{S}}$. Therefore, when using Cartesian coordinates $\{z^i\}$ and $\{\tilde{z}^{\tilde{i}}\}$ for \mathcal{S} and $\tilde{\mathcal{S}}$, respectively, \mathbf{w} would have constant components $\mathbf{w}^{\tilde{i}}_i$. However, knowing that on the outer boundary of the cloak (and also outside the cloak) forces on the corresponding infinitesimal areas are equal one concludes that \mathbf{w} is the identity, i.e., $\mathbf{w}^{\tilde{i}}_i = \delta^{\tilde{i}}_i$. Hence, \mathbf{w} is the shifter when general coordinate charts are used for \mathcal{S} and $\tilde{\mathcal{S}}$.

Now in the virtual body $\tilde{\mathbf{P}} = J_{\tilde{\mathcal{E}}}^{-1} \mathbf{s} \circ \varphi \mathbf{P} \tilde{\mathbf{F}}^{\tilde{\mathcal{E}}}$ is the shifted Piola transform of \mathbf{P} and

$$\mathbf{P} = \mathbf{g}^\sharp \frac{\partial W}{\partial \mathbf{F}}, \quad \tilde{\mathbf{P}} = \mathbf{g}^\sharp \frac{\partial \tilde{W}}{\partial \tilde{\mathbf{F}}}, \quad (4.18)$$

where $\tilde{W} = \tilde{W}(\tilde{\mathbf{G}}, \mathbf{g} \circ \tilde{\varphi}, \tilde{\mathbf{F}})$ as we assume that the virtual body is isotropic and homogeneous. To ensure that the relation $\tilde{\mathbf{P}} = J_{\tilde{\mathcal{E}}}^{-1} \mathbf{s} \circ \varphi \mathbf{P} \tilde{\mathbf{F}}^{\tilde{\mathcal{E}}}$ holds one needs to define W in $\mathcal{B} \setminus \mathcal{C}$ as $W|_{\mathcal{B} \setminus \mathcal{C}} = \tilde{W} \circ \tilde{\mathcal{E}}, \forall X \in \mathcal{B} \setminus \mathcal{C}$. This is because $\tilde{\mathcal{E}}$ is defined to be the identity map in $\mathcal{B} \setminus \mathcal{C}$, and hence, the material outside the cloak will be isotropic and identical to that of the virtual body. The cloaking region \mathcal{C} , nevertheless, will be anisotropic, in general, i.e., $W_{\mathcal{C}} = W_{\mathcal{C}}(X, \mathbf{G}, \mathbf{g} \circ \varphi, \mathbf{F}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_m), \forall X \in \mathcal{C}$, where $\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_m$ are the structural tensors that characterize the symmetry group of the material in the cloaking region.²¹ Note that $W_{\mathcal{C}}$ must satisfy the relation

$$(\mathbf{g}^\sharp \circ \varphi_t) \frac{\partial W_{\mathcal{C}}}{\partial \mathbf{F}} = J_{\tilde{\mathcal{E}}}(\mathbf{s}^{-1} \circ \tilde{\varphi}) (\mathbf{g}^\sharp \circ \tilde{\varphi}_t \circ \tilde{\mathcal{E}}) \left(\frac{\partial \tilde{W}}{\partial \tilde{\mathbf{F}}} \circ \tilde{\mathcal{E}} \right) \tilde{\mathbf{F}}^{-\star}. \quad (4.19)$$

One can now rewrite the balance of linear momentum $\text{Div } \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}$ as

$$\widetilde{\text{Div}} \tilde{\mathbf{P}} + \tilde{\rho}_0 \tilde{\mathbf{B}} = \tilde{\rho}_0 \tilde{\mathbf{A}}, \quad (4.20)$$

where $\tilde{\rho}_0 = J_{\tilde{\mathcal{E}}}^{-1} \rho_0 \circ \tilde{\mathcal{E}}^{-1}$, $\tilde{\mathbf{B}} = (\mathbf{s} \circ \varphi) \mathbf{B} \circ \tilde{\mathcal{E}}^{-1}$, and $\tilde{\mathbf{A}} = (\mathbf{s} \circ \varphi) \mathbf{A} \circ \tilde{\mathcal{E}}^{-1}$. In other words, assuming that in the virtual body the first Piola-Kirchhoff stress is given by (4.13), the virtual body has the mass density $\tilde{\rho}_0$, is under the body force $\tilde{\mathbf{B}}$, and has the material acceleration $\tilde{\mathbf{A}}$, the balance of linear momentum is satisfied. Conversely, if the balance of linear momentum is satisfied for the virtual body with the above transformed fields, it is satisfied for the physical body as well.

Balance of angular momentum in the physical and virtual bodies. In the physical body the balance of angular momentum is equivalent to $S^{AB} = S^{BA}$, or $\mathbf{S}^* = \mathbf{S}$. The balance of angular momentum must hold in the virtual body as well, i.e., $\tilde{\mathbf{S}}^* = \tilde{\mathbf{S}}$, which holds if and only if

$$(\mathbf{RS})^* = \mathbf{RS}, \quad (4.21)$$

where $\mathbf{R} = (\tilde{\mathbf{F}}^{-1} \circ \tilde{\mathcal{E}})(\tilde{\mathbf{F}}^{-1} \circ \tilde{\varphi} \circ \tilde{\mathcal{E}})(\mathbf{s} \circ \varphi) \mathbf{F}$. In terms of \mathbf{R} , the second Piola-Kirchhoff stress in the physical and virtual bodies are related as

$$\tilde{\mathbf{S}} = J_{\tilde{\mathcal{E}}}^{-1} \tilde{\mathbf{F}} \mathbf{R} \tilde{\mathbf{S}}^{\tilde{\mathcal{E}}}, \quad \text{or} \quad \mathbf{S} = J_{\tilde{\mathcal{E}}} \mathbf{R}^{-1} \tilde{\mathbf{F}}^{-1} \tilde{\mathbf{S}}^{\tilde{\mathcal{E}}}. \quad (4.22)$$

Note that assuming that the balance of angular momentum is satisfied in one configuration it may or may not be satisfied in the other configuration. We will see in Section 4.3 and Section 4.4 that the balance of angular momentum is the obstruction to transformation cloaking in both classical linear elastodynamics and the

²¹ Note that we do not know a priori how many structural tensors are needed and what they are.

small-on-large theory. It happens to be the obstruction to transformation cloaking in linear gradient elasticity and linear generalized Cosserat elasticity as well as we will show in Section 5.

Remark 4. Note that although we use the same metric for the ambient space for the two bodies, after deformation, a physical point and its corresponding virtual point will be mapped to two different points in the ambient space, in general. For example, in cylindrical coordinates, $r(R, \Theta, \Phi) \neq \tilde{r}(\tilde{R}, \tilde{\Theta}, \tilde{\Phi})$, in general, where $\mathcal{E}(R, \Theta, \Phi) = (\tilde{R}, \tilde{\Theta}, \tilde{\Phi})$. Therefore, when calculating J and \tilde{J} , $\det \mathbf{g}$ at two different points is used.

Remark 5. We showed that if $\tilde{\mathbf{G}} = \mathcal{E}_* \mathbf{G}$, then the symmetry class of the material is preserved under \mathcal{E} (cf. (2.28) in the nonlinear case and Prop. 1 in the linear case). However, for general $\tilde{\mathcal{E}}$ and $\tilde{\mathbf{G}}$, the type of the symmetry class of the body $\tilde{\mathcal{B}}$ with respect to the metric $\tilde{\mathbf{G}}$ can be quite non-trivial and different from that of the body \mathcal{B} with respect to the metric \mathbf{G} . This is a geometric interpretation of the expected anisotropy of an elastic cloak.

Remark 6. The relation $\tilde{\rho}_0 = J_{\tilde{\mathcal{E}}}^{-1} \rho_0 \circ \mathcal{E}^{-1}$ implies that the total mass of the cloak is equal to the mass of the corresponding transformed region in the virtual structure with uniform density ρ_0 , i.e., $m(\mathcal{C}) = \rho_0 \text{vol}_{\tilde{\mathbf{G}}}(\tilde{\mathcal{C}})$. For example, the mass of a cylindrical cloak, which is an annulus with inner radius R_i and outer radius R_o in the physical body is equal to the mass of the annulus in the virtual body (filled with a homogeneous solid) with inner radius $f(R_i) = \epsilon$ and outer radius $f(R_o) = R_o$. Therefore, as the virtual and physical bodies are identical outside the cloak, one concludes that the two bodies have the same total mass.

Even if the balance of angular momentum is satisfied in both configurations, it turns out that hiding a hole in a nonlinear elastic medium from arbitrary finite-amplitude excitations is not possible.

Proposition 2. *Nonlinear elastodynamic transformation cloaking is not possible. In other words, it is not possible to design a cloak that would hide a cavity from any (finite) time-dependent elastic disturbance (or wave).*

Proof. The idea of nonlinear elastodynamic transformation cloaking can be summarized as follows: starting from the balance of linear momentum one assumes the relation (4.13) between the first Piola-Kirchhoff stresses in the virtual and physical bodies. The energy function of the virtual body is given while that of the cloak in the physical body is not known. This implies that to be able to find the energy function of the cloak one would need to relate the kinematics of the physical and virtual bodies. This kinematic constraint is the relation between accelerations: $\tilde{\mathbf{A}} \circ \mathcal{E} = (\mathbf{s} \circ \varphi) \mathbf{A}$.²² All these will have to be consistent with the balance of angular momentum (4.21). We start from the relation $\tilde{\mathbf{A}} \circ \mathcal{E} = (\mathbf{s} \circ \varphi) \mathbf{A}$ and write it in Cartesian coordinates: $\tilde{A}^{\tilde{i}} = \mathbf{s}^{\tilde{i}}_j A^j = \delta^{\tilde{i}}_j A^j$. Thus, $\tilde{V}^{\tilde{i}}(\mathcal{E}(X), t) = \delta^{\tilde{i}}_j V^j(X, t) + a_1$,

²² The kinematic relationship in linear elastodynamic cloaking turns out to be $\tilde{\mathbf{U}} \circ \mathcal{E} = \mathbf{U}$.

where a_1 is a constant. Assuming that $\tilde{V}^i(\mathcal{E}(X), 0) = 0$ and $V^i(X, 0) = 0$, $a_1 = 0$, and hence, $\tilde{\varphi}^i(\mathcal{E}(X), t) = \delta_t^i \varphi^i(X, t) + a_0$, where a_0 is a constant. Knowing that $\tilde{\varphi}$ and φ satisfy the same essential boundary conditions on $\partial_d \tilde{\mathcal{B}} = \partial_d \mathcal{B}$, $a_0 = 0$ (boundary conditions will be discussed in detail in Section 5.2). Therefore, we have concluded that

$$\tilde{\varphi}^i(\mathcal{E}(X), t) = \delta_t^i \varphi^i(X, t). \quad (4.23)$$

In the virtual body there is a circular (spherical) hole of radius $\epsilon \rightarrow 0$. (4.23) is telling us that the deformed configuration of $\partial \mathcal{H}$ is identical to that of the infinitesimal cavity in the virtual body. This is true for any loading of the physical body, and even in the limit of vanishing loads. This implies that the hole surrounded by a cloak in the physical body would collapse to a cavity of radius $\epsilon \rightarrow 0$ (anti-cavitation), i.e., the physical body is unstable. Therefore, nonlinear elastodynamic transformation cloaking is not possible. \square

However, nonlinear elastostatic transformation cloaking may be possible as discussed next.

4.2. Nonlinear Elastostatic Cloaking

We ignore inertial forces and explore the possibility of static cloaking in nonlinear elasticity. Note that the virtual body is assumed to be isotropic, and therefore, its constitutive equation is given by

$$\tilde{\mathbf{S}} = 2 \frac{\partial \tilde{W}}{\partial \tilde{\mathbf{C}}^b} = 2 \tilde{W}_{\tilde{I}_1} \tilde{\mathbf{G}}^\sharp + 2 \tilde{W}_{\tilde{I}_2} (\tilde{I}_2 \tilde{\mathbf{C}}^{-1} - \tilde{I}_3 \tilde{\mathbf{C}}^{-2}) + 2 \tilde{W}_{\tilde{I}_3} \tilde{I}_3 \tilde{\mathbf{C}}^{-1}, \quad (4.24)$$

where $\tilde{\mathbf{S}}$ is the second Piola-Kirchhoff stress and

$$\begin{aligned} \tilde{I}_1 &= \text{tr } \tilde{\mathbf{C}}, \quad \tilde{I}_2 = \det \tilde{\mathbf{C}} \text{tr } \tilde{\mathbf{C}}^{-1}, \quad \tilde{I}_3 = \det \tilde{\mathbf{C}}, \\ \text{and } \tilde{W}_{\tilde{I}_n} &:= \frac{\partial \tilde{W}}{\partial \tilde{I}_n}, \quad n = 1, 2, 3. \end{aligned} \quad (4.25)$$

Note that

$$\tilde{\mathbf{C}}^{-1} = \tilde{\mathbf{F}}^{-1} \tilde{\mathbf{g}}^\sharp \tilde{\mathbf{F}}^{-\star}, \quad \tilde{\mathbf{C}}^{-2} = \tilde{\mathbf{C}}^{-1} \tilde{\mathbf{G}} \tilde{\mathbf{C}}^{-1}, \quad s^{-1} \tilde{\mathbf{g}}^\sharp = \mathbf{g}^\sharp s^\star, \quad \mathbf{C}^{-1} = \mathbf{F}^{-1} \mathbf{g}^\sharp \mathbf{F}^{-\star}. \quad (4.26)$$

Hence, from (4.22), (4.24), and (4.26) one can write

$$J_{\tilde{\mathcal{E}}}^{-1} \mathbf{S}_C = \alpha \mathbf{R}^{-1} \tilde{\mathcal{E}}^\star \tilde{\mathbf{G}}^\sharp + (\tilde{I}_2 \beta + \tilde{I}_3 \gamma) \mathbf{C}^{-1} \mathbf{R}^\star - \tilde{I}_3 \beta \mathbf{C}^{-1} \mathbf{R}^\star (\tilde{\mathcal{E}}^\star \tilde{\mathbf{G}}) \mathbf{R} \mathbf{C}^{-1} \mathbf{R}^\star, \quad (4.27)$$

where $\alpha = 2 \tilde{W}_{\tilde{I}_1}$, $\beta = 2 \tilde{W}_{\tilde{I}_2}$, and $\gamma = 2 \tilde{W}_{\tilde{I}_3}$. In components

$$\begin{aligned} J_{\tilde{\mathcal{E}}}^{-1} S_C^{AB} &= \alpha (R^{-1})^A_M (\tilde{\mathcal{E}}^\star \tilde{\mathbf{G}})^{MB} + (\tilde{I}_2 \beta + \tilde{I}_3 \gamma) (C^{-1})^{AM} R^B_M \\ &\quad - \tilde{I}_3 \beta (C^{-1})^{AM} R^K_M (\tilde{\mathcal{E}}^\star \tilde{\mathbf{G}})_{KL} R^L_P (C^{-1})^{PQ} R^B_Q, \end{aligned} \quad (4.28)$$

where $(\mathcal{E}^* \tilde{G})^{MB} = (\tilde{F}^{-1})^M_{\tilde{M}} (\tilde{F}^{-1})^B_{\tilde{N}} \tilde{G}^{\tilde{M}\tilde{N}}$, and $(\mathcal{E}^* \tilde{G})_{KL} = \tilde{F}^{\tilde{K}}_{\tilde{K}} \tilde{F}^{\tilde{L}}_{\tilde{L}} \tilde{G}_{\tilde{K}\tilde{L}}$.

Next we consider an infinitely-long cylindrical bar (or an infinite body) with a cylindrical hole and a spherical ball (or an infinite body) with a spherical cavity. In both examples we assume radial deformations. Under this assumption we will find the constitutive equations of the corresponding cloaks. It should be emphasized that these static cloaks are the nonlinear analogues of Mansfield [66]’s neutral holes.

Example 1: A static nonlinear cylindrical cloak. Let us consider radial deformations of an infinitely-long solid cylinder with a cylindrical hole of radius R_i . This hole is covered by a cloak with the outer radius $R_o > R_i$. For the physical body in the cylindrical coordinates (R, Θ, Z) and (r, θ, z) for the reference configuration and the ambient space, respectively, we consider deformations of the form $(r, \theta, z) = (r(R), \Theta, Z)$. Similarly, for the virtual body in the cylindrical coordinates $(\tilde{R}, \tilde{\Theta}, \tilde{Z})$ and $(\tilde{r}, \tilde{\theta}, \tilde{z})$ we have $(\tilde{r}, \tilde{\theta}, \tilde{z}) = (\tilde{r}(\tilde{R}), \tilde{\Theta}, \tilde{Z})$. The cloaking map is assumed radial, i.e., $(\tilde{R}, \tilde{\Theta}, \tilde{Z}) = \mathcal{E}(R, \Theta, Z) = (f(R), \Theta, Z)$. We will design a nonlinear elastic cloak of inner and outer radii R_i and R_o in the physical body such that the static response of the body with a hole in radial deformations outside the cloak is identical to that of an isotropic and homogeneous elastic body with an infinitesimal hole. We assume that $f(R_i) = \epsilon$, and $f(R_o) = R_o$. As we will see in §5.2, the function f must satisfy the extra condition $f'(R_o) = 1$. We also assume that $f(R) = R$, for $R \geq R_o$. Let us assume the following relation between the kinematics of deformations in the physical and virtual bodies²³

$$\frac{\tilde{r}(\tilde{R})}{\tilde{R}} = \frac{r(R)}{R}. \tag{4.29}$$

Therefore

$$(\tilde{r}, \tilde{\theta}, \tilde{z}) = (\tilde{r}(\tilde{R}), \tilde{\Theta}, \tilde{Z}) = \left(\frac{f(R)}{R} r(R), \Theta, Z \right). \tag{4.30}$$

Hence

$$\tilde{r}'(\tilde{R}) = \frac{1}{f'(R)} \frac{d}{dR} \left[\frac{f(R)}{R} r(R) \right] = \frac{f(R)}{R f'(R)} r'(R) + \left[1 - \frac{f(R)}{R f'(R)} \right] \frac{r(R)}{R}. \tag{4.31}$$

²³ In the case of elastodynamic cloaking the kinematic relation between the physical and virtual problems was the equality (up to a shift) of acceleration vectors. In elastostatics there is no such constraint and one has freedom in choosing a kinematic relation between the two problems. Note that (4.29) is just one choice and one may assume

$$\frac{\tilde{r}(\tilde{R})}{\tilde{R}} = h \left(\frac{r(R)}{R} \right),$$

for any positive and strictly increasing function h such that $h(1) = 1$. Note also that (4.29) is a nonlinear analogue of Olsson and Wall [83]’s kinematic assumption.

In particular, $\tilde{r}(\tilde{R}_o) = r(R_o)$, and $\tilde{r}'(\tilde{R}_o) = r'(R_o)$. Note that the spatial shifter map has the coordinate representation

$$\mathbf{s} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r(R)/\tilde{r}(\tilde{R}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R/f(R) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (4.32)$$

Therefore

$$\mathbf{R} = \begin{bmatrix} r'(R)/[f'(R)\tilde{r}'(\tilde{R})] & 0 & 0 \\ 0 & R/f(R) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C}^{-1} = \begin{bmatrix} 1/r'(R)^2 & 0 & 0 \\ 0 & 1/r(R)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (4.33)$$

Using (4.27) the constitutive equations of the cloak read

$$\mathbf{S} = \alpha \begin{bmatrix} f\tilde{r}'/(Rr') & 0 & 0 \\ 0 & f'/R^2 & 0 \\ 0 & 0 & ff'/R \end{bmatrix} + (\tilde{I}_2\beta + \tilde{I}_3\gamma) \begin{bmatrix} f/(Rr'\tilde{r}') & 0 & 0 \\ 0 & f'/r^2 & 0 \\ 0 & 0 & ff'/R \end{bmatrix} - \tilde{I}_3\beta \begin{bmatrix} f/(Rr\tilde{r}'^3) & 0 & 0 \\ 0 & fR^2/r^4 & 0 \\ 0 & 0 & ff'/R \end{bmatrix}, \quad (4.34)$$

where

$$\tilde{I}_1 = 1 + \tilde{r}'^2(f(R)) + \frac{r^2(R)}{R^2}, \quad \tilde{I}_2 = \frac{1}{2}\tilde{r}'^2(1 - \tilde{r}'^2) + \frac{r^2(R)}{2R^2} \left[1 - \frac{r^2(R)}{R^2} \right],$$

$$\tilde{I}_3 = \tilde{J}^2 = \left[\frac{r(R)}{R}\tilde{r}'(f(R)) \right]^2. \quad (4.35)$$

It is seen that \mathbf{S} is symmetric, and hence, the balance of angular momentum is satisfied. The surface of the cavity is traction free. The boundary conditions in the physical and virtual bodies will be discussed in detail in Section 5.2. In particular, it will be shown that the boundary of the hole is traction-free in the virtual body if and only if the boundary of the physical hole is traction-free. As an example we can assume that the physical and virtual bodies have finite outer radii both equal to R_e on which a radial traction is specified.

Note that radial deformations cannot be maintained in an arbitrary compressible isotropic solid only by applying boundary tractions. This is a result of Ericksen's theorem, which states that the only universal deformations in compressible isotropic solids are homogeneous deformations [26] (see [120] for an extension of this result to compressible solids with finite eigenstrains). Note, however, that if the radial stretch is uniform, i.e., $\tilde{r}(\tilde{R})/\tilde{R} = r(R)/R = \lambda$, then the deformation will be homogeneous, and hence, universal, i.e., it can be maintained solely by applying boundary tractions in any compressible isotropic solid.

Example 2: A nonlinear static spherical cloak. Let us consider radial deformations of a finite solid sphere with a spherical cavity of radius R_i . For the physical body in the spherical coordinates (R, Θ, Φ) and (r, θ, ϕ) for the reference configuration and the ambient space, respectively, we consider deformations of the form $(r, \theta, \phi) = (r(R), \Theta, \Phi)$. Similarly, for the virtual body in the cylindrical coordinates $(\tilde{R}, \tilde{\Theta}, \tilde{\Phi})$ and $(\tilde{r}, \tilde{\theta}, \tilde{\phi})$ we have $(\tilde{r}, \tilde{\theta}, \tilde{\phi}) = (\tilde{r}(\tilde{R}), \tilde{\Theta}, \tilde{\Phi})$. Notice that $(\tilde{R}, \tilde{\Theta}, \tilde{\Phi}) = \mathcal{E}(R, \Theta, Z) = (f(R), \Theta, \Phi)$. We again assume the following relation between the kinematics of deformations in the physical and virtual bodies: $\tilde{r}(\tilde{R})/\tilde{R} = r(R)/R$. Therefore

$$(\tilde{r}, \tilde{\theta}, \tilde{\phi}) = (\tilde{r}(\tilde{R}), \tilde{\Theta}, \tilde{\Phi}) = \left(\frac{f(R)}{R} r(R), \Theta, \Phi \right). \quad (4.36)$$

Note that $\tilde{r}'(\tilde{R})$ is given in (4.31). We will design a nonlinear elastic cloak of inner and outer radii R_i and R_o in the physical body such that the static response of the body with a cavity in radial deformations outside the cloak be identical to that of an isotropic and homogeneous elastic body with an infinitesimal cavity. The function f satisfies the following conditions: $f(R_i) = \epsilon$, $f(R_o) = R_o$, and $f'(R_o) = 1$. Note that the spatial shifter map has the coordinate representation

$$\mathbf{s} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r(R)/\tilde{r}(\tilde{R}) & 0 \\ 0 & 0 & r(R)/\tilde{r}(\tilde{R}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R/f(R) & 0 \\ 0 & 0 & R/f(R) \end{bmatrix}. \quad (4.37)$$

Therefore

$$\mathbf{R} = \begin{bmatrix} r'(R)/(f'(R)\tilde{r}'(\tilde{R})) & 0 & 0 \\ 0 & R/f(R) & 0 \\ 0 & 0 & R/f(R) \end{bmatrix}, \quad (4.38)$$

$$\mathbf{C}^{-1} = \begin{bmatrix} 1/r'(R)^2 & 0 & 0 \\ 0 & 1/r(R)^2 & 0 \\ 0 & 0 & 1/(r(R)^2 \sin^2 \Theta) \end{bmatrix}.$$

The constitutive equations of the cloak read

$$\mathbf{S} = \alpha \begin{bmatrix} f^2 \tilde{r}'/(R^2 r') & 0 & 0 \\ 0 & f f'/R^3 & 0 \\ 0 & 0 & f f'/(R^3 \sin^2 \Theta) \end{bmatrix}$$

$$+ (\tilde{I}_2 \beta + \tilde{I}_3 \gamma) \begin{bmatrix} f^2/(R^2 r' \tilde{r}') & 0 & 0 \\ 0 & f f'/(R r^2) & 0 \\ 0 & 0 & f f'/(R r^2 \sin^2 \Theta) \end{bmatrix} \quad (4.39)$$

$$- \tilde{I}_3 \beta \begin{bmatrix} f^2/(R^2 r' \tilde{r}'^3) & 0 & 0 \\ 0 & R f f'/r^4 & 0 \\ 0 & 0 & R f f'/(r^4 \sin^2 \Theta) \end{bmatrix},$$

where

$$\tilde{I}_1 = \tilde{r}'^2(f(R)) + \frac{2r^2(R)}{R^2}, \quad \tilde{I}_2 = \frac{1}{2} \tilde{r}'^2(1 - \tilde{r}'^2) + \frac{r^2(R)}{R^2} \left[1 - \frac{r^2(R)}{R^2} \right],$$

$$\tilde{I}_3 = \tilde{J}^2 = \left[\frac{r(R)}{R} \tilde{r}'(f(R)) \right]^4. \quad (4.40)$$

It is seen that \mathbf{S} is symmetric, and hence, the balance of angular momentum is satisfied.

4.3. Linear Elastodynamic Transformation Cloaking

In this section we explore the possibility of transformation cloaking in classical linear elasticity. Let us assume that the reference motion is static, i.e., $\tilde{\mathbf{V}}(X, t) = \mathbf{0}$, and the ambient space is Euclidean, that is, $\mathbf{Ric}_{\mathbf{g}} = \mathbf{0}$. In this special case, the linearized material balance of linear momentum (3.48) is simplified to read

$$\text{Div}(\mathbf{A} : \nabla_0^{\mathbf{g}} \mathbf{U}) + \rho_0 \nabla_0^{\mathbf{g}} \mathring{\mathbf{B}} = \rho_0 \ddot{\mathbf{U}}. \quad (4.41)$$

Now suppose that the reference configuration of the physical body \mathcal{B} is mapped to the reference configuration of the virtual body $\tilde{\mathcal{B}}$, where \mathcal{B} and $\tilde{\mathcal{B}}$ are endowed with the Euclidean metrics $\mathbf{G}(X)$ and $\tilde{\mathbf{G}}(\tilde{X})$, respectively. Let us denote the map between the two stress-free reference configurations by $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$. The Jacobian of this cloaking map is calculated as

$$J_{\mathcal{E}} = \sqrt{\frac{\det \tilde{\mathbf{G}}(\tilde{X}(X))}{\det \mathbf{G}(X)}} \det \tilde{\mathbf{F}}. \quad (4.42)$$

Linearized balance of linear momentum in the physical and virtual bodies. Using the Piola identity, the divergence term in (4.41) can be rewritten as

$$\begin{aligned} \left(\mathbf{A}^{aA}_b{}^B U^b|_B \right)_{|A} &= J_{\mathcal{E}} \left[J_{\mathcal{E}}^{-1} \tilde{\mathbf{F}}^{\tilde{A}}_{\tilde{A}} \mathbf{A}^{aA}_b{}^B U^b|_B \right]_{|\tilde{A}} \\ &= J_{\mathcal{E}} \left(\tilde{\mathbf{A}}^{a\tilde{A}}_{\tilde{b}}{}^{\tilde{B}} U^b|_{\tilde{B}} \right)_{|\tilde{A}}, \end{aligned} \quad (4.43)$$

where

$$U^b|_{\tilde{B}} = (\tilde{\mathbf{F}}^{-1})^B_{\tilde{B}} U^b|_B, \quad (\tilde{\mathbf{A}}^{a\tilde{A}}_{\tilde{b}}{}^{\tilde{B}}) \circ \mathcal{E} = J_{\mathcal{E}}^{-1} \tilde{\mathbf{F}}^{\tilde{A}}_{\tilde{A}} \tilde{\mathbf{F}}^{\tilde{B}}_{\tilde{B}} \mathbf{A}^{aA}_b{}^B. \quad (4.44)$$

Hence, using the shifter map and knowing that it is covariantly constant one can write

$$\mathbf{s}^{\tilde{a}}_{\tilde{a}} \left(\mathbf{A}^{aA}_b{}^B U^b|_B \right)_{|A} = J_{\mathcal{E}} \left(\tilde{\mathbf{A}}^{\tilde{a}\tilde{A}}_{\tilde{b}}{}^{\tilde{B}} \tilde{U}^{\tilde{b}}|_{\tilde{B}} \right)_{|\tilde{A}}, \quad (4.45)$$

where²⁴

$$\tilde{\mathbf{A}}^{\tilde{a}\tilde{A}}_{\tilde{b}}{}^{\tilde{B}} \circ \mathcal{E} = J_{\mathcal{E}}^{-1} \tilde{\mathbf{F}}^{\tilde{A}}_{\tilde{A}} \tilde{\mathbf{F}}^{\tilde{B}}_{\tilde{B}} \mathbf{s}^{\tilde{a}}_{\tilde{a}} (\mathbf{s}^{-1})^b_{\tilde{b}} \mathbf{A}^{aA}_b{}^B, \quad \tilde{U}^{\tilde{a}} \circ \mathcal{E} = \mathbf{s}^{\tilde{a}}_{\tilde{a}} \circ \varphi U^a. \quad (4.46)$$

²⁴ Note that transformation of the elastic constants under a cloaking map is different from that under a material change of coordinates (3.72).

Therefore

$$\mathbf{s} \circ \varphi \operatorname{Div}(\mathbf{A} : \nabla_0^{\mathbf{g}} \mathbf{U}) = J_{\mathcal{E}} \widetilde{\operatorname{Div}}(\tilde{\mathbf{A}} : \tilde{\nabla}_0^{\mathbf{g}} \tilde{\mathbf{U}}) \circ \mathcal{E}, \quad (4.47)$$

where $\tilde{\mathbf{U}} \circ \mathcal{E} = \mathbf{s} \circ \varphi \mathbf{U}$, i.e., one assumes that the displacements in the two bodies are equal at the corresponding points. Again, knowing that the shifter is covariantly constant, one obtains $\dot{\tilde{\mathbf{U}}} = (\mathbf{s} \circ \varphi) \dot{\mathbf{U}} \circ \mathcal{E}^{-1}$. Similarly, $\ddot{\tilde{\mathbf{U}}} = (\mathbf{s} \circ \dot{\varphi}) \ddot{\mathbf{U}} \circ \mathcal{E}^{-1}$. Substituting (4.47) into (4.41)₁, one can write the linearized balance of linear momentum in the virtual body as

$$\widetilde{\operatorname{Div}}(\tilde{\mathbf{A}} : \tilde{\nabla}_0^{\mathbf{g}} \tilde{\mathbf{U}}) + \tilde{\rho}_0 \tilde{\nabla}_0^{\mathbf{g}} \tilde{\mathbf{B}} = \tilde{\rho}_0 \ddot{\tilde{\mathbf{U}}}, \quad (4.48)$$

where $\tilde{\rho}_0 = J_{\mathcal{E}}^{-1} \rho_0 \circ \mathcal{E}^{-1}$ and $\tilde{\mathbf{B}} = \mathbf{s} \circ \dot{\varphi} \mathbf{B} \circ \mathcal{E}^{-1}$. In summary, the linearized balance of linear momentum is form-invariant under the following *cloaking transformations*:

$$\begin{aligned} \tilde{X} &= \mathcal{E}(X), \quad \tilde{\mathbf{U}} = \mathbf{s} \circ \dot{\varphi} \mathbf{U} \circ \mathcal{E}^{-1}, \quad \tilde{\mathbf{B}} = \mathbf{s} \circ \dot{\varphi} \mathbf{B} \circ \mathcal{E}^{-1}, \\ \tilde{\rho}_0 &= J_{\mathcal{E}}^{-1} \rho_0 \circ \mathcal{E}^{-1}, \quad \tilde{\mathbf{A}} = (J_{\mathcal{E}}^{-1} \mathbf{s}^{-1} \circ \dot{\varphi} \tilde{\mathbf{F}} \mathbf{s} \circ \dot{\varphi} \mathbf{A} \tilde{\mathbf{F}}^*) \circ \mathcal{E}^{-1}. \end{aligned} \quad (4.49)$$

Linearized balance of angular momentum in the physical and virtual bodies. Suppose that the reference motions of both the physical and virtual bodies are their corresponding undeformed configurations, i.e., both $\mathring{\mathbf{F}}$ and $\mathring{\tilde{\mathbf{F}}}$ are identity maps (there are no initial stresses). Therefore, for any $\mathbf{W} \in T_X \mathcal{B}$, one has $\mathring{\mathbf{F}} \mathbf{S} \mathbf{W} = \mathbf{s} \mathring{\tilde{\mathbf{F}}} \mathbf{W}$, and hence, $\mathbf{S} = \mathring{\mathbf{F}}^{-1} \mathring{\mathbf{S}} \mathring{\mathbf{F}}$ (in components, $S^{\tilde{A}}_A = \delta^{\tilde{A}}_a S^{\tilde{a}}_a \delta^a_A$). Let us next linearize (4.21) with respect to $\dot{\varphi}$ and $\dot{\tilde{\varphi}}$, which reads $(\mathring{\mathbf{R}} \delta \mathbf{S})^* = \mathring{\mathbf{R}} \delta \mathbf{S}$, where $\mathring{\mathbf{R}} = (\mathring{\tilde{\mathbf{F}}}^{-1} \circ \mathcal{E}) \mathbf{S}$. In components

$$(\mathring{\tilde{\mathbf{F}}}^{-1})^A_{\tilde{A}} \mathring{\mathbf{S}}^{\tilde{A}}_M \mathbf{C}^{MBKL} = (\mathring{\tilde{\mathbf{F}}}^{-1})^B_{\tilde{B}} \mathring{\mathbf{S}}^{\tilde{A}}_M \mathbf{C}^{MAKL}. \quad (4.50)$$

Clearly, the balance of angular momentum may not be satisfied for a given cloaking map \mathcal{E} . In terms of the first Piola-Kirchhoff stress, the balance of angular momentum reads $P^{[aA F^c]}_A = 0$, where $2P^{[aA F^c]}_A = P^{aA} F^c_A - P^{cA} F^a_A$. Assuming that there is no initial stress, the linearized balance of angular momentum reads $\delta P^{[aA F^c]}_A = 0$, or $\mathbf{A}^{[aAbB \hat{F}^c]}_A = 0$. Assuming that the balance of angular momentum in the virtual body is satisfied, i.e., $\tilde{\mathbf{A}}^{[\tilde{a}\tilde{A}\tilde{b}\tilde{B} \tilde{\hat{F}}^c]}_{\tilde{A}} = 0$, the balance of angular momentum in the physical body requires that

$$(\mathbf{s}^{-1})^{[a}_{\tilde{a}} \hat{F}^c]_A (\mathbf{s}^{-1})^b_{\tilde{b}} (\mathring{\tilde{\mathbf{F}}}^{-1})^A_{\tilde{A}} (\mathring{\tilde{\mathbf{F}}}^{-1})^B_{\tilde{B}} \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = 0. \quad (4.51)$$

Note that (4.51) is not necessarily satisfied.

Transformed second elasticity tensor. Using (3.16) and the relation $\mathbf{S} = \overset{\circ}{\mathbf{F}}^{-1} \mathbf{s} \overset{\circ}{\mathbf{F}}$, it is straightforward to show that $\tilde{\mathbf{C}} = J_{\overset{\circ}{\mathbf{E}}}^{-1} \overset{\circ}{\mathbf{F}} \mathbf{S} \overset{\circ}{\mathbf{F}} \mathbf{S} \overset{\circ}{\mathbf{C}} \circ \overset{\circ}{\mathbf{E}}$. In components one has

$$\tilde{C}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = J_{\overset{\circ}{\mathbf{E}}}^{-1} \overset{\circ}{F}^{\tilde{A}}_{\tilde{A}} \mathbf{S}^{\tilde{B}}_{\tilde{B}} \overset{\circ}{F}^{\tilde{C}}_{\tilde{C}} \mathbf{S}^{\tilde{D}}_{\tilde{D}} C^{ABCD}, \quad (4.52)$$

or²⁵

$$C^{ABCD} = J_{\overset{\circ}{\mathbf{E}}} (\overset{\circ}{F}^{-1})^A_{\tilde{A}} (\mathbf{S}^{-1})^B_{\tilde{B}} (\overset{\circ}{F}^{-1})^C_{\tilde{C}} (\mathbf{S}^{-1})^D_{\tilde{D}} \tilde{C}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}}. \quad (4.53)$$

We assume that the virtual structure is isotropic, and hence, $\tilde{\mathbf{C}}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = \lambda \tilde{G}^{\tilde{A}\tilde{B}} \tilde{G}^{\tilde{C}\tilde{D}} + \mu (\tilde{G}^{\tilde{A}\tilde{C}} \tilde{G}^{\tilde{B}\tilde{D}} + \tilde{G}^{\tilde{A}\tilde{D}} \tilde{G}^{\tilde{B}\tilde{C}})$ are given. Note that the transformed second elasticity tensor \mathbf{C} possesses the major symmetries, i.e., $C^{ABCD} = C^{CDAB}$. However, the minor symmetries are not satisfied, in general. This is in agreement with the previous works in the literature.

Proposition 3. *Classical linear elasticity does not allow for elastodynamic transformation cloaking regardless of the shape of the hole and the cloak. In other words, elastodynamic transformation cloaking is not possible if the cloak is required to be made of a classical linear elastic solid.*

Proof. The balance of angular momentum in the physical body (4.51) is expanded to read

$$\begin{aligned} & (\mathbf{s}^{-1})^a_{\tilde{a}} \overset{\circ}{F}^c_A (\mathbf{s}^{-1})^b_{\tilde{b}} (\overset{\circ}{F}^{-1})^A_{\tilde{A}} (\overset{\circ}{F}^{-1})^B_{\tilde{B}} \tilde{A}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \\ & = (\mathbf{s}^{-1})^c_{\tilde{a}} \overset{\circ}{F}^a_A (\mathbf{s}^{-1})^b_{\tilde{b}} (\overset{\circ}{F}^{-1})^A_{\tilde{A}} (\overset{\circ}{F}^{-1})^B_{\tilde{B}} \tilde{A}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}. \end{aligned} \quad (4.54)$$

Thus, knowing that $\overset{\circ}{F}^a_A = \delta^a_A$, one obtains

$$\begin{aligned} & (\mathbf{s}^{-1})^a_{\tilde{a}} (\mathbf{s}^{-1})^b_{\tilde{b}} (\overset{\circ}{F}^{-1})^c_{\tilde{A}} (\overset{\circ}{F}^{-1})^B_{\tilde{B}} \tilde{A}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \\ & = (\mathbf{s}^{-1})^c_{\tilde{a}} (\mathbf{s}^{-1})^b_{\tilde{b}} (\overset{\circ}{F}^{-1})^a_{\tilde{A}} (\overset{\circ}{F}^{-1})^B_{\tilde{B}} \tilde{A}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}, \quad \forall a, b, c, B \in \{1, 2, 3\}. \end{aligned} \quad (4.55)$$

Without loss of generality, we may represent (4.55) in Cartesian coordinates in which the shifter is the identity, i.e., $\mathbf{s}^{\tilde{a}}_{\tilde{a}} = \delta^{\tilde{a}}_{\tilde{a}}$. Therefore

$$\begin{aligned} & (\overset{\circ}{F}^{-1})^B_{\tilde{B}} \left[(\overset{\circ}{F}^{-1})^c_{\tilde{A}} \tilde{A}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} - (\overset{\circ}{F}^{-1})^a_{\tilde{A}} \tilde{A}^{\tilde{c}\tilde{A}\tilde{b}\tilde{B}} \right] = 0, \\ & \forall a, b, c, B \in \{1, 2, 3\}. \end{aligned} \quad (4.56)$$

Knowing that in Cartesian coordinates $\tilde{A}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = \lambda \delta^{\tilde{a}\tilde{A}} \delta^{\tilde{b}\tilde{B}} + \mu (\delta^{\tilde{a}\tilde{b}} \delta^{\tilde{A}\tilde{B}} + \delta^{\tilde{a}\tilde{B}} \delta^{\tilde{b}\tilde{A}})$, for an arbitrary cloaking transformation with components $F_{ij} = (\overset{\circ}{F}^{-1})^i_j$, $i, j \in \{1, 2, 3\}$, (4.56) is simplified for $i \neq j \neq k \in \{1, 2, 3\}$ to read

$$(\lambda + \mu) F_{ii} (F_{ij} - F_{ji}) - \mu (F_{ij} F_{jj} + F_{ik} F_{jk} + F_{ii} F_{ji}) = 0, \quad (4.57)$$

²⁵ When Cartesian coordinates are used in both the physical and virtual bodies one writes $\tilde{C}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = J_{\overset{\circ}{\mathbf{E}}}^{-1} \overset{\circ}{F}^{\tilde{A}}_{\tilde{A}} \overset{\circ}{F}^{\tilde{B}}_{\tilde{B}} \overset{\circ}{F}^{\tilde{C}}_{\tilde{C}} \overset{\circ}{F}^{\tilde{D}}_{\tilde{D}} C^{ABCD}$. This is identical to NORRIS and PARNELL [79]'s Eq.(2.6). See also AL-ATTAR and CRAWFORD [1]'s Eq.(129).

$$\mu(\mathbf{F}_{ij}\mathbf{F}_{ik} - \mathbf{F}_{ii}\mathbf{F}_{kj}) + \lambda\mathbf{F}_{ij}(\mathbf{F}_{ik} - \mathbf{F}_{ki}) = 0, \quad (4.58)$$

$$\mu(\mathbf{F}_{ii}^2 + \mathbf{F}_{ik}^2) + (\lambda + 2\mu)\mathbf{F}_{ij}^2 - \mu\mathbf{F}_{ii}\mathbf{F}_{jj} - \lambda\mathbf{F}_{ij}\mathbf{F}_{ji} = 0. \quad (4.59)$$

(4.57) can be rewritten as ($i \leftrightarrow j$)

$$(\lambda + \mu)\mathbf{F}_{jj}(\mathbf{F}_{ji} - \mathbf{F}_{ij}) - \mu(\mathbf{F}_{ji}\mathbf{F}_{ii} + \mathbf{F}_{jk}\mathbf{F}_{ik} + \mathbf{F}_{jj}\mathbf{F}_{ij}) = 0. \quad (4.60)$$

Using (4.57) and (4.60), one obtains

$$(\lambda + \mu)(\mathbf{F}_{ii} + \mathbf{F}_{jj})(\mathbf{F}_{ij} - \mathbf{F}_{ji}) = 0. \quad (4.61)$$

Note that $\lambda + \mu = \frac{1}{3}(3\lambda + 2\mu) + \frac{1}{3}\mu > 0$. Thus, either $\mathbf{F}_{ij} = \mathbf{F}_{ji}$, or $\mathbf{F}_{ii} = -\mathbf{F}_{jj}$. Note that $\mathbf{F}_{ii} = -\mathbf{F}_{jj}$ immediately implies that $\mathbf{F}_{ii} = 0$, and from (4.57), $\mathbf{F}_{jk} = 0$, resulting in $\bar{\mathbf{F}}^{-1} = \mathbf{0}$, which is not possible as the tangent map of a cloaking transformation cannot be singular. Thus, $\mathbf{F}_{ij} = \mathbf{F}_{ji}$, which simplifies (4.57)–(4.59) to read

$$\mathbf{F}_{ii}\mathbf{F}_{ij}(\mathbf{F}_{jj} + \mathbf{F}_{ii}) + \mathbf{F}_{ik}\mathbf{F}_{ii}\mathbf{F}_{kj} = 0, \quad (4.62)$$

$$\mathbf{F}_{ij}\mathbf{F}_{ik} = \mathbf{F}_{ii}\mathbf{F}_{kj}, \quad (4.63)$$

$$\mathbf{F}_{ii}^2 + \mathbf{F}_{ik}^2 + 2\mathbf{F}_{ij}^2 - \mathbf{F}_{ii}\mathbf{F}_{jj} = 0. \quad (4.64)$$

Now, substituting (4.63) into (4.62), one obtains

$$\mathbf{F}_{ij}(\mathbf{F}_{ii}^2 + \mathbf{F}_{ii}\mathbf{F}_{jj} + \mathbf{F}_{ik}^2) = 0. \quad (4.65)$$

Using this relation and (4.64), one obtains $\mathbf{F}_{ij}(\mathbf{F}_{ij}^2 - \mathbf{F}_{ii}\mathbf{F}_{jj}) = 0$, which implies that either $\mathbf{F}_{ij} = 0$, or $\mathbf{F}_{ij}^2 = \mathbf{F}_{ii}\mathbf{F}_{jj}$. However, $\mathbf{F}_{ij}^2 = \mathbf{F}_{ii}\mathbf{F}_{jj}$ is not acceptable because from (4.64) one concludes that $\bar{\mathbf{F}}^{-1} = \mathbf{0}$. Therefore, $\mathbf{F}_{ij} = 0$, for $i \neq j \in \{1, 2, 3\}$.

Using (4.64) one concludes that $\bar{\mathbf{F}}_{ii} = \mathbf{F}_{jj}$, i.e., $\bar{\mathbf{F}} = \beta\mathbf{I}$, where \mathbf{I} denotes the identity and β is a scalar. As the cloaking transformation \mathcal{E} must be the identity on the outer boundary of the cloak $\partial_o\mathcal{C}$ one concludes that $\beta = 1$, and thus, $\mathcal{E} = id$. Therefore, transformation cloaking is not possible in classical linear elasticity. \square

Remark 7. We observe that the balance of angular momentum is the obstruction to cloaking, and not the acceleration term. This implies that transformation cloaking is not possible in classical linear elasticity at fixed frequency or classical linear elastostatics either.

Remark 8. Note that when $\mu = 0$ (a pentamode material [71]) transformation cloaking may be possible. When $\mu = 0$, the only constraint imposed by the balance of angular momentum for the physical body (cf. (4.57), (4.58), and (4.59)) is that $\bar{\mathbf{F}}$ be symmetric. One should note that $\bar{\mathbf{F}}$ is a two-point tensor. However, as the reference configuration of both the virtual and the physical bodies are embedded in the Euclidean space, symmetry of $\bar{\mathbf{F}}$ makes sense. For an arbitrary hole surrounded by a cloak (with an arbitrary shape), if the derivative of a cloaking map is symmetric, the elastic constants of the cloak are fully symmetric. This happens to be the case for cylindrical and spherical cloaks. As we will see in §5.3, for the virtual and physical boundary-value problems to be equivalent outside the cloak, $\bar{\mathbf{F}}$ must be the identity map on the outer boundary of the cloak, i.e., $\bar{\mathbf{F}}|_{\partial_o\mathcal{C}} = id$.

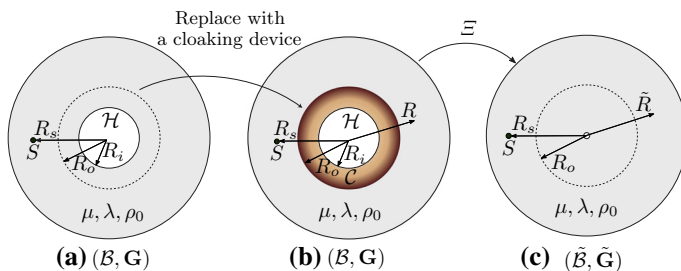


Fig. 6. Cloaking an object inside a hole \mathcal{H} from elastic waves generated by a line source located at a distance R_s from the center of the hole. The system **a** is an isotropic and homogeneous medium containing a finite hole \mathcal{H} . The system **c** is isotropic and homogeneous with the same elastic properties as the medium in the reference configuration **b** outside the cloaking region \mathcal{C} . The configuration **b** is mapped to the new reference configuration **c** such that the hole \mathcal{H} is mapped to an infinitesimal hole with negligible effects on the elastic waves. The cloaking transformation is the identity mapping in $\mathcal{B} \setminus \mathcal{C}$

Next we critically examine a cloaking geometry that has been suggested in several previous works in the literature. Consider an infinitely-long hollow solid cylinder that in its stress-free reference configuration has inner and outer radii R_i and R_o , respectively (see Fig. 6). Let us transform the reference configuration to the reference configuration of another body (virtual body) that is a hollow cylinder with inner and outer radii ϵ and R_o , respectively, using a cloaking map $\mathcal{E}(R, \Theta, Z) = (f(R), \Theta, Z)$ such that $f(R_o) = R_o$. For such a map we have

$$\tilde{\mathbf{F}} = \begin{bmatrix} f'(R) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.66}$$

Let us consider an infinitely extended isotropic homogeneous elastic medium with Lamé constants μ and λ , and mass density ρ_0 containing an infinitely-long cylindrical cavity \mathcal{H} with radius R_i in its stress-free reference configuration. Let (R, Θ, Z) be the cylindrical coordinates such that $R = 0$ corresponds to the centerline of the hole. The cloaking device \mathcal{C} is an infinitely-long hollow solid cylinder with inner radius R_i (radius of the hole) and outer radius R_o , where $R_o < R_s$ (R_s is the distance of a line source from the origin) surrounding the hole. The elastic properties of the cloaking device are to be determined. In doing so, the reference configuration of the physical body \mathcal{B} is mapped to that of the virtual body $\tilde{\mathcal{B}}$ via a mapping $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$, where for $R_i \leq R \leq R_o$, it is defined as, $(\tilde{R}, \tilde{\Theta}, \tilde{Z}) = \mathcal{E}(R, \Theta, Z) = (f(R), \Theta, Z)$ such that $f(R_i) = \epsilon$ and $f(R_o) = R_o$, and for $R \geq R_o$ it is the identity map. We assume that the transformed reference configuration is also isotropic and homogeneous with the same elastic properties as the medium in the physical reference configuration. Let the reference configurations of \mathcal{B} and $\tilde{\mathcal{B}}$ be endowed with the flat metric in the cylindrical coordinates, i.e., $\mathbf{G} = \text{diag}(1, R^2, 1)$ and $\tilde{\mathbf{G}} = \text{diag}(1, \tilde{R}^2, 1)$ in the cylindrical coordinates (R, Θ, Z) and $(\tilde{R}, \tilde{\Theta}, \tilde{Z})$, respectively. Also, let the ambient space be endowed

with the Euclidean metric $\mathbf{g} = \text{diag}(1, r^2, 1)$ in the coordinates (r, θ, z) . Therefore, the elasticity tensor for the material in the virtual body is written as

$$\tilde{\mathbf{C}}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = \lambda \tilde{G}^{\tilde{A}\tilde{B}} \tilde{G}^{\tilde{C}\tilde{D}} + \mu (\tilde{G}^{\tilde{A}\tilde{C}} \tilde{G}^{\tilde{B}\tilde{D}} + \tilde{G}^{\tilde{A}\tilde{D}} \tilde{G}^{\tilde{C}\tilde{B}}). \quad (4.67)$$

In Voigt representation, $\tilde{\mathbf{C}}$ reads²⁶

$$\tilde{\mathbf{C}} = \begin{bmatrix} \lambda + 2\mu & \lambda/\tilde{R}^2 & \lambda & 0 & 0 & 0 \\ \lambda/\tilde{R}^2 & (\lambda + 2\mu)/\tilde{R}^4 & \lambda/\tilde{R}^2 & 0 & 0 & 0 \\ \lambda & \lambda/\tilde{R}^2 & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu/\tilde{R}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu/\tilde{R}^2 \end{bmatrix}. \quad (4.68)$$

As \mathcal{E} is the identity mapping for $R \geq R_o$, we note that $\mathbf{C} = \tilde{\mathbf{C}}$ in $\mathcal{B} \setminus \mathcal{C}$, for which $R \geq R_o$. For $R_i \leq R \leq R_o$ (the cloaking device region \mathcal{C}), we have

$$\mathbf{C}^{ABCD} = J_{\mathcal{E}}^{\mathcal{C}} (\tilde{\mathbf{F}}^{-1})^A_{\tilde{A}} (\mathbf{S}^{-1})^B_{\tilde{B}} (\tilde{\mathbf{F}}^{-1})^C_{\tilde{C}} (\mathbf{S}^{-1})^D_{\tilde{D}} \tilde{\mathbf{C}}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} \circ \mathcal{E}, \quad (4.69)$$

where $\tilde{\mathbf{F}}^{-1} = \text{diag}(f'(R)^{-1}, 1, 1)$, and hence

$$J_{\mathcal{E}}^{\mathcal{C}} = \sqrt{\frac{\det \tilde{\mathbf{G}}(\tilde{X}(X))}{\det \mathbf{G}(X)}} \det \tilde{\mathbf{F}} = \frac{f(R)f'(R)}{R}, \quad R_i \leq R \leq R_o. \quad (4.70)$$

Note that

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R/f(R) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (4.71)$$

Therefore, in the cloaking device ($R_i \leq R \leq R_o$), the elasticity tensor has the following non-zero physical components²⁷

$$\begin{aligned} \hat{\mathbf{C}}^{RRRR} &= \frac{(\lambda + 2\mu)f(R)}{Rf'(R)}, \quad \hat{\mathbf{C}}^{\theta\theta\theta\theta} = \frac{(\lambda + 2\mu)Rf'(R)}{f(R)}, \\ \hat{\mathbf{C}}^{RR\theta\theta} &= \hat{\mathbf{C}}^{\theta\theta RR} = \lambda, \quad \hat{\mathbf{C}}^{R\theta R\theta} = \frac{\mu f(R)}{Rf'(R)}, \quad \hat{\mathbf{C}}^{\theta R \theta R} = \frac{\mu Rf'(R)}{f(R)}, \\ \hat{\mathbf{C}}^{\theta RR\theta} &= \hat{\mathbf{C}}^{R\theta\theta R} = \mu, \quad \hat{\mathbf{C}}^{RRZZ} = \hat{\mathbf{C}}^{ZZRR} = \frac{\lambda f(R)}{R}, \\ \hat{\mathbf{C}}^{\theta\theta ZZ} &= \hat{\mathbf{C}}^{ZZ\theta\theta} = \lambda f'(R), \quad \hat{\mathbf{C}}^{ZZZZ} = \frac{(\lambda + 2\mu)f(R)f'(R)}{R}, \end{aligned}$$

²⁶ Note that in the Voigt notation one has the following bijection between indices: $\{11, 22, 33, 23, 13, 12\} \rightarrow \{1, 2, 3, 4, 5, 6\}$.

²⁷ Note that the physical components of the elasticity tensor $\hat{\mathbf{C}}^{ABCD}$ are related to the components of the elasticity tensor as $\hat{\mathbf{C}}^{ABCD} = \sqrt{G_{AA}}\sqrt{G_{BB}}\sqrt{G_{CC}}\sqrt{G_{DD}}\mathbf{C}^{ABCD}$ (no summation) [114].

$$\begin{aligned}
 \hat{C}^{RZRZ} &= \frac{\mu f(R)}{Rf'(R)}, \quad \hat{C}^{ZRZR} = \frac{\mu f(R)f'(R)}{R}, \\
 \hat{C}^{ZRRZ} &= \hat{C}^{RZZR} = \frac{\mu f(R)}{R}, \quad \hat{C}^{Z\theta Z\theta} = \frac{\mu f(R)f'(R)}{R}, \\
 \hat{C}^{\theta Z\theta Z} &= \frac{\mu Rf'(R)}{f(R)}, \quad \hat{C}^{\theta ZZ\theta} = \hat{C}^{Z\theta\theta Z} = \mu f'(R).
 \end{aligned} \tag{4.72}$$

Remark 9. The balance of angular momentum reads $\delta S^{AB} = \delta S^{BA}$, where $\delta S^{AB} = \mathbf{C}^{ABCD} e_{CD}$, and $e_{CD} = (\hat{\varphi}^* \epsilon)_{CD}$. From (4.72), it is seen that the balance of angular momentum is satisfied if and only if $e_{R\theta} = e_{\theta Z} = e_{RZ} = 0$. In other words, for the class of infinitesimal deformations with diagonal linearized strains cloaking is possible. This is the linear analogue of what was discussed for nonlinear elastostatic cloaking in Section 4.2.

Remark 10. When $\mu = 0$ (energy density is not positive-definite anymore), the elastic constants of the cloak can be written as

$$\begin{aligned}
 \hat{C}^{RRRR} &= \frac{\lambda f(R)}{Rf'(R)}, \quad \hat{C}^{\theta\theta\theta\theta} = \frac{\lambda Rf'(R)}{f(R)}, \quad \hat{C}^{RR\theta\theta} = \hat{C}^{\theta\theta RR} = \lambda, \\
 \hat{C}^{R\theta R\theta} &= \hat{C}^{\theta R\theta R} = \hat{C}^{\theta RR\theta} = \hat{C}^{R\theta\theta R} = 0, \\
 \hat{C}^{RRZZ} &= \hat{C}^{ZZRR} = \frac{\lambda f(R)}{R}, \quad \hat{C}^{\theta\theta ZZ} = \hat{C}^{ZZ\theta\theta} = \lambda f'(R), \\
 \hat{C}^{ZZZZ} &= \frac{\lambda f(R)f'(R)}{R}, \quad \hat{C}^{RZRZ} = \hat{C}^{ZRZR} = \hat{C}^{ZRRZ} = \hat{C}^{RZZR} = 0, \\
 \hat{C}^{Z\theta Z\theta} &= \hat{C}^{\theta Z\theta Z} = \hat{C}^{\theta ZZ\theta} = \hat{C}^{Z\theta\theta Z} = 0.
 \end{aligned} \tag{4.73}$$

It is seen that in this special case the elastic constants of the cloak are fully symmetric. One should note that this can be seen as equivalent to acoustics [78].

Note that $\rho = J_{\mathcal{E}} \tilde{\rho} \circ \mathcal{E}$, and therefore, the mass density in $\mathcal{B} \setminus \mathcal{C}$ is homogeneous and is equal to ρ_0 . The mass density in the cloaking device is inhomogeneous and is given by

$$\rho_{\mathcal{C}}(R) = \frac{f(R)f'(R)}{R} \rho_0, \quad R_i \leq R \leq R_o. \tag{4.74}$$

In the case of anti-plane waves, the only non-trivial equilibrium equation is the one in the z -direction, which using (3.16) for the physical body in the cloaking region is written as

$$\begin{aligned}
 &\frac{1}{R} \frac{\partial}{\partial R} \left(R \hat{C}^{RZRZ}(R) \frac{\partial W}{\partial R} \right) + \frac{1}{R^2} \hat{C}^{\theta Z\theta Z}(R) \frac{\partial^2 W}{\partial \theta^2} + \rho_{\mathcal{C}}(R) \omega^2 W \\
 &= \frac{a_m}{R_s} \delta(R - R_s) \delta(\theta - \theta_s),
 \end{aligned} \tag{4.75}$$

where a_m is the amplitude of a time-harmonic line source with frequency ω located at (R_s, θ_s) such that the induced displacement is given as $\delta\varphi_t(R, \theta) =$

$\mathbf{U}(R, \Theta, t) = (0, 0, \Re[W(R, \Theta)e^{-i\omega t}])$.²⁸ The only non-zero stress components in the cloak read

$$\begin{aligned} \delta S^{RZ} &= \mathbf{C}^{ZRRZ} W_{|R}, & \delta S^{ZR} &= \mathbf{C}^{RZRR} W_{|R}, \\ \delta S^{\Theta Z} &= \mathbf{C}^{Z\Theta\Theta Z} W_{|\Theta}, & \delta S^{Z\Theta} &= \mathbf{C}^{\Theta Z\Theta Z} W_{|\Theta}. \end{aligned} \quad (4.76)$$

As $\hat{\mathbf{C}}^{ZRRZ} \neq \hat{\mathbf{C}}^{RZRR}$ and $\hat{\mathbf{C}}^{Z\Theta\Theta Z} \neq \hat{\mathbf{C}}^{\Theta Z\Theta Z}$ (cf. (4.72)), the balance of angular momentum is not satisfied in the simple case of anti-plane waves. One should note that the imbalance of angular momentum is actually significant as $\hat{\mathbf{C}}^{Z\Theta\Theta Z} - \hat{\mathbf{C}}^{\Theta Z\Theta Z} = \mu f'(R) \left[1 - \frac{R}{f(R)}\right]$ will be unbounded at the inner boundary of the cloak ($R = R_i$) in the limit of $f(R_i) = \epsilon \rightarrow 0$. Thus, the loss of the minor symmetry of the elasticity tensor cannot be ignored. For a linear mapping one has²⁹

$$\begin{aligned} \hat{\mathbf{C}}^{RZRR}(R) &= \mu \frac{R - R_i}{R}, & \hat{\mathbf{C}}^{\Theta Z\Theta Z}(R) &= \mu \frac{R}{R - R_i}, \\ \rho_C(R) &= \frac{R_o^2}{(R_o - R_i)^2} \left(1 - \frac{R_i}{R}\right) \rho_0. \end{aligned} \quad (4.77)$$

These are identical to Parnell [87]’s Eq.(2.5). In the next section we will show that even in the presence of pre-stress elastodynamic cloaking is not possible for anti-plane deformations. However, pre-stress will allow for elastodynamic cloaking for in-plane deformations.

BRUN et al. [8] and many other researchers have used a linear function for $f(R)$, which has been borrowed from electromagnetism. However, a linear $f(R)$ is not acceptable for elasticity as will be explained in Section 5.2. Let us ignore this condition and use a linear $f(R)$, i.e.,

$$f(R) = \frac{R_o(R - R_i)}{R_o - R_i}. \quad (4.78)$$

²⁸ Using (3.16), one obtains

$$\left(\mathbf{A}^{aA} {}^C W^z_{|C}\right)_{|A} = \left(\mathbf{C}^{ABCN} \hat{F}^a_B \hat{F}^n_N g_{zn} W^z_{|C}\right)_{|A} = \left(\mathbf{C}^{ABCZ} \delta^a_B \delta^z_Z g_{zz} W^z_{|C}\right)_{|A}.$$

Note that

$$\begin{aligned} \left(\mathbf{C}^{ABCZ} W_{|C}\right)_{|A} &= \left(\mathbf{C}^{RBRZ} W_{|R} + \mathbf{C}^{RB\Theta Z} W_{|\Theta}\right)_{|R} \\ &\quad + \left(\mathbf{C}^{\Theta BRZ} W_{|R} + \mathbf{C}^{\Theta B\Theta Z} W_{|\Theta}\right)_{|\Theta} \\ &= \left(\mathbf{C}^{RZRR} W_{|R}\right)_{|R} + \left(\mathbf{C}^{\Theta Z\Theta Z} W_{|\Theta}\right)_{|\Theta}. \end{aligned}$$

²⁹ This map does not satisfy the required traction continuity condition, i.e., $f'(R_o) = 1$, but nevertheless has been extensively used in the literature (see Section 5.2).

For this (inappropriate) choice of $f(R)$ the mass density in the cloak reads

$$\rho_{\mathcal{C}}(R) = \frac{R_o^2}{(R_o - R_i)^2} \left(1 - \frac{R_i}{R}\right) \rho_0, \quad (4.79)$$

which is identical to the mass density in [8]. Our elastic constants in the cloak read

$$\begin{aligned} \hat{\mathbf{C}}^{RRRR} &= (\lambda + 2\mu) \frac{R - R_i}{R}, & \hat{\mathbf{C}}^{\Theta\Theta\Theta\Theta} &= (\lambda + 2\mu) \frac{R}{R - R_i}, \\ \hat{\mathbf{C}}^{RR\Theta\Theta} &= \hat{\mathbf{C}}^{\Theta\Theta RR} = \lambda, & \hat{\mathbf{C}}^{\Theta RR\Theta} &= \hat{\mathbf{C}}^{R\Theta\Theta R} = \mu, \\ \hat{\mathbf{C}}^{R\Theta R\Theta} &= \frac{R - R_i}{R} \mu, & \hat{\mathbf{C}}^{\Theta R\Theta R} &= \frac{R}{R - R_i} \mu, \end{aligned} \quad (4.80)$$

which are identical to those in [8]. BRUN et al. [8] and many other authors (e.g., NORRIS and SHUVALOV [80]) have claimed that the lack of minor symmetries in the elastic constants of the cloak implies that a cloak is made of a Cosserat solid. This claim is, unfortunately, incorrect. We will show in Section 5.2 that transformation cloaking is not possible in (generalized) Cosserat solids.

4.4. Small-on-Large Transformation Cloaking

In this section we formulate transformation cloaking in the small-on-large theory of GREEN et al. [38] and explore the possibility of transformation cloaking when a cloak can be pre-stressed.

Linearized balance of angular momentum with respect to a stressed initial configuration. Note that the form of the linearized balance of linear momentum (4.41) in the case of a flat ambient space is independent of the initial stress $\mathring{\mathbf{P}}$, i.e., the balance of linear momentum when the initial configuration is pre-stressed is still (4.41). Next, we explore the possibility of transformation cloaking in linearized elasticity with respect to a pre-stressed configuration. Let us assume that the initial configuration of the virtual body is its undeformed stress-free reference configuration, while the initial configuration of the physical body is stressed within the cloak with the initial second Piola-Kirchhoff stress $\mathring{\mathbf{S}}$. Let us denote the cloak in the reference configuration by $\mathring{\mathcal{C}}$ and in the stressed initial configuration by \mathcal{C} . We assume that the stressed initial configuration is the result of a deformation $\mathring{\varphi} : \mathcal{B} \rightarrow \mathcal{S}$ from the stress-free configuration $(\mathcal{B}, \mathbf{G})$. We consider linearized elasticity with respect to the two reference configurations $(\mathcal{B}, \mathbf{G})$ and $(\tilde{\mathcal{B}}, \tilde{\mathbf{G}})$ and the initial motions $\mathring{\varphi} : \mathcal{B} \rightarrow \mathcal{S}$ and $\mathring{\tilde{\varphi}} = id : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{S}}$. Assuming that the physical body has an energy function W with respect to its stress-free configuration the pre-stress $\mathring{\mathbf{P}}$ is expressed as

$$\mathring{P}^{aA} = g^{ab} \frac{\partial W}{\partial F^b_B} \Big|_{\mathring{\mathbf{F}}}. \quad (4.81)$$

We still assume that the two corresponding variation fields are related through shifters. The balance of angular momentum $S^{AB} = S^{BA}$ when linearized about any (stress-free or stressed) initial configuration reads $\delta S^{AB} = \delta S^{BA}$, which is equivalent to $\mathbf{C}^{ABCD} = \mathbf{C}^{BACD}$. We assume that $\mathring{\varphi}|_{\mathcal{B} \setminus \mathring{\mathcal{C}}} = id$, and $\mathring{\mathbf{F}}|_{\partial_o \mathcal{C}} = \mathbf{I}$ (note that $\partial_o \mathcal{C} = \partial_o \mathring{\mathcal{C}}$).

Transformed second elasticity tensor. As the form of the balance of linear momentum is not affected by pre-stress, the transformation relation for the first elasticity tensor is still (4.49), i.e., $\tilde{\mathbf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \circ \mathcal{E} = J_{\mathcal{E}}^{-1} \tilde{\mathbf{F}}^{\tilde{A}} \tilde{\mathbf{F}}^{\tilde{B}} \mathbf{s}^{\tilde{a}} \mathbf{s}^{\tilde{b}} \mathbf{A}^{aAbB}$. From (3.16) we have the following two relations:

$$\begin{aligned} \mathbf{A}^{aAbB} &= \mathbf{C}^{AMBN} \hat{\mathbf{F}}^a_M \hat{\mathbf{F}}^b_N + \hat{\mathbf{S}}^{AB} g^{ab}, \\ \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} &= \tilde{\mathbf{C}}^{\tilde{A}\tilde{M}\tilde{B}\tilde{N}} \hat{\mathbf{F}}^{\tilde{a}}_{\tilde{M}} \hat{\mathbf{F}}^{\tilde{b}}_{\tilde{N}}. \end{aligned} \quad (4.82)$$

When the reference configuration of the physical body is pre-stressed the cloaking map is not $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$; it is the map $\xi : \hat{\varphi}(\mathcal{B}) \rightarrow \hat{\mathcal{S}}$ (see Fig. 4). Note that $\tilde{\mathbf{F}} = \hat{\mathbf{F}}^{-1} \xi \hat{\mathbf{F}}$, where $\hat{\mathbf{F}} = id$, i.e., $(\tilde{\mathbf{F}})^{\tilde{a}}_{\tilde{A}} = \delta^{\tilde{a}}_{\tilde{A}}$. Using (4.49) and (4.82), one can calculate the second elasticity constants of the physical body as

$$\begin{aligned} \mathbf{C}^{ABCD} &= J_{\mathcal{E}} \left[(\hat{\mathbf{F}}^{-1})^B_k (\mathbf{s}^{-1})^k_{\tilde{a}} \right] \left[(\hat{\mathbf{F}}^{-1})^D_l (\mathbf{s}^{-1})^l_{\tilde{b}} \right] \\ &\quad \times \left[(\hat{\mathbf{F}}^{-1})^A_m (\hat{\mathbf{F}}^{-1})^m_{\tilde{m}} \delta^{\tilde{m}}_{\tilde{A}} \right] \left[(\hat{\mathbf{F}}^{-1})^C_n (\hat{\mathbf{F}}^{-1})^n_{\tilde{n}} \delta^{\tilde{n}}_{\tilde{B}} \right] \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \\ &\quad - \hat{\mathbf{S}}^{AC} (\hat{\varphi}^* \mathbf{g}^{\#})^{BD}. \end{aligned} \quad (4.83)$$

It is observed that under a cloaking map the major symmetries are preserved. Pushing forward the elastic constants to the initial configuration, one obtains

$$\begin{aligned} (\hat{\varphi}_* \mathbf{C})^{abcd} &= J_{\mathcal{E}} (\mathbf{s}^{-1})^b_{\tilde{a}} (\mathbf{s}^{-1})^d_{\tilde{b}} \left[(\hat{\mathbf{F}}^{-1})^a_{\tilde{m}} \delta^{\tilde{m}}_{\tilde{A}} \right] \left[(\hat{\mathbf{F}}^{-1})^c_{\tilde{n}} \delta^{\tilde{n}}_{\tilde{B}} \right] \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \\ &\quad - (\hat{\varphi}_* \hat{\mathbf{S}})^{ac} g^{bd}. \end{aligned} \quad (4.84)$$

Note that $J_{\mathcal{E}} = J_{\xi} \hat{J}$, and hence

$$\begin{aligned} \mathbf{c}^{abcd} &= \frac{1}{\hat{J}} (\hat{\varphi}_* \mathbf{C})^{abcd} \\ &= J_{\xi} \left[(\hat{\mathbf{F}}^{-1})^a_{\tilde{m}} \delta^{\tilde{m}}_{\tilde{A}} \right] (\mathbf{s}^{-1})^b_{\tilde{a}} \left[(\hat{\mathbf{F}}^{-1})^c_{\tilde{n}} \delta^{\tilde{n}}_{\tilde{B}} \right] (\mathbf{s}^{-1})^d_{\tilde{b}} \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \\ &\quad - \hat{\sigma}^{ac} g^{bd}. \end{aligned} \quad (4.85)$$

The balance of angular momentum in the physical body is equivalent to $\mathbf{c}^{[ab]cd} = 0$, which is written as

$$\hat{\sigma}^{[ac} g^{b]d} = J_{\xi} \left[(\hat{\mathbf{F}}^{-1})^{[a}_{\tilde{m}} \delta^{\tilde{m}}_{\tilde{A}} \right] (\mathbf{s}^{-1})^b]_{\tilde{a}} \left[(\hat{\mathbf{F}}^{-1})^c_{\tilde{n}} \delta^{\tilde{n}}_{\tilde{B}} \right] (\mathbf{s}^{-1})^d_{\tilde{b}} \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}. \quad (4.86)$$

Note that $\hat{\sigma}^{ac} = \hat{\sigma}^{ca}$ is not specified a priori.

From (4.49), the referential mass densities of the physical and virtual bodies are related as $\rho_0 = J_{\mathcal{E}} \tilde{\rho}_0 \circ \mathcal{E}$. Conservation of mass for the virtual and physical bodies implies that $\rho_0 = \rho \hat{J}$ and $\tilde{\rho}_0 = \tilde{\rho}$. Therefore, the spatial mass density of the cloak is given by $\rho = J_{\xi} \tilde{\rho}$.

Proposition 4. *In the small-on-large theory, i.e., linearized elasticity with respect to a pre-stressed configuration, elastodynamic transformation cloaking is not possible regardless of the shape of the hole and the cloak.*

Proof. Let us assume that the physical body is pre-stressed with the initial Cauchy stress $\hat{\sigma}$ and the initial body force $\hat{\mathbf{b}}$. The balance of linear and angular momenta for the physical body in its initial configuration read

$$\hat{\sigma}^{ab}|_b + \rho \hat{b}^a = 0, \quad \text{and} \quad \hat{\sigma}^{[ab]} = 0. \quad (4.87)$$

Note that $\hat{\sigma}$ and $\hat{\mathbf{b}}$ are supported on the interior of the cloaking region, i.e., on $\mathcal{C} \setminus \{\partial\mathcal{H} \cup \partial_o\mathcal{C}\}$. Without loss of generality, we may work in Cartesian coordinates, where the shifter and the metrics have trivial representations. In Cartesian coordinates, the first elasticity tensor for the virtual body is given by $\tilde{\mathbf{A}}^{\tilde{a}\tilde{b}\tilde{B}\tilde{B}} = [\lambda \delta^{\tilde{a}\tilde{b}} \delta^{\tilde{B}\tilde{B}} + \mu (\delta^{\tilde{A}\tilde{B}} \delta^{\tilde{a}\tilde{b}} + \delta^{\tilde{A}\tilde{b}} \delta^{\tilde{B}\tilde{a}})]$. With an abuse of notation let $\mathbf{F}_{ij} = (\tilde{\mathbf{F}}^{-1})^i_j$, $i, j \in \{1, 2, 3\}$, and thus, $J_\xi = [\epsilon_{lmn} \epsilon_{ijk} \mathbf{F}_{il} \mathbf{F}_{jm} \mathbf{F}_{kn}]^{-1}$. Expanding (4.86), one obtains the following relations for $i \neq j \neq k \in \{1, 2, 3\}$

$$\hat{\sigma}^{ii} = J_\xi \left[\mathbf{F}_{ij} (\mathbf{F}_{ij} - \mathbf{F}_{ji}) \lambda + \left(\mathbf{F}_{ii}^2 + 2\mathbf{F}_{ij}^2 + \mathbf{F}_{ik}^2 - \mathbf{F}_{ii} \mathbf{F}_{jj} \right) \mu \right], \quad (4.88)$$

$$\hat{\sigma}^{ii} = J_\xi \left[\mathbf{F}_{ik} (\mathbf{F}_{ik} - \mathbf{F}_{ki}) \lambda + \left(\mathbf{F}_{ii}^2 + 2\mathbf{F}_{ik}^2 + \mathbf{F}_{ij}^2 - \mathbf{F}_{ii} \mathbf{F}_{kk} \right) \mu \right], \quad (4.89)$$

$$\begin{aligned} \hat{\sigma}^{ij} &= J_\xi \left[\mathbf{F}_{ii} (\mathbf{F}_{ji} - \mathbf{F}_{ij}) \lambda \right. \\ &\quad \left. + \left(\mathbf{F}_{ij} \mathbf{F}_{jj} + 2\mathbf{F}_{ii} \mathbf{F}_{ji} + \mathbf{F}_{ik} \mathbf{F}_{jk} - \mathbf{F}_{ii} \mathbf{F}_{ij} \right) \mu \right], \end{aligned} \quad (4.90)$$

$$\begin{aligned} \hat{\sigma}^{ij} &= J_\xi \left[\mathbf{F}_{jj} (\mathbf{F}_{ij} - \mathbf{F}_{ji}) \lambda + \left(\mathbf{F}_{ii} \mathbf{F}_{ji} + 2\mathbf{F}_{jj} \mathbf{F}_{ij} + \mathbf{F}_{ik} \mathbf{F}_{jk} \right. \right. \\ &\quad \left. \left. - \mathbf{F}_{jj} \mathbf{F}_{ji} \right) \mu \right], \end{aligned} \quad (4.91)$$

$$\mathbf{F}_{ii} (\mathbf{F}_{kj} - \mathbf{F}_{jk}) \lambda + \left(\mathbf{F}_{ij} \mathbf{F}_{ki} - \mathbf{F}_{ik} \mathbf{F}_{ji} \right) \mu = 0, \quad (4.92)$$

$$\mathbf{F}_{ik} (\mathbf{F}_{ji} - \mathbf{F}_{ij}) \lambda + \left(\mathbf{F}_{ii} \mathbf{F}_{jk} - \mathbf{F}_{ij} \mathbf{F}_{ik} \right) \mu = 0. \quad (4.93)$$

Using (4.90) and (4.91), one has

$$J_\xi (\lambda + \mu) (\mathbf{F}_{ij} - \mathbf{F}_{ji}) (\mathbf{F}_{ii} + \mathbf{F}_{jj}) = 0. \quad (4.94)$$

Noting that $\lambda + \mu = \frac{1}{3}(3\lambda + 2\mu) + \frac{1}{3}\mu > 0$, either $\mathbf{F}_{ij} = \mathbf{F}_{ji}$, or $\mathbf{F}_{ii} = -\mathbf{F}_{jj}$. If $\mathbf{F}_{ii} = -\mathbf{F}_{jj}$, then it follows that $\mathbf{F}_{ii} = 0$, and from (4.92), $\mathbf{F}_{ij} \mathbf{F}_{ki} = \mathbf{F}_{ik} \mathbf{F}_{ji}$. Using this relation and (4.93), one obtains $\lambda \mathbf{F}_{ij} \mathbf{F}_{ki} = (\lambda + \mu) \mathbf{F}_{ij} \mathbf{F}_{ik}$. This relation holds if $\mathbf{F}_{ij} = 0$. If $\mathbf{F}_{ij} \neq 0$, then $\lambda \mathbf{F}_{ki} = (\lambda + \mu) \mathbf{F}_{ik}$. This can be rewritten as $\lambda \mathbf{F}_{ik} = (\lambda + \mu) \mathbf{F}_{ki}$, which implies that $\mathbf{F}_{ik} = 0$. Therefore, $\tilde{\mathbf{F}}^{-1} = \mathbf{0}$, which is not possible. Thus, $\mathbf{F}_{ij} = \mathbf{F}_{ji}$, and hence, using (4.93), $\mathbf{F}_{ii} \mathbf{F}_{jk} = \mathbf{F}_{ij} \mathbf{F}_{ik}$. Therefore, either $\mathbf{F}_{jk} = 0$, or $\mathbf{F}_{ii} = \mathbf{F}_{ij} \mathbf{F}_{ik} / \mathbf{F}_{jk}$. If $\mathbf{F}_{jk} = 0$, then either $\mathbf{F}_{ij} = 0$ or $\mathbf{F}_{ik} = 0$. Without loss of generality, let us assume that $\mathbf{F}_{ij} = 0$ for some $i \neq j$ and $\mathbf{F}_{ik} \neq 0$ ($\mathbf{F}_{jk} = \mathbf{F}_{ij} = 0$, $\mathbf{F}_{ik} \neq 0$). As $\mathbf{F}_{ik} \neq 0$, from $\mathbf{F}_{jj} \mathbf{F}_{ik} = \mathbf{F}_{ji} \mathbf{F}_{jk}$, we note that $\mathbf{F}_{jj} = 0$. Therefore, $\tilde{\mathbf{F}}^{-1}$ has a zero column vector, i.e., it is singular, which is not possible. Our conclusion is that either all the off-diagonal terms of $\tilde{\mathbf{F}}^{-1}$ are zero or they are all non-zero. If $\mathbf{F}_{ij} = \mathbf{F}_{jk} = \mathbf{F}_{ik} = 0$, then $\tilde{\mathbf{F}} = \beta \mathbf{I}$ (cf. (4.95))

for some nonzero scalar β , which means that cloaking is not possible.³⁰ If the off-diagonal terms are non-zero, then $F_{ii} = F_{ij}F_{ik}/F_{jk}$, $F_{jj} = F_{ji}F_{jk}/F_{ik}$, and $F_{kk} = F_{kj}F_{ki}/F_{ji}$. Substituting these relations into $J_{\xi}^{-1} = \epsilon_{lmn}\epsilon_{ijk}F_{il}F_{jm}F_{kn}$, it is straightforward to see that $J_{\xi}^{-1} = 0$, i.e., the inverse cloaking map is singular, which is not possible. \square

If one uses linearized elasticity with respect to a pre-stressed configuration the best one can hope for is cloaking for a restricted class of deformations. In the following, we will find such an example in the case of a cylindrical cloak.

An example of a pre-stressed linear elastic cloak. Let us consider the cylindrical cloak example in the presence of the initial stress $\hat{\sigma}$. The cloaking map ξ transforms a pre-stressed cylindrical annulus in the physical body with inner and outer radii r_i and r_o , respectively, to a cylindrical annulus in the virtual body with inner and outer radii ϵ and r_o , respectively. In cylindrical coordinates let $(\tilde{r}, \tilde{\theta}, \tilde{z}) = \xi(r, \theta, z) = (f(r), \theta, z)$. The tangent map of the cloaking transformation is given by

$$\mathbf{F}^{\xi} = \begin{bmatrix} f'(r) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (4.96)$$

where $f(r_o) = r_o$ and $f(r_i) = \epsilon$. Thus

$$J_{\xi} = \sqrt{\frac{\det \tilde{\mathbf{g}}(\tilde{x}(x))}{\det \mathbf{g}(x)}} \det \mathbf{F}^{\xi} = \frac{f(r)f'(r)}{r}, \quad (4.97)$$

where $\mathbf{g} = \text{diag}(1, r^2, 1)$ and $\tilde{\mathbf{g}} = \text{diag}(1, \tilde{r}^2, 1)$ are, respectively, the Euclidean metrics of the physical and virtual bodies in their current configurations. Note that $\mathbf{s} = \text{diag}(1, r/f(r), 1)$ and

$$\tilde{\mathbf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = \left[\lambda \tilde{g}^{\tilde{A}\tilde{M}} \tilde{g}^{\tilde{B}\tilde{N}} + \mu \left(\tilde{g}^{\tilde{A}\tilde{B}} \tilde{g}^{\tilde{M}\tilde{N}} + \tilde{g}^{\tilde{A}\tilde{N}} \tilde{g}^{\tilde{B}\tilde{M}} \right) \right] \tilde{F}^{\tilde{a}}_{\tilde{M}} \tilde{F}^{\tilde{b}}_{\tilde{N}}, \quad (4.98)$$

where $\tilde{F}^{\tilde{a}}_{\tilde{A}} = \delta^{\tilde{a}}_{\tilde{A}}$. Noting that $\rho = J_{\xi} \tilde{\rho}$, the mass density of the cloaking device in its current configuration is given by

$$\rho(r) = \frac{f(r)f'(r)}{r} \tilde{\rho}, \quad r_i \leq r \leq r_o. \quad (4.99)$$

Using (4.85), the non-zero physical components of the elasticity tensor of the cloaking device in the current configuration are written as ($r_i \leq r \leq r_o$)

$$\hat{\mathbf{C}}^{rrrr} = \frac{(\lambda + 2\mu)f(r)}{rf'(r)} - \hat{\sigma}^{rr}, \quad \hat{\mathbf{C}}^{\theta\theta\theta\theta} = \frac{(\lambda + 2\mu)rf'(r)}{f(r)} - \hat{\sigma}^{\theta\theta},$$

³⁰ Knowing that $F_{ij} = F_{ji}$, from (4.88) and (4.89), one obtains

$$0 = F_{ij}^2 - F_{ik}^2 = F_{ii} (F_{jj} - F_{kk}). \quad (4.95)$$

$$\begin{aligned}
\hat{\mathbf{c}}^{rr\theta\theta} &= \hat{\mathbf{c}}^{\theta\theta rr} = \lambda, \quad \hat{\mathbf{c}}^{r\theta r\theta} = \frac{\mu f(r)}{r f'(r)} - \hat{\sigma}^{rr}, \quad \hat{\mathbf{c}}^{\theta r \theta r} = \frac{\mu r f'(r)}{f(r)} - \hat{\sigma}^{\theta\theta}, \\
\hat{\mathbf{c}}^{\theta rr\theta} &= \hat{\mathbf{c}}^{r\theta\theta r} = \mu, \quad \hat{\mathbf{c}}^{r r z z} = \hat{\mathbf{c}}^{z z r r} = \frac{\lambda f(r)}{r}, \quad \hat{\mathbf{c}}^{\theta\theta z z} = \hat{\mathbf{c}}^{z z \theta\theta} = \lambda f'(r), \\
\hat{\mathbf{c}}^{z z z z} &= \frac{(\lambda + 2\mu) f(r) f'(r)}{r} - \hat{\sigma}^{z z}, \\
\hat{\mathbf{c}}^{r z r z} &= \frac{\mu f(r)}{r f'(r)} - \hat{\sigma}^{r r}, \quad \hat{\mathbf{c}}^{z r z r} = \frac{\mu f(r) f'(r)}{r} - \hat{\sigma}^{z z}, \quad \hat{\mathbf{c}}^{z r r z} = \hat{\mathbf{c}}^{r z z r} = \frac{\mu f(r)}{r}, \\
\hat{\mathbf{c}}^{z\theta z\theta} &= \frac{\mu f(r) f'(r)}{r} - \hat{\sigma}^{z z}, \quad \hat{\mathbf{c}}^{\theta z \theta z} = \frac{\mu r f'(r)}{f(r)} - \hat{\sigma}^{\theta\theta}, \quad \hat{\mathbf{c}}^{\theta z z \theta} = \hat{\mathbf{c}}^{z \theta \theta z} = \mu f'(r), \\
\hat{\mathbf{c}}^{r r \theta r} &= \hat{\mathbf{c}}^{\theta r r r} = -\hat{\sigma}^{r\theta}, \quad \hat{\mathbf{c}}^{r r z r} = \hat{\mathbf{c}}^{z r r r} = -\hat{\sigma}^{r z}, \quad \hat{\mathbf{c}}^{r \theta \theta \theta} = \hat{\mathbf{c}}^{\theta \theta r \theta} = -\hat{\sigma}^{r\theta}, \\
\hat{\mathbf{c}}^{r \theta z \theta} &= \hat{\mathbf{c}}^{z \theta r \theta} = -\hat{\sigma}^{r z}, \quad \hat{\mathbf{c}}^{r z \theta z} = \hat{\mathbf{c}}^{\theta z r z} = -\hat{\sigma}^{r\theta}, \quad \hat{\mathbf{c}}^{r z z z} = \hat{\mathbf{c}}^{z z r z} = -\hat{\sigma}^{r z}, \\
\hat{\mathbf{c}}^{\theta r z r} &= \hat{\mathbf{c}}^{z r \theta r} = -\hat{\sigma}^{\theta z}, \quad \hat{\mathbf{c}}^{\theta \theta z \theta} = \hat{\mathbf{c}}^{z \theta \theta \theta} = -\hat{\sigma}^{\theta z}, \quad \hat{\mathbf{c}}^{\theta z z z} = \hat{\mathbf{c}}^{z z \theta z} = -\hat{\sigma}^{\theta z}.
\end{aligned} \tag{4.100}$$

Let us examine the possibility of designing a cylindrical cloak using pre-stress for the special cases of in-plane and anti-plane deformations.

- (i) Anti-plane deformations: In this case, the only non-zero components of the strain tensor are ϵ_{rz} and $\epsilon_{\theta z}$. Therefore, from (4.100), the balance of angular momentum, i.e., $\mathbf{c}^{[ab]rz} = \mathbf{c}^{[ab]zr} = \mathbf{c}^{[ab]\theta z} = \mathbf{c}^{[ab]z\theta} = 0$, for $a = r$, $b = \theta$ implies that $\hat{\sigma}^{rz} = \hat{\sigma}^{\theta z} = 0$. For $a = r$, $b = z$, $\hat{\sigma}^{r\theta} = 0$, and

$$\hat{\sigma}^{rr} = \frac{\mu f(r)}{r} \left[\frac{1}{f'(r)} - 1 \right], \quad \hat{\sigma}^{zz} = \frac{\mu f(r)}{r} [f'(r) - 1]. \tag{4.101}$$

For $a = \theta$, $b = z$:

$$\hat{\sigma}^{\theta\theta} = \mu f'(r) \left[\frac{r}{f(r)} - 1 \right], \quad \hat{\sigma}^{zz} = \mu f'(r) \left[\frac{f(r)}{r} - 1 \right]. \tag{4.102}$$

Thus, from the two expressions for $\hat{\sigma}^{zz}$ it follows that $r f'(r) = f(r)$. Knowing that $f(r_o) = r_o$, one obtains $f(r) = r$, which implies that cloaking is not possible for anti-plane deformations.

- (ii) In-plane deformations: For the in-plane case, the non-zero components of the linearized strain are ϵ_{rr} , $\epsilon_{\theta\theta}$, and $\epsilon_{r\theta}$. The balance of angular momentum $\mathbf{c}^{[ab]rr} = \mathbf{c}^{[ab]\theta\theta} = \mathbf{c}^{[ab]r\theta} = \mathbf{c}^{[ab]\theta r} = 0$, for $a = r$, $b = \theta$ gives

$$\hat{\sigma}^{r\theta} = 0, \quad \hat{\sigma}^{rr} = \mu \left[\frac{f(r)}{r f'(r)} - 1 \right], \quad \hat{\sigma}^{\theta\theta} = \mu \left[\frac{r f'(r)}{f(r)} - 1 \right]. \tag{4.103}$$

For $a = r$, $b = z$, and $a = \theta$, $b = z$, it gives: $\hat{\sigma}^{rz} = \hat{\sigma}^{\theta z} = 0$. The traction vector in the physical and virtual bodies are given by

$$T^a = \hat{P}^{aA} N_A + \delta P^{aA} N_A, \quad \text{and} \quad \tilde{T}^{\tilde{a}} = \delta \tilde{P}^{\tilde{a}\tilde{A}} \tilde{N}_{\tilde{A}}, \tag{4.104}$$

where $\delta\mathbf{P}$ and $\delta\tilde{\mathbf{P}}$ are, respectively, the linearized first Piola-Kirchhoff stress tensors for the physical and virtual bodies related by $\delta P^{aA} = J_{\Xi}(\tilde{F}^{-1})^A_{\tilde{A}}(\mathbf{s}^{-1})^a_{\tilde{a}} \delta\tilde{P}^{\tilde{a}\tilde{A}}$, and $\hat{\mathbf{P}}$ is the initial first Piola-Kirchhoff stress in the physical body. On the outer boundary of the cloak $\partial_o\mathcal{C}$, the traction vector in the two problems must be identical. Therefore, the initial traction in the physical body must vanish on the outer boundary of the cloak, viz., $\hat{\mathbf{t}}|_{\partial_o\mathcal{C}} = \langle\langle \hat{\boldsymbol{\sigma}}, \mathbf{n} \rangle\rangle_{\mathbf{g}}|_{\partial_o\mathcal{C}} = \mathbf{0}$, where \mathbf{n} is the normal to the outer boundary of the cloak in its current configuration. Thus, $\hat{\boldsymbol{\sigma}}^{rr}(r_o) = \hat{\boldsymbol{\sigma}}^{r\theta}(r_o) = 0$, which holds if $f'(r_o) = 1$ (i.e., $\hat{\mathbf{F}}|_{\partial_o\mathcal{C}} = \mathbf{I}$). To ensure that the hole surface is traction-free in the physical body, the initial traction needs to vanish on $\partial\mathcal{H}$ as well, i.e., $\hat{\mathbf{t}}|_{\partial\mathcal{H}} = \mathbf{0}$.³¹ Hence, it follows that $\hat{\boldsymbol{\sigma}}^{rr}(r_i) = \hat{\boldsymbol{\sigma}}^{r\theta}(r_i) = 0$, amounting to $f'(r_i) = \epsilon/r_i$. Moreover, the linearized tractions must also be identical on the outer boundary of the cloak in the two systems, i.e., one must have, $\tilde{\mathbf{F}}|_{\partial_o\mathcal{C}} = \mathbf{I}$, but note that $\tilde{\mathbf{F}} = \hat{\mathbf{F}}^{-1}\hat{\mathbf{F}}\hat{\mathbf{F}}^{\xi}$, and hence, $\tilde{\mathbf{F}}|_{\partial_o\mathcal{C}} = \hat{\mathbf{F}}^{\xi}|_{\partial_o\mathcal{C}} \hat{\mathbf{F}}|_{\partial_o\mathcal{C}} = \mathbf{I}$ already holds. Equilibrium equations in the initial configuration $\hat{\boldsymbol{\sigma}}^{ab}|_b + \rho\hat{b}^a = 0$, have only one non-trivial component, which in terms of the physical components of the Cauchy stress reads

$$\frac{d}{dr}\hat{\boldsymbol{\sigma}}^{rr} + \frac{1}{r}(\hat{\boldsymbol{\sigma}}^{rr} - \hat{\boldsymbol{\sigma}}^{\theta\theta}) + \rho(r)\hat{b}^r = 0. \tag{4.105}$$

Therefore

$$\begin{aligned} \rho(r)\hat{b}^r(r) &= -\mu \left[\frac{f(r)}{rf'(r)} - 1 \right]' + \frac{\mu}{r} \left[\frac{rf'(r)}{f(r)} - \frac{f(r)}{rf'(r)} \right] \\ &= \mu \left[\frac{f(r)f''(r)}{rf'(r)^2} + \frac{f'(r)}{f(r)} - \frac{1}{r} \right]. \end{aligned} \tag{4.106}$$

Note that the simplest polynomial that satisfies the boundary conditions $f(r_o) = r_o$, $f(r_i) = \epsilon$, $f'(r_i) = \epsilon/r_i$, and $f'(r_o) = 1$ is

$$\begin{aligned} f(r) &= \frac{(r_i + r_o)(r_i - \epsilon)}{r_i(r_i - r_o)^3}r^3 - \frac{2(r_i^2 + r_i r_o + r_o^2)(r_i - \epsilon)}{r_i(r_i - r_o)^3}r^2 \\ &\quad + \frac{r_i^2(r_i^2 + r_i r_o + 4r_o^2) - r_o\epsilon(4r_i^2 + r_i r_o + r_o^2)}{r_i(r_i - r_o)^3}r + \frac{2r_i r_o^2(\epsilon - r_i)}{(r_i - r_o)^3}. \end{aligned} \tag{4.107}$$

5. Elastodynamic Transformation Cloaking in Solids with Microstructure

Having established that classical linear elasticity is not flexible enough to allow for transformation cloaking a possible solution would be to see if transformation

³¹ The inner boundary of the virtual and physical holes must be traction-free. This condition for the virtual hole reads $\delta\tilde{\mathbf{t}} = \delta\tilde{P}^{\tilde{a}\tilde{A}}\tilde{N}_{\tilde{A}}|_{\partial\tilde{\mathcal{H}}} = 0$. However, note that $\mathbf{s}\delta\mathbf{t}dA = \delta\tilde{\mathbf{t}}d\tilde{A} = \mathbf{0}$, i.e., the linearized traction in the physical body vanishes, and thus, in order for the hole to be traction-free in the physical body, the initial traction must vanish, i.e., $\hat{P}^{aA}N_A|_{\partial\mathcal{H}} = 0$.

cloaking may be achieved in solids with microstructure. In such continua the Cauchy stress does not need to be symmetric. This is the reason that in the literature of elastodynamic transformation cloaking it has been suggested that a cloak should be made of a Cosserat solid. The first systematic formulation of generalized continua goes back to the seminal work of Cosserat brothers [17], which remained unnoticed until the interest in continua with microstructure was revived in the 1950s, 1960s, and 1970s [27, 28, 74, 75, 112, 113] and now there is a vast literature on generalized continua.

In formulating the cloaking problem, we assume that both the virtual and physical bodies are made of solids with microstructure. We can follow two paths: i) Assume that energy depends on $\nabla \mathbf{F}$ with components $F^a{}_{A|B}$, i.e., gradient elasticity. This seems to be more natural for our purposes because first there is no ambiguity in the physical meaning of microstructure, and second there are well-established connections between strain gradient elasticity and atomistic calculations (see [67] and references therein); ii) In addition to the deformation mapping φ assume a set of director fields that have their own independent kinematics, i.e., (generalized) Cosserat elasticity. Energy depends on these extra fields as well. One should note that there are many different choices for describing microstructure. This has been discussed in some detail in the monograph [10]; see also [124]. We discuss elastodynamic transformation cloaking in both gradient and generalized Cosserat solids. The goal of this section is to show that adding microstructure does not remedy the obstruction to elastodynamic transformation cloaking.

5.1. Elastodynamic Transformation Cloaking in Gradient Elastic Solids

In gradient elasticity (or strain-gradient elasticity) energy function depends on the (covariant) derivative of the deformation gradient as well, i.e., [113]

$$W = W(X, \mathbf{F}, \nabla \mathbf{F}, \mathbf{G}, \mathbf{g} \circ \varphi). \tag{5.1}$$

Note that (bulk) compatibility equations are written as $F^a{}_{A|B} = F^a{}_{B|A}$ [122]. Objectivity of an energy function means invariance under (rigid-body) rotations in the ambient space. One can think of \mathbf{F} as a vector-valued 1-form [121]. The 1-form part would not be affected by changes of coordinates in the ambient space. Now the question is how should W depend on \mathbf{F} in order to be isotropic? Suppose $f = f(\mathbf{u})$, where \mathbf{u} is a vector. We know that for f to be isotropic it should have the form $f = \hat{f}(\mathbf{u} \cdot \mathbf{u})$, where $\mathbf{u} \cdot \mathbf{u} = u^a u^b g_{ab}$ [109]. This new variable in the case of deformation gradient is $F^a{}_A F^b{}_B g_{ab} = C_{AB}$. In gradient elasticity one has an extra independent variable. Let us first consider a scalar function of two vectors $f = f(\mathbf{u}, \mathbf{v})$. For f to be isotropic, one must have $f = \hat{f}(\mathbf{u} \cdot \mathbf{u}, \mathbf{u} \cdot \mathbf{v}, \mathbf{v} \cdot \mathbf{v})$. In the case of energy function these three variables are

$$F^a{}_A F^b{}_B g_{ab} = C_{AB}, \quad F^a{}_A F^b{}_B|_C g_{ab} =: D_{ABC}, \quad F^a{}_A|B F^b{}_C|D g_{ab} =: E_{ABCD}. \tag{5.2}$$

Note the symmetries of the new measures of strain: $D_{ABC} = D_{ACB}$, $E_{ABCD} = E_{CDAB} = E_{BACD} = E_{ABDC} = E_{BADC}$. The strain measure \mathbf{E} is, however,

functionally dependent on \mathbf{C}^b and \mathbf{D} as noticed by Toupin [113]. More specifically, given \mathbf{D} , one can write $F^b{}_{B|C} = g^{ab}(F^{-1})^A{}_a D_{ABC}$. Hence, $E_{ABCD} = C^{-MN} D_{MAB} D_{NCD}$. Therefore, $W = \hat{W}(X, C_{AB}, D_{ABC}, G_{AB})$. Note that $C_{AB|C} = F^a{}_{A|C} F^b{}_{B} g_{ab} + F^a{}_{A} F^b{}_{B|C} g_{ab} = D_{BAC} + D_{ABC}$. Thus, $D_{ABC} = \frac{1}{2} (C_{AB|C} + C_{AC|B} - C_{BC|A})$. Therefore, one can assume that energy density depends on \mathbf{C}^b and $\nabla \mathbf{G}^b$, i.e., $W = \hat{W}(X, C_{AB}, C_{AB|C}, G_{AB})$, which is exactly the way Toupin expressed the energy density.

Balance of linear momentum. We next derive the governing equations of gradient elasticity using Hamilton's principle of least action. Assuming the Lagrangian density $\mathcal{L} = W - T$, where $T = \frac{1}{2} \rho_0 \langle \mathbf{V}, \mathbf{V} \rangle_{\mathbf{g}} = \frac{1}{2} \rho_0 V^a V^b g_{ab}$ is the classical kinetic energy density, the action is defined as

$$S = \int_{t_1}^{t_2} \int_{\mathcal{B}} (W - T) dV dt, \tag{5.3}$$

where $t_1 < t_2$ are arbitrary time instances, and dV is the Riemannian volume element of $(\mathcal{B}, \mathbf{G})$. Hamilton's principle is written as

$$\delta S = \int_{t_1}^{t_2} \int_{\mathcal{B}} (\delta W - \delta T) dV dt = \int_{\partial_t \mathcal{B}} T^a \delta \varphi^b g_{abd} dA, \tag{5.4}$$

where T^a is traction, and $\partial_t \mathcal{B} \subset \partial \mathcal{B}$ is part of the boundary on which tractions are specified. The only term that is different from that of classical nonlinear elasticity is δW , which is calculated as

$$\begin{aligned} \delta W &= \nabla_{\frac{\partial}{\partial \epsilon}}^{\varphi_\epsilon} W \Big|_{\epsilon=0} = \frac{\partial W}{\partial \mathbf{F}} : \delta \mathbf{F} + \frac{\partial W}{\partial \nabla \mathbf{F}} : \delta \nabla \varphi \mathbf{F} \\ &= \frac{\partial W}{\partial F^a{}_A} \delta F^a{}_A + \frac{\partial W}{\partial F^a{}_{A|B}} \delta F^a{}_{A|B}, \end{aligned} \tag{5.5}$$

where we used the fact that $\nabla \mathbf{g} = \mathbf{0}$. Variation of the covariant derivative of the deformation gradient is calculated as follows:

$$\begin{aligned} \delta F^a{}_{A|B} &= \nabla_{\frac{\partial}{\partial \epsilon}} \nabla_{\frac{\partial}{\partial X^B}} \frac{\partial \varphi_\epsilon^a}{\partial X^A} \Big|_{\epsilon=0} = \nabla_{\frac{\partial}{\partial X^B}} \nabla_{\frac{\partial}{\partial \epsilon}} \frac{\partial \varphi_\epsilon^a}{\partial X^A} \Big|_{\epsilon=0} \\ &= \nabla_{\frac{\partial}{\partial X^B}} \delta F^a{}_A = U^a{}_{|A|B}, \end{aligned} \tag{5.6}$$

where use was made of the fact that ∇ is a flat connection, and hence, order of covariant differentiation can be interchanged (see the appendix). Therefore

$$\delta W = \frac{\partial W}{\partial F^a{}_A} \delta \varphi^a{}_{|A} + \frac{\partial W}{\partial F^a{}_{A|B}} \delta \varphi^a{}_{|A|B}. \tag{5.7}$$

Note that $\frac{\partial W}{\partial F^a{}_A} \delta \varphi^a{}_{|A} = \left(\frac{\partial W}{\partial F^a{}_A} \delta \varphi^a \right)_{|A} - \left(\frac{\partial W}{\partial F^a{}_A} \right)_{|A} \delta \varphi^a$. Thus

$$\int_{\mathcal{B}} \frac{\partial W}{\partial F^a{}_A} \delta \varphi^a{}_{|A} dV = - \int_{\mathcal{B}} \left(\frac{\partial W}{\partial F^a{}_A} \right)_{|A} \delta \varphi^a dV + \int_{\partial \mathcal{B}} \frac{\partial W}{\partial F^a{}_A} N_A \delta \varphi^a dA, \tag{5.8}$$

where \mathbf{N} is the unit normal vector to $\partial\mathcal{B}$, and N_A are components of the corresponding 1-form \mathbf{N}^\flat . Similarly, for the second term one can write

$$\begin{aligned} \int_{\mathcal{B}} \frac{\partial W}{\partial F^a_{A|B}} \delta\varphi^a|_{A|B} dV &= - \int_{\mathcal{B}} \left(\frac{\partial W}{\partial F^a_{A|B}} \right)_{|B} \delta\varphi^a|_A dV \\ &+ \int_{\partial\mathcal{B}} \frac{\partial W}{\partial F^a_{A|B}} N_B \delta\varphi^a|_A dA = \int_{\mathcal{B}} \left(\frac{\partial W}{\partial F^a_{A|B}} \right)_{|B|_A} \delta\varphi^a dV \\ &- \int_{\partial\mathcal{B}} \left[\left(\frac{\partial W}{\partial F^a_{A|B}} N_B \right)_{|A} + \left(\frac{\partial W}{\partial F^a_{A|B}} \right)_{|B} N_A \right] \delta\varphi^a dA. \end{aligned} \tag{5.9}$$

In deriving the above relation we used the topological fact that boundary of a boundary is empty, i.e., $\partial\partial\mathcal{B} = \emptyset$. Therefore

$$\begin{aligned} \delta W &= - \int_{\mathcal{B}} \left[\frac{\partial W}{\partial F^a_A} - \left(\frac{\partial W}{\partial F^a_{A|B}} \right)_{|B} \right]_{|A} \delta\varphi^a dV \\ &+ \int_{\partial\mathcal{B}} \left[\frac{\partial W}{\partial F^a_A} - \left(\frac{\partial W}{\partial F^a_{A|B}} \right)_{|B} \right] N_A \delta\varphi^a dA \\ &- \int_{\partial\mathcal{B}} \left(\frac{\partial W}{\partial F^a_{A|B}} N_B \right)_{|A} \delta\varphi^a dA. \end{aligned} \tag{5.10}$$

The first term in (5.10) implies that the first Piola-Kirchhoff stress in gradient elasticity has the following representation:

$$P^{aA} = g^{ab} \left[\frac{\partial W}{\partial F^b_A} - \left(\frac{\partial W}{\partial F^b_{A|B}} \right)_{|B} \right]. \tag{5.11}$$

Following Toupin [113] we define a *hyper-stress* $H_a^{AB} = H_a^{BA} = \frac{\partial W}{\partial F^a_{A|B}}$. The integrand of the third term in (5.10) is simplified as

$$(H_a^{AB} N_B)_{|A} = H_a^{AB}|_A N_B - \mathfrak{B}_{AB} H_a^{AB}, \tag{5.12}$$

where $\mathfrak{B}_{AB} = \mathfrak{B}_{BA} = -N_{A|B}$ is the second fundamental form of the surface $\partial\mathcal{B}$ embedded in the Euclidean space (we assume that the undeformed body is embedded in a Euclidean ambient space). Therefore, traction in gradient elasticity is written as

$$T^a = P^{aA} N_A - H^{aAB}|_B N_A + H^{aAB} \mathfrak{B}_{AB}. \tag{5.13}$$

Similar to classical nonlinear elasticity, the kinetic energy variation reads

$$\delta T = \rho_0 \langle \langle \mathbf{V}, D_t^{\mathbf{g}} \delta\varphi \rangle \rangle_{\mathbf{g}} = \frac{d}{dt} \rho_0 \langle \langle \mathbf{V}, \delta\varphi \rangle \rangle_{\mathbf{g}} - \rho_0 \langle \langle \mathbf{A}, \delta\varphi \rangle \rangle_{\mathbf{g}}. \tag{5.14}$$

Assuming that $\delta\varphi(X, t_1) = \delta\varphi(X, t_2) = 0$, one obtains

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{B}} -T dV dt = \int_{t_1}^{t_2} \int_{\mathcal{B}} \rho_0 \langle \langle \mathbf{A}, \delta\varphi \rangle \rangle_{\mathbf{g}} dV dt. \tag{5.15}$$

Therefore, the balance of linear momentum (the Euler-Lagrange equations) reads $P^{aA}|_A = \rho_0 A^a$ (inclusion of body forces would be straightforward using the Lagrange-D'Alembert principle) and traction vector is given in (5.13).

The Cauchy stress $\sigma^{ab} = J^{-1} F^a{}_A P^{bA}$ in gradient elasticity has the following representation (note the typo in Toupin [113]'s Eq.(10.18)):

$$\sigma^{ab} = J^{-1} g^{bc} \left[F^a{}_A \frac{\partial W}{\partial F^c{}_A} + F^a{}_{A|B} \frac{\partial W}{\partial F^c{}_{A|B}} \right] - h^{bac}|_c, \tag{5.16}$$

where $h^{bac} = J^{-1} F^a{}_A F^c{}_B H^{bAB}$.

Remark 11. When a gradient elastic solid is in a stress-free state the traction vector at every point on any surface vanishes. From (5.13) this implies that $P^{aA} - H^{aAB}|_B = 0$, and $H^{aAB} = 0$. Therefore, in a stress-free state both the (total) first Piola-Kirchhoff stress and hyper-stress vanish.

Balance of angular momentum. In the setting of Lagrangian mechanics of continua the balance of angular momentum is derived using Noether's theorem. Consider a flow $\psi_s : \mathcal{S} \rightarrow \mathcal{S}$ on the (Euclidean) ambient space. According to Noether's theorem any symmetry of the Lagrangian density corresponds to a conserved quantity. In particular, invariance of a Lagrangian density under rotations of the ambient space corresponds to the balance of angular momentum. For the balance of angular momentum in a Euclidean ambient space $\psi_s(x) = x + s\mathbf{\Omega}x$, where $\mathbf{\Omega}$ is an anti-symmetric matrix. As the kinetic energy density is invariant under rotations one only needs to require invariance of the energy function under flows of rotations of the ambient space, i.e.,

$$W(X, \mathbf{F}, \nabla \mathbf{F}, \mathbf{G}, \mathbf{g}) = W(X, \psi_{s*}\mathbf{F}, \psi_{s*}\nabla \mathbf{F}, \mathbf{G}, \psi_{s*}\mathbf{g}). \tag{5.17}$$

Taking derivative with respect to s of both sides and evaluating at $s = 0$, one obtains

$$g^{ac} \left[F^b{}_A \frac{\partial W}{\partial F^c{}_A} + F^b{}_{A|B} \frac{\partial W}{\partial F^c{}_{A|B}} \right] \Omega_{ab} = 0. \tag{5.18}$$

As $\Omega_{ba} = -\Omega_{ab}$, one concludes that $\Pi^{ab} = \Pi^{ba}$, or $\Pi^{[ab]} = 0$, where³²

$$\begin{aligned} \Pi^{ab} &= g^{ac} \left[F^b{}_A \frac{\partial W}{\partial F^c{}_A} + F^b{}_{A|B} \frac{\partial W}{\partial F^c{}_{A|B}} \right] \\ &= g^{ac} F^b{}_A \frac{\partial W}{\partial F^c{}_A} + H^{aAB} F^b{}_{A|B}. \end{aligned} \tag{5.19}$$

In terms of the first Piola-Kirchhoff stress the balance of angular momentum reads

$$P^{[aA} F^{b]}_A + \left(H^{[aAB} F^{b]}_A \right)|_B = 0. \tag{5.20}$$

In terms of the Cauchy stress, $\sigma^{[ab]} + m^{bac}|_c = 0$, where $m^{bac} = h^{[ba]c}$ is Toupin's couple-stress.

³² We use the standard notation $\Pi^{[ab]} = \frac{1}{2} (\Pi^{ab} - \Pi^{ba})$.

Linearized balance of linear momentum. Linearizing the balance of linear momentum about a motion $\hat{\phi}$ one obtains $(\delta P^{aA})|_A + \rho_0 \delta B^a = \rho_0 \ddot{U}^a$. Note that $\delta(P^{aA}|_A) = \delta P^{aA}|_A$, where

$$\delta P^{aA} = \frac{\partial P^{aA}}{\partial F^b_B} \delta F^b_B + \frac{\partial P^{aA}}{\partial F^b_{B|C}} \delta F^b_{B|C} = \mathbf{A}^{aA}_b{}^B U^b|_B + \mathbf{B}^{aA}_b{}^{BC} U^b|_{B|C}, \quad (5.21)$$

and

$$\mathbf{A}^{aA}_b{}^B = \frac{\partial P^{aA}}{\partial F^b_B}, \quad \mathbf{B}^{aA}_b{}^{BC} = \frac{\partial P^{aA}}{\partial F^b_{B|C}}. \quad (5.22)$$

Following DiVincenzo [21] we call \mathbf{A} and \mathbf{B} *dynamic elastic constants*. Notice that $\mathbf{B}^{aAbc} = \mathbf{B}^{aAbc}$. Noting that $P^{aA} = g^{am} \frac{\partial W}{\partial F^m_A} - H^{aAM}|_M$, one can write

$$\begin{aligned} \delta P^{aA} &= g^{am} \frac{\partial^2 W}{\partial F^m_A \partial F^n_N} \delta F^n_N + g^{am} \frac{\partial^2 W}{\partial F^m_A \partial F^n_{N|M}} \delta F^n_{N|M} - \delta(H^{aAM}|_M) \\ &= g^{am} \frac{\partial^2 W}{\partial F^m_A \partial F^n_N} U^n|_N + g^{am} \frac{\partial^2 W}{\partial F^m_A \partial F^n_{N|M}} U^n|_{N|M} - (\delta H^{aAM})|_M. \end{aligned} \quad (5.23)$$

However

$$\begin{aligned} \delta H^{aAM} &= \frac{\partial H^{aAM}}{\partial F^c_C} \delta F^c_C + \frac{\partial H^{aAM}}{\partial F^c_{C|D}} \delta F^c_{C|D} \\ &= g^{am} \frac{\partial^2 W}{\partial F^c_C \partial F^m_{A|M}} U^c|_C + g^{am} \frac{\partial^2 W}{\partial F^c_{C|D} \partial F^m_{A|M}} U^c|_{C|D}. \end{aligned} \quad (5.24)$$

Let us define the following three *static elastic constants* [21]

$$\begin{aligned} \mathbb{A}^{aA}_b{}^B &= \frac{\partial^2 W}{\partial F^a_A \partial F^b_B}, \quad \mathbb{B}^{aA}_b{}^{BC} = \frac{\partial^2 W}{\partial F^a_A \partial F^b_{B|C}}, \\ \mathbb{C}^{aAB}_b{}^{CD} &= \frac{\partial^2 W}{\partial F^a_{A|B} \partial F^b_{C|D}}. \end{aligned} \quad (5.25)$$

Therefore, the static elastic constants satisfy the following symmetries: $\mathbb{A}^{aA}_b{}^B = \mathbb{A}^B_{b a A}$, $\mathbb{B}^{aA}_b{}^{BC} = \mathbb{B}^{aA}_b{}^{CB}$, and $\mathbb{C}^{aAB}_b{}^{CD} = \mathbb{C}^{aAB}_b{}^{CD} = \mathbb{C}^{aBA}_b{}^{DC} = \mathbb{C}^{bDC}_a{}^{BA}$ (note that \mathbb{C} has 21 and 171 independent components in 2D and 3D, respectively). Thus, $\delta H^{aAM} = \mathbb{B}^{BaAM} U^b|_B + \mathbb{C}^{BCaAM} U^b|_{B|C}$, and hence

$$(\delta H^{aAM})|_M = (\mathbb{B}^{BaAM} U^b|_B + \mathbb{C}^{BCaAM} U^b|_{B|C})|_M. \quad (5.26)$$

Therefore

$$\begin{aligned} \mathbf{A}^{aA}_b{}^B &= \mathbb{A}^{aA}_b{}^B - \mathbb{B}^{BaAM}|_M, \\ \mathbf{B}^{aA}_b{}^{BC} &= \mathbb{B}^{aA}_b{}^{BC} - \mathbb{B}^{BaAC} - \mathbb{C}^{aAM}_b{}^{BC}|_M. \end{aligned} \quad (5.27)$$

Or, equivalently

$$\begin{aligned} \mathbf{A}^{aAbB} &= \mathbb{A}^{aAbB} - \mathbb{B}^{bBaAM}|_M, \\ \mathbf{B}^{aAbBC} &= \mathbb{B}^{aAbBC} - \mathbb{B}^{bBaAC} - \mathbb{C}^{aAMbBC}|_M. \end{aligned} \quad (5.28)$$

In deriving the second relation we ignored the term $U^b{}_{|B|C|M}$ in $\delta H^{aAM}{}_{|M}$ as we are assuming a second-gradient elasticity in which displacement derivatives of orders three or higher are neglected. Note that in [21] homogeneous solids were considered. Here, for cloaking purposes, one needs to consider inhomogeneous solids and that is why derivatives of the static elastic constants appear in (5.28).

Note that (5.28)₂ requires that $\mathbb{B}^{bBaAC} = \mathbb{B}^{bCaAB} = \mathbb{B}^{bAaBC}$, reducing the number of independent components of \mathbb{B} to 16 and 90 in 2D and 3D, respectively. After some manipulations and using the major symmetries of \mathbb{C} , one also finds that

$$\begin{aligned} \mathbb{B}^{aAbBC} + \mathbb{B}^{bBaAC} &= -\left(\mathbb{C}^{bBMaAC} + \mathbb{C}^{aAMbBC}\right)_{|M} \\ &= -\left(\mathbb{C}^{aACbBM} + \mathbb{C}^{bBCaAM}\right)_{|M}. \end{aligned} \tag{5.29}$$

Similarly, from (5.28)₁, and knowing that \mathbb{A} has the major symmetries, one obtains

$$\mathbb{A}^{aAbB} - \mathbb{A}^{bBaA} = \left(\mathbb{B}^{aAbBM} - \mathbb{B}^{bBaAM}\right)_{|M} = \left(\mathbb{B}^{aBbAM} - \mathbb{B}^{bBaAM}\right)_{|M}. \tag{5.30}$$

Linearized balance of angular momentum. The balance of angular momentum is equivalent to $\Pi^{ab} = \Pi^{ba}$, or $\Pi^{[ab]} = 0$. Suppose the reference motion is an isometric embedding of an initially stress-free body into the Euclidean space, i.e., $\mathring{F}^a{}_A = \delta^a_A$, which implies that $\mathring{F}^a{}_{A|B} = 0$. Assuming that $\mathring{P}^{aA} = 0$, one concludes that

$$\mathring{H}^{aAB}{}_{|B} = g^{ab} \frac{\partial W}{\partial F^a{}_A} \Big|_{\mathbf{F}=\mathring{\mathbf{F}}} =: \mathring{P}^{aA}_{\text{cl}}. \tag{5.31}$$

Therefore

$$\begin{aligned} \delta \Pi^{ab} &= \mathring{P}^{aA}_{\text{cl}} U^b{}_{|A} + \mathring{H}^{aAB} U^b{}_{|A|B} + \mathbb{A}^{aA}{}_m M \mathring{F}^b{}_A U^m{}_{|M} \\ &\quad + \mathbb{B}^{aA}{}_{mN} \mathring{F}^b{}_A U^m{}_{|M|N} \\ &= \left(\mathbb{A}^{aM}{}_m{}^A \mathring{F}^b{}_A + \mathring{H}^{aAB}{}_{|B} \delta_m^b\right) U^m{}_{|A} \\ &\quad + \left(\mathbb{B}^{aM}{}_m{}^{AB} \mathring{F}^b{}_M + \mathring{H}^{aAB} \delta_m^b\right) U^m{}_{|A|B}. \end{aligned} \tag{5.32}$$

Knowing that $\delta \Pi^{[ab]} = 0$, and that the first and the second covariant derivatives of the displacement field are independent, one concludes that

$$\begin{aligned} \mathbb{A}^{[aM}{}_m{}^A \mathring{F}^b]_M + \mathring{H}^{[aAB}{}_{|B} \delta_m^b] &= 0, \\ \mathbb{B}^{[aM}{}_m{}^{AB} \mathring{F}^b]_M + \mathring{H}^{[aAB} \delta_m^b] &= 0. \end{aligned} \tag{5.33}$$

Note that the issue with the acceleration term that was discussed in classical nonlinear elasticity in Section 4.1 persists even in nonlinear gradient elasticity, and hence, we do not discuss transformation cloaking in nonlinear gradient elastodynamics.

Transformation cloaking in linearized gradient elastodynamics. We start from a virtual body that is made of a homogeneous, isotropic, and centro-symmetric gradient elastic solid. We then consider a cloaking transformation and try to find the elastic constants of the physical body induced from the cloaking transformation such that the balance of linear and angular momenta are respected in both the virtual and physical bodies. Let us start from the balance of linear momentum in the physical body, i.e., $\text{Div } \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}$. Its linearization reads $\delta(\text{Div } \mathbf{P}) + \rho_0 \delta \mathbf{B} = \rho_0 \mathbf{A}$, where

$$\begin{aligned} \delta(\text{Div } \mathbf{P}) &= \text{Div } \delta \mathbf{P} = \text{Div} (\mathbf{A} : \nabla \mathbf{U} + \mathbf{B} : \nabla \nabla \mathbf{U}) \\ &= \left(\mathbf{A}^{aA}{}_b{}^B U^b|_B + \mathbf{B}^{aA}{}_b{}^{BC} U^b|_{B|C} \right)_{|A} \frac{\partial}{\partial x^a}. \end{aligned} \quad (5.34)$$

Under a cloaking transformation $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ and using the shifter map, $\text{Div } \delta \mathbf{P}$ is transformed to

$$J_{\mathcal{E}} \left(\tilde{\mathbf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} \tilde{U}^{\tilde{b}}|_{\tilde{B}} + \tilde{\mathbf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} \tilde{U}^{\tilde{b}}|_{\tilde{B}|\tilde{C}} \right)_{|\tilde{A}} \frac{\partial}{\partial \tilde{x}^{\tilde{a}}}, \quad (5.35)$$

where

$$\begin{aligned} \tilde{U}^{\tilde{a}} &= \mathbf{s}^{\tilde{a}}{}_{\tilde{a}} U^a, \\ \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} &= J_{\mathcal{E}}^{-1} \mathbf{s}^{\tilde{a}}{}_{\tilde{a}} \tilde{F}^{\tilde{A}}{}_{\tilde{A}} (\mathbf{s}^{-1})^b{}_{\tilde{b}} \tilde{F}^{\tilde{B}}{}_{\tilde{B}} \mathbf{A}^{aA}{}_b{}^B \\ &\quad + J_{\mathcal{E}}^{-1} \mathbf{s}^{\tilde{a}}{}_{\tilde{a}} \tilde{F}^{\tilde{A}}{}_{\tilde{A}} (\mathbf{s}^{-1})^b{}_{\tilde{b}} \tilde{F}^{\tilde{B}}{}_{\tilde{B}}{}_{|C} \mathbf{B}^{aA}{}_b{}^{BC}, \\ \tilde{\mathbf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} &= J_{\mathcal{E}}^{-1} \mathbf{s}^{\tilde{a}}{}_{\tilde{a}} \tilde{F}^{\tilde{A}}{}_{\tilde{A}} (\mathbf{s}^{-1})^b{}_{\tilde{b}} \tilde{F}^{\tilde{B}}{}_{\tilde{B}} \tilde{F}^{\tilde{C}}{}_{\tilde{C}} \mathbf{B}^{aA}{}_b{}^{BC}, \end{aligned} \quad (5.36)$$

or, equivalently

$$\begin{aligned} \mathbf{A}^{aA}{}_b{}^B &= J_{\mathcal{E}} (\mathbf{s}^{-1})^a{}_{\tilde{a}} (\tilde{F}^{-1})^A{}_{\tilde{A}} \mathbf{s}^{\tilde{b}}{}_{\tilde{b}} (\tilde{F}^{-1})^B{}_{\tilde{B}} \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} \\ &\quad + J_{\mathcal{E}} (\mathbf{s}^{-1})^a{}_{\tilde{a}} (\tilde{F}^{-1})^A{}_{\tilde{A}} \mathbf{s}^{\tilde{b}}{}_{\tilde{b}} (\tilde{F}^{-1})^B{}_{\tilde{B}}{}_{|\tilde{C}} \tilde{\mathbf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}}, \\ \mathbf{B}^{aA}{}_b{}^{BC} &= J_{\mathcal{E}} (\mathbf{s}^{-1})^a{}_{\tilde{a}} (\tilde{F}^{-1})^A{}_{\tilde{A}} \mathbf{s}^{\tilde{b}}{}_{\tilde{b}} (\tilde{F}^{-1})^B{}_{\tilde{B}} (\tilde{F}^{-1})^C{}_{\tilde{C}} \tilde{\mathbf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}}. \end{aligned} \quad (5.37)$$

To see this, note that

$$\begin{aligned} &\left(\mathbf{A}^{aA}{}_b{}^B U^b|_B + \mathbf{B}^{aA}{}_b{}^{BC} U^b|_{B|C} \right)_{|A} \\ &= J_{\mathcal{E}} \left[J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_{\tilde{A}} \mathbf{A}^{aA}{}_b{}^B U^b|_B + J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_{\tilde{A}} \mathbf{B}^{aA}{}_b{}^{BC} U^b|_{B|C} \right]_{|\tilde{A}} \\ &= J_{\mathcal{E}} \left[J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_{\tilde{A}} \mathbf{A}^{aA}{}_b{}^B \tilde{F}^{\tilde{B}}{}_{\tilde{B}} U^b|_{\tilde{B}} + J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_{\tilde{A}} \mathbf{B}^{aA}{}_b{}^{BC} \tilde{F}^{\tilde{B}}{}_{\tilde{B}} \tilde{F}^{\tilde{C}}{}_{\tilde{C}} U^b|_{\tilde{B}|\tilde{C}} \right. \\ &\quad \left. + J_{\mathcal{E}}^{-1} \tilde{F}^{\tilde{A}}{}_{\tilde{A}} \mathbf{B}^{aA}{}_b{}^{BC} \tilde{F}^{\tilde{B}}{}_{\tilde{B}}{}_{|C} U^b|_{\tilde{B}} \right]_{|\tilde{A}} \\ &= (\mathbf{s}^{-1})^a{}_{\tilde{a}} J_{\mathcal{E}} \left(\tilde{\mathbf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} \tilde{U}^{\tilde{b}}|_{\tilde{B}} + \tilde{\mathbf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} \tilde{U}^{\tilde{b}}|_{\tilde{B}|\tilde{C}} \right)_{|\tilde{A}}, \end{aligned} \quad (5.38)$$

where the following relations were used.

$$\begin{aligned} U^b|_B &= (\mathbf{s}^{-1})^b_{\tilde{b}} \tilde{F}^{\tilde{b}}_{\tilde{B}} \tilde{U}^{\tilde{b}}_{|\tilde{B}}, \\ U^b|_B|_C &= (\mathbf{s}^{-1})^b_{\tilde{b}} \left[\tilde{F}^{\tilde{b}}_{\tilde{B}} \tilde{U}^{\tilde{b}}_{|\tilde{B}} + \tilde{F}^{\tilde{b}}_{\tilde{B}} \tilde{F}^{\tilde{c}}_{\tilde{C}} \tilde{U}^{\tilde{b}}_{|\tilde{B}|\tilde{C}} \right]. \end{aligned} \quad (5.39)$$

We assume that the reference motion for both the physical and virtual bodies are isometric embeddings in the Euclidean ambient space, i.e., $\hat{F}^a_A = \delta^a_A$ and $\hat{\tilde{F}}^{\tilde{a}}_{\tilde{A}} = \delta^{\tilde{a}}_{\tilde{A}}$. This implies that $\hat{F}^a_{A|B} = 0$, and $\hat{\tilde{F}}^{\tilde{a}}_{\tilde{A}|\tilde{B}} = 0$. It is also assumed that there is no initial stress in either configuration, i.e., $\hat{P}^a_A = 0$, $\hat{H}^{aAB} = 0$, and $\hat{\tilde{P}}^{\tilde{a}}_{\tilde{A}} = 0$, $\hat{\tilde{H}}^{\tilde{a}\tilde{A}\tilde{B}} = 0$. Therefore, from (5.33), the balance of angular momentum in the physical and virtual bodies reads as (minor symmetries of \mathbf{A} , \mathbf{B} , $\tilde{\mathbf{A}}$, and $\tilde{\mathbf{B}}$)

$$\mathbf{A}^{[aM}_m{}^A \hat{F}^b]}_M = 0, \quad \mathbf{B}^{[aM}_m{}^{AB} \hat{F}^b]}_M = 0, \quad (5.40)$$

$$\tilde{\mathbf{A}}^{[\tilde{a}\tilde{M}}_{\tilde{m}}{}^{\tilde{A}} \hat{\tilde{F}}^{\tilde{b}}]}_{\tilde{M}} = 0, \quad \tilde{\mathbf{B}}^{[\tilde{a}\tilde{M}}_{\tilde{m}}{}^{\tilde{A}\tilde{B}} \hat{\tilde{F}}^{\tilde{b}}]}_{\tilde{M}} = 0. \quad (5.41)$$

The virtual body being uniform its elastic constants are (covariantly) constant, and hence, from (5.27) one concludes that

$$\tilde{\mathbf{A}}^{\tilde{a}\tilde{A}}_{\tilde{b}}{}^{\tilde{B}} = \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}}_{\tilde{b}}{}^{\tilde{B}}, \quad \tilde{\mathbf{B}}^{\tilde{a}\tilde{A}}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} = \tilde{\mathbf{B}}^{\tilde{a}\tilde{A}}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} - \tilde{\mathbf{B}}^{\tilde{a}}_{\tilde{b}}{}^{\tilde{B}\tilde{A}\tilde{C}}. \quad (5.42)$$

Therefore, from (5.41)₁ one obtains

$$\tilde{\mathbf{A}}^{[\tilde{a}\tilde{M}}_{\tilde{m}}{}^{\tilde{A}} \hat{\tilde{F}}^{\tilde{b}}]}_{\tilde{M}} = 0. \quad (5.43)$$

The virtual body is assumed to be isotropic and non-chiral (centro-symmetric).³³ Knowing that an odd-order tensor cannot be isotropic and centro-symmetric [3, 74] one concludes that $\tilde{\mathbf{B}}^{\tilde{a}\tilde{A}}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} = 0$, and hence $\tilde{\mathbf{B}}^{\tilde{a}\tilde{A}}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} = 0$. In particular, (5.41)₂ is trivially satisfied. From (5.37)₂ one obtains $\mathbf{B}^{aA}_b{}^{BC} = 0$, and hence from (5.28)₂

$$\begin{aligned} \mathbf{B}^{aA}_b{}^{BC} - \mathbf{B}_b{}^{BaAC} &= \mathbb{C}^{aAM}_b{}^{BC}|_M, \\ \mathbf{A}^{aA}_b{}^B &= J_{\mathcal{E}} (\mathbf{s}^{-1})^a_{\tilde{a}} (\hat{F}^{-1})^A_{\tilde{A}} \mathbf{s}^{\tilde{b}}_b (\hat{F}^{-1})^B_{\tilde{B}} \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}}_{\tilde{b}}{}^{\tilde{B}}. \end{aligned} \quad (5.44)$$

Notice that the dynamic elastic constants \mathbf{A} of the cloak possess the major symmetries, i.e., $\mathbf{A}^{aAbB} = \mathbf{A}^bBaA = J_{\mathcal{E}} (\mathbf{s}^{-1})^a_{\tilde{a}} (\hat{F}^{-1})^A_{\tilde{A}} (\mathbf{s}^{-1})^b_{\tilde{b}} (\hat{F}^{-1})^B_{\tilde{B}} \tilde{\mathbf{A}}^{\tilde{a}\tilde{A}}_{\tilde{b}}{}^{\tilde{B}}$. Thus, using (5.30), one obtains $\mathbb{B}^{[aBb]AM}|_M = 0$, i.e.,

$$\mathbb{B}^{aBbAM}|_M = \mathbb{B}^bBaAM}|_M. \quad (5.45)$$

From (5.44)₁ and (5.40)₂, one obtains

$$\mathbb{C}^{[aAMbBC}|_M \hat{F}^c]}_A = -\mathbb{B}^bB[aAC \hat{F}^c]}_A. \quad (5.46)$$

Also, from (5.29) and knowing that \mathbf{B} vanishes identically, one obtains

$$\left(\mathbb{C}^{aACbBM} + \mathbb{C}^bBCaAM \right)|_M = 0. \quad (5.47)$$

³³ Note that only the virtual body is assumed to be centro-symmetric. There is no such constraint on the physical body; it can be both non-centro-symmetric and anisotropic.

Remark 12. From (5.44)₁ and (5.40)₂, one can write

$$\begin{aligned} \mathbb{B}^{[aAbBC]}_{|C\dot{F}^c]}_A - \mathbb{B}^{bB[aAC]}_{|C\dot{F}^c]}_A &= -\mathbb{B}^{bB[aAC]}_{|C\dot{F}^c]}_A \\ &= \mathbb{C}^{[aAMbBC]}_{|M|C\dot{F}^c]}_A. \end{aligned} \quad (5.48)$$

Therefore

$$\mathbb{B}^{bB[aAC]}_{|C\dot{F}^c]}_A = -\mathbb{C}^{[aAMbBC]}_{|M|C\dot{F}^c]}_A. \quad (5.49)$$

Note that from (5.28)₁ and (5.44)₁ we have

$$\begin{aligned} \mathbb{A}^{aAbB} &= \mathbb{B}^{bBaAM}_{|M} + J_{\mathcal{E}}(\mathbf{s}^{-1})^a_{\tilde{a}}(\tilde{F}^{-1})^A_{\tilde{A}}(\mathbf{s}^{-1})^b_{\tilde{b}}(\tilde{F}^{-1})^B_{\tilde{B}} \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \\ &= \mathbb{B}^{aAbBM}_{|M} - \mathbb{C}^{aAMbBN}_{|M|N} \\ &\quad + J_{\mathcal{E}}(\mathbf{s}^{-1})^a_{\tilde{a}}(\tilde{F}^{-1})^A_{\tilde{A}}(\mathbf{s}^{-1})^b_{\tilde{b}}(\tilde{F}^{-1})^B_{\tilde{B}} \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}. \end{aligned} \quad (5.50)$$

From the above relation and (5.40) one obtains

$$\begin{aligned} 0 &= \mathbb{A}^{aAbB} \dot{F}^c]_A = -\mathbb{C}^{[aAMbBN]}_{|M|N} \dot{F}^c]_A \\ &\quad + J_{\mathcal{E}}(\mathbf{s}^{-1})^a_{\tilde{a}}(\tilde{F}^{-1})^A_{\tilde{A}}(\mathbf{s}^{-1})^b_{\tilde{b}}(\tilde{F}^{-1})^B_{\tilde{B}} \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \dot{F}^c]_A. \end{aligned} \quad (5.51)$$

Therefore

$$\mathbb{C}^{[aAMbBN]}_{|M|N} \dot{F}^c]_A = J_{\mathcal{E}}(\mathbf{s}^{-1})^a_{\tilde{a}}(\tilde{F}^{-1})^A_{\tilde{A}}(\mathbf{s}^{-1})^b_{\tilde{b}}(\tilde{F}^{-1})^B_{\tilde{B}} \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \dot{F}^c]_A. \quad (5.52)$$

Note that in classical linearized elasticity the right-hand side had to vanish, e.g., Eq. (4.51), which was an obstruction to transformation cloaking.

From (5.52) and (5.49), the system of first-order PDEs for \mathbb{B} read

$$\mathbb{B}^{bB[aAC]}_{|C\dot{F}^c]}_A = -J_{\mathcal{E}}(\mathbf{s}^{-1})^a_{\tilde{a}}(\tilde{F}^{-1})^A_{\tilde{A}}(\mathbf{s}^{-1})^b_{\tilde{b}}(\tilde{F}^{-1})^B_{\tilde{B}} \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \dot{F}^c]_A. \quad (5.53)$$

In summary, one is given the homogeneous static elastic constants of the virtual body. Because of isotropy and centro-symmetry only $\tilde{\mathbb{A}}$ and $\tilde{\mathbb{C}}$ are nonzero. These are both (covariantly) constant tensors. The balance of angular momentum implies that $\tilde{\mathbb{A}}$ has the classical minor symmetries. Thus, it will have the same form as $\tilde{\mathbb{A}}$ in the classical linear elasticity. There is no relation between $\tilde{\mathbb{C}}$ and \mathbb{C} , and there are no constraints on $\tilde{\mathbb{C}}$ other than being positive-definite. The balance of linear momentum relates the dynamic elastic constants of the two problems. From $\tilde{\mathbb{B}} = \mathbf{0}$ in the virtual body one concludes that $\mathbb{B} = \mathbf{0}$ in the physical body. This gives a relation between \mathbb{B} and \mathbb{C} in the form of a system of PDEs. The balance of angular momentum in the physical body is written as a set of constraints in the form of a system of second-order PDEs for \mathbb{C} .

The impossibility of transformation cloaking in gradient elasticity. Next we show that transformation cloaking is not possible in gradient elasticity. We first show this for cylindrical and spherical cloaks. From (5.40)₂ and knowing that $\tilde{F}^b_M = \delta^b_M$, with an abuse of notation, one writes

$$\mathbb{B}^{abmAB} = \mathbb{B}^{bamAB}, \quad (5.54)$$

i.e., \mathbb{B} must be symmetric with respect to the first two indices. This symmetry and (5.53), imply that the right-hand side of (5.53) is symmetric with respect to indices b and B as well. Thus

$$\begin{aligned} & (\mathbf{s}^{-1})^{[a}_{\tilde{a}}(\tilde{F}^{-1})^A_{\tilde{A}}(\mathbf{s}^{-1})^b_{\tilde{b}}(\tilde{F}^{-1})^B_{\tilde{B}}\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{F}^c]_A \\ &= (\mathbf{s}^{-1})^{[a}_{\tilde{a}}(\tilde{F}^{-1})^A_{\tilde{A}}(\mathbf{s}^{-1})^B_{\tilde{B}}(\tilde{F}^{-1})^b_{\tilde{b}}\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{F}^c]_A. \end{aligned} \quad (5.55)$$

This is simplified and rearranged to read ($\forall b, B, c, a \in \{1, 2, 3\}$)

$$\begin{aligned} & (\mathbf{s}^{-1})^b_{\tilde{b}}(\tilde{F}^{-1})^B_{\tilde{B}}\left[(\mathbf{s}^{-1})^c_{\tilde{a}}(\tilde{F}^{-1})^a_{\tilde{A}} - (\mathbf{s}^{-1})^a_{\tilde{a}}(\tilde{F}^{-1})^c_{\tilde{A}}\right]\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \\ &= (\mathbf{s}^{-1})^B_{\tilde{B}}(\tilde{F}^{-1})^b_{\tilde{b}}\left[(\mathbf{s}^{-1})^c_{\tilde{a}}(\tilde{F}^{-1})^a_{\tilde{A}} - (\mathbf{s}^{-1})^a_{\tilde{a}}(\tilde{F}^{-1})^c_{\tilde{A}}\right]\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}. \end{aligned} \quad (5.56)$$

In particular, it is straightforward to verify that this condition cannot be satisfied for either a cylindrical or a spherical cloak. To see this, let us expand (5.56) for $b = 1, B = 3, a = 1$, and $c = 3$:

$$\begin{aligned} & (\mathbf{s}^{-1})^1_{\tilde{b}}(\tilde{F}^{-1})^3_{\tilde{B}}\left[(\mathbf{s}^{-1})^3_{\tilde{a}}(\tilde{F}^{-1})^1_{\tilde{A}} - (\mathbf{s}^{-1})^1_{\tilde{a}}(\tilde{F}^{-1})^3_{\tilde{A}}\right]\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \\ &= (\mathbf{s}^{-1})^3_{\tilde{B}}(\tilde{F}^{-1})^1_{\tilde{b}}\left[(\mathbf{s}^{-1})^3_{\tilde{a}}(\tilde{F}^{-1})^1_{\tilde{A}} - (\mathbf{s}^{-1})^1_{\tilde{a}}(\tilde{F}^{-1})^3_{\tilde{A}}\right]\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}. \end{aligned} \quad (5.57)$$

Knowing that in the spherical (or cylindrical) coordinates and for a radial cloak \mathbf{s}^{-1} and $\tilde{\mathbf{F}}^{-1}$ have diagonal representations, this is further simplified and reads

$$\begin{aligned} & (\mathbf{s}^{-1})^1_1(\tilde{F}^{-1})^3_3\left[(\mathbf{s}^{-1})^3_3(\tilde{F}^{-1})^1_1\tilde{\mathbb{A}}^{3113} - (\mathbf{s}^{-1})^1_1(\tilde{F}^{-1})^3_3\tilde{\mathbb{A}}^{1313}\right] \\ &= (\mathbf{s}^{-1})^3_3(\tilde{F}^{-1})^1_1\left[(\mathbf{s}^{-1})^3_3(\tilde{F}^{-1})^1_1\tilde{\mathbb{A}}^{3131} - (\mathbf{s}^{-1})^1_1(\tilde{F}^{-1})^3_3\tilde{\mathbb{A}}^{1331}\right]. \end{aligned} \quad (5.58)$$

Note, however that $\tilde{\mathbb{A}}$ (the elastic constants of the virtual body) possesses both the minor and major symmetries. Moreover, $\tilde{\mathbb{A}}^{1331} = \mu > 0$. Hence, one obtains

$$\begin{aligned} & (\mathbf{s}^{-1})^1_1(\tilde{F}^{-1})^3_3\left[(\mathbf{s}^{-1})^3_3(\tilde{F}^{-1})^1_1 - (\mathbf{s}^{-1})^1_1(\tilde{F}^{-1})^3_3\right] \\ &= (\mathbf{s}^{-1})^3_3(\tilde{F}^{-1})^1_1\left[(\mathbf{s}^{-1})^3_3(\tilde{F}^{-1})^1_1 - (\mathbf{s}^{-1})^1_1(\tilde{F}^{-1})^3_3\right]. \end{aligned} \quad (5.59)$$

Therefore, $(\tilde{F}^{-1})^3_3(\mathbf{s}^{-1})^1_1 = (\mathbf{s}^{-1})^3_3(\tilde{F}^{-1})^1_1$. Recalling that $\tilde{\mathbf{F}} = \text{diag}(f'(R), 1, 1)$, and $\mathbf{s} = \text{diag}(1, R/f(R), 1)$ and $\mathbf{s} = \text{diag}(1, R/f(R), R/f(R))$, in the cylindrical and spherical coordinates, respectively, one must have $f(R) = R$, i.e., $\tilde{\mathcal{E}} = id$, which is not acceptable for a cloaking map.

One may now ask whether choosing a less symmetric cloaking map may make transformation cloaking possible. We next show that this is not the case.

Proposition 5. *Assuming that the virtual body is isotropic and centro-symmetric, elastodynamic transformation cloaking is not possible for gradient elastic solids in either 2D or 3D for a hole (cavity) of any shape.*

Proof. Without loss of generality, one may write (5.56) in Cartesian coordinates for which the shifter has a trivial representation, i.e., $\tilde{\mathbf{s}}^a = \delta_a^a$. In 2D, one has ($\forall b, B, c, a \in \{1, 2\}$)

$$\begin{aligned} & (\tilde{F}^{-1})^B_{\tilde{B}} \left[(\tilde{F}^{-1})^a_{\tilde{A}} \tilde{\mathbb{A}}^{c\tilde{A}b\tilde{B}} - (\tilde{F}^{-1})^c_{\tilde{A}} \tilde{\mathbb{A}}^{a\tilde{A}b\tilde{B}} \right] \\ &= (\tilde{F}^{-1})^b_{\tilde{B}} \left[(\tilde{F}^{-1})^a_{\tilde{A}} \tilde{\mathbb{A}}^{c\tilde{A}B\tilde{B}} - (\tilde{F}^{-1})^c_{\tilde{A}} \tilde{\mathbb{A}}^{a\tilde{A}B\tilde{B}} \right]. \end{aligned} \quad (5.60)$$

Let us consider an arbitrary cloaking transformation with the following components:

$$\tilde{\mathbf{F}}^{-1} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix}. \quad (5.61)$$

Equation (5.60) is expanded to read

$$\lambda (\mathbf{F}_{12} - \mathbf{F}_{21})^2 + \mu \left[(\mathbf{F}_{11} - \mathbf{F}_{22})^2 + 2 (\mathbf{F}_{12}^2 + \mathbf{F}_{21}^2) \right] = 0. \quad (5.62)$$

We know that $\mu > 0$, and $2\mu + 3\lambda > 0$ for the energy function to be positive-definite.³⁴ Thus, multiplying the above identity by 3, one obtains $3\lambda (\mathbf{F}_{12} - \mathbf{F}_{21})^2 + 3\mu [(\mathbf{F}_{11} - \mathbf{F}_{22})^2 + 2(\mathbf{F}_{12}^2 + \mathbf{F}_{21}^2)] = 0$, which can be rewritten as

$$\begin{aligned} & (3\lambda + 2\mu) (\mathbf{F}_{12} - \mathbf{F}_{21})^2 + 3\mu (\mathbf{F}_{11} - \mathbf{F}_{22})^2 \\ &+ \mu \left[3 (\mathbf{F}_{12} + \mathbf{F}_{21})^2 + (\mathbf{F}_{12} - \mathbf{F}_{21})^2 \right] = 0. \end{aligned} \quad (5.63)$$

Note that the coefficient of each term is positive. Therefore, $\mathbf{F}_{12} = \mathbf{F}_{21} = 0$, and $\mathbf{F}_{11} = \mathbf{F}_{22}$, i.e., $\tilde{\mathbf{F}} = \alpha \mathbf{I}$, where \mathbf{I} is the identity matrix and $\alpha > 0$ is a scalar.

In 3D, the balance of angular momentum in the physical body reads

$$\begin{aligned} & (\tilde{F}^{-1})^B_{\tilde{B}} \left[(\tilde{F}^{-1})^a_{\tilde{A}} \tilde{\mathbb{A}}^{c\tilde{A}b\tilde{B}} - (\tilde{F}^{-1})^c_{\tilde{A}} \tilde{\mathbb{A}}^{a\tilde{A}b\tilde{B}} \right] \\ &= (\tilde{F}^{-1})^b_{\tilde{B}} \left[(\tilde{F}^{-1})^a_{\tilde{A}} \tilde{\mathbb{A}}^{c\tilde{A}B\tilde{B}} - (\tilde{F}^{-1})^c_{\tilde{A}} \tilde{\mathbb{A}}^{a\tilde{A}B\tilde{B}} \right], \quad \forall b, B, c, a \in \{1, 2, 3\}. \end{aligned} \quad (5.64)$$

³⁴ Note that

$$\begin{aligned} \delta W &= \frac{1}{2} \frac{\partial W}{\partial F^a_A \partial F^b_B} U^a_{|A} U^b_{|B} + \frac{\partial W}{\partial F^a_A \partial F^b_{B|C}} U^a_{|A} U^b_{|B|C} \\ &+ \frac{1}{2} \frac{\partial W}{\partial F^a_{A|B} \partial F^b_{C|D}} U^a_{|A|B} U^b_{|C|D} \\ &= \frac{1}{2} \mathbb{A}^{aAbB} U_{a|A} U_{b|B} + \mathbb{B}^{aAbBC} U_{a|A} U_{b|B|C} + \frac{1}{2} \mathbb{C}^{aABbCD} U_{a|A|B} U_{b|C|D}. \end{aligned}$$

Positive-definiteness of energy requires that $\delta W > 0$ for any pair $(U_{a|A}, U_{a|A|B}) \neq (0, 0)$. In particular, when $U_{a|A} \neq 0$, and $U_{a|A|B} = 0$, $\mathbb{A}^{aAbB} U_{a|A} U_{b|B} > 0$, which implies that \mathbb{A} must be positive-definite. In the case of isotropic solids this is equivalent to $\mu > 0$, and $3\lambda + 2\mu > 0$.

This relation for $\{b \neq B\}$ and $\{c \neq a\}$ is nontrivial and gives six linearly independent algebraic equations. Let us consider an arbitrary cloaking transformation with the following components:

$$\bar{\mathbf{F}}^{-1} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}. \quad (5.65)$$

Equation (5.64) is expanded for $i \neq j \neq k \in \{1, 2, 3\}$ and reads

$$\lambda (F_{ij} - F_{ji})^2 + \mu \left[2 (F_{ij}^2 + F_{ji}^2) + F_{ik}^2 + F_{jk}^2 + (F_{ii} - F_{jj})^2 \right] = 0, \quad (5.66)$$

$$\begin{aligned} &\lambda (F_{ij} - F_{ji}) (F_{ik} - F_{ki}) \\ &+ \mu [F_{ij}F_{ik} + F_{jj}F_{kj} + F_{kk}F_{jk} + 2F_{ji}F_{ki} - F_{ii} (F_{jk} + F_{kj})] = 0. \end{aligned} \quad (5.67)$$

After some algebraic manipulations (5.66) is rewritten as

$$\begin{aligned} &(3\lambda + 2\mu) (F_{ij} - F_{ji})^2 + \mu \left[3 (F_{ij} + F_{ji})^2 + (F_{ij} - F_{ji})^2 \right] \\ &+ 3\mu \left[F_{ik}^2 + F_{jk}^2 + (F_{ii} - F_{jj})^2 \right] = 0. \end{aligned} \quad (5.68)$$

Note that the coefficient of each term is positive. Thus, $F_{ij} = 0$ for $i \neq j$, and $F_{11} = F_{22} = F_{33}$. These trivially satisfy (5.67). One concludes that $\bar{\mathbf{F}} = \alpha \mathbf{I}$, where \mathbf{I} is the identity matrix and $\alpha > 0$ is a scalar. Knowing that the cloaking map restricted to the outer boundary of the cloak is the identity map, one concludes that $\alpha = 1$, and hence, $\bar{\mathcal{E}} = id$, which is clearly not an acceptable cloaking transformation. \square

5.2. Elastodynamic Transformation Cloaking in Generalized Cosserat Solids

We assume that the microstructure in the deformed configuration is described by three (or two in 2D) linearly independent vectors $\{\mathbf{d}_\alpha(x, t), \alpha = 1, 2, 3\}$, which are called director fields or directors [27]. The directors in the reference configuration are denoted by $\{\mathbf{D}_\alpha(X), \alpha = 1, 2, 3\}$. These are not material vectors in the sense that $\mathbf{d}_\alpha(x, t) \neq (\varphi_* \mathbf{D}_\alpha)(x, t)$. The kinematics of an oriented body is described by the pair $(\varphi(X, t), \mathbf{d}_\alpha(X, t))$, where $\mathbf{d}_\alpha(X, t)$ is short for $\mathbf{d}_\alpha(\varphi(X, t), \mathbf{D}_\alpha(X))$ [111, 113]. An oriented body with (deformable) directors is called a generalized Cosserat solid. If the directors are rigid, i.e.,

$$\mathbf{d}_\alpha^a(X, t) \mathbf{d}_\beta^b(X, t) g_{ab}(\varphi(X, t)) = \mathbf{g}_{\alpha\beta}^a(X), \quad (5.69)$$

for some symmetric, positive-definite, and time-independent matrix $\mathbf{g}_{\alpha\beta}^a$, the oriented body is referred to as a Cosserat solid. The reciprocal of \mathbf{D}_α is denoted by $\overset{\circ}{\Theta}_\alpha$ such that $\mathbf{D}_\alpha^A \overset{\circ}{\Theta}_B = \delta_B^A$ and $\mathbf{D}_\alpha^A \overset{\circ}{\Theta}_A = \delta_\alpha^b$. Similarly, the reciprocal of \mathbf{d}_α is denoted by $\overset{\circ}{\vartheta}$ such that $\mathbf{d}_\alpha^a \overset{\circ}{\vartheta}_b = \delta_b^a$ and $\mathbf{d}_\alpha^a \overset{\circ}{\vartheta}_a = \delta_\alpha^b$. The referential directors vary from point to

point and $\mathbf{D}_a^A|_B = W^A{}_{CB} \mathbf{D}_a^C$, where $W^A{}_{CB}$ is called the wryness of the director field and is defined as

$$W^A{}_{BC} = \mathbf{D}_a^A|_C \overset{a}{\Theta}_B = -\mathbf{D}_a^A \overset{a}{\Theta}_{B|C}. \quad (5.70)$$

The director gradient $\mathbf{F}_a = \nabla^G \mathbf{d}_a$ with components $F_a^A = \mathbf{d}_a^A|_A$ is related to the relative wryness as

$$F_a^A = w^a{}_{bA} \mathbf{d}_a^b, \quad (5.71)$$

where

$$w^a{}_{bA} = \mathbf{d}_a^A|_A \overset{a}{\vartheta}_b. \quad (5.72)$$

One can define the following director metrics: $\mathfrak{G} = \mathbf{D}_a^A \mathbf{D}_b^B G_{AB}$, and $\mathfrak{g} = \mathbf{d}_a^a \mathbf{d}_b^b g_{ab}$.

Using these metrics, $\overset{a}{\Theta}_A = \mathfrak{G}_{AB} \mathbf{D}_b^B$, and $\overset{a}{\vartheta}_a = \mathfrak{g}_{ab} \mathbf{d}_b^b$.

The kinetic energy density of a Cosserat solid is written as [113]

$$T = \frac{1}{2} \rho V^a V^b g_{ab} + \frac{1}{2} \overset{ab}{\mathfrak{v}} \dot{\mathbf{d}}_a^a \dot{\mathbf{d}}_b^b g_{ab}, \quad (5.73)$$

where $\overset{ab}{\mathfrak{v}} = \overset{ba}{\mathfrak{v}}$ is the micro-mass moment of inertia, and $\dot{\mathbf{d}}_a^a(X, t) = \frac{\partial}{\partial t} \mathbf{d}_a^a(X, t)$ is the director velocity. The energy density is written as $W = W(X, \mathbf{F}, \mathbf{F}_a, \mathbf{G}, \mathbf{g})$.³⁵

Balance of linear momentum. We next derive the governing equations of generalized Cosserat elasticity using Hamilton's principle of least action. The variation of the kinetic energy is written as

$$\begin{aligned} \delta T &= \rho_0 \langle \langle \mathbf{V}, D_t^{\mathfrak{G}} \delta \varphi \rangle \rangle_{\mathfrak{g}} + \overset{ab}{\mathfrak{v}} \langle \langle \dot{\mathbf{d}}_a^a, D_t^{\mathfrak{g}} \delta \mathbf{d}_b^b \rangle \rangle_{\mathfrak{g}} \\ &= \frac{d}{dt} \left(\rho_0 \langle \langle \mathbf{V}, \delta \varphi \rangle \rangle_{\mathfrak{g}} + \overset{ab}{\mathfrak{v}} \langle \langle \dot{\mathbf{d}}_a^a, \delta \mathbf{d}_b^b \rangle \rangle_{\mathfrak{g}} \right) - \rho_0 \langle \langle \mathbf{A}, \delta \varphi \rangle \rangle_{\mathfrak{g}} - \overset{ab}{\mathfrak{v}} \langle \langle \ddot{\mathbf{d}}_a^a, \delta \mathbf{d}_b^b \rangle \rangle_{\mathfrak{g}}, \end{aligned} \quad (5.74)$$

where $\ddot{\mathbf{d}}_a^a = D_t^{\mathfrak{g}} \dot{\mathbf{d}}_a^a$ is the director acceleration. Assuming that $\delta \varphi(X, t_1) = \delta \varphi(X, t_2) = 0$, and $\delta \mathbf{d}_b^b(X, t_1) = \delta \mathbf{d}_b^b(X, t_2) = \mathbf{0}$ one obtains

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{B}} -T dV dt = \int_{t_1}^{t_2} \int_{\mathcal{B}} \left(\rho_0 \langle \langle \mathbf{A}, \delta \varphi \rangle \rangle_{\mathfrak{g}} + \overset{ab}{\mathfrak{v}} \langle \langle \ddot{\mathbf{d}}_a^a, \delta \mathbf{d}_b^b \rangle \rangle_{\mathfrak{g}} \right) dV dt. \quad (5.75)$$

The variation of the energy density is calculated as

$$\delta W = \nabla_{\frac{\partial}{\partial \epsilon}} W \Big|_{\epsilon=0} = \frac{\partial W}{\partial \mathbf{F}} : \delta \mathbf{F} + \frac{\partial W}{\partial \mathbf{F}_a} : \delta \mathbf{F}_a = \frac{\partial W}{\partial F^a{}_A} \delta F^a{}_A + \frac{\partial W}{\partial F^a{}_A} \delta F^a{}_A. \quad (5.76)$$

³⁵ Note that the energy function can have an explicit dependence on the director field, i.e., $W = W(X, \mathbf{F}, \mathbf{d}_a^a, \mathbf{F}_a, \mathbf{G}, \mathbf{g})$. The partial derivative $\frac{\partial W}{\partial \mathbf{d}_a^a}$ is a micro body force that we do not consider in this paper.

Note that $\delta F^a_A = U^a|_A$ and $\delta \mathbf{F}^a_A = \delta \mathbf{d}^a|_A$. Thus

$$\begin{aligned} \int_{\mathcal{B}} \delta W dV &= \int_{\mathcal{B}} \left(\frac{\partial W}{\partial F^a_A} \delta \varphi^a|_A + \frac{\partial W}{\partial \mathbf{F}^a_A} \delta \mathbf{d}^a|_A \right) dV \\ &= - \int_{\mathcal{B}} \left[\left(\frac{\partial W}{\partial F^a_A} \right) |_A \delta \varphi^a + \left(\frac{\partial W}{\partial \mathbf{F}^a_A} \right) |_A \delta \mathbf{d}^a \right] dV \quad (5.77) \\ &\quad + \int_{\partial \mathcal{B}} \left(\frac{\partial W}{\partial F^a_A} N_A \delta \varphi^a + \frac{\partial W}{\partial \mathbf{F}^a_A} N_A \delta \mathbf{d}^a \right) dA. \end{aligned}$$

Therefore, the balance of linear momentum and the balance of micro linear momentum read

$$P^{aA}|_A = \rho_0 A^a, \quad \mathring{H}^{aA}|_A = \mathring{v}^b \mathring{d}^a, \quad (5.78)$$

where

$$P^{aA} = g^{ab} \frac{\partial W}{\partial F^b_A}, \quad \mathring{H}^{aA} = g^{ab} \frac{\partial W}{\partial \mathbf{F}^b_A}. \quad (5.79)$$

\mathring{H}^{aA} is called the hyperstress tensor. The traction and micro-traction are defined as $T^a = P^{aA} N_A$ and $\mathring{T}^a = \mathring{H}^{aA} N_A$, respectively.

Balance of angular momentum. In order to derive the balance of angular momentum in a Euclidean ambient space consider the flow $\psi_s(x) = x + s\Omega x$, where Ω is an anti-symmetric matrix. As the kinetic energy density (5.73) is invariant under rotations one only needs to require invariance of the energy function under flows of rotations of the ambient space, i.e.,

$$W(X, \mathbf{F}, \mathbf{F}_a, \mathbf{G}, \mathbf{g}) = W(X, \psi_{s*} \mathbf{F}, \psi_{s*} \mathbf{F}_a, \mathbf{G}, \psi_{s*} \mathbf{g}). \quad (5.80)$$

Note that under a rotation R^a_b of the Euclidean ambient space the directors are transformed as $\mathbf{d}^{a'} = R^a_b \mathbf{d}^b$. Taking derivative with respect to s of both sides of (5.80) and evaluating at $s = 0$, one obtains

$$g^{ac} \left[F^b_A \frac{\partial W}{\partial F^c_A} + \mathbf{F}^b_A \frac{\partial W}{\partial \mathbf{F}^c_A} \right] \Omega_{ab} = 0. \quad (5.81)$$

As $\Omega_{ba} = -\Omega_{ab}$, one concludes that $\Pi^{ab} = \Pi^{ba}$, or $\Pi^{[ab]} = 0$, where

$$\Pi^{ab} = g^{ac} \left[F^b_A \frac{\partial W}{\partial F^c_A} + \mathbf{F}^b_A \frac{\partial W}{\partial \mathbf{F}^c_A} \right] = F^b_A P^{aA} + \mathbf{F}^b_A \mathring{H}^{aA}. \quad (5.82)$$

Thus, the balance of angular momentum reads

$$P^{[aA} F^{b]}_A + \mathring{H}^{[aA} \mathbf{F}^{b]}_A = 0. \quad (5.83)$$

Linearized balance of linear momentum. Linearizing the balance of linear momentum and the balance of micro linear momentum about a pair $(\mathring{\varphi}, \mathring{\mathbf{d}})$, one obtains

$$(\delta P^{aA})|_A = \rho_0 \ddot{U}^a, \quad (\delta \mathring{H}^{aA})|_A = \mathring{v}^{\text{ab}} \mathring{\mathbf{u}}^a, \quad (5.84)$$

where $\mathring{\mathbf{u}} = \delta \mathring{\mathbf{d}}$ is the director displacement field. Note that

$$\begin{aligned} \delta P^{aA} &= \mathbb{A}^{aA}_b{}^B U^b|_B + \mathbb{B}^{aA}_b{}^B \mathring{\mathbf{u}}^b|_B, \\ \delta \mathring{H}^{aA} &= \mathbb{B}^{aA}_b{}^B U^b|_B + \mathbb{C}^{aA}_b{}^B \mathring{\mathbf{u}}^b|_B, \end{aligned} \quad (5.85)$$

where

$$\mathbb{A}^{aA}_b{}^B = \frac{\partial^2 W}{\partial F^a_A \partial F^b_B}, \quad \mathbb{B}^{aA}_b{}^B = \frac{\partial^2 W}{\partial F^a_A \partial \mathring{F}^b_B}, \quad \mathbb{C}^{aA}_b{}^B = \frac{\partial^2 W}{\partial \mathring{F}^a_A \partial \mathring{F}^b_B}. \quad (5.86)$$

The elastic constants satisfy the following symmetries: $\mathbb{A}^{aAbB} = \mathbb{A}^{bBaA}$ and $\mathbb{C}^{aAbB} = \mathbb{C}^{bBaA}$.

Linearized balance of angular momentum. Suppose the reference motion is an isometric embedding of an initially stress-free body into the Euclidean space, i.e., $\mathring{F}^a_A = \delta^a_A$, $\mathring{P}^{aA} = 0$, and $\mathring{H}^{aA} = 0$. We will linearize the balance of angular momentum about this motion and about a director field $\mathring{\mathbf{d}}^a = \delta^a_A \mathring{D}^A$. Thus, $\mathring{F}^a_A = \delta^a_B \mathring{D}^B|_A = \mathring{w}^a_{mA} \mathring{d}^m = \delta^a_C W^C{}_{MA} \mathring{D}^M$. Linearization of Π^{ab} in (5.82) one obtains

$$\begin{aligned} \delta \Pi^{ab} &= \left(\mathbb{A}^{aA}_c{}^B \mathring{F}^b_A + \mathbb{B}^{aA}_c{}^B \mathring{F}^b_A \right) U^c|_B \\ &\quad + \left(\mathbb{B}^{aA}_c{}^B \mathring{F}^b_A + \mathbb{C}^{aA}_c{}^B \mathring{F}^b_A \right) \mathring{\mathbf{u}}^c|_B. \end{aligned} \quad (5.87)$$

Knowing that $\delta \Pi^{[ab]} = 0$ and that the displacement and director displacement fields are independent, one obtains

$$\begin{aligned} \mathbb{A}^{[aA}_c{}^B \mathring{F}^b]_A + \mathbb{B}^{[aA}_c{}^B \mathring{F}^b]_A &= 0, \\ \mathbb{B}^{[aA}_c{}^B \mathring{F}^b]_A + \mathbb{C}^{[aA}_c{}^B \mathring{F}^b]_A &= 0. \end{aligned} \quad (5.88)$$

Or equivalently

$$\begin{aligned} \mathbb{A}^{[aAbB} \mathring{F}^c]_A + \mathbb{B}^{[aAbB} \mathring{F}^c]_A &= 0, \\ \mathbb{B}^{[aAbB} \mathring{F}^c]_A + \mathbb{C}^{[aAbB} \mathring{F}^c]_A &= 0. \end{aligned} \quad (5.89)$$

It is seen that the wryness of the reference configuration enters the linearized balance of angular momentum. In other words, in addition to the elastic constants, one needs some information on the non-uniformity of the director field in the stress-free reference configuration.

Transformation cloaking in generalized Cosserat elastodynamics. It is straightforward to show that the issue with the acceleration term observed in classical nonlinear elasticity persists even in nonlinear Cosserat elasticity, and hence, we do not discuss transformation cloaking in nonlinear Cosserat elastodynamics. We start from a virtual body that is made of a homogeneous, isotropic Cosserat elastic solid. We consider a cloaking transformation and try to calculate the elastic constants of the physical body induced from the cloaking transformation such that the balance of linear and angular momenta are respected in both the virtual and physical bodies.

The divergence term $(\delta P^{aA})|_A$ in the balance of linear momentum in the physical body is written as

$$\left(\mathbb{A}^{aA}{}_b{}^B U^b|_B + \mathbb{B}^{aA}{}_b{}^B \mathbb{Y}^b|_B \right) |_A \frac{\partial}{\partial x^a}. \tag{5.90}$$

Under a cloaking transformation $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ and using the shifter map this is transformed to

$$J_{\mathcal{E}} \left(\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} \tilde{U}^{\tilde{b}}|_{\tilde{B}} + \tilde{\mathbb{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} \tilde{\mathbb{Y}}^{\tilde{b}}|_{\tilde{B}} \right) |_{\tilde{A}} \frac{\partial}{\partial \tilde{x}^{\tilde{a}}}, \tag{5.91}$$

where

$$\begin{aligned} \tilde{U}^{\tilde{a}} &= \mathbf{s}^{\tilde{a}}{}_a U^a, \quad \tilde{\mathbb{Y}}^{\tilde{a}} = \mathbf{s}^{\tilde{a}}{}_a \mathbb{Y}^a, \\ \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} &= J_{\mathcal{E}}^{-1} \mathbf{s}^{\tilde{a}}{}_a \tilde{F}^{\tilde{A}}{}_A (\mathbf{s}^{-1})^b{}_{\tilde{b}} \tilde{F}^{\tilde{B}}{}_B \mathbb{A}^{aA}{}_b{}^B, \\ \tilde{\mathbb{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} &= J_{\mathcal{E}}^{-1} \mathbf{s}^{\tilde{a}}{}_a \tilde{F}^{\tilde{A}}{}_A (\mathbf{s}^{-1})^b{}_{\tilde{b}} \tilde{F}^{\tilde{B}}{}_B \mathbb{B}^{aA}{}_b{}^B. \end{aligned} \tag{5.92}$$

Or, equivalently

$$\begin{aligned} \mathbb{A}^{aA}{}_b{}^B &= J_{\mathcal{E}} (\mathbf{s}^{-1})^a{}_{\tilde{a}} (\tilde{F}^{-1})^A{}_{\tilde{A}} \mathbf{s}^{\tilde{b}}{}_{\tilde{b}} (\tilde{F}^{-1})^B{}_{\tilde{B}} \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}, \\ \mathbb{B}^{aA}{}_b{}^B &= J_{\mathcal{E}} (\mathbf{s}^{-1})^a{}_{\tilde{a}} (\tilde{F}^{-1})^A{}_{\tilde{A}} \mathbf{s}^{\tilde{b}}{}_{\tilde{b}} (\tilde{F}^{-1})^B{}_{\tilde{B}} \tilde{\mathbb{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}. \end{aligned} \tag{5.93}$$

The divergence term $(\delta \mathbb{H}^{aA})|_A$ in the balance of micro linear momentum in the physical body is written as

$$\left(\mathbb{B}^{aA}{}_b{}^B U^b|_B + \mathbb{C}^{ab}{}_b{}^B \mathbb{Y}^b|_B \right) |_A \frac{\partial}{\partial x^a}. \tag{5.94}$$

Under the cloaking transformation and using the shifter map this is transformed to read

$$J_{\mathcal{E}} \left(\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} \tilde{U}^{\tilde{b}}|_{\tilde{B}} + \tilde{\mathbb{C}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} \tilde{\mathbb{Y}}^{\tilde{b}}|_{\tilde{B}} \right) |_{\tilde{A}} \frac{\partial}{\partial \tilde{x}^{\tilde{a}}}, \tag{5.95}$$

where

$$\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} = J_{\mathcal{E}}^{-1} \mathbf{s}^{\tilde{a}}{}_a \tilde{F}^{\tilde{A}}{}_A (\mathbf{s}^{-1})^b{}_{\tilde{b}} \tilde{F}^{\tilde{B}}{}_B \mathbb{C}^{ab}{}_b{}^B, \tag{5.96}$$

and the other transformed quantities are given in (5.98). Equivalently

$$\overset{ab}{\mathbb{C}}{}^aA{}_b{}^B = J_{\mathcal{E}}(\mathbf{s}^{-1})^a{}_{\tilde{a}}(\tilde{F}^{-1})^A{}_{\tilde{A}}\tilde{\mathbf{s}}^{\tilde{b}}{}_b(\tilde{F}^{-1})^B{}_{\tilde{B}}\overset{ab}{\mathbb{C}}{}^{\tilde{a}}\tilde{A}{}_{\tilde{b}}{}^{\tilde{B}}. \quad (5.97)$$

The micro-mass moment of inertia is transformed as $\overset{ab}{\mathbb{v}} = J_{\mathcal{E}}^{-1}\overset{ab}{\mathbb{v}}$. Similar to classical linear elasticity the mass density is transformed as $\tilde{\rho}_0 = J_{\mathcal{E}}^{-1}\rho_0$. In summary, the linearized balance of linear momentum and micro linear momentum are form-invariant under the following *cloaking transformations*:

$$\begin{aligned} \tilde{X} &= \mathcal{E}(X), \quad \tilde{\mathbf{U}} = \mathbf{s} \circ \hat{\phi} \mathbf{U} \circ \mathcal{E}^{-1}, \quad \tilde{\mathbf{u}}_{\tilde{\alpha}} = \mathbf{s} \circ \hat{\phi} \mathbf{u}_{\alpha} \circ \mathcal{E}^{-1}, \quad \tilde{\rho}_0 = J_{\mathcal{E}}^{-1}\rho_0 \circ \mathcal{E}^{-1}, \\ \overset{ab}{\mathbb{v}} &= J_{\mathcal{E}}^{-1}\overset{ab}{\mathbb{v}} \circ \mathcal{E}^{-1}, \quad \tilde{\mathbb{A}} = (J_{\mathcal{E}}^{-1}\mathbf{s}^{-1} \circ \hat{\phi} \overset{\tilde{\tilde{F}}}{\mathbf{F}} \mathbf{s} \circ \hat{\phi} \mathbb{A} \overset{\tilde{\tilde{F}}}{\mathbf{F}}^{\star}) \circ \mathcal{E}^{-1}, \\ \overset{a}{\mathbb{B}} &= (J_{\mathcal{E}}^{-1}\mathbf{s}^{-1} \circ \hat{\phi} \overset{\tilde{\tilde{F}}}{\mathbf{F}} \mathbf{s} \circ \hat{\phi} \overset{a}{\mathbb{B}} \overset{\tilde{\tilde{F}}}{\mathbf{F}}^{\star}) \circ \mathcal{E}^{-1}, \\ \overset{ab}{\mathbb{C}} &= (J_{\mathcal{E}}^{-1}\mathbf{s}^{-1} \circ \hat{\phi} \overset{\tilde{\tilde{F}}}{\mathbf{F}} \mathbf{s} \circ \hat{\phi} \overset{ab}{\mathbb{C}} \overset{\tilde{\tilde{F}}}{\mathbf{F}}^{\star}) \circ \mathcal{E}^{-1}. \end{aligned} \quad (5.98)$$

Note that under a cloaking transformation, the stress and hyperstress tensors are transformed as

$$\tilde{P}^{\tilde{a}}\tilde{A} = J_{\mathcal{E}}^{-1}\mathbf{s}^{\tilde{a}}{}_a\tilde{F}^{\tilde{A}}{}_A P^{aA}, \quad \overset{a}{\mathbb{H}}^{\tilde{a}}\tilde{A} = J_{\mathcal{E}}^{-1}\mathbf{s}^{\tilde{a}}{}_a\tilde{F}^{\tilde{A}}{}_A \overset{a}{\mathbb{H}}^{aA}. \quad (5.99)$$

We assume that in the stress-free reference configuration of the virtual body the director field is uniform, i.e., $\overset{\circ}{\mathbb{D}}{}^{\tilde{A}}{}_{|\tilde{B}} = 0$, and hence, $\overset{\circ}{\mathbb{F}}{}^{\tilde{a}}{}_{\tilde{A}} = 0$. In other words, in the virtual body the wryness vanishes. Therefore, the balance of angular momentum (5.89) in the virtual body reads

$$\tilde{\mathbb{A}}[\tilde{a}\tilde{A}\tilde{b}\tilde{B} \overset{\circ}{\mathbb{F}}{}^{\tilde{c}}{}_{\tilde{A}}]_{\tilde{A}} = 0, \quad \overset{a}{\mathbb{B}}[\tilde{a}\tilde{A}\tilde{b}\tilde{B} \overset{\circ}{\mathbb{F}}{}^{\tilde{c}}{}_{\tilde{A}}]_{\tilde{A}} = 0. \quad (5.100)$$

Assuming that in the virtual body the balance of angular momentum (5.100) is satisfied, the balance of angular momentum in the physical body (5.89) requires that

$$\begin{aligned} (\mathbf{s}^{-1})^{[a}{}_{\tilde{a}}(\mathbf{s}^{-1})^{b}{}_{\tilde{b}}(\tilde{F}^{-1})^A{}_{\tilde{A}}(\tilde{F}^{-1})^B{}_{\tilde{B}} \left\{ \overset{\circ}{\mathbb{F}}{}^{c|}{}_A \tilde{\mathbb{A}}^{\tilde{a}}\tilde{A}\tilde{b}\tilde{B} + \overset{\circ}{\mathbb{F}}{}^{c|}{}_A \overset{a}{\mathbb{B}}^{\tilde{a}}\tilde{A}\tilde{b}\tilde{B} \right\} &= 0, \\ (\mathbf{s}^{-1})^{[a}{}_{\tilde{a}}(\mathbf{s}^{-1})^{b}{}_{\tilde{b}}(\tilde{F}^{-1})^A{}_{\tilde{A}}(\tilde{F}^{-1})^B{}_{\tilde{B}} \left\{ \overset{\circ}{\mathbb{F}}{}^{c|}{}_A \overset{a}{\mathbb{B}}^{\tilde{a}}\tilde{A}\tilde{b}\tilde{B} + \overset{\circ}{\mathbb{F}}{}^{c|}{}_A \overset{ba}{\mathbb{C}}{}^{\tilde{a}}\tilde{A}\tilde{b}\tilde{B} \right\} &= 0, \end{aligned} \quad (5.101)$$

where $\overset{\circ}{\mathbb{F}}{}^c{}_A = \delta^C_C W^C{}_{MA} \overset{\circ}{\mathbb{D}}{}^M$. Note that $(\tilde{F}^{\tilde{A}})_{A|B} = (\tilde{F}^{\tilde{A}})_{B|A}$.

Remark 13. Note that $\overset{\circ}{\mathbb{F}}{}^a{}_{A|B} = \overset{\circ}{\mathbb{F}}{}^a{}_{|A|B}$. Knowing that the reference configuration of the physical body is flat, covariant derivatives commute. Therefore, a necessary condition for a field $\overset{\circ}{\mathbb{F}}{}^a{}_{A}$ to be the director gradient of a director field is

$$\overset{\circ}{\mathbb{F}}{}^a{}_{A|B} = \overset{\circ}{\mathbb{F}}{}^a{}_{B|A}. \quad (5.102)$$

When designing a cloak the field $\overset{\circ}{\mathbb{F}}{}^a{}_{A}$ is not known a priori. It must satisfy both (5.101) and the compatibility equations (5.102).

Remark 14. In the literature it has been assumed that the virtual body is made of a classical linear elastic solid while the cloak is suggested to be made of a Cosserat elastic solid (without any discussion on how to calculate the Cosserat elastic constants). Note, however, that the virtual body cannot be made of a classical solid. If one assumes that the virtual body is classical, i.e., $\overset{a}{\mathbb{B}}$ and $\overset{ab}{\mathbb{C}}$ vanish, (5.101) reduces to the classical balance of angular momentum for the physical body (4.51), which is an obstruction to transformation cloaking. In other words, both the virtual body and the physical body (even outside the cloak) must be made of Cosserat linear elastic solids in order to achieve elastodynamic transformation cloaking.

Elastic constants of the virtual body. We assume that the virtual body is made of an elastic, homogeneous, and isotropic generalized Cosserat solid. The most general form of a fourth-order isotropic tensor is given by $a_1\delta_{ij}\delta_{kl} + a_2\delta_{ik}\delta_{jl} + a_3\delta_{il}\delta_{jk}$ for some scalars $a_1, a_2,$ and a_3 . Therefore, the elastic constants of the virtual body are written as

$$\begin{aligned} \tilde{\mathbb{A}}_{\tilde{a}}^{\tilde{A}} \tilde{\mathbb{B}}_{\tilde{b}}^{\tilde{B}} &= \lambda \tilde{G}^{\tilde{M}\tilde{A}} \tilde{G}^{\tilde{N}\tilde{B}} \tilde{F}^{\tilde{m}}_{\tilde{M}} \tilde{F}^{\tilde{n}}_{\tilde{N}} g_{\tilde{a}\tilde{m}} g_{\tilde{b}\tilde{n}} \\ &+ \mu \left(\tilde{G}^{\tilde{M}\tilde{N}} \tilde{G}^{\tilde{A}\tilde{B}} \tilde{F}^{\tilde{m}}_{\tilde{M}} \tilde{F}^{\tilde{n}}_{\tilde{N}} g_{\tilde{a}\tilde{m}} g_{\tilde{b}\tilde{n}} + \tilde{G}^{\tilde{M}\tilde{B}} \tilde{G}^{\tilde{N}\tilde{A}} \tilde{F}^{\tilde{m}}_{\tilde{M}} \tilde{F}^{\tilde{n}}_{\tilde{N}} g_{\tilde{a}\tilde{m}} g_{\tilde{b}\tilde{n}} \right). \end{aligned} \quad (5.103)$$

This is a consequence of the minor symmetries. Also

$$\begin{aligned} \overset{a}{\mathbb{B}}_{\tilde{a}}^{\tilde{A}} \tilde{\mathbb{B}}_{\tilde{b}}^{\tilde{B}} &= \overset{a}{b}_1 G^{\tilde{M}\tilde{A}} G^{\tilde{N}\tilde{B}} \tilde{F}^{\tilde{m}}_{\tilde{M}} \tilde{F}^{\tilde{n}}_{\tilde{N}} g_{\tilde{a}\tilde{m}} g_{\tilde{b}\tilde{n}} \\ &+ \overset{a}{b}_2 G^{\tilde{M}\tilde{N}} G^{\tilde{A}\tilde{B}} \tilde{F}^{\tilde{m}}_{\tilde{M}} \tilde{F}^{\tilde{n}}_{\tilde{N}} g_{\tilde{a}\tilde{m}} g_{\tilde{b}\tilde{n}} \\ &+ \overset{a}{b}_3 G^{\tilde{M}\tilde{B}} G^{\tilde{N}\tilde{A}} \tilde{F}^{\tilde{m}}_{\tilde{M}} \tilde{F}^{\tilde{n}}_{\tilde{N}} g_{\tilde{a}\tilde{m}} g_{\tilde{b}\tilde{n}}, \end{aligned} \quad (5.104)$$

but note that $\overset{a}{\mathbb{B}}$ has the minor symmetries from (5.100)₂, and hence, $\overset{a}{b}_2 = \overset{a}{b}_3$. Similarly

$$\begin{aligned} \overset{ab}{\mathbb{C}}_{\tilde{a}}^{\tilde{A}} \tilde{\mathbb{B}}_{\tilde{b}}^{\tilde{B}} &= \overset{ab}{c}_1 G^{\tilde{M}\tilde{A}} G^{\tilde{N}\tilde{B}} \tilde{F}^{\tilde{m}}_{\tilde{M}} \tilde{F}^{\tilde{n}}_{\tilde{N}} g_{\tilde{a}\tilde{m}} g_{\tilde{b}\tilde{n}} \\ &+ \overset{ab}{c}_2 G^{\tilde{M}\tilde{N}} G^{\tilde{A}\tilde{B}} \tilde{F}^{\tilde{m}}_{\tilde{M}} \tilde{F}^{\tilde{n}}_{\tilde{N}} g_{\tilde{a}\tilde{m}} g_{\tilde{b}\tilde{n}} \\ &+ \overset{ab}{c}_3 G^{\tilde{M}\tilde{B}} G^{\tilde{N}\tilde{A}} \tilde{F}^{\tilde{m}}_{\tilde{M}} \tilde{F}^{\tilde{n}}_{\tilde{N}} g_{\tilde{a}\tilde{m}} g_{\tilde{b}\tilde{n}}, \end{aligned} \quad (5.105)$$

where $\overset{ab}{c}_i = \overset{ba}{c}_i, i = 1, 2, 3,$ because $\overset{ab}{\mathbb{C}}_{\tilde{a}}^{\tilde{A}} \tilde{\mathbb{B}}_{\tilde{b}}^{\tilde{B}} = \overset{ba}{\mathbb{C}}_{\tilde{b}}^{\tilde{B}} \tilde{\mathbb{A}}_{\tilde{a}}^{\tilde{A}}$. Knowing that $\tilde{F}^a_A = \delta^a_A$, one can identify the spatial and material manifolds with the same metric, and thus, with a slight abuse of notation, the elastic constants are simplified to read

$$\begin{aligned} \tilde{\mathbb{A}}^{\tilde{A}\tilde{B}} &= \lambda \tilde{G}^{\tilde{a}\tilde{B}} \tilde{G}^{\tilde{b}\tilde{B}} + \mu (\tilde{G}^{\tilde{a}\tilde{b}} \tilde{G}^{\tilde{A}\tilde{B}} + \tilde{G}^{\tilde{a}\tilde{B}} \tilde{G}^{\tilde{b}\tilde{A}}), \\ \overset{a}{\mathbb{B}}^{\tilde{A}\tilde{B}} &= \overset{a}{b}_1 G^{\tilde{a}\tilde{A}} G^{\tilde{b}\tilde{B}} + \overset{a}{b}_2 (G^{\tilde{a}\tilde{b}} G^{\tilde{A}\tilde{B}} + G^{\tilde{a}\tilde{B}} G^{\tilde{b}\tilde{A}}), \\ \overset{ab}{\mathbb{C}}^{\tilde{A}\tilde{B}} &= \overset{ab}{c}_1 G^{\tilde{a}\tilde{A}} G^{\tilde{b}\tilde{B}} + \overset{ab}{c}_2 G^{\tilde{a}\tilde{b}} G^{\tilde{A}\tilde{B}} + \overset{ab}{c}_3 G^{\tilde{a}\tilde{B}} G^{\tilde{b}\tilde{A}}. \end{aligned} \quad (5.106)$$

Therefore, one has 15 and 26 independent elastic constants in dimensions two and three, respectively.

Positive-definiteness of the energy density. Knowing that in the initial configuration $\dot{P}^{aA} = 0$ and $\dot{H}^{aA} = 0$, one can write

$$\begin{aligned} \delta W &= \frac{1}{2} \frac{\partial W}{\partial F^a_A \partial F^b_B} U^a_{|A} U^b_{|B} + \frac{\partial W}{\partial F^a_A \partial \mathbb{F}^b_B} U^a_{|A} \mathbb{U}^b_{|B} + \frac{1}{2} \frac{\partial W}{\partial \mathbb{F}^a_A \partial \mathbb{F}^b_B} \mathbb{U}^a_{|A} \mathbb{U}^b_{|B} \\ &= \frac{1}{2} \mathbb{A}^A B U^a_{|A} U^b_{|B} + \mathbb{B}^a B U^a_{|A} \mathbb{U}^b_{|B} + \frac{1}{2} \mathbb{C}^{ab} U^a_{|A} \mathbb{U}^b_{|B} \quad (5.107) \\ &= \frac{1}{2} \mathbb{A}^{aAbB} U_{a|A} U_{b|B} + \mathbb{B}^{aAbB} U_{a|A} \mathbb{U}_{b|B} + \frac{1}{2} \mathbb{C}^{ab} \mathbb{U}_{a|A} \mathbb{U}_{b|B}. \end{aligned}$$

Let us introduce the new indices $\gamma = \{aA\}$, $\Gamma = \{aA\}$, and define the new variables $X_\gamma = U_{a|A}$ and $Y_\Gamma = \mathbb{U}_{a|A}$.³⁶ Thus, the energy density is rewritten as

$$\delta W = \frac{1}{2} \mathbb{A}^{\gamma\lambda} X_\gamma X_\lambda + \mathbb{B}^{\gamma\Gamma} X_\gamma Y_\Gamma + \frac{1}{2} \mathbb{C}^{\Gamma\Lambda} Y_\Gamma Y_\Lambda. \quad (5.108)$$

Next let us define a new variable $\mathbf{Z} = \begin{Bmatrix} \mathbf{X} \\ \mathbf{Y} \end{Bmatrix}$. It is straightforward to show that

$$\delta W = \frac{1}{2} \mathbf{Z}^T \cdot \mathbb{D} \mathbf{Z}, \quad \mathbb{D} = \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B}^T & \mathbb{C} \end{bmatrix}. \quad (5.109)$$

Note that $\mathbb{A}^{\gamma\lambda} = \mathbb{A}^{\lambda\gamma}$, and $\mathbb{C}^{\Gamma\Lambda} = \mathbb{C}^{\Lambda\Gamma}$, but $\mathbb{B}^{\gamma\Gamma} \neq \mathbb{B}^{\Gamma\gamma}$, in general. Note that \mathbb{D} is symmetric, and hence, $\delta W > 0$, $\forall \mathbf{Z} \neq \mathbf{0}$ if and only if all the eigenvalues of \mathbb{D} are positive (\mathbb{D} is a square matrix of size 12 and 36 in dimensions two and three, respectively). For the energy density to be positive-definite it is necessary that \mathbb{A} and \mathbb{C} be positive-definite because \mathbf{X} and \mathbf{Y} are independent. \mathbb{A} is positive-definite if and only if $\mu > 0$ and $3\lambda + 2\mu > 0$. Note that \mathbb{U}_a^a , $a = 1, 2, 3$ are independent, and hence, as a consequence of positive-definiteness of \mathbb{C} , $3c_1^{aa} + c_2^{aa} + c_3^{aa} > 0$, $c_2^{aa} + c_3^{aa} > 0$, and $c_2^{aa} - c_3^{aa} > 0$, (no summation on a). In particular, $c_2^{aa} > 0$. \mathbb{A} and \mathbb{C} are positive-definite, and hence, invertible. From Schur's complement condition [20], positive-definiteness of \mathbb{D} is now equivalent to positive-definiteness of either $\mathbb{C} - \mathbb{B}^T \mathbb{A}^{-1} \mathbb{B}$ or $\mathbb{A} - \mathbb{B} \mathbb{C}^{-1} \mathbb{B}^T$.

Suppose $\delta W > 0$ for any $(U^a_{|A}, \mathbb{U}_a^a_{|A}) \neq (0, 0)$. In the virtual body

$$\delta \tilde{W} = \frac{1}{2} \tilde{\mathbb{A}}^{\tilde{a}\tilde{b}\tilde{B}} \tilde{U}_{\tilde{a}|\tilde{A}} \tilde{U}_{\tilde{b}|\tilde{B}} + \tilde{\mathbb{B}}^{\tilde{a}\tilde{b}\tilde{B}} \tilde{U}_{\tilde{a}|\tilde{A}} \tilde{\mathbb{U}}_{\tilde{b}|\tilde{B}} + \frac{1}{2} \tilde{\mathbb{C}}^{\tilde{a}\tilde{b}\tilde{B}} \tilde{\mathbb{U}}_{\tilde{a}|\tilde{A}} \tilde{\mathbb{U}}_{\tilde{b}|\tilde{B}}. \quad (5.110)$$

Using (5.98), the relations $\tilde{U}_{\tilde{b}|\tilde{B}} = \mathbf{s}_{\tilde{b}}^{\tilde{B}} (\tilde{F}^{-1})^B_{\tilde{B}} U^b_{|B}$, $\tilde{\mathbb{U}}_{\tilde{b}|\tilde{B}} = \mathbf{s}_{\tilde{b}}^{\tilde{B}} (\tilde{F}^{-1})^B_{\tilde{B}} \mathbb{U}_b^b_{|B}$, and the relation $\mathbf{s}_{\tilde{a}}^{\tilde{a}} \mathbf{s}_{\tilde{b}}^{\tilde{b}} \tilde{g}_{\tilde{a}\tilde{b}} = g_{ab}$, one can easily show that $\delta \tilde{W} = J_{\tilde{\Sigma}}^{-1} \delta W > 0$. Therefore, the elastic constants of the physical body are positive-definite if and only if those of the virtual body are positive-definite. In other words, the cloaking transformations (5.98) preserve the positive-definiteness of the elastic constants.

³⁶ More specifically, $\{11, 12, \dots, 1n, 21, 22, \dots, nn\} \rightarrow \{1, 2, \dots, n^2\}$ and $\{111, 112, \dots, 1nn, 211, \dots, nnn\} \rightarrow \{1, 2, \dots, n^3\}$.

Boundary conditions in the physical and virtual problems. Let $\partial\mathcal{B} = \mathcal{H} \cup \partial_o\mathcal{B}$, where \mathcal{H} is the boundary of the hole and $\partial_o\mathcal{B}$ is the outer boundary of \mathcal{B} . Suppose $\partial_o\mathcal{B} = \partial_o\mathcal{B}_t \cup \partial_o\mathcal{B}_d$ such that the Neumann and Dirichlet boundary conditions are written as

$$\begin{cases} \mathbf{PN} = \bar{\mathbf{t}}(X, t) \\ \overset{\mathfrak{a}}{\mathbf{HN}} = \overset{\mathfrak{a}}{\mathbf{m}}(X, t) \end{cases} \quad \text{on } \partial_o\mathcal{B}_t, \tag{5.111}$$

$$\begin{cases} \varphi(X, t) = \bar{\varphi}(X, t) \\ \mathbf{d}_\alpha(X, t) = \bar{\mathbf{d}}_\alpha(X, t) \end{cases} \quad \text{on } \partial_o\mathcal{B}_d,$$

where \mathbf{N} is the unit normal vector on $\partial_o\mathcal{B}_t$ and $\bar{\mathbf{t}}$, $\bar{\mathbf{m}}$, $\bar{\varphi}$, and $\bar{\mathbf{d}}_\alpha$ are given. Under the mapping $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$, $\partial_o\tilde{\mathcal{B}} = \mathcal{E}(\partial_o\mathcal{B}) = \mathcal{E}(\partial_o\mathcal{B}_t) \cup \mathcal{E}(\partial_o\mathcal{B}_d)$. Suppose on $\partial\mathcal{H}$, $\mathbf{t} = \mathbf{PN} = \bar{\mathbf{t}}$, and $\overset{\mathfrak{a}}{\mathbf{HN}} = \overset{\mathfrak{a}}{\mathbf{m}}$. Note that

$$\mathbf{st}dA = \bar{\mathbf{t}}d\tilde{A} = \bar{\mathbf{t}}d\tilde{A}, \quad \mathbf{sm}dA = \overset{\mathfrak{a}}{\mathbf{sm}}dA = \overset{\mathfrak{a}}{\mathbf{m}}d\tilde{A}. \tag{5.112}$$

Therefore, on $\partial_o\mathcal{B}_t$, $\tilde{\mathbf{t}} = \tilde{\mathbf{P}}\tilde{\mathbf{N}} = (dA/d\tilde{A})\bar{\mathbf{t}}$, and $\overset{\mathfrak{a}}{\mathbf{m}} = \overset{\mathfrak{a}}{\mathbf{H}}\tilde{\mathbf{N}} = (dA/d\tilde{A})\overset{\mathfrak{a}}{\mathbf{m}}$. We know that $\tilde{N}_{\tilde{A}}d\tilde{A} = J_{\mathcal{E}}(\tilde{F}^{-1})^A_{\tilde{A}}N_A dA$. Therefore (note that $\tilde{\mathbf{N}}$ is a $\tilde{\mathbf{G}}$ -unit one-form)

$$\tilde{G}^{\tilde{A}\tilde{B}}\tilde{N}_{\tilde{A}}\tilde{N}_{\tilde{B}}d\tilde{A}^2 = d\tilde{A}^2 = J_{\mathcal{E}}^2 \left[(\tilde{F}^{-1})^A_{\tilde{A}}(\tilde{F}^{-1})^B_{\tilde{B}}\tilde{G}^{\tilde{A}\tilde{B}}N_A N_B \right] dA^2. \tag{5.113}$$

Thus

$$\frac{dA}{d\tilde{A}} = J_{\mathcal{E}}^{-1} \left[(\tilde{C}^{-1})^{AB}N_A N_B \right]^{-\frac{1}{2}}, \tag{5.114}$$

where $\tilde{C}_{AB} = \tilde{F}^{\tilde{A}}_{\tilde{A}}\tilde{F}^{\tilde{B}}_{\tilde{B}}\tilde{G}^{\tilde{A}\tilde{B}}$, and $(\tilde{C}^{-1})^{AB} = (\tilde{F}^{-1})^A_{\tilde{A}}(\tilde{F}^{-1})^B_{\tilde{B}}\tilde{G}^{\tilde{A}\tilde{B}}$. Therefore, the traction boundary condition on both $\partial\tilde{\mathcal{H}}$ and $\partial_o\tilde{\mathcal{B}}_t$ reads

$$\begin{aligned} \tilde{\mathbf{t}} &= J_{\mathcal{E}}^{-1} \left[(\tilde{C}^{-1})^{AB}N_A N_B \right]^{-\frac{1}{2}} \bar{\mathbf{t}}, \\ \overset{\mathfrak{a}}{\mathbf{m}} &= J_{\mathcal{E}}^{-1} \left[(\tilde{C}^{-1})^{AB}N_A N_B \right]^{-\frac{1}{2}} \overset{\mathfrak{a}}{\mathbf{m}}. \end{aligned} \tag{5.115}$$

Knowing that in $\mathcal{B} \setminus \mathcal{C}$, $\mathcal{E} = id$, we have

$$\tilde{\mathbf{t}} = \bar{\mathbf{t}}, \quad \overset{\mathfrak{a}}{\mathbf{m}} = \overset{\mathfrak{a}}{\mathbf{m}} \quad \text{on } \partial_o\tilde{\mathcal{B}}_t. \tag{5.116}$$

The hole in the physical body is assumed to be traction free, i.e.,

$$\mathbf{PN} = \mathbf{t} = \mathbf{0}, \quad \overset{\mathfrak{a}}{\mathbf{HN}} = \overset{\mathfrak{a}}{\mathbf{m}} = \mathbf{0}, \quad \text{on } \partial\mathcal{H}. \tag{5.117}$$

Using (5.115) in the virtual body one has

$$\tilde{\mathbf{P}}\tilde{\mathbf{N}} = \tilde{\mathbf{t}} = \mathbf{0}, \quad \overset{\mathfrak{a}}{\mathbf{H}}\tilde{\mathbf{N}} = \overset{\mathfrak{a}}{\mathbf{m}} = \mathbf{0} \quad \text{on } \partial\tilde{\mathcal{H}}. \tag{5.118}$$

In other words, the traction-free boundary condition on the surface of the hole \mathcal{H} implies that the surface of the transformed (infinitesimal) hole $\tilde{\mathcal{H}}$ in the virtual body is traction-free as well.

On $\partial_o \tilde{\mathcal{B}}_d = \mathcal{E}(\partial_o \mathcal{B}_d)$, one assumes that the virtual and physical problems have the same Dirichlet boundary condition, i.e.,

$$\tilde{\varphi}(\tilde{X}, t) = \bar{\varphi} \circ \mathcal{E}^{-1}(\tilde{X}, t), \quad \tilde{\mathbf{d}}_{\mathfrak{a}}(\tilde{X}, t) = \bar{\mathbf{d}}_{\mathfrak{a}} \circ \mathcal{E}^{-1}(\tilde{X}, t) \quad \text{on } \partial_o \tilde{\mathcal{B}}_d. \quad (5.119)$$

Moreover, we note that $\partial(\mathcal{B} \setminus \mathcal{C}) = \partial_o \mathcal{B} \cup \partial_o \mathcal{C}$, and similarly, $\partial(\tilde{\mathcal{B}} \setminus \tilde{\mathcal{C}}) = \partial_o \tilde{\mathcal{B}} \cup \partial_o \tilde{\mathcal{C}}$, where $\partial_o \mathcal{C}$ is the outer boundary of the cloak. As \mathcal{E} is defined to be the identity map in $\mathcal{B} \setminus \mathcal{C}$, from (5.119), $\partial_o \mathcal{B}_d$ and $\partial_o \tilde{\mathcal{B}}_d$ have the same displacement boundary conditions. Notice that $T\mathcal{E}|_{\partial_o \mathcal{B}_t} = \tilde{\mathbf{F}}|_{\partial_o \mathcal{B}_t} = id$, and hence $J_{\mathcal{E}}|_{\partial_o \mathcal{B}_t} = 1$, which implies that the traction boundary conditions on $\partial_o \mathcal{B}_t$ and $\partial_o \tilde{\mathcal{B}}_t$ are identical. Thus, $\partial_o \mathcal{B}$ and $\partial_o \tilde{\mathcal{B}}$ have the same traction and displacement boundary conditions. Note that $\partial \mathcal{C} = \partial \mathcal{H} \cup \partial_o \mathcal{C}$, where $\partial \mathcal{H}$ is the boundary of the hole. On $\partial_o \mathcal{C}$ one can write

$$\tilde{\mathbf{t}} = J_{\mathcal{E}}^{-1} \left[(\tilde{\mathcal{C}}^{-1})^{AB} N_A N_B \right]^{-\frac{1}{2}} \mathbf{st}. \quad (5.120)$$

Let us assume that in addition to $\mathcal{E}|_{\partial_o \mathcal{C}} = id$, the tangent map is the identity map as well, i.e., $T\mathcal{E}|_{\partial_o \mathcal{C}} = \tilde{\mathbf{F}}|_{\partial_o \mathcal{C}} = id$.³⁷ For such maps $J_{\mathcal{E}}|_{\partial_o \mathcal{C}} = 1$, and hence

$$\tilde{\mathbf{t}} = \left[\delta_A^A \delta_B^B \tilde{G}^{\tilde{A}\tilde{B}} N_A N_B \right]^{-\frac{1}{2}} \mathbf{st} \quad \text{on } \partial_o \mathcal{C}. \quad (5.121)$$

Note that \mathbf{G} and $\tilde{\mathbf{G}}$ are induced Euclidean metrics and because \mathcal{E} and $T\mathcal{E}$ are both identity on $\partial_o \mathcal{C}$, $\delta_A^A \delta_B^B \tilde{G}^{\tilde{A}\tilde{B}} N_A N_B = G^{AB} N_A N_B = 1$, and hence

$$\tilde{\mathbf{t}} = \mathbf{st} \quad \text{on } \partial_o \tilde{\mathcal{C}}. \quad (5.122)$$

Similarly

$$\mathfrak{a} \tilde{\mathbf{m}} = \mathfrak{a} \mathbf{m} \quad \text{on } \partial_o \tilde{\mathcal{C}}, \quad (5.123)$$

that is, $\partial_o \mathcal{C}$ and $\partial_o \tilde{\mathcal{C}}$ have identical traction boundary conditions. Outside the cloak the elastic constants of the two problems are obviously identical from (5.98). One needs to assume that outside the cloak the two bodies have identical mechanical properties. In particular, outside the cloak $\tilde{\mathbf{F}}_{\mathfrak{a}} = \mathbf{0}$, i.e., the director field in the physical body outside the cloak is uniform. This means that on $\mathcal{B} \setminus \mathcal{C}$, $\mathbf{D}_{\mathfrak{a}} = \tilde{\mathbf{D}}_{\mathfrak{a}}$. This holds on the outer boundary of the cloak as well, i.e., $\tilde{\mathbf{F}}_{\mathfrak{a}} = \mathbf{0}$, and $\mathbf{D}_{\mathfrak{a}} = \tilde{\mathbf{D}}_{\mathfrak{a}}$ on $\partial_o \mathcal{C}$. Therefore, $\mathcal{B} \setminus \mathcal{C}$ and $\tilde{\mathcal{B}} \setminus \tilde{\mathcal{C}}$ are made of the same generalized Cosserat solid (have identical elastic constants and have the same director fields in their undeformed configurations), and are subject to the same body forces and boundary conditions.

³⁷ This condition has been ignored in the existing works on elastodynamic transformation cloaking. In particular, borrowing the cloaking transformation of PENDRY et al. [89] from electromagnetism is not acceptable as it does not satisfy this condition.

It then immediately follows that $\varphi(X, t) = \tilde{\varphi}(\tilde{X}, t)$ and $\mathbf{d}_q(X, t) = \tilde{\mathbf{d}}_q(\tilde{X}, t)$ on $\mathcal{B} \setminus \mathcal{C}$, and hence, the current configurations of the two bodies are identical outside the cloak. This, in turn, renders cloaking possible as the virtual body is isotropic and homogeneous, and contains an infinitesimal hole, which has a low scattering effect on the incident waves.

The following summarizes the construction of an elastic cloak in a linear elastic generalized Cosserat solid. Consider a diffeomorphism $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ that shrinks a hole in the physical body \mathcal{B} to an infinitesimal hole in the virtual body $\tilde{\mathcal{B}}$. The hole is surrounded by a cloak \mathcal{C} in the physical body \mathcal{B} . Assume that $\mathcal{E}|_{\mathcal{B} \setminus \mathcal{C}} = id$ and that on the outer boundary of the cloak $T\mathcal{E} = id$. Assume that the displacement vectors in the physical and virtual bodies, mass densities, body forces, and the elastic constants are related as given in (5.98). Outside the cloak the physical body is homogeneous and isotropic and has a constitutive equation identical to that of the virtual body. One assumes that the two problems have identical boundary conditions on the outer boundaries of \mathcal{B} and $\tilde{\mathcal{B}}$, and the physical hole and the virtual hole are both traction free. Under the above assumptions the two boundary-value problems are equivalent. In other words, the governing equations of the physical problem are satisfied if and only if those of the virtual problem are satisfied. In addition, the two bodies have identical current configurations outside the cloak \mathcal{C} .

The impossibility of transformation cloaking in generalized Cosserat elasticity. Next we prove that in dimension two transformation cloaking is not possible in linear generalized Cosserat elasticity. A corollary of this result is that transformation cloaking cannot be possible in any subclass of generalized Cosserat solids, and in particular, transformation cloaking is not possible in Cosserat elasticity. We start our discussion by first looking at the example of a cylindrical hole covered by a cylindrical cloak. It has been claimed in the literature that a cylindrical cloak would have to be made of a Cosserat solid. We show that this is not possible. In other words, (generalized) Cosserat elasticity allowing for a non-symmetric Cauchy stress does not imply that transformation cloaking can be achieved in (generalized) Cosserat elasticity.

Example (A generalized Cosserat cylindrical cloak). Consider an infinitely-long hollow solid cylinder that in its stress-free reference configuration has inner and outer radii R_i and R_o , respectively (see Fig. 7). Let us transform the reference configuration to the reference configuration of another body (virtual body) that is a hollow cylinder with inner and outer radii ϵ and R_o , respectively, using a cloaking map $\mathcal{E}(R, \Theta, Z) = (f(R), \Theta, Z)$ such that $f(R_o) = R_o$. For such a map we have

$$\overset{\mathcal{E}}{\mathbf{F}} = \begin{bmatrix} f'(R) & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.124}$$

Note that the shifter map is given as

$$\mathbf{s} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{R}{f(R)} \end{bmatrix}. \tag{5.125}$$

Following our discussion in Section 5.2, we note that \mathcal{E} must satisfy $T\mathcal{E}|_{\partial_o\mathcal{C}} = \overset{\mathcal{E}}{\mathbf{F}}|_{\partial_o\mathcal{C}} = id$, i.e., $f'(R_o) = 1$. Therefore, the simplest form of $f(R)$ is a quadratic

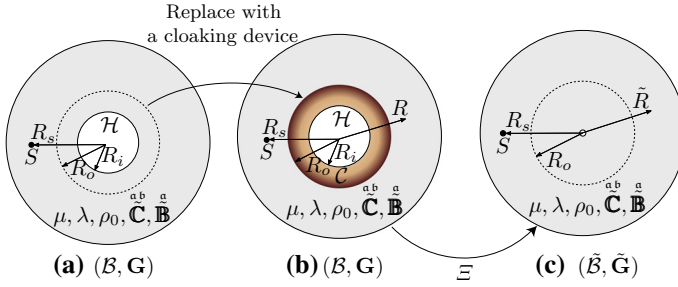


Fig. 7. Cloaking an object inside a hole \mathcal{H} from elastic waves generated by a source S located at a distance R_s from the center of the hole. The system **a** is an isotropic, homogeneous, generalized Cosserat medium with elastic properties $\mu, \lambda, \rho_0, \overset{ab}{\mathbf{B}}, \overset{ab}{\mathbf{C}}$, and $\overset{ab}{\mathbf{v}}$ containing a finite hole \mathcal{H} . The system **c** is an isotropic and homogeneous generalized Cosserat solid with the same elastic properties as the medium in the system **a**. The cloak \mathcal{C} in **b** has the elastic properties $\overset{ab}{\mathbf{A}}, \overset{ab}{\mathbf{B}}, \overset{ab}{\mathbf{C}}$, and $\overset{ab}{\mathbf{v}}$. The configuration **b** is mapped to the new reference configuration **c** such that the hole \mathcal{H} is mapped to an infinitesimal hole. The cloaking transformation is the identity mapping in $\mathcal{B} \setminus \mathcal{C}$

polynomial given by

$$f(R) = -\frac{R_o^2(R_i - \epsilon)}{(R_o - R_i)^2} + \frac{R_o^2 + R_i^2 - 2R_o\epsilon}{(R_o - R_i)^2}R - \frac{R_i - \epsilon}{(R_o - R_i)^2}R^2. \quad (5.126)$$

Note that $\rho = J_{\mathcal{E}} \tilde{\rho} \circ \mathcal{E}$ and $\overset{ab}{\mathbf{v}} = J_{\mathcal{E}} \overset{ab}{\tilde{\mathbf{v}}} \circ \mathcal{E}$, $\mathbf{a}, \mathbf{b} = 1, 2$, and therefore, the mass density and the micro-mass moment of inertia in $\mathcal{B} \setminus \mathcal{C}$ are homogeneous and are equal to ρ_0 and $\overset{ab}{\tilde{\mathbf{v}}}$, respectively. The mass density in the cloaking device is inhomogeneous and is given by

$$\rho_{\mathcal{C}}(R) = \frac{f(R)f'(R)}{R} \rho_0, \quad R_i \leq R \leq R_o. \quad (5.127)$$

Hence, (5.127) implies that for $R_i \leq R \leq R_o$,

$$\rho_{\mathcal{C}}(R) = \frac{[(R_o^2 + R_i^2 - 2R_o\epsilon)R - (R_o^2 + R_i^2)(R_i - \epsilon)]}{[R_o^2 + R_i^2 - 2R_o\epsilon - 2R(R_i - \epsilon)]^{-1} R(R_o - R_i)^4} \rho_0. \quad (5.128)$$

Similarly, for $R_i \leq R \leq R_o$ ($\mathbf{a}, \mathbf{b} \in \{1, 2\}$),

$$\overset{ab}{\mathbf{v}}_{\mathcal{C}}(R) = \frac{[(R_o^2 + R_i^2 - 2R_o\epsilon)R - (R_o^2 + R_i^2)(R_i - \epsilon)]}{[R_o^2 + R_i^2 - 2R_o\epsilon - 2R(R_i - \epsilon)]^{-1} R(R_o - R_i)^4} \overset{ab}{\tilde{\mathbf{v}}}. \quad (5.129)$$

From (5.93), we find the Cosserat elastic constants of the cloak as follows $(\mathbf{a}, \mathbf{b} \in \{1, 2\})$:³⁸

$$\hat{\mathbf{A}} = [\hat{\mathbf{A}}^{aAbB}] = \begin{bmatrix} \left[\begin{array}{cc} \frac{(\lambda+2\mu)f(R)}{Rf'(R)} & 0 \\ 0 & \lambda \end{array} \right] & \left[\begin{array}{c} 0 \\ \mu \end{array} \right] \\ \left[\begin{array}{c} 0 \\ \frac{\mu f(R)}{Rf'(R)} \end{array} \right] & \left[\begin{array}{cc} \lambda & 0 \\ 0 & \frac{(\lambda+2\mu)Rf'(R)}{f(R)} \end{array} \right] \end{bmatrix}, \quad (5.130)$$

$$\overset{a}{\mathbf{B}} = [\overset{a}{\mathbf{B}}^{aAbB}] = \begin{bmatrix} \left[\begin{array}{cc} \frac{(\overset{a}{b}_1+2\overset{a}{b}_2)f(R)}{Rf'(R)} & 0 \\ 0 & \overset{a}{b}_1 \end{array} \right] & \left[\begin{array}{c} 0 \\ \overset{a}{b}_2 \end{array} \right] \\ \left[\begin{array}{c} 0 \\ \frac{\overset{a}{b}_2 f(R)}{Rf'(R)} \end{array} \right] & \left[\begin{array}{cc} \overset{a}{b}_1 & 0 \\ 0 & \frac{(\overset{a}{b}_1+2\overset{a}{b}_2)Rf'(R)}{f(R)} \end{array} \right] \end{bmatrix}, \quad (5.131)$$

$$\overset{ab}{\mathbf{C}} = [\overset{ab}{\mathbf{C}}^{aAbB}] = \begin{bmatrix} \left[\begin{array}{cc} \frac{(\overset{ab}{c}_1+\overset{ab}{c}_2+\overset{ab}{c}_3)f(R)}{Rf'(R)} & 0 \\ 0 & \overset{ab}{c}_1 \end{array} \right] & \left[\begin{array}{c} 0 \\ \overset{ab}{c}_2 \end{array} \right] \\ \left[\begin{array}{c} 0 \\ \frac{\overset{ab}{c}_2 f(R)}{Rf'(R)} \end{array} \right] & \left[\begin{array}{cc} \overset{ab}{c}_2 & 0 \\ \overset{ab}{c}_1 & 0 \end{array} \right] \\ \left[\begin{array}{c} 0 \\ \frac{\overset{ab}{c}_3 f(R)}{Rf'(R)} \end{array} \right] & \left[\begin{array}{c} 0 \\ \frac{(\overset{ab}{c}_1+\overset{ab}{c}_2+\overset{ab}{c}_3)Rf'(R)}{f(R)} \end{array} \right] \end{bmatrix}. \quad (5.132)$$

Note that the first two indices specify the submatrix and the last two identify the components of that submatrix. We next consider a cylindrical cloak and find the distribution of those (initial) director gradient fields that are compatible with the balance of angular momentum. (5.101)₁ gives the following equations for $\overset{c}{\mathbf{F}}_A = \overset{c}{\mathbf{F}}_A(R, \Theta)$:

$$Rf(R) \left[(\overset{1}{b}_1 + 2\overset{1}{b}_2) \overset{c}{\mathbf{F}}_1^\theta R + (\overset{2}{b}_1 + 2\overset{2}{b}_2) \overset{c}{\mathbf{F}}_2^\theta R \right] - \left[\overset{1}{b}_1 \overset{c}{\mathbf{F}}_1^r \Theta + \overset{2}{b}_1 \overset{c}{\mathbf{F}}_2^r \Theta \right] f'(R) = 0, \quad (5.133)$$

$$Rf(R) \left[\overset{1}{b}_1 \overset{c}{\mathbf{F}}_1^\theta R + \overset{2}{b}_1 \overset{c}{\mathbf{F}}_2^\theta R \right] - \left[(\overset{1}{b}_1 + 2\overset{1}{b}_2) \overset{c}{\mathbf{F}}_1^r \Theta + (\overset{2}{b}_1 + 2\overset{2}{b}_2) \overset{c}{\mathbf{F}}_2^r \Theta \right] f'(R) = 0, \quad (5.134)$$

$$Rf'(R) \left[\overset{1}{b}_2 \overset{c}{\mathbf{F}}_1^\theta \Theta + \overset{2}{b}_2 \overset{c}{\mathbf{F}}_2^\theta \Theta \right] - f(R) \left[\overset{1}{b}_2 \overset{c}{\mathbf{F}}_1^r R + \overset{2}{b}_2 \overset{c}{\mathbf{F}}_2^r R \right] = \mu[f(R) - Rf'(R)]. \quad (5.135)$$

(5.101)₂ gives the following equations

$$Rf(R) \left[(\overset{11}{c}_1 + \overset{12}{c}_2 + \overset{13}{c}_3) \overset{c}{\mathbf{F}}_1^\theta R + (\overset{21}{c}_1 + \overset{22}{c}_2 + \overset{23}{c}_3) \overset{c}{\mathbf{F}}_2^\theta R \right] - \left[\overset{11}{c}_1 \overset{c}{\mathbf{F}}_1^r \Theta + \overset{12}{c}_1 \overset{c}{\mathbf{F}}_2^r \Theta \right] f'(R) = 0, \quad (5.136)$$

$$Rf(R) \left[(\overset{12}{c}_1 + \overset{12}{c}_2 + \overset{13}{c}_3) \overset{c}{\mathbf{F}}_1^\theta R + (\overset{22}{c}_1 + \overset{22}{c}_2 + \overset{23}{c}_3) \overset{c}{\mathbf{F}}_2^\theta R \right] - \left[\overset{12}{c}_1 \overset{c}{\mathbf{F}}_1^r \Theta + \overset{22}{c}_1 \overset{c}{\mathbf{F}}_2^r \Theta \right] f'(R) = 0, \quad (5.137)$$

³⁸ Note that the physical components of the elasticity tensor $\hat{\mathbf{A}}^{aAbB}$ are related to the components of the elasticity tensor as $\hat{\mathbf{A}}^{aAbB} = \sqrt{g_{aa}}\sqrt{G_{AA}}\sqrt{g_{bb}}\sqrt{G_{BB}}\mathbb{A}^{aAbB}$ (no summation).

$$Rf(R) \left[\overset{11}{c}_1 \overset{\circ}{F}{}^{\theta}_R + \overset{12}{c}_1 \overset{\circ}{F}{}^{\theta}_R \right] - f'(R) \left[(\overset{11}{c}_1 + \overset{11}{c}_2 + \overset{11}{c}_3) \overset{\circ}{F}{}^r{}_{\theta} + (\overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3) \overset{\circ}{F}{}^r{}_{\theta} \right] = 0, \quad (5.138)$$

$$Rf(R) \left[\overset{12}{c}_1 \overset{\circ}{F}{}^{\theta}_R + \overset{22}{c}_1 \overset{\circ}{F}{}^{\theta}_R \right] - f'(R) \left[(\overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3) \overset{\circ}{F}{}^r{}_{\theta} + (\overset{22}{c}_1 + \overset{22}{c}_2 + \overset{22}{c}_3) \overset{\circ}{F}{}^r{}_{\theta} \right] = 0, \quad (5.139)$$

$$Rf'(R) \left[\overset{11}{c}_3 \overset{\circ}{F}{}^{\theta}_{\theta} + \overset{12}{c}_3 \overset{\circ}{F}{}^{\theta}_{\theta} \right] - f(R) \left[\overset{11}{c}_2 \overset{\circ}{F}{}^r{}_R + \overset{12}{c}_2 \overset{\circ}{F}{}^r{}_R \right] = \overset{1}{b}_2 [f(R) - Rf'(R)], \quad (5.140)$$

$$Rf'(R) \left[\overset{12}{c}_3 \overset{\circ}{F}{}^{\theta}_{\theta} + \overset{12}{c}_2 \overset{\circ}{F}{}^{\theta}_{\theta} \right] - f(R) \left[\overset{11}{c}_3 \overset{\circ}{F}{}^r{}_R + \overset{12}{c}_3 \overset{\circ}{F}{}^r{}_R \right] = \overset{1}{b}_2 [f(R) - Rf'(R)], \quad (5.141)$$

$$Rf'(R) \left[\overset{12}{c}_3 \overset{\circ}{F}{}^{\theta}_{\theta} + \overset{22}{c}_3 \overset{\circ}{F}{}^{\theta}_{\theta} \right] - f(R) \left[\overset{12}{c}_2 \overset{\circ}{F}{}^r{}_R + \overset{22}{c}_2 \overset{\circ}{F}{}^r{}_R \right] = \overset{2}{b}_2 [f(R) - Rf'(R)], \quad (5.142)$$

$$Rf'(R) \left[\overset{12}{c}_2 \overset{\circ}{F}{}^{\theta}_{\theta} + \overset{22}{c}_2 \overset{\circ}{F}{}^{\theta}_{\theta} \right] - f(R) \left[\overset{12}{c}_3 \overset{\circ}{F}{}^r{}_R + \overset{22}{c}_3 \overset{\circ}{F}{}^r{}_R \right] = \overset{2}{b}_2 [f(R) - Rf'(R)]. \quad (5.143)$$

Note that the algebraic equations governing the diagonal and off-diagonal director gradients are uncoupled. We have six equations (5.133)–(5.134), and (5.136)–(5.139) for the four off-diagonal terms and five equations (5.135), and (5.140)–(5.143) for the diagonal terms. We first show that the determinant of the coefficient matrix of the linear system (5.136)–(5.139) is non-zero, and hence, $\overset{\circ}{F}{}^r{}_{\theta} = \overset{\circ}{F}{}^{\theta}{}_{\theta} = 0$, and $\overset{\circ}{F}{}^{\theta}_R = \overset{\circ}{F}{}^r{}_R = 0$. Note that the determinant of the coefficient matrix reads

$$R^2 f^2(R) f'^2(R) \left[-(\overset{12}{c}_2 + \overset{12}{c}_3)^2 + (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) \right] \times \left[-(2\overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3)^2 + (2\overset{11}{c}_1 + \overset{11}{c}_2 + \overset{11}{c}_3)(2\overset{22}{c}_1 + \overset{22}{c}_2 + \overset{22}{c}_3) \right]. \quad (5.144)$$

It turns out that this determinant cannot vanish. This is a consequence of the positive-definiteness of \mathbb{C} . To see this let us assume that $U^a|_A = 0$, and $\mathbb{U}^1|_2 = \mathbb{U}^2|_1 = 0$. For this class of deformations $\delta W > 0$ implies positive-definiteness of the following matrix:

$$\begin{bmatrix} \overset{11}{c}_1 + \overset{11}{c}_2 + \overset{11}{c}_3 & \overset{11}{c}_1 & \overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3 & \overset{12}{c}_1 \\ \overset{11}{c}_1 & \overset{11}{c}_1 + \overset{11}{c}_2 + \overset{11}{c}_3 & \overset{12}{c}_1 & \overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3 \\ \overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3 & \overset{12}{c}_1 & \overset{22}{c}_1 + \overset{22}{c}_2 + \overset{22}{c}_3 & \overset{22}{c}_1 \\ \overset{12}{c}_1 & \overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3 & \overset{22}{c}_1 & \overset{22}{c}_1 + \overset{22}{c}_2 + \overset{22}{c}_3 \end{bmatrix}. \quad (5.145)$$

In particular, its determinant must be positive, i.e.,

$$\left[-(\overset{12}{c}_2 + \overset{12}{c}_3)^2 + (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) \right] \times \left[-(2\overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3)^2 + (2\overset{11}{c}_1 + \overset{11}{c}_2 + \overset{11}{c}_3)(2\overset{22}{c}_1 + \overset{22}{c}_2 + \overset{22}{c}_3) \right] > 0. \quad (5.146)$$

Therefore

$$\overset{\circ}{F}{}^r{}_{\theta} = \overset{\circ}{F}{}^{\theta}{}_{\theta} = 0, \quad \overset{\circ}{F}{}^{\theta}_R = \overset{\circ}{F}{}^r{}_R = 0. \quad (5.147)$$

Note that (5.133) and (5.134) are now trivially satisfied.

The determinant of the coefficient matrix of the linear system (5.140)–(5.143) reads

$$\begin{aligned}
 & -R^2 f^2(R) f'^2(R) \left[(\overset{12}{c}_2 - \overset{12}{c}_3)^2 - (\overset{11}{c}_2 - \overset{11}{c}_3)(\overset{22}{c}_2 - \overset{22}{c}_3) \right] \\
 & \times \left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) \right].
 \end{aligned} \tag{5.148}$$

Positive-definiteness of energy requires that this determinant be non-vanishing. To see this let us assume that $U^a|_A = 0$, and $\underline{\mu}_a^1|_1 = \underline{\mu}_a^2|_2 = 0$. For this class of deformations $\delta W > 0$ implies positive-definiteness of the following matrix.

$$\begin{bmatrix} \overset{11}{c}_2 & \overset{11}{c}_3 & \overset{12}{c}_2 & \overset{12}{c}_3 \\ \overset{11}{c}_3 & \overset{11}{c}_2 & \overset{12}{c}_3 & \overset{12}{c}_2 \\ \overset{12}{c}_2 & \overset{12}{c}_3 & \overset{22}{c}_2 & \overset{22}{c}_3 \\ \overset{12}{c}_3 & \overset{12}{c}_2 & \overset{22}{c}_3 & \overset{22}{c}_2 \end{bmatrix}. \tag{5.149}$$

In particular, its determinant must be positive, and hence

$$\left[(\overset{12}{c}_2 - \overset{12}{c}_3)^2 - (\overset{11}{c}_2 - \overset{11}{c}_3)(\overset{22}{c}_2 - \overset{22}{c}_3) \right] \left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) \right] > 0 \tag{5.150}$$

Therefore

$$\overset{\circ}{\mathbb{F}}_1^r R = - \frac{\left[\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{1}{b}_2(\overset{22}{c}_2 + \overset{22}{c}_3) \right] (f(R) - Rf'(R))}{f(R) \left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) \right]}, \tag{5.151}$$

$$\overset{\circ}{\mathbb{F}}_1^\theta \ominus = \frac{\left[\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{1}{b}_2(\overset{22}{c}_2 + \overset{22}{c}_3) \right] (f(R) - Rf'(R))}{Rf'(R) \left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) \right]}, \tag{5.152}$$

$$\overset{\circ}{\mathbb{F}}_2^r R = \frac{\left[\overset{2}{b}_2(\overset{11}{c}_2 + \overset{11}{c}_3) - \overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) \right] (f(R) - Rf'(R))}{f(R) \left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) \right]}, \tag{5.153}$$

$$\overset{\circ}{\mathbb{F}}_2^\theta \ominus = \frac{\left[\overset{2}{b}_2(\overset{11}{c}_2 + \overset{11}{c}_3) - \overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) \right] (Rf'(R) - f(R))}{Rf'(R) \left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) \right]}. \tag{5.154}$$

Note that

$$\overset{\circ}{\mathbb{F}}_1^r R = - \frac{Rf'(R)}{f(R)} \overset{\circ}{\mathbb{F}}_1^\theta \ominus, \quad \overset{\circ}{\mathbb{F}}_2^r R = - \frac{Rf'(R)}{f(R)} \overset{\circ}{\mathbb{F}}_2^\theta \ominus. \tag{5.155}$$

From (5.147), it is immediate to see that (5.133) and (5.134) are automatically satisfied, while (5.135) imposes the following constraint on the elastic constants of the virtual body:

$$\begin{aligned}
 & 2(\overset{1}{b}_2)^2(\overset{22}{c}_2 + \overset{22}{c}_3) - 4\overset{2}{b}_2\overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) + 2(\overset{2}{b}_2)^2(\overset{11}{c}_2 + \overset{11}{c}_3) \\
 & + \mu \left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) \right] = 0.
 \end{aligned} \tag{5.156}$$

Let us assume that $U^a|_A = \delta_1^a \delta_A^2$, and $\underline{\underline{\Upsilon}}_a^1|_1 = \underline{\underline{\Upsilon}}_a^2|_2 = 0$. For this class of deformations $\delta W > 0$ implies positive-definiteness of the following matrix.

$$\begin{bmatrix} \mu & \overset{1}{b}_2 & \overset{1}{b}_2 & \overset{2}{b}_2 & \overset{2}{b}_2 \\ \overset{1}{b}_2 & \overset{1}{c}_2 & \overset{1}{c}_3 & \overset{1}{c}_2 & \overset{1}{c}_3 \\ \overset{1}{b}_2 & \overset{1}{c}_3 & \overset{1}{c}_2 & \overset{1}{c}_3 & \overset{1}{c}_2 \\ \overset{2}{b}_2 & \overset{2}{c}_2 & \overset{2}{c}_3 & \overset{2}{c}_2 & \overset{2}{c}_3 \\ \overset{2}{b}_2 & \overset{2}{c}_3 & \overset{2}{c}_2 & \overset{2}{c}_3 & \overset{2}{c}_2 \end{bmatrix}. \quad (5.157)$$

Therefore, its determinant must be positive, and hence

$$\begin{aligned} & \left[(\overset{1}{c}_2 - \overset{1}{c}_3)^2 - (\overset{1}{c}_2 - \overset{1}{c}_3)(\overset{2}{c}_2 - \overset{2}{c}_3) \right] \left\{ \mu \left[(\overset{1}{c}_2 + \overset{1}{c}_3)^2 - (\overset{1}{c}_2 + \overset{1}{c}_3)(\overset{2}{c}_2 + \overset{2}{c}_3) \right] \right. \\ & \left. - 2 \left[2\overset{1}{b}_2\overset{2}{b}_2(\overset{1}{c}_2 + \overset{1}{c}_3) - (\overset{2}{b}_2)^2(\overset{1}{c}_2 + \overset{1}{c}_3) - (\overset{1}{b}_2)^2(\overset{2}{c}_2 + \overset{2}{c}_3) \right] \right\} > 0. \quad (5.158) \end{aligned}$$

In particular

$$\begin{aligned} & \mu \left[(\overset{1}{c}_2 + \overset{1}{c}_3)^2 - (\overset{1}{c}_2 + \overset{1}{c}_3)(\overset{2}{c}_2 + \overset{2}{c}_3) \right] \\ & - 2 \left[2\overset{1}{b}_2\overset{2}{b}_2(\overset{1}{c}_2 + \overset{1}{c}_3) - (\overset{2}{b}_2)^2(\overset{1}{c}_2 + \overset{1}{c}_3) - (\overset{1}{b}_2)^2(\overset{2}{c}_2 + \overset{2}{c}_3) \right] \neq 0. \quad (5.159) \end{aligned}$$

Therefore, (5.156) violates positive-definiteness of the energy, and hence cloaking is not possible.

Remark 15. Even if one accepts a positive-semidefinite energy, the director gradient field given by (5.151) - (5.154) is not compatible. In other words, the director gradient given by (5.151) - (5.154) does not correspond to a single-valued director field in the physical body, in general. Necessary compatibility equations for the director gradients are written as

$$\begin{aligned} \overset{\circ}{\mathbb{F}}_1^r|_{\Theta|R} &= \overset{\circ}{\mathbb{F}}_1^r|_{R|\Theta} = 0, & \overset{\circ}{\mathbb{F}}_1^\theta|_{\Theta|R} &= \overset{\circ}{\mathbb{F}}_1^\theta|_{R|\Theta} = 0, \\ \overset{\circ}{\mathbb{F}}_2^r|_{\Theta|R} &= \overset{\circ}{\mathbb{F}}_2^r|_{R|\Theta} = 0, & \overset{\circ}{\mathbb{F}}_2^\theta|_{\Theta|R} &= \overset{\circ}{\mathbb{F}}_2^\theta|_{R|\Theta} = 0. \end{aligned} \quad (5.160)$$

Note that $\overset{\circ}{\mathbb{F}}_a^a|_B = \partial \overset{\circ}{\mathbb{F}}_a^a|_A / \partial X^B - \Gamma^C_{AB} \overset{\circ}{\mathbb{F}}_a^a|_C + \gamma^a_{bc} \overset{\circ}{\mathbb{F}}_B^b \overset{\circ}{\mathbb{F}}_A^c$. Thus

$$\overset{\circ}{\mathbb{F}}_a^r|_{R|\Theta} = \overset{\circ}{\mathbb{F}}_a^r|_{R,\Theta} - R \overset{\circ}{\mathbb{F}}_a^\theta|_R - \frac{1}{R} \overset{\circ}{\mathbb{F}}_a^r|_\Theta, \quad (5.161)$$

$$\overset{\circ}{\mathbb{F}}_a^r|_{\Theta|R} = \overset{\circ}{\mathbb{F}}_a^r|_{\Theta,R} - \frac{1}{R} \overset{\circ}{\mathbb{F}}_a^r|_\Theta, \quad (5.162)$$

$$\overset{\circ}{\mathbb{F}}_a^\theta|_{\Theta|R} = \overset{\circ}{\mathbb{F}}_a^\theta|_{\Theta,R} + \frac{1}{R} \overset{\circ}{\mathbb{F}}_a^\theta|_\Theta - \frac{1}{R} \overset{\circ}{\mathbb{F}}_a^\theta|_\Theta = \overset{\circ}{\mathbb{F}}_a^\theta|_{\Theta,R}, \quad (5.163)$$

$$\overset{\circ}{\mathbb{F}}_a^\theta|_{R|\Theta} = \overset{\circ}{\mathbb{F}}_a^\theta|_{R,\Theta} + \frac{1}{R} \left[\overset{\circ}{\mathbb{F}}_a^r|_R - \overset{\circ}{\mathbb{F}}_a^\theta|_\Theta \right]. \quad (5.164)$$

Hence, $\overset{\circ}{\mathbb{F}}_a^r|_{R|\Theta} = \overset{\circ}{\mathbb{F}}_a^r|_{\Theta|R}$, and $\overset{\circ}{\mathbb{F}}_a^\theta|_{\Theta|R} = \overset{\circ}{\mathbb{F}}_a^\theta|_{R|\Theta}$ imply that

$$\begin{aligned} \overset{\circ}{\mathbb{F}}_a^r|_{R,\Theta} - \overset{\circ}{\mathbb{F}}_a^r|_{\Theta,R} &= R \overset{\circ}{\mathbb{F}}_a^\theta|_R, \\ \overset{\circ}{\mathbb{F}}_a^\theta|_{\Theta,R} - \overset{\circ}{\mathbb{F}}_a^\theta|_{R,\Theta} &= \frac{1}{R} \left[\overset{\circ}{\mathbb{F}}_a^r|_R - \overset{\circ}{\mathbb{F}}_a^\theta|_\Theta \right]. \end{aligned} \quad (5.165)$$

Now using (5.147), one obtains

$$\begin{aligned} \overset{\circ}{\mathbf{F}}_{\alpha}^r R, \vartheta &= 0, \\ (R \overset{\circ}{\mathbf{F}}_{\alpha}^{\vartheta})_{,R} &= \overset{\circ}{\mathbf{F}}_{\alpha}^r R. \end{aligned} \tag{5.166}$$

Note that in this example (5.166)₁ is trivially satisfied. From (5.155) and (5.166)₂ one obtains

$$\overset{\circ}{\mathbf{F}}_{\alpha}^{\vartheta} = \frac{C_{\alpha}}{Rf(R)}, \tag{5.167}$$

for constants C_{α} . If $C_{\alpha} \neq 0$, (5.167) gives the following ODEs for $f(R)$:

$$Rf(R)f'(R) + k_{\alpha}f'(R) - f^2(R) = 0, \tag{5.168}$$

where k_{α} are constant. Knowing that $f(R_0) = R_0$, and $f'(R_0) = 1$, one concludes that $k_{\alpha} = 0$, and thus, $f(R) = R$. If $C_{\alpha} = 0$, then either $f(R) = R$, or

$$\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{1}{b}_2(\overset{22}{c}_2 + \overset{22}{c}_3) = 0, \quad \overset{2}{b}_2(\overset{11}{c}_2 + \overset{11}{c}_3) - \overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) = 0. \tag{5.169}$$

Knowing that $\overset{11}{c}_2 + \overset{11}{c}_3 > 0$, and $\overset{22}{c}_2 + \overset{22}{c}_3 > 0$, the above relations are written as

$$\overset{1}{b}_2 = \frac{\overset{12}{c}_2 + \overset{12}{c}_3}{\overset{22}{c}_2 + \overset{22}{c}_3} \overset{2}{b}_2, \quad \overset{2}{b}_2 = \frac{\overset{12}{c}_2 + \overset{12}{c}_3}{\overset{11}{c}_2 + \overset{11}{c}_3} \overset{1}{b}_2. \tag{5.170}$$

Thus

$$\begin{aligned} \overset{1}{b}_2 \frac{(\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) - (\overset{12}{c}_2 + \overset{12}{c}_3)^2}{(\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)} &= 0, \\ \overset{2}{b}_2 \frac{(\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) - (\overset{12}{c}_2 + \overset{12}{c}_3)^2}{(\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)} &= 0. \end{aligned} \tag{5.171}$$

From (5.146), one concludes that $\overset{1}{b}_2 = \overset{2}{b}_2 = 0$. Substituting this into (5.135), one concludes that $f(R) = R$. Therefore, transformation cloaking is not possible in this example for any linear generalized Cosserat solid.

We next prove that the impossibility of transformation cloaking in dimension two for linear generalized Cosserat elasticity is independent of the shape of the hole (cavity). We write (5.101) in Cartesian coordinates for which the shifter is written as $\mathbf{s}^{\tilde{a}}_a = \delta^{\tilde{a}}_a$. Knowing that $\overset{\circ}{\mathbf{F}}^c_A = \delta^c_A$, (5.101) is simplified to read $\forall b, B, c, a \in \{1, 2, 3\}$

$$\begin{aligned} (\overset{\circ}{\mathbf{F}}^{-1})^c_{\tilde{A}} (\overset{\circ}{\mathbf{F}}^{-1})^B_{\tilde{B}} \tilde{\mathbb{A}}^a \tilde{\mathbb{A}}^b \tilde{B} + (\overset{\circ}{\mathbf{F}}^{-1})^A_{\tilde{A}} (\overset{\circ}{\mathbf{F}}^{-1})^B_{\tilde{B}} \overset{\circ}{\mathbf{F}}^c_{\tilde{a}} \overset{\circ}{\mathbb{B}}^a \tilde{\mathbb{A}}^b \tilde{B} &= 0, \\ (\overset{\circ}{\mathbf{F}}^{-1})^c_{\tilde{A}} (\overset{\circ}{\mathbf{F}}^{-1})^B_{\tilde{B}} \overset{\circ}{\mathbb{B}}^a \tilde{\mathbb{A}}^b \tilde{B} + (\overset{\circ}{\mathbf{F}}^{-1})^A_{\tilde{A}} (\overset{\circ}{\mathbf{F}}^{-1})^B_{\tilde{B}} \overset{\circ}{\mathbf{F}}^c_{\tilde{a}} \overset{\circ}{\mathbb{C}}^a \tilde{\mathbb{A}}^b \tilde{B} &= 0. \end{aligned} \tag{5.172}$$

First let us consider an arbitrary cloaking map and director gradient with the following components in 2D:

$$\overset{\circ}{\mathbf{F}}^{-1} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad \overset{\circ}{\mathbf{F}}_{\alpha} = \begin{bmatrix} \overset{\circ}{\mathbf{F}}_{\alpha}^{11} & \overset{\circ}{\mathbf{F}}_{\alpha}^{12} \\ \overset{\circ}{\mathbf{F}}_{\alpha}^{21} & \overset{\circ}{\mathbf{F}}_{\alpha}^{22} \end{bmatrix}. \tag{5.173}$$

Note that $\bar{\mathbf{F}}^{-1}$ is compatible, and hence, $F_{11,2} = F_{12,1}$, and $F_{22,1} = F_{21,2}$. Similarly, the compatibility equations for the director gradient are $\bar{\mathbf{F}}_{\alpha}^{11},_2 = \bar{\mathbf{F}}_{\alpha}^{12},_1$, and $\bar{\mathbf{F}}_{\alpha}^{22},_1 = \bar{\mathbf{F}}_{\alpha}^{21},_2$. Expanding (5.172)₁, for $i \neq j \in \{1, 2\}$ one obtains

$$\begin{aligned} & -F_{ii}F_{ij}\bar{\mathbf{F}}_1^{ii}(\dot{b}_1 + \dot{b}_2) + \bar{\mathbf{F}}_1^{jj} \left[F_{ii}F_{ji}(\dot{b}_1 + 2\dot{b}_2) + F_{ij}F_{jj}\dot{b}_2 \right] - F_{ii}F_{ij}\bar{\mathbf{F}}_2^{ii}(\dot{b}_1 + \dot{b}_2) \\ & + \bar{\mathbf{F}}_2^{jj} \left[F_{ii}F_{ji}(\dot{b}_1 + 2\dot{b}_2) + F_{ij}F_{jj}\dot{b}_2 \right] - \bar{\mathbf{F}}_1^{ij} (F_{ii}F_{jj}\dot{b}_1 + F_{ij}F_{ji}\dot{b}_2) \\ & + \bar{\mathbf{F}}_1^{ji} \left[F_{ii}^2(\dot{b}_1 + 2\dot{b}_2) + F_{ij}^2\dot{b}_2 \right] - \bar{\mathbf{F}}_2^{ij} (F_{ii}F_{jj}\dot{b}_1 + F_{ij}F_{ji}\dot{b}_2) \\ & + \bar{\mathbf{F}}_2^{ji} \left[F_{ii}^2(\dot{b}_1 + 2\dot{b}_2) + F_{ij}^2\dot{b}_2 \right] = F_{ii}(F_{ij} - F_{ji})\lambda - (F_{ij}F_{jj} - F_{ii}F_{ij} + 2F_{ii}F_{ji})\mu, \end{aligned} \quad (5.174)$$

$$\begin{aligned} & -\bar{\mathbf{F}}_1^{ii} (F_{ii}F_{jj}\dot{b}_2 + F_{ij}F_{ji}\dot{b}_1) + \bar{\mathbf{F}}_1^{jj} \left[F_{ji}^2(\dot{b}_1 + 2\dot{b}_2) + F_{jj}^2\dot{b}_2 \right] \\ & - \bar{\mathbf{F}}_2^{ii} (F_{ii}F_{jj}\dot{b}_2 + F_{ij}F_{ji}\dot{b}_1) \\ & + \bar{\mathbf{F}}_2^{jj} \left[F_{ji}^2(\dot{b}_1 + 2\dot{b}_2) + F_{jj}^2\dot{b}_2 \right] - F_{ji}F_{jj}\bar{\mathbf{F}}_1^{ij}(\dot{b}_1 + \dot{b}_2) \\ & + \bar{\mathbf{F}}_1^{ji} \left[F_{ii}F_{ji}(\dot{b}_1 + 2\dot{b}_2) + F_{ij}F_{jj}\dot{b}_2 \right] \\ & - F_{ji}F_{jj}\bar{\mathbf{F}}_2^{ij}(\dot{b}_1 + \dot{b}_2) + \bar{\mathbf{F}}_2^{ji} \left[F_{ii}F_{ji}(\dot{b}_1 + 2\dot{b}_2) + F_{ij}F_{jj}\dot{b}_2 \right] \\ & = F_{ji}(F_{ij} - F_{ji})\lambda - (2F_{ji}^2 + F_{jj}^2 - F_{ii}F_{jj})\mu. \end{aligned} \quad (5.175)$$

On the other hand, expanding (5.172)₂ one obtains

$$\begin{aligned} & -F_{11}F_{12}\bar{\mathbf{F}}_1^{11}(\dot{c}_1 + \dot{c}_3) + \bar{\mathbf{F}}_1^{22} \left[F_{11}F_{21}\dot{c}_\Sigma + F_{12}F_{22}\dot{c}_2 \right] - F_{11}F_{12}\bar{\mathbf{F}}_2^{11}(\dot{c}_1 + \dot{c}_3) \\ & + \bar{\mathbf{F}}_2^{22} \left[F_{11}F_{21}\dot{c}_\Sigma + F_{12}F_{22}\dot{c}_2 \right] - \bar{\mathbf{F}}_1^{12} (F_{11}F_{22}\dot{c}_1 + F_{12}F_{21}\dot{c}_3) + \bar{\mathbf{F}}_1^{21} \left[F_{11}^2\dot{c}_\Sigma + F_{12}^2\dot{c}_2 \right] \\ & - \bar{\mathbf{F}}_2^{12} (F_{11}F_{22}\dot{c}_1 + F_{12}F_{21}\dot{c}_3) + \bar{\mathbf{F}}_2^{21} \left[F_{11}^2\dot{c}_\Sigma + F_{12}^2\dot{c}_2 \right] \\ & = F_{11}(F_{12} - F_{21})\dot{b}_1 + (F_{11}F_{12} - 2F_{11}F_{21} - F_{12}F_{22})\dot{b}_2, \end{aligned} \quad (5.176)$$

$$\begin{aligned} & -\bar{\mathbf{F}}_1^{11} \left[F_{11}^2\dot{c}_2 + F_{12}^2\dot{c}_\Sigma \right] + \bar{\mathbf{F}}_1^{22} (F_{11}F_{22}\dot{c}_3 + F_{12}F_{21}\dot{c}_1) - \bar{\mathbf{F}}_2^{11} \left[F_{11}^2\dot{c}_2 + F_{12}^2\dot{c}_\Sigma \right] \\ & + \bar{\mathbf{F}}_2^{22} (F_{11}F_{22}\dot{c}_3 + F_{12}F_{21}\dot{c}_1) - \bar{\mathbf{F}}_1^{12} [F_{11}F_{21}\dot{c}_2 + F_{12}F_{22}\dot{c}_\Sigma] + F_{11}F_{12}\bar{\mathbf{F}}_1^{21}(\dot{c}_1 + \dot{c}_3) \\ & - \bar{\mathbf{F}}_2^{12} [F_{11}F_{21}\dot{c}_2 + F_{12}F_{22}\dot{c}_\Sigma] + F_{11}F_{12}\bar{\mathbf{F}}_2^{21}(\dot{c}_1 + \dot{c}_3) \\ & = F_{12}(F_{12} - F_{21})\dot{b}_1 + (F_{11}^2 + 2F_{12}^2 - F_{11}F_{22})\dot{b}_2, \end{aligned} \quad (5.177)$$

$$\begin{aligned} & -\bar{\mathbf{F}}_1^{11} [F_{11}F_{22}\dot{c}_3 + F_{12}F_{21}\dot{c}_1] + \bar{\mathbf{F}}_1^{22} \left[F_{21}^2\dot{c}_\Sigma + F_{22}^2\dot{c}_2 \right] - \bar{\mathbf{F}}_2^{11} [F_{11}F_{22}\dot{c}_3 + F_{12}F_{21}\dot{c}_1] \\ & + \bar{\mathbf{F}}_2^{22} \left(F_{21}^2\dot{c}_\Sigma + F_{22}^2\dot{c}_2 \right) - F_{21}F_{22}\bar{\mathbf{F}}_1^{12}(\dot{c}_1 + \dot{c}_3) + \bar{\mathbf{F}}_1^{21} [F_{11}F_{21}\dot{c}_\Sigma + F_{12}F_{22}\dot{c}_2] \\ & - F_{21}F_{22}\bar{\mathbf{F}}_2^{12}(\dot{c}_1 + \dot{c}_3) + \bar{\mathbf{F}}_2^{21} [F_{11}F_{21}\dot{c}_\Sigma + F_{12}F_{22}\dot{c}_2] \\ & = F_{21}(F_{12} - F_{21})\dot{b}_1 + (F_{11}F_{22} - 2F_{21}^2 - F_{22}^2)\dot{b}_2, \end{aligned} \quad (5.178)$$

$$\begin{aligned} & -\bar{\mathbf{F}}_1^{11} [F_{11}F_{21}\dot{c}_2 + F_{12}F_{22}\dot{c}_\Sigma] + F_{21}F_{22}\bar{\mathbf{F}}_1^{22}(\dot{c}_1 + \dot{c}_3) - \bar{\mathbf{F}}_2^{11} [F_{11}F_{21}\dot{c}_2 + F_{12}F_{22}\dot{c}_\Sigma] \\ & + F_{21}F_{22}\bar{\mathbf{F}}_2^{22}(\dot{c}_1 + \dot{c}_3) - \bar{\mathbf{F}}_1^{12} \left(F_{21}^2\dot{c}_2 + F_{22}^2\dot{c}_\Sigma \right) + \bar{\mathbf{F}}_1^{21} (F_{11}F_{22}\dot{c}_1 + F_{12}F_{21}\dot{c}_3) \\ & - \bar{\mathbf{F}}_2^{12} \left(F_{21}^2\dot{c}_2 + F_{22}^2\dot{c}_\Sigma \right) + \bar{\mathbf{F}}_2^{21} (F_{11}F_{22}\dot{c}_1 + F_{12}F_{21}\dot{c}_3) \end{aligned}$$

$$= F_{22}(F_{12} - F_{21})\overset{1}{b}_1 + (F_{11}F_{21} - F_{21}F_{22} + 2F_{12}F_{22})\overset{1}{b}_2, \tag{5.179}$$

where $\overset{ab}{c}_\Sigma = \overset{ab}{c}_1 + \overset{ab}{c}_2 + \overset{ab}{c}_3$. Another independent set of four equations are generated from (5.176)–(5.179) using the transformation $\{\overset{1}{c} \rightarrow \overset{22}{c}, \overset{11}{c} \rightarrow \overset{12}{c}, \overset{1}{b} \rightarrow \overset{2}{b}\}$. The determinant of the coefficient matrix reads

$$\begin{aligned} & (F_{12}F_{21} - F_{11}F_{22})^8 \left[(\overset{22}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) \right]^2 \\ & \times \left[(\overset{12}{c}_2 - \overset{12}{c}_3)^2 - (\overset{11}{c}_2 - \overset{11}{c}_3)(\overset{22}{c}_2 - \overset{22}{c}_3) \right] \left[(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma) \right]. \end{aligned} \tag{5.180}$$

Note that from the positive-definiteness of energy, this determinant is non-zero, and hence the system (5.176)–(5.179) (along with four more equations) has a unique solution, which reads

$$\begin{aligned} \overset{1}{F}_1^{11} &= \frac{F_{22}(F_{11} - F_{22})\overset{1}{t}_3 - F_{12}F_{21}\overset{1}{t}_2 + F_{21}^2\overset{1}{t}_1}{F_{11}F_{22} - F_{12}F_{21}}, \\ \overset{1}{F}_1^{22} &= \frac{F_{11}(F_{22} - F_{11})\overset{1}{t}_3 - F_{12}F_{21}\overset{1}{t}_2 + F_{12}^2\overset{1}{t}_1}{F_{11}F_{22} - F_{12}F_{21}}, \\ \overset{1}{F}_1^{12} &= \frac{F_{12}F_{22}\overset{1}{t}_3 - F_{11}F_{21}\overset{1}{t}_1 + F_{11}F_{12}\overset{1}{t}_4}{F_{11}F_{22} - F_{12}F_{21}}, \\ \overset{1}{F}_1^{21} &= \frac{F_{21}F_{11}\overset{1}{t}_3 - F_{12}F_{22}\overset{1}{t}_1 + F_{21}F_{22}\overset{1}{t}_4}{F_{11}F_{22} - F_{12}F_{21}}, \\ \overset{2}{F}_2^{11} &= \frac{F_{22}(F_{11} - F_{22})\overset{2}{t}_3 - F_{12}F_{21}\overset{2}{t}_2 + F_{21}^2\overset{2}{t}_1}{F_{11}F_{22} - F_{12}F_{21}}, \\ \overset{2}{F}_2^{22} &= \frac{F_{11}(F_{22} - F_{11})\overset{2}{t}_3 - F_{12}F_{21}\overset{2}{t}_2 + F_{12}^2\overset{2}{t}_1}{F_{11}F_{22} - F_{12}F_{21}}, \\ \overset{2}{F}_2^{12} &= \frac{F_{12}F_{22}\overset{2}{t}_3 - F_{11}F_{21}\overset{2}{t}_1 + F_{11}F_{12}\overset{2}{t}_4}{F_{11}F_{22} - F_{12}F_{21}}, \\ \overset{2}{F}_2^{21} &= \frac{F_{21}F_{11}\overset{2}{t}_3 - F_{12}F_{22}\overset{2}{t}_1 + F_{21}F_{22}\overset{2}{t}_4}{F_{11}F_{22} - F_{12}F_{21}}, \end{aligned} \tag{5.181}$$

where $\overset{a}{t}_i, i = 1, \dots, 4, a = 1, 2$ are constants given as

$$\overset{1}{t}_1 = \frac{(\overset{1}{b}_1 + \overset{1}{b}_2)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma) - (\overset{2}{b}_1 + \overset{2}{b}_2)(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)}{(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)} + \frac{\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{1}{b}_2(\overset{22}{c}_2 + \overset{22}{c}_3)}{(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)}, \tag{5.182}$$

$$\overset{2}{t}_1 = \frac{(\overset{2}{b}_1 + \overset{2}{b}_2)(\overset{11}{c}_1 + \overset{11}{c}_\Sigma) - (\overset{1}{b}_1 + \overset{1}{b}_2)(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)}{(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)} + \frac{\overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{2}{b}_2(\overset{11}{c}_2 + \overset{11}{c}_3)}{(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)}, \tag{5.183}$$

$$\begin{aligned} \dot{\mathbf{t}}_2 = & \frac{\dot{b}_1(\dot{c}_1^2 + \dot{c}_\Sigma^2) - \dot{b}_1(\dot{c}_1^2 + \dot{c}_\Sigma^2)}{(\dot{c}_1^2 + \dot{c}_\Sigma^2)^2 - (\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)} \\ & - 2\dot{b}_2 \frac{\dot{c}_\Sigma^2(\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_2^2 + \dot{c}_3^2) - (\dot{c}_1^2 \dot{c}_\Sigma^2 + \dot{c}_1^2 \dot{c}_\Sigma^2)(\dot{c}_2^2 + \dot{c}_3^2) - \dot{c}_\Sigma^2(\dot{c}_2^2 + \dot{c}_3^2)(\dot{c}_2^2 + \dot{c}_3^2)}{[(\dot{c}_2^2 + \dot{c}_3^2)^2 - (\dot{c}_2^2 + \dot{c}_3^2)(\dot{c}_2^2 + \dot{c}_3^2)][(\dot{c}_1^2 + \dot{c}_\Sigma^2)^2 - (\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)]} \\ & + 2\dot{b}_2 \frac{[2\dot{c}_1^2 \dot{c}_\Sigma^2 - \dot{c}_\Sigma^2(\dot{c}_1^2 + \dot{c}_\Sigma^2)](\dot{c}_2^2 + \dot{c}_3^2) + \dot{c}_\Sigma^2(\dot{c}_2^2 + \dot{c}_3^2)^2}{[(\dot{c}_2^2 + \dot{c}_3^2)^2 - (\dot{c}_2^2 + \dot{c}_3^2)(\dot{c}_2^2 + \dot{c}_3^2)][(\dot{c}_1^2 + \dot{c}_\Sigma^2)^2 - (\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)]}, \end{aligned} \quad (5.184)$$

$$\begin{aligned} \dot{\mathbf{t}}_2 = & \frac{\dot{b}_1(\dot{c}_1^2 + \dot{c}_\Sigma^2) - \dot{b}_1(\dot{c}_1^2 + \dot{c}_\Sigma^2)}{(\dot{c}_1^2 + \dot{c}_\Sigma^2)^2 - (\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)} \\ & + 2\dot{b}_2 \frac{[2\dot{c}_1^2 \dot{c}_\Sigma^2 - \dot{c}_\Sigma^2(\dot{c}_1^2 + \dot{c}_\Sigma^2)](\dot{c}_2^2 + \dot{c}_3^2) + \dot{c}_\Sigma^2(\dot{c}_2^2 + \dot{c}_3^2)^2}{[(\dot{c}_2^2 + \dot{c}_3^2)^2 - (\dot{c}_2^2 + \dot{c}_3^2)(\dot{c}_2^2 + \dot{c}_3^2)][(\dot{c}_1^2 + \dot{c}_\Sigma^2)^2 - (\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)]} \\ & - 2\dot{b}_2 \frac{\dot{c}_\Sigma^2(\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_2^2 + \dot{c}_3^2) - (\dot{c}_1^2 \dot{c}_\Sigma^2 + \dot{c}_1^2 \dot{c}_\Sigma^2)(\dot{c}_2^2 + \dot{c}_3^2) - \dot{c}_\Sigma^2(\dot{c}_2^2 + \dot{c}_3^2)(\dot{c}_2^2 + \dot{c}_3^2)}{[(\dot{c}_2^2 + \dot{c}_3^2)^2 - (\dot{c}_2^2 + \dot{c}_3^2)(\dot{c}_2^2 + \dot{c}_3^2)][(\dot{c}_1^2 + \dot{c}_\Sigma^2)^2 - (\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)]}, \end{aligned} \quad (5.185)$$

$$\dot{\mathbf{t}}_3 = \frac{\dot{b}_2(\dot{c}_2^2 + \dot{c}_3^2) - \dot{b}_2(\dot{c}_2^2 + \dot{c}_3^2)}{(\dot{c}_2^2 + \dot{c}_3^2)^2 - (\dot{c}_2^2 + \dot{c}_3^2)(\dot{c}_2^2 + \dot{c}_3^2)}, \quad \dot{\mathbf{t}}_3 = \frac{\dot{b}_2(\dot{c}_2^2 + \dot{c}_3^2) - \dot{b}_2(\dot{c}_2^2 + \dot{c}_3^2)}{(\dot{c}_2^2 + \dot{c}_3^2)^2 - (\dot{c}_2^2 + \dot{c}_3^2)(\dot{c}_2^2 + \dot{c}_3^2)}, \quad (5.186)$$

$$\dot{\mathbf{t}}_4 = \frac{(\dot{b}_1 + \dot{b}_2)(\dot{c}_1^2 + \dot{c}_\Sigma^2) - (\dot{b}_1 + \dot{b}_2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)}{(\dot{c}_1^2 + \dot{c}_\Sigma^2)^2 - (\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)}, \quad \dot{\mathbf{t}}_4 = \frac{(\dot{b}_1 + \dot{b}_2)(\dot{c}_1^2 + \dot{c}_\Sigma^2) - (\dot{b}_1 + \dot{b}_2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)}{(\dot{c}_1^2 + \dot{c}_\Sigma^2)^2 - (\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)}. \quad (5.187)$$

Substituting this solution into (5.174) and (5.175) one obtains the following two conditions to be satisfied by the elastic constants

$$\begin{aligned} \mu = & 2 \left[\frac{2\dot{b}_2 \dot{b}_2 (\dot{c}_2^2 + \dot{c}_3^2) - (\dot{b}_2)^2 (\dot{c}_2^2 + \dot{c}_3^2) - (\dot{b}_2)^2 (\dot{c}_2^2 + \dot{c}_3^2)}{(\dot{c}_2^2 + \dot{c}_3^2)^2 - (\dot{c}_2^2 + \dot{c}_3^2)(\dot{c}_2^2 + \dot{c}_3^2)} \right], \quad (5.188) \\ \lambda = & 2 \left[\frac{2(\dot{b}_1 + \dot{b}_2)(\dot{b}_1 + \dot{b}_2)(\dot{c}_1^2 + \dot{c}_\Sigma^2) - (\dot{b}_1 + \dot{b}_2)^2 (\dot{c}_1^2 + \dot{c}_\Sigma^2) - (\dot{b}_1 + \dot{b}_2)^2 (\dot{c}_1^2 + \dot{c}_\Sigma^2)}{(\dot{c}_1^2 + \dot{c}_\Sigma^2)^2 - (\dot{c}_1^2 + \dot{c}_\Sigma^2)(\dot{c}_1^2 + \dot{c}_\Sigma^2)} \right] \\ & + 2 \left[\frac{(\dot{b}_2)^2 (\dot{c}_2^2 + \dot{c}_3^2) + (\dot{b}_2)^2 (\dot{c}_2^2 + \dot{c}_3^2) - 2\dot{b}_2 \dot{b}_2 (\dot{c}_2^2 + \dot{c}_3^2)}{(\dot{c}_2^2 + \dot{c}_3^2)^2 - (\dot{c}_2^2 + \dot{c}_3^2)(\dot{c}_2^2 + \dot{c}_3^2)} \right]. \end{aligned} \quad (5.189)$$

Note that (5.188) is identical to what we obtained for the cylindrical cloak example in (5.156), i.e., (5.188) violates positive-definiteness of the energy, and hence, cloaking is not possible.

The following proposition summarizes the discussions and calculations of this section:

Proposition 6. *Elastodynamic transformation cloaking is not possible for linear generalized Cosserat elastic solids in dimension two.*

Corollary 7. *Elastodynamic transformation cloaking is not possible for linear Cosserat elastic solids in dimension two.*

Example (A generalized Cosserat spherical cloak). Let us next consider a finite spherical cavity \mathcal{H} with radius R_i embedded in an infinite isotropic homogeneous generalized Cosserat elastic medium with the elastic properties given in (5.106). Let (R, Θ, Φ) be the spherical coordinates, for which $R \geq 0$ with $R = 0$ corresponding to the center of the cavity, $0 \leq \Theta \leq \pi$, and $0 \leq \Phi \leq 2\pi$. Suppose that there is a wave source located at (R_p, Θ_p, Φ_p) . The cloak \mathcal{C} is a spherical shell with inner radius R_i and outer radius R_o surrounding the hole such that the source is located outside the cloaking region, i.e., $R_o < R_p$. Similar to the cylindrical cloak example, the reference configuration of the body \mathcal{B} is mapped to that of $\tilde{\mathcal{B}}$ (the virtual body) using a mapping $\mathcal{E} : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$. Note that \mathcal{E} is the identity outside the cloak, i.e., $R \geq R_o$ and is defined as $(\tilde{R}, \tilde{\Theta}, \tilde{\Phi}) = \mathcal{E}(R, \Theta, \Phi) = (f(R), \Theta, \Phi)$ for $R_i \leq R \leq R_o$ such that $f(R_i) = \epsilon$, $f(R_o) = R_o$, and $f'(R_o) = 1$.

The reference configurations of \mathcal{B} and $\tilde{\mathcal{B}}$ are endowed with the induced metrics from \mathbb{R}^3 , i.e., $\mathbf{G} = \text{diag}(1, R^2, R^2 \sin^2 \Theta)$, and $\tilde{\mathbf{G}} = \text{diag}(1, \tilde{R}^2, \tilde{R}^2 \sin^2 \tilde{\Theta})$, respectively. In the coordinates (r, θ, ϕ) , the ambient space metric reads $\mathbf{g} = \text{diag}(1, r^2, r^2 \sin^2 \theta)$. We assume that the virtual body $\tilde{\mathcal{B}}$ is isotropic and has the same elastic properties as the primary medium in the region $\mathcal{B} \setminus \mathcal{C}$. For the cloaking map $\tilde{\mathbf{F}} = \text{diag}(f'(R), 1, 1)$ in $R_i \leq R \leq R_o$. Note also that

$$\mathbf{s} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{R}{f(R)} & 0 \\ 0 & 0 & \frac{R}{f(R)} \end{bmatrix}. \tag{5.190}$$

Noting that $\rho = J_{\mathcal{E}} \tilde{\rho} \circ \mathcal{E} = J_{\mathcal{E}} \rho_0$, the mass density of the cloak is obtained as

$$\rho_{\mathcal{C}}(R) = \frac{f(R)^2 f'(R)}{R^2} \rho_0, \quad R_i \leq R \leq R_o. \tag{5.191}$$

Also

$${}^{\text{ab}}\nu_{\mathcal{C}}(R) = \frac{f(R)^2 f'(R)}{R^2} {}^{\text{ab}}\tilde{\nu}, \quad R_i \leq R \leq R_o. \tag{5.192}$$

Using (5.93) and (5.97), the elastic constants of the cloak are calculated and read $(\mathbf{a}, \mathbf{b} \in \{1, 2, 3\})$

$$\hat{\mathbf{A}} = \begin{bmatrix} \begin{bmatrix} \frac{(\lambda+2\mu)f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & \frac{\lambda f(R)}{R} & 0 \\ 0 & 0 & \frac{\lambda f(R)}{R} \end{bmatrix} \begin{bmatrix} 0 & \mu f'(R) & 0 \\ \frac{\mu f(R)}{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \mu f'(R) \\ 0 & 0 & 0 \\ \frac{\mu f(R)}{R} & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \frac{\mu f(R)}{R} & 0 \\ \frac{\mu f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\lambda f(R)}{R} & 0 & 0 \\ 0 & (\lambda + 2\mu) f'(R) & 0 \\ 0 & 0 & \lambda f'(R) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu f'(R) \\ 0 & \mu f'(R) & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \frac{\mu f(R)}{R} \\ 0 & 0 & 0 \\ \frac{\mu f(R)^2}{R^2 f'(R)} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu f'(R) \\ 0 & \mu f'(R) & 0 \end{bmatrix} \begin{bmatrix} \frac{\lambda f(R)}{R} & 0 & 0 \\ 0 & \lambda f'(R) & 0 \\ 0 & 0 & (\lambda + 2\mu) f'(R) \end{bmatrix} \end{bmatrix}, \tag{5.193}$$

$$\hat{\mathbb{B}}^a = \begin{bmatrix} \begin{bmatrix} \frac{a}{b_\Sigma} \frac{f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & \frac{a}{b_1} \frac{f(R)}{R} & 0 \\ 0 & 0 & \frac{a}{b_1} \frac{f(R)}{R} \end{bmatrix} & \begin{bmatrix} 0 & \frac{a}{b_2} f'(R) & 0 \\ \frac{a}{b_2} \frac{f(R)}{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \frac{a}{b_2} f'(R) \\ 0 & 0 & 0 \\ \frac{a}{b_2} \frac{f(R)}{R} & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \frac{a}{b_2} \frac{f(R)}{R} & 0 \\ \frac{a}{b_2} \frac{f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{a}{b_1} \frac{f(R)}{R} & 0 & 0 \\ 0 & \frac{a}{b_\Sigma} f'(R) & 0 \\ 0 & 0 & \frac{a}{b_1} f'(R) \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{a}{b_2} f'(R) \\ 0 & \frac{a}{b_2} f'(R) & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \frac{a}{b_2} \frac{f(R)}{R} \\ 0 & 0 & 0 \\ \frac{a}{b_2} \frac{f(R)^2}{R^2 f'(R)} & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{a}{b_2} f'(R) \\ 0 & \frac{a}{b_2} f'(R) & 0 \end{bmatrix} & \begin{bmatrix} \frac{a}{b_1} \frac{f(R)}{R} & 0 & 0 \\ 0 & \frac{a}{b_1} f'(R) & 0 \\ 0 & 0 & \frac{a}{b_\Sigma} f'(R) \end{bmatrix} \end{bmatrix}, \quad (5.194)$$

$$\hat{\mathbb{C}}^{ab} = \begin{bmatrix} \begin{bmatrix} \frac{ab}{c_\Sigma} \frac{f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & \frac{ab}{c_1} \frac{f(R)}{R} & 0 \\ 0 & 0 & \frac{ab}{c_1} \frac{f(R)}{R} \end{bmatrix} & \begin{bmatrix} 0 & \frac{ab}{c_2} f'(R) & 0 \\ \frac{ab}{c_2} \frac{f(R)}{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \frac{ab}{c_2} f'(R) \\ 0 & 0 & 0 \\ \frac{ab}{c_2} \frac{f(R)}{R} & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \frac{ab}{c_2} \frac{f(R)}{R} & 0 \\ \frac{ab}{c_2} \frac{f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{ab}{c_1} \frac{f(R)}{R} & 0 & 0 \\ 0 & \frac{ab}{c_\Sigma} f'(R) & 0 \\ 0 & 0 & \frac{ab}{c_1} f'(R) \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{ab}{c_2} f'(R) \\ 0 & \frac{ab}{c_2} f'(R) & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \frac{ab}{c_2} \frac{f(R)}{R} \\ 0 & 0 & 0 \\ \frac{ab}{c_2} \frac{f(R)^2}{R^2 f'(R)} & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{ab}{c_2} f'(R) \\ 0 & \frac{ab}{c_2} f'(R) & 0 \end{bmatrix} & \begin{bmatrix} \frac{ab}{c_1} \frac{f(R)}{R} & 0 & 0 \\ 0 & \frac{ab}{c_1} f'(R) & 0 \\ 0 & 0 & \frac{ab}{c_\Sigma} f'(R) \end{bmatrix} \end{bmatrix}, \quad (5.195)$$

where $\hat{c}_\Sigma^{ab} = \hat{c}_1^{ab} + \hat{c}_2^{ab} + \hat{c}_3^{ab}$, and $\hat{b}_\Sigma^a = \hat{b}_1^a + 2\hat{b}_2^a$.

Similar to the cylindrical cloak example, the balance of angular momentum (5.101) for the diagonal and off-diagonal components of the director gradients are uncoupled. We consider the equations for the diagonal components of the director gradients in (5.101)₂ and show that they force the cloaking map to be the identity. For the choice $(a, c) = (2, 3)$, (5.101)₂ is expanded and gives the following equations for the (circumferential and azimuthal) diagonal director gradient components:

$${}^{11}c_2^1 F_1^\phi + {}^{12}c_2^2 F_2^\phi + {}^{13}c_2^3 F_3^\phi - {}^{11}c_3^1 F_1^\theta - {}^{12}c_3^2 F_2^\theta - {}^{13}c_3^3 F_3^\theta = 0, \quad (5.196)$$

$${}^{11}c_3^1 F_1^\phi + {}^{12}c_3^2 F_2^\phi + {}^{13}c_3^3 F_3^\phi - {}^{11}c_2^1 F_1^\theta - {}^{12}c_2^2 F_2^\theta - {}^{13}c_2^3 F_3^\theta = 0, \quad (5.197)$$

$${}^{12}c_2^1 F_1^\phi + {}^{22}c_2^2 F_2^\phi + {}^{23}c_2^3 F_3^\phi - {}^{12}c_3^1 F_1^\theta - {}^{22}c_3^2 F_2^\theta - {}^{23}c_3^3 F_3^\theta = 0, \quad (5.198)$$

$${}^{12}c_3^1 F_1^\phi + {}^{22}c_3^2 F_2^\phi + {}^{23}c_3^3 F_3^\phi - {}^{12}c_2^1 F_1^\theta - {}^{22}c_2^2 F_2^\theta - {}^{23}c_2^3 F_3^\theta = 0, \quad (5.199)$$

$${}^{13}c_2^1 F_1^\phi + {}^{23}c_2^2 F_2^\phi + {}^{33}c_2^3 F_3^\phi - {}^{13}c_3^1 F_1^\theta - {}^{23}c_3^2 F_2^\theta - {}^{33}c_3^3 F_3^\theta = 0, \quad (5.200)$$

$${}^{13}c_3^1 F_1^\phi + {}^{23}c_3^2 F_2^\phi + {}^{33}c_3^3 F_3^\phi - {}^{13}c_2^1 F_1^\theta - {}^{23}c_2^2 F_2^\theta - {}^{33}c_2^3 F_3^\theta = 0. \quad (5.201)$$

Choosing a class of deformations for which $U^a|_A = 0$, and $\underline{\underline{U}}_a^1|_1 = \underline{\underline{U}}_a^2|_2 = \underline{\underline{U}}_a^3|_3 = 0$, the positive-definiteness of the energy function implies the positive definiteness

of the following matrix:

$$\begin{bmatrix} c_2^{11} & c_3^{11} & c_2^{12} & c_3^{12} & c_2^{13} & c_3^{13} \\ c_3^{11} & c_2^{11} & c_3^{12} & c_2^{12} & c_3^{13} & c_2^{13} \\ c_2^{12} & c_3^{12} & c_2^{22} & c_3^{22} & c_2^{23} & c_3^{23} \\ c_3^{12} & c_2^{12} & c_3^{22} & c_2^{22} & c_3^{23} & c_2^{23} \\ c_2^{13} & c_3^{13} & c_2^{23} & c_3^{23} & c_2^{33} & c_3^{33} \\ c_3^{13} & c_2^{13} & c_3^{23} & c_2^{23} & c_3^{33} & c_2^{33} \end{bmatrix}. \tag{5.202}$$

In particular, the determinant of (5.202) is non-vanishing, and thus, so is the determinant of the coefficient matrix of the system (5.196)–(5.201). Therefore, $F_1^\phi \phi = F_2^\phi \phi = F_3^\phi \phi = 0$, and $F_1^\theta \theta = F_2^\theta \theta = F_3^\theta \theta = 0$. Using this, (5.101)₂ for choices $(a, c) = (1, 2)$, and $(a, c) = (1, 3)$ gives the following equations for $F_1^r R$, $F_2^r R$, and $F_3^r R$:

$$f(R) \left(c_3^{11} F_1^r R + c_2^{12} F_2^r R + c_3^{13} F_3^r R \right) = b_2 \left(R f'(R) - f(R) \right), \tag{5.203}$$

$$f(R) \left(c_2^{11} F_1^r R + c_3^{12} F_2^r R + c_2^{13} F_3^r R \right) = b_2 \left(R f'(R) - f(R) \right), \tag{5.204}$$

$$f(R) \left(c_3^{12} F_1^r R + c_2^{22} F_2^r R + c_3^{23} F_3^r R \right) = b_2 \left(R f'(R) - f(R) \right), \tag{5.205}$$

$$f(R) \left(c_2^{12} F_1^r R + c_3^{22} F_2^r R + c_2^{23} F_3^r R \right) = b_2 \left(R f'(R) - f(R) \right), \tag{5.206}$$

$$f(R) \left(c_3^{13} F_1^r R + c_2^{23} F_2^r R + c_3^{33} F_3^r R \right) = b_2 \left(R f'(R) - f(R) \right), \tag{5.207}$$

$$f(R) \left(c_2^{13} F_1^r R + c_3^{23} F_2^r R + c_2^{33} F_3^r R \right) = b_2 \left(R f'(R) - f(R) \right). \tag{5.208}$$

The coefficient matrix of the system of homogeneous algebraic equations that is obtained from (5.203)–(5.204), (5.205)–(5.206), and (5.207)–(5.208) reads

$$\begin{bmatrix} c_3^{11} - c_2^{11} & c_3^{12} - c_2^{12} & c_3^{13} - c_2^{13} \\ c_2^{11} - c_3^{11} & c_2^{12} - c_3^{12} & c_2^{13} - c_3^{13} \\ c_3^{12} - c_2^{12} & c_3^{22} - c_2^{22} & c_3^{23} - c_2^{23} \\ c_2^{12} - c_3^{12} & c_2^{22} - c_3^{22} & c_2^{23} - c_3^{23} \\ c_3^{13} - c_2^{13} & c_3^{23} - c_2^{23} & c_3^{33} - c_2^{33} \\ c_2^{13} - c_3^{13} & c_2^{23} - c_3^{23} & c_2^{33} - c_3^{33} \end{bmatrix}. \tag{5.209}$$

It is straightforward to see that the determinant of (5.202) being non-zero implies that the determinant of (5.209) is non-zero as well. Thus, $F_1^r R = F_2^r R = F_3^r R = 0$, and from (5.203)–(5.208), one concludes that $f(R) = R$, which, in turn, means that cloaking is not possible. Therefore, we have proved the following result:

Proposition 8. *Elastodynamic transformation cloaking is not possible for a spherical cavity using a spherical cloak in linear generalized Cosserat elastic solids.*

Corollary 9. *Elastodynamic transformation cloaking is not possible for a spherical cavity using a spherical cloak in linear Cosserat elastic solids.*

We suspect that transformation cloaking in dimension three is not possible for a cavity of any shape. The idea of proof is similar to that of 2D. However, in this case there are 108 equations for 27 unknown director gradients F_n^a . We have not been able to solve this large system of equations but expect that they would violate positive-definiteness of the energy and also force the cloaking map to be the identity.

Conjecture 1. *Elastodynamic transformation cloaking is not possible for linear generalized Cosserat elastic solids in dimension three.*

Acknowledgements. This research was supported by ARO W911NF-16-1-0064 and ARO W911NF-18-1-0003 (Dr. David Stepp).

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix A Riemannian Geometry

To make the paper self-contained, in this appendix some basic concepts of Riemannian geometry are tersely reviewed. It should be emphasized that only those geometric concepts that have been used in the paper are discussed here.

For a smooth n -dimensional manifold \mathcal{B} , the tangent space of \mathcal{B} at a point $X \in \mathcal{B}$ is denoted by $T_X\mathcal{B}$. Assume that \mathcal{S} is another n -dimensional manifold and $\varphi : \mathcal{B} \rightarrow \mathcal{S}$ is a diffeomorphism (smooth and invertible map with a smooth inverse) between the two manifolds. A smooth vector field \mathbf{W} on \mathcal{B} assigns a vector $\mathbf{W}_X \in T_X\mathcal{B}$ for every $X \in \mathcal{B}$ such that the mapping $X \mapsto \mathbf{W}_X$ is smooth. If \mathbf{W} is a vector field on \mathcal{B} , then the push-forward of \mathbf{W} by φ is a vector field on $\varphi(\mathcal{B})$ defined as $\varphi_*\mathbf{W} = T\varphi \cdot \mathbf{W} \circ \varphi^{-1}$. Similarly, if \mathbf{w} is a vector field on $\varphi(\mathcal{B}) \subset \mathcal{S}$, the pull-back of \mathbf{w} by φ is a vector field on \mathcal{B} defined as $\varphi^*\mathbf{w} = T(\varphi^{-1}) \cdot \mathbf{w} \circ \varphi$. Let us denote the tangent map of φ by \mathbf{F} , i.e., $\mathbf{F} = T\varphi$. Let $\{X^A\}$ and $\{x^a\}$ be the local charts for \mathcal{B} and \mathcal{S} , respectively. More specifically, a local chart for \mathcal{B} at $X \in \mathcal{B}$ is a homeomorphism from an open subset $\mathcal{U} \subset \mathcal{B}$ ($X \in \mathcal{U}$) to an open subset $\mathcal{V} \subset \mathbb{R}^n$. $\{X^A\}$ are components of this map. The derivative map \mathbf{F} is a two-point tensor with the following representation in the local charts: $\mathbf{F} = F^a{}_A \frac{\partial}{\partial X^A} \otimes dx^a$, $F^a{}_A = \frac{\partial \varphi^a}{\partial X^A}$, where $\{\frac{\partial}{\partial X^A}\}$ and $\{dx^a\}$ are bases for $T_X\mathcal{B}$ and $T_{\varphi(X)}^*\varphi(\mathcal{B})$, respectively. Recall that $T_{\varphi(X)}^*\varphi(\mathcal{B})$ denotes the cotangent space (or the dual space) of $T_{\varphi(X)}\varphi(\mathcal{B})$. The push-forward and pull-back of vectors have the following coordinate representations: $(\varphi_*\mathbf{W})^a = F^a{}_A W^A$, and $(\varphi^*\mathbf{w})^A = (F^{-1})^A{}_a w^a$.

A type $\binom{0}{2}$ -tensor at $X \in \mathcal{B}$ is a bilinear map $\mathbf{T} : T_X\mathcal{B} \times T_X\mathcal{B} \rightarrow \mathbb{R}$, where in a local coordinate chart $\{X^A\}$ for \mathcal{B} it reads $\mathbf{T}(\mathbf{U}, \mathbf{V}) = T_{AB}U^AV^B$, $\forall \mathbf{U}, \mathbf{V} \in T_X\mathcal{B}$. A Riemannian manifold $(\mathcal{B}, \mathbf{G})$ is a smooth manifold \mathcal{B} endowed with an inner product \mathbf{G}_X (a symmetric $\binom{0}{2}$ -tensor field) on the tangent space $T_X\mathcal{B}$ that smoothly varies in the sense that if \mathbf{U} and \mathbf{V} are smooth vector fields on \mathcal{B} , then $X \mapsto \mathbf{G}_X(\mathbf{U}_X, \mathbf{V}_X) =: \langle\langle \mathbf{U}_X, \mathbf{V}_X \rangle\rangle_{\mathbf{G}_X}$, is a smooth function. Let $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ be Riemannian manifolds and let $\varphi : \mathcal{B} \rightarrow \mathcal{S}$ be a diffeomorphism (smooth map with smooth inverse). The push-forward of the metric \mathbf{G} is a metric on $\varphi(\mathcal{B}) \subset \mathcal{S}$, which is denoted by $\varphi_*\mathbf{G}$ defined as

$$(\varphi_*\mathbf{G})_{\varphi(X)}(\mathbf{u}_{\varphi(X)}, \mathbf{v}_{\varphi(X)}) := \mathbf{G}_X((\varphi^*\mathbf{u})_X, (\varphi^*\mathbf{v})_X). \quad (\text{A.1})$$

In components, $(\varphi_*\mathbf{G})_{ab} = (F^{-1})^A{}_a(F^{-1})^B{}_b G_{AB}$. Similarly, the pull-back of the metric \mathbf{g} is a metric in \mathcal{B} , which is denoted by $\varphi^*\mathbf{g}$ defined as

$$(\varphi^*\mathbf{g})_X(\mathbf{U}_X, \mathbf{V}_X) := \mathbf{g}_{\varphi(X)}((\varphi_*\mathbf{U})_{\varphi(X)}, (\varphi_*\mathbf{V})_{\varphi(X)}). \quad (\text{A.2})$$

In components, $(\varphi^*\mathbf{g})_{AB} = F^a{}_A F^b{}_B g_{ab}$. The diffeomorphism φ is an isometry between two Riemannian manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ if $\mathbf{g} = \varphi_*\mathbf{G}$, or equivalently, $\mathbf{G} = \varphi^*\mathbf{g}$. An isometry, by definition, preserves distances.

Affine connections, and their torsion and curvature tensors. A linear (affine) connection on a manifold \mathcal{B} is an operation $\nabla : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \rightarrow \mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ is the set of vector fields on \mathcal{B} , such that $\forall \mathbf{X}, \mathbf{Y}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{X}(\mathcal{B}), \forall f, f_1, f_2 \in C^\infty(\mathcal{B}), \forall a_1, a_2 \in \mathbb{R}$: i) $\nabla_{f_1\mathbf{X}_1+f_2\mathbf{X}_2}\mathbf{Y} = f_1\nabla_{\mathbf{X}_1}\mathbf{Y} + f_2\nabla_{\mathbf{X}_2}\mathbf{Y}$, ii) $\nabla_{\mathbf{X}}(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2) = a_1\nabla_{\mathbf{X}}(\mathbf{Y}_1) + a_2\nabla_{\mathbf{X}}(\mathbf{Y}_2)$, and iii) $\nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y}$. $\nabla_{\mathbf{X}}\mathbf{Y}$ is called the covariant derivative of \mathbf{Y} along \mathbf{X} . In a local coordinate chart $\{X^A\}$, $\nabla_{\partial_A}\partial_B = \Gamma^C{}_{AB}\partial_C$, where $\Gamma^C{}_{AB}$ are Christoffel symbols of the connection, and $\partial_A = \frac{\partial}{\partial x^A}$ are the natural bases for the tangent space corresponding to a coordinate chart $\{x^A\}$. A linear connection is said to be compatible with a metric \mathbf{G} on the manifold if

$$\nabla_{\mathbf{X}}\langle\langle \mathbf{Y}, \mathbf{Z} \rangle\rangle_{\mathbf{G}} = \langle\langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle\rangle_{\mathbf{G}} + \langle\langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z} \rangle\rangle_{\mathbf{G}}, \tag{A.3}$$

where $\langle\langle \cdot, \cdot \rangle\rangle_{\mathbf{G}}$ is the inner product induced by the metric \mathbf{G} . It can be shown that ∇ is compatible with \mathbf{G} if and only if $\nabla\mathbf{G} = \mathbf{0}$, or in components

$$G_{AB|C} = \frac{\partial G_{AB}}{\partial X^C} - \Gamma^S{}_{CA}G_{SB} - \Gamma^S{}_{CB}G_{AS} = 0. \tag{A.4}$$

Suppose $\mathbf{V}, \mathbf{W} \in \mathcal{X}(\mathcal{B})$ are vector fields and $\alpha : I \rightarrow \mathcal{B}$ is a smooth curve. The restriction of the vector fields to α , i.e., $\mathbf{V} \circ \alpha$ and $\mathbf{W} \circ \alpha$ are called vector fields along the curve α . The set of all vector fields along α is denoted by $\mathcal{X}(\alpha)$. Covariant derivative along the curve α is a map $D_t : \mathcal{X}(\alpha) \rightarrow \mathcal{X}(\alpha)$ with the following properties: $D_t(\mathbf{V} + \mathbf{W}) = D_t\mathbf{V} + D_t\mathbf{W}$, and $D_t(f\mathbf{W}) = \frac{df}{dt}\mathbf{W} + fD_t\mathbf{W}$. If $\mathbf{W} \in \mathcal{X}(\alpha)$ is the restriction of $\tilde{\mathbf{W}} \in \mathcal{X}(\mathcal{B})$ to α , then, $D_t\mathbf{W} = \nabla_{\alpha'(t)}\tilde{\mathbf{W}}$. If the connection ∇ is \mathbf{G} -compatible, then

$$\frac{d}{dt}\langle\langle \mathbf{X}, \mathbf{Y}(X, t) \rangle\rangle_{\mathbf{G}} = \langle\langle D_t\mathbf{X}, \mathbf{Y} \rangle\rangle_{\mathbf{G}} + \langle\langle \mathbf{X}, D_t\mathbf{Y} \rangle\rangle_{\mathbf{G}}. \tag{A.5}$$

The covariant derivative of a two-point tensor \mathbf{T} is given by

$$\begin{aligned} T^{AB\dots F}{}_{G\dots Q}{}^{ab\dots f}{}_{g\dots q}|_K &= \frac{\partial}{\partial X^k} T^{AB\dots F}{}_{G\dots Q}{}^{ab\dots f}{}_{g\dots q} \\ &+ T^{RB\dots F}{}_{G\dots Q}{}^{ab\dots f}{}_{g\dots q} \Gamma^A{}_{RK} + (\text{all upper referential indices}) \\ &- T^{AB\dots F}{}_{R\dots Q}{}^{ab\dots f}{}_{g\dots q} \Gamma^R{}_{GK} - (\text{all lower referential indices}) \\ &+ T^{RB\dots F}{}_{G\dots Q}{}^{lb\dots f}{}_{g\dots q} \gamma^a{}_{lr} F^r{}_K + (\text{all upper spatial indices}) \\ &- T^{AB\dots F}{}_{G\dots Q}{}^{ab\dots f}{}_{l\dots q} \gamma^l{}_{gr} F^r{}_K - (\text{all lower spatial indices}). \end{aligned} \tag{A.6}$$

The torsion of a connection is defined as $T(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]$, where $[\mathbf{X}, \mathbf{Y}](F) = \mathbf{X}(\mathbf{Y}(F)) - \mathbf{Y}(\mathbf{X}(F)), \forall F \in C^\infty(\mathcal{S})$, is the commutator of \mathbf{X} and \mathbf{Y} . In components, in a local chart $\{X^A\}$, $T^A{}_{BC} = \Gamma^A{}_{BC} - \Gamma^A{}_{CB}$, and $[\mathbf{X}, \mathbf{Y}]^a = \frac{\partial Y^a}{\partial x^b} X^b - \frac{\partial X^a}{\partial x^b} Y^b$. ∇ is symmetric if it is torsion-free, i.e., $\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]$. On any Riemannian manifold $(\mathcal{B}, \mathbf{G})$ there is a unique linear connection $\nabla^{\mathbf{G}}$ that is compatible with \mathbf{G} and is torsion-free. This is the Levi-Civita connection. If

the Levi-Civita connection $\nabla^{\mathbf{G}}$ is used, the covariant time derivative is denoted by $D_t^{\mathbf{G}}$. In a manifold with a connection the curvature is a map $\mathcal{R} : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \rightarrow \mathcal{X}(\mathcal{B})$ defined by $\mathcal{R}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X}, \mathbf{Y}]}\mathbf{Z}$, or in components $\mathcal{R}^A{}_{BCD} = \frac{\partial \Gamma^A{}_{CD}}{\partial X^B} - \frac{\partial \Gamma^A{}_{BD}}{\partial X^C} + \Gamma^A{}_{BM}\Gamma^M{}_{CD} - \Gamma^A{}_{CM}\Gamma^M{}_{BD}$. The Riemannian curvature is the curvature tensor of the Levi-Civita connection $\nabla^{\mathbf{G}}$ and is denoted by $\mathcal{R}_{\mathbf{G}}$. The Ricci identity for a vector field \mathbf{U} with components W^A reads $U^A|_{BC} - U^A|_{CB} = \mathcal{R}^A{}_{BCD}U^D$. Ricci identity for a 1-form α with components α_A reads $\alpha_A|_{BC} - \alpha_A|_{CB} = \mathcal{R}^D{}_{BCA}\alpha_D$. The Ricci curvature \mathbf{Ric} is defined as $\mathbf{Ric}_{CD} = \mathcal{R}^A{}_{ACD}$, and is a symmetric tensor. The Ricci curvature of the Levi-Civita connection $\nabla^{\mathbf{G}}$ is denoted by $\mathbf{Ric}_{\mathbf{G}}$.

Vector bundles. Suppose \mathcal{E} and \mathcal{B} are sets and consider a map $\pi : \mathcal{E} \rightarrow \mathcal{B}$. The fiber over $X \in \mathcal{B}$ is the set $\mathcal{E}_X := \pi^{-1}(X) \subset \mathcal{E}$. For an onto map π fibers are non-empty and $\mathcal{E} = \sqcup_{X \in \mathcal{B}} \mathcal{E}_X$, where \sqcup denoted disjoint union of sets. Now suppose \mathcal{E} and \mathcal{B} are manifolds and assume that for any $X \in \mathcal{B}$, there exists a neighborhood $\mathcal{U} \subset \mathcal{B}$ of X , a manifold \mathcal{F} , and a diffeomorphism $\psi : \pi^{-1}(\mathcal{U}) \rightarrow \mathcal{U} \times \mathcal{F}$ such that $\pi = \text{pr}_1 \circ \psi$, where $\text{pr}_1 : \mathcal{U} \times \mathcal{F} \rightarrow \mathcal{U}$ is projection onto the first factor. $(\mathcal{E}, \pi, \mathcal{B})$ is called a fiber bundle and \mathcal{E}, π , and \mathcal{B} are called the total space, the projection, and the base space, respectively. If for any $X \in \mathcal{B}$, $\pi^{-1}(X)$ is a vector space, $(\mathcal{E}, \pi, \mathcal{B})$ is called a vector bundle. The set of sections of this bundle $\Gamma(\mathcal{E})$ is the set of all smooth maps $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ such that $\sigma(X) \in \mathcal{E}_X, \forall X \in \mathcal{B}$. An important example of a vector bundle is the tangent bundle of a manifold for which $\mathcal{E} = T\mathcal{B}$.

Induced bundle and connection. Consider a map between Riemannian manifolds $\varphi : \mathcal{B} \rightarrow \mathcal{S}$. The tangent bundles of \mathcal{B} and \mathcal{S} are denoted by $T\mathcal{B} = \sqcup_{X \in \mathcal{B}} T_X\mathcal{B}$ and $T\mathcal{S} = \sqcup_{X \in \mathcal{S}} T_X\mathcal{S}$, respectively. We define an induced vector bundle $\varphi^{-1}T\mathcal{S}$, which is a vector bundle over \mathcal{B} whose fiber over $X \in \mathcal{B}$ is $T_{\varphi(X)}\mathcal{S}$ [76]. The connection $\nabla^{\mathbf{g}}$ induces a unique connection ∇^{φ} on $\varphi^{-1}T\mathcal{S}$ defined as

$$\nabla_{\mathbf{W}}^{\varphi} \mathbf{w} \circ \varphi = \nabla_{\varphi_* \mathbf{W}}^{\mathbf{g}} \mathbf{w}, \quad \mathbf{W} \in T_X\mathcal{B}, \mathbf{w} \in \Gamma(T\mathcal{S}). \quad (\text{A.7})$$

∇^{φ} is called the induced connection. It can be shown that its connection coefficients with respect to the coordinate charts $\{X^A\}$ and $\{x^a\}$ of \mathcal{B} and \mathcal{S} , respectively, are $\frac{\partial \varphi^b}{\partial X^A} \gamma^a{}_{bc}$. In particular, the variation field $\delta\varphi$ defined in §3 is a section of $\Gamma(\varphi^{-1}T\mathcal{S})$, i.e., $\delta\varphi$ defines a vector field in \mathcal{S} along the map φ . For a two-point tensor, e.g., deformation gradient, covariant derivative involves both $\nabla^{\mathbf{g}}$ and $\nabla^{\mathbf{G}}$: $F^a{}_{A|B} = \frac{\partial F^a{}_{A|B}}{\partial X^B} + (F^b{}_{B|Y} \gamma^a{}_{bc}) F^c{}_{A|} - \Gamma^C{}_{AB} F^a{}_{C|} = \frac{\partial F^a{}_{A|B}}{\partial X^B} + \gamma^a{}_{bc} F^b{}_{B|} F^c{}_{A|} - \Gamma^C{}_{AB} F^a{}_{C|}$. We denote the covariant derivative of the deformation gradient by $\nabla \mathbf{F} = F^a{}_{A|B} dX^B \otimes dX^A \otimes \frac{\partial}{\partial x^a}$. It is straightforward to show that [76]

$$\nabla^{\varphi} \mathbf{F}(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}^{\varphi} \varphi_* \mathbf{Y} - \varphi_* \nabla_{\mathbf{X}}^{\mathbf{G}} \mathbf{Y}, \quad \nabla_{\mathbf{X}}^{\varphi} \varphi_* \mathbf{Y} - \nabla_{\mathbf{Y}}^{\varphi} \varphi_* \mathbf{X} = \varphi_* [\mathbf{X}, \mathbf{Y}]. \quad (\text{A.8})$$

The metrics \mathbf{G} and \mathbf{g} induce an inner product $\langle \cdot, \cdot \rangle_X$ in $T_{\varphi(X)}\mathcal{S} \otimes T_X^* \mathcal{B}$. This is defined first for the basis $\left\{ \frac{\partial}{\partial x^a} \otimes dX^A, 1 \leq a \leq n, 1 \leq A \leq n \right\}$ as $\left\langle \frac{\partial}{\partial x^a} \otimes dX^A, \frac{\partial}{\partial x^b} \otimes dX^B \right\rangle_X = g_{ab} G^{AB}$, and then one extends it linearly to arbitrary elements in $T_{\varphi(X)}\mathcal{S} \otimes T_X^* \mathcal{B}$. $\varphi^{-1}T\mathcal{S} \otimes T^* \mathcal{B}$ is the vector bundle whose fiber at $X \in \mathcal{B}$ is

$T_{\varphi(X)}\mathcal{S} \otimes T_X^*\mathcal{B}$. The two-point tensor $\mathbf{F} = T\varphi : \mathcal{B} \rightarrow \varphi^{-1}T\mathcal{S} \otimes T^*\mathcal{B}$, is a section of this bundle, i.e., $\mathbf{F} \in \Gamma(\varphi^{-1}T\mathcal{S} \otimes T^*\mathcal{B})$. One can define a fiber metric $\langle\langle, \rangle\rangle$ on $\varphi^{-1}T\mathcal{S} \otimes T^*\mathcal{B}$ using the inner product \langle, \rangle_X in $T_{\varphi(X)}\mathcal{S} \otimes T_X^*\mathcal{B}$ as follows: for $\sigma, \tau \in \Gamma(\varphi^{-1}T\mathcal{S} \otimes T^*\mathcal{B})$, define $\langle\langle\sigma, \tau\rangle\rangle(X) = \langle\sigma(X), \tau(X)\rangle_X$, $X \in \mathcal{B}$. One can define a connection ∇ in $\varphi^{-1}T\mathcal{S} \otimes T^*\mathcal{B}$ using the Levi-Civita connections $\nabla^{\mathbf{G}}$ and $\nabla^{\mathbf{g}}$: consider a section $\mathbf{W} \otimes \boldsymbol{\alpha} \in \Gamma(\varphi^{-1}T\mathcal{S} \otimes T^*\mathcal{B})$ and let $\nabla(\mathbf{W} \otimes \boldsymbol{\alpha}) = \nabla^\varphi\mathbf{W} \otimes \boldsymbol{\alpha} + \mathbf{W} \otimes \nabla^{\mathbf{G}}\boldsymbol{\alpha}$. This connection is compatible with the fiber metric $\langle\langle, \rangle\rangle$ in $\varphi^{-1}T\mathcal{S} \otimes T^*\mathcal{B}$.

The Piola transform. The Piola transform of a vector $\mathbf{w} \in T_{\varphi(X)}\mathcal{S}$ is a vector $\mathbf{W} \in T_X\mathcal{B}$ given by $\mathbf{W} = J\varphi^*\mathbf{w} = \mathbf{J}\mathbf{F}^{-1}\mathbf{w}$. In coordinates, $W^A = J(F^{-1})^A{}_b w^b$, where $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}$ is the Jacobian of φ with \mathbf{G} and \mathbf{g} the Riemannian metrics of \mathcal{B} and \mathcal{S} , respectively. It can be shown that $\text{Div } \mathbf{W} = J(\text{div } \mathbf{w}) \circ \varphi$. In coordinates, $W^A|_A = Jw^a|_a$; this is also known as the Piola identity. Another way of writing the Piola identity is in terms of the unit normal vectors of a surface in \mathcal{B} and its corresponding surface in \mathcal{S} and the area elements. It is written as $\hat{\mathbf{n}}da = \mathbf{J}\mathbf{F}^{-*}\hat{\mathbf{N}}dA$, or in components, $n_a da = J(F^{-1})^A{}_a N_A dA$. In the literature of continuum mechanics, this is called Nanson's formula.

Lie derivative. Let $\mathbf{w} : \mathcal{U} \rightarrow T\mathcal{S}$ be a vector field, where $\mathcal{U} \subset \mathcal{S}$ is open. A curve $\alpha : I \rightarrow \mathcal{S}$, where I is an open interval, is an integral curve of \mathbf{w} if $\frac{d\alpha(t)}{dt} = \mathbf{w}(\alpha(t))$, $\forall t \in I$. For a time-dependent vector field $\mathbf{w} : \mathcal{S} \times I \rightarrow T\mathcal{S}$, where I is some open interval, the collection of maps $\psi_{\tau,t}$ is the flow of \mathbf{w} if for each t and x , $\tau \mapsto \psi_{\tau,t}(x)$ is an integral curve of \mathbf{w}_t , i.e., $\frac{d}{d\tau}\psi_{\tau,t}(x) = \mathbf{w}(\psi_{\tau,t}(x), \tau)$, and $\psi_{t,t}(x) = x$. Let \mathbf{t} be a time-dependent tensor field on \mathcal{S} , i.e., $\mathbf{t}_t(x) = \mathbf{t}(x, t)$ is a tensor. The Lie derivative of \mathbf{t} with respect to \mathbf{w} is defined as $\mathbf{L}_\mathbf{w}\mathbf{t} = \frac{d}{d\tau}\psi_{\tau,t}^*\mathbf{t}_\tau|_{\tau=t}$. Note that $\psi_{\tau,t}$ maps \mathbf{t}_t to \mathbf{t}_τ . Hence, to calculate the Lie derivative one drags \mathbf{t} along the flow of \mathbf{w} from τ to t and then differentiates the Lie dragged tensor with respect to τ . The autonomous Lie derivative of \mathbf{t} with respect to \mathbf{w} is defined as $\mathfrak{L}_\mathbf{w}\mathbf{t} = \frac{d}{dt}\psi_{\tau,t}^*\mathbf{t}_t|_{\tau=t}$. Thus, $\mathbf{L}_\mathbf{w}\mathbf{t} = \partial\mathbf{t}/\partial t + \mathfrak{L}_\mathbf{w}\mathbf{t}$. For a scalar f , $\mathbf{L}_\mathbf{w}f = \partial f/\partial t + \mathbf{w}[f]$. In a coordinate chart $\{x^a\}$ this reads, $\mathbf{L}_\mathbf{w}f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^a}w^a$. For a vector \mathbf{u} , one can show that $\mathbf{L}_\mathbf{w}\mathbf{u} = \frac{\partial \mathbf{w}}{\partial t} + [\mathbf{w}, \mathbf{u}]$. If ∇ is a torsion-free connection, then $[\mathbf{w}, \mathbf{u}] = \nabla_\mathbf{w}\mathbf{u} - \nabla_\mathbf{u}\mathbf{w}$. Thus, $\mathbf{L}_\mathbf{w}\mathbf{u} = \frac{\partial \mathbf{w}}{\partial t} + \nabla_\mathbf{w}\mathbf{u} - \nabla_\mathbf{u}\mathbf{w}$.

When linearizing nonlinear elasticity one starts with a one-parameter family of motions $\varphi_{t,\epsilon} : \mathcal{B} \rightarrow \mathcal{S}$. By definition of the variation field $\mathbf{U}_t = \delta\varphi_t$, $\varphi_{t,\epsilon}$ is the flow of the variation field. Given a tensor field \mathbf{t} in \mathcal{S} , $\bar{\mathbf{T}}_\epsilon = \varphi_{t,\epsilon}^*\mathbf{t} \circ \varphi_{t,\epsilon}$ is a vector field on \mathcal{B} . Its linearization is defined as

$$\begin{aligned} \delta\bar{\mathbf{T}} &= \frac{d}{d\epsilon}\bar{\mathbf{T}}_\epsilon \Big|_{\epsilon=0} = \left(\frac{d}{d\epsilon}\varphi_{t,\epsilon}^*\mathbf{t} \circ \varphi_{t,\epsilon} \right) \Big|_{\epsilon=0} \\ &= \left(\varphi_{t,\epsilon}^*\mathbf{L}_{\mathbf{U}_t}\mathbf{t} \circ \varphi_{t,\epsilon} \right) \Big|_{\epsilon=0} = \dot{\varphi}_t^* (\mathbf{L}_{\mathbf{U}_t}\mathbf{t} \circ \dot{\varphi}_t). \end{aligned} \tag{A.9}$$

Thus, $\delta\mathbf{t} = \mathbf{L}_{\mathbf{u}_t}\mathbf{t}$, where $\mathbf{u}_t = \mathbf{U}_t \circ \dot{\varphi}_t^{-1}$.

References

1. AL-ATTAR, D., CRAWFORD, O.: Particle relabelling transformations in elastodynamics. *Geophys. J. Int.* **205**(1), 575–593, 2016
2. AMIRKHIZI, A.V., TEHRANIAN, A., NEMAT-NASSER, S.: Stress-wave energy management through material anisotropy. *Wave Motion* **47**(8), 519–536, 2010
3. AUFRAY, N., LE QUANG, H., HE, Q.-C.: Matrix representations for 3D strain-gradient elasticity. *J. Mech. Phys. Solids* **61**(5), 1202–1223, 2013
4. BANERJEE, B.: *An Introduction to Metamaterials and Waves in Composites*. CRC Press, Boca Raton 2011
5. BENVENISTE, Y., MILTON, G.W.: New exact results for the effective electric, elastic, piezoelectric and other properties of composite ellipsoid assemblages. *J. Mech. Phys. Solids* **51**(10), 1773–1813, 2003
6. BOEHLER, J.-P.: A simple derivation of representations for non-polynomial constitutive equations in some cases of anisotropy. *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)* **59**(4), 157–167, 1979
7. BOEHLER, J.-P.: *Applications of Tensor Functions in Solid Mechanics*, vol. 292. Springer, New York 1987
8. BRUN, M., GUENNEAU, S., MOVCHAN, A.B.: Achieving control of in-plane elastic waves. *Appl. Phys. Lett.* **94**(6), 061903, 2009
9. BRYAN, K., LEISE, T.: Impedance imaging, inverse problems, and Harry Potter's cloak. *SIAM Rev.* **52**(2), 359–377, 2010
10. CAPRIZ, G.: *Continua with Microstructure*, vol. 35. Springer, New York 2013
11. CHADWICK, P., VIANELLO, M., COWIN, S.C.: A new proof that the number of linear elastic symmetries is eight. *J. Mech. Phys. Solids* **49**(11), 2471–2492, 2001
12. CHANG, Z., GUO, H.-Y., LI, B., FENG, X.-Q.: Disentangling longitudinal and shear elastic waves by neo-Hookean soft devices. *Appl. Phys. Lett.* **106**(16), 161903, 2015
13. CHEN, H., CHAN, C.T.: Acoustic cloaking in three dimensions using acoustic metamaterials. *Appl. Phys. Lett.* **91**(18), 183518, 2007
14. COLEMAN, B.D., NOLL, W.: On the thermostatics of continuous media. *Arch. Ration. Mech. Anal.* **4**(1), 97–128, 1959
15. COLEMAN, B.D., NOLL, W.: The thermodynamics of elastic materials with heat conduction and viscosity. *Arch. Ration. Mech. Anal.* **13**(1), 167–178, 1963
16. COLEMAN, B.D., NOLL, W.: Material symmetry and thermostatic inequalities in finite elastic deformations. *Arch. Ration. Mech. Anal.* **15**(2), 87–111, 1964
17. COSSERAT, Eugène: *François Cosserat*. Théorie des corps déformables, 1909
18. CUMMER, S.A., POPA, B.-I., SCHURIG, D., SMITH, D.R., PENDRY, J., RAHM, M., STARR, A.: Scattering theory derivation of a 3D acoustic cloaking shell. *Phys. Rev. Lett.* **100**(2), 024301, 2008a
19. CUMMER, S.A., RAHM, M., SCHURIG, D.: Material parameters and vector scaling in transformation acoustics. *N. J. Phys.* **10**(11), 115025, 2008b
20. De KLERK, E.: *Aspects of Semidefinite Programming: Interior Point Algorithms and Selected Applications*, vol. 65. Springer, New York 2006
21. DiVINCENZO, D.P.: Dispersive corrections to continuum elastic theory in cubic crystals. *Phys. Rev. B* **34**(8), 5450, 1986
22. DOYLE, T.C., ERICKSEN, J.L.: Nonlinear elasticity. *Adv. Appl. Mech.* **4**, 53–115, 1956
23. EFRATI, E., SHARON, E., KUPFERMAN, R.: The metric description of elasticity in residually stressed soft materials. *Soft Matter* **9**(34), 8187–8197, 2013
24. EHRET, A.E., ITSKOV, M.: Modeling of anisotropic softening phenomena: application to soft biological tissues. *Int. J. Plast.* **25**(5), 901–919, 2009
25. ERGIN, T., STENGER, N., BRENNER, P., PENDRY, J.B., WEGENER, M.: Three-dimensional invisibility cloak at optical wavelengths. *Science* **328**(5976), 337–339, 2010
26. ERICKSEN, J.L.: Deformations possible in every compressible, isotropic, perfectly elastic material. *Stud. Appl. Math.* **34**(1–4), 126–128, 1955

27. ERICKSEN, J.L., TRUESDELL, C.: Exact theory of stress and strain in rods and shells. *Arch. Ration. Mech. Anal.* **1**(1), 295–323, 1957
28. ERINGEN, A.C.: *Microcontinuum Field Theories: I. Foundations and Solids*. Springer, New York 2012
29. FARHAT, M., ENOCH, S., GUENNEAU, S., MOVCHAN, A.B.: Broadband cylindrical acoustic cloak for linear surface waves in a fluid. *Phys. Rev. Lett.* **101**(13), 134501, 2008
30. FARHAT, M., GUENNEAU, S., ENOCH, S.: Ultrabroadband elastic cloaking in thin plates. *Phys. Rev. Lett.* **103**(2), 024301, 2009a
31. FARHAT, M., GUENNEAU, S., ENOCH, S., MOVCHAN, A.B.: Cloaking bending waves propagating in thin elastic plates. *Phys. Rev. B* **79**(3), 033102, 2009b
32. FORTE, S., VIANELLO, M.: Symmetry classes for elasticity tensors. *J. Elast.* **43**(2), 81–108, 1996
33. GOLGOON, A., YAVARI, A.: On the stress field of a nonlinear elastic solid torus with a toroidal inclusion. *J. Elast.* **128**(1), 115–145, 2017
34. GOLGOON, A., YAVARI, A.: Line and point defects in nonlinear anisotropic solids. *Zeitschrift für angewandte Mathematik und Physik* **69**(3), 81, 2018a
35. GOLGOON, A., YAVARI, A.: Nonlinear elastic inclusions in anisotropic solids. *J. Elast.* **130**(2), 239–269, 2018b
36. GOLGOON, A., SADIK, S., YAVARI, A.: Circumferentially-symmetric finite eigenstrains in incompressible isotropic nonlinear elastic wedges. *Int. J. Non-Linear Mech.* **84**, 116–129, 2016
37. GORIELY, A.: *The Mathematics and Mechanics of Biological Growth*, vol. 45. Springer, New York 2017
38. GREEN, A.E., RIVLIN, R.S., SHIELD, R.T.: General theory of small elastic deformations superposed on finite elastic deformations. *Proc. R. Soc. Lond. A* **211**(1104), 128–154, 1952
39. GREEN, A.E., RIVLIN, R.S.: On Cauchy's equations of motion. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)* **15**(3), 290–292, 1964
40. GREENLEAF, A., LASSAS, M., UHLMANN, G.: On nonuniqueness for Calderón's inverse problem. *Math. Res. Lett.* **10**, 2003a
41. GREENLEAF, A., LASSAS, M., UHLMANN, G.: Anisotropic conductivities that cannot be detected by EIT. *Physiol. Meas.* **24**(2), 413, 2003b
42. GREENLEAF, A., KURYLEV, Y., LASSAS, M., UHLMANN, G.: Full-wave invisibility of active devices at all frequencies. *Commun. Math. Phys.* **275**(3), 749–789, 2007
43. GREENLEAF, A., KURYLEV, Y., LASSAS, M., UHLMANN, G.: Cloaking devices, electromagnetic wormholes, and transformation optics. *SIAM Rev.* **51**(1), 3–33, 2009a
44. GREENLEAF, A., KURYLEV, Y., LASSAS, M., UHLMANN, G.: Invisibility and inverse problems. *Bull. Am. Math. Soc.* **46**(1), 55–97, 2009b
45. GUENNEAU, S., PUVIRAJESINGHE, T.M.: Fick's second law transformed: one path to cloaking in mass diffusion. *J. R. Soc. Interface* **10**(83), 20130106, 2013
46. GUENNEAU, S., AMRA, C., VEYNANTE, D.: Transformation thermodynamics: cloaking and concentrating heat flux. *Opt. Express* **20**(7), 8207–8218, 2012
47. GURNEY, C.: *An Analysis of the Stresses in a Flat Plate with a Reinforced Circular Hole Under Edge Forces*. Reports and Memoranda. H.M. Stationery Office, 1938
48. HASHIN, Z.: The elastic moduli of heterogeneous materials. *J. Appl. Mech.* **29**(1), 143–150, 1962
49. HASHIN, Z.: Large isotropic elastic deformation of composites and porous media. *Int. J. Solids Struct.* **21**(7), 711–720, 1985
50. HASHIN, Z., ROSEN, B.W.: The elastic moduli of fiber-reinforced materials. *J. Appl. Mech.* **31**(2), 223–232, 1964
51. HASHIN, Z., SHTRIKMAN, S.: A variational approach to the theory of the elastic behaviour of multiphase materials. *J. Mech. Phys. Solids* **11**(2), 127–140, 1963
52. HUGHES, T.J.R., MARSDEN, J.E.: Some applications of geometry in continuum mechanics. *Rep. Math. Phys.* **12**(1), 35–44, 1977

53. KADIC, M., BÜCKMANN, T., SCHITTNY, R., WEGENER, M.: Metamaterials beyond electromagnetism. *Rep. Prog. Phys.* **76**(12), 126501, 2013
54. KADIC, M., BÜCKMANN, T., SCHITTNY, R., WEGENER, M.: Experiments on cloaking in optics, thermodynamics and mechanics. *Philos. Trans. A* **373**(2049), 20140357, 2015
55. KHLOPOTIN, A., OLSSON, P., LARSSON, F.: Transformational cloaking from seismic surface waves by micropolar metamaterials with finite couple stiffness. *Wave Motion* **58**, 53–67, 2015
56. KOHN, R.V., SHEN, H., VOGELIUS, M.S., WEINSTEIN, M.I.: Cloaking via change of variables in electric impedance tomography. *Inverse Probl.* **24**(1), 015016, 2008
57. LANG, H.A.: The affine transformation for orthotropic plane-stress and plane-strain problems. *J. Appl. Mech.* **23**(1), 1–6, 1956
58. LEE, J.M.: *Riemannian Manifold An Introduction to Curves*. Springer, New York 1997
59. LEONHARDT, U.: Optical conformal mapping. *Science* **312**(5781), 1777–1780, 2006
60. LEONHARDT, U., PHILBIN, T.: *Geometry and Light: The Science of Invisibility*. Courier Corporation, New York, 2012
61. LIU, I.: On representations of anisotropic invariants. *Int. J. Eng. Sci.* **20**(10), 1099–1109, 1982
62. LIU, R., JI, C., MOCK, J.J., CHIN, J.Y., CUI, T.J., SMITH, D.R.: Broadband ground-plane cloak. *Science* **323**(5912), 366–369, 2009
63. LODGE, A.S.: A new theorem in the classical theory of elasticity. *Nature* **169**, 926–927, 1952
64. LODGE, A.S.: The transformation to isotropic form of the equilibrium equations for a class of anisotropic elastic solids. *Quart. J. Mech. Appl. Math.* **8**(2), 211–225, 1955
65. JIA, L., PAPADOPOULOS, P.: A covariant constitutive description of anisotropic nonlinear elasticity. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)* **51**(2), 204–217, 2000
66. MANSFIELD, E.H.: Neutral holes in plane sheet-reinforced holes which are elastically equivalent to the uncut sheet. *Quart. J. Mech. Appl. Math.* **6**(3), 370, 1953
67. MARANGANTI, R., SHARMA, P.: A novel atomistic approach to determine strain-gradient elasticity constants: tabulation and comparison for various metals, semiconductors, silica, polymers and their relevance for nanotechnologies. *J. Mech. Phys. Solids* **55**(9), 1823–1852, 2007
68. MARSDEN, J.E., HUGHES, T.J.R.: *Mathematical Foundations of Elasticity*. Dover, New York 1983
69. MAZZUCATO, A.L., RACHELE, L.V.: Partial uniqueness and obstruction to uniqueness in inverse problems for anisotropic elastic media. *J. Elast.* **83**(3), 205–245, 2006
70. MILTON, G.W.: New metamaterials with macroscopic behavior outside that of continuum elastodynamics. *N. J. Phys.* **9**(10), 359, 2007
71. MILTON, G.W., CHERKAEV, A.V.: Which elasticity tensors are realizable? *J. Eng. Mater. Technol.* **117**(4), 483–493, 1995
72. MILTON, G.W., WILLIS, J.R.: On modifications of Newton's second law and linear continuum elastodynamics. *Proc. R. Soc. A* **463**(2079), 855–880, 2007
73. MILTON, G.W., BRIANE, M., WILLIS, J.R.: On cloaking for elasticity and physical equations with a transformation invariant form. *N. J. Phys.* **8**(10), 248, 2006
74. MINDLIN, R.D.: Micro-structure in linear elasticity. *Arch. Ration Mech. Anal.* **16**(1), 51–78, 1964
75. MINDLIN, R.D., TIERSTEN, H.F.: Effects of couple-stresses in linear elasticity. *Arch. Ration Mech. Anal.* **11**(1), 415–448, 1962
76. NISHIKAWA, S.: *Variational Problems in Geometry*. American Mathematical Society, New York 2002
77. NOLL, W.: A mathematical theory of the mechanical behavior of continuous media. *Arch. Ration Mech. Anal.* **2**(1), 197–226, 1958
78. NORRIS, A.N.: Acoustic cloaking theory. *Proc. R. Soc. Lond. A* **464**(2097), 2411–2434, 2008

79. NORRIS, A.N., PARNELL, W.J.: Hyperelastic cloaking theory: transformation elasticity with pre-stressed solids. *Proc. R. Soc. A* **468**(2146), 2881–2903, 2012
80. NORRIS, A.N., SHUVALOV, A.L.: Elastic cloaking theory. *Wave Motion* **48**(6), 525–538, 2011
81. OGDEN, R.W.: *Non-linear Elastic Deformations*. Courier Corporation, New York 1997
82. OLIVE, M., AUFRAY, N.: Symmetry classes for even-order tensors. *Math. Mech. Complex Syst.* **1**(2), 177–210, 2013
83. OLSSON, P., WALL, D.J.N.: Partial elastodynamic cloaking by means of fiber-reinforced composites. *Inverse Probl.* **27**(4), 045010, 2011
84. OLVER, P.J.: Canonical elastic moduli. *J. Elast.* **19**(3), 189–212, 1988
85. OSTROSABLIN, N.I.: Affine transformations of the equations of the linear theory of elasticity. *J. Appl. Mech. Tech. Phys.* **47**(4), 564–572, 2006
86. OZAKIN, A., YAVARI, A.: A geometric theory of thermal stresses. *J. Math. Phys.* **51**, 032902, 2010
87. PARNELL, W.J.: Nonlinear pre-stress for cloaking from antiplane elastic waves. *Proc. R. Soc. A* **468**(2138), 563–580, 2012
88. PARNELL, W.J., NORRIS, A.N., SHEARER, T.: Employing pre-stress to generate finite cloaks for antiplane elastic waves. *Appl. Phys. Lett.* **100**(17), 171907, 2012
89. PENDRY, J.B., SCHURIG, D., SMITH, D.R.: Controlling electromagnetic fields. *Science* **312**(5781), 1780–1782, 2006
90. POST, E.J.: *Formal Structure of Electromagnetics: General Covariance and Electromagnetics*. Courier Corporation, New York 1997
91. REISSNER, H., MORDUCHOW, M.: Reinforced circular cutouts in plane sheets. Technical Report Technical Note No. 1852, 1949
92. RIVLIN, R.S.: Large elastic deformations of isotropic materials. I. Fundamental concepts. *Philos. Trans. R. Soc. Lond. Ser. A* **240**, 459–490, 1948
93. RIVLIN, R.S.: Large elastic deformations of isotropic materials. II. Some uniqueness theorems for pure, homogeneous deformation. *Philos. Trans. R. Soc. Lond. A* **240**(822), 491–508, 1948a
94. RIVLIN, R.S.: Large elastic deformations of isotropic materials. III. Some simple problems in cylindrical polar co-ordinates. *Philos. Trans. R. Soc. Lond. A* **240**(823), 509–525, 1948b
95. RIVLIN, R.S.: Large elastic deformations of isotropic materials. IV. Further developments of the general theory. *Philos. Trans. R. Soc. Lond. A* **241**(835), 379–397, 1948c
96. RIVLIN, R.S.: Large elastic deformations of isotropic materials. V. The problem of flexure. *Philos. Trans. R. Soc. Lond. A* **195**(1043), 463–473, 1949a
97. RIVLIN, R.S.: Large elastic deformations of isotropic materials. VI. Further results in the theory of torsion, shear and flexure. *Philos. Trans. R. Soc. Lond. A* **242**(845), 173–195, 1949b
98. RIVLIN, R.S., SAUNDERS, D.W.: Large elastic deformations of isotropic materials. VII. Experiments on the deformation of rubber. *Philos. Trans. R. Soc. Lond. A* **243**(865), 251–288, 1951
99. SADIK, S., YAVARI, A.: Geometric nonlinear thermoelasticity and the time evolution of thermal stresses. *Math. Mech. Solids* **1**, 42, 2015
100. SADIK, S., YAVARI, A.: Small-on-large geometric anelasticity. *Proc. R. Soc. A* **472**(2195), 20160659, 2016
101. SADIK, S., ANGOSHTARI, A., GORIELY, A., YAVARI, A.: A geometric theory of nonlinear morphoelastic shells. *J. Nonlinear Sci.* **26**, 929–978, 2016
102. SCHURIG, D., MOCK, J.J., JUSTICE, B.J., CUMMER, S.A., PENDRY, J.B., STARR, A.F., SMITH, D.R.: Metamaterial electromagnetic cloak at microwave frequencies. *Science* **314**(5801), 977–980, 2006
103. SIMO, J.C., MARSDEN, J.E.: Stress tensors, Riemannian metrics and the alternative descriptions in elasticity. In: *Trends and Applications of Pure Mathematics to Mechanics*, pp. 369–383. Springer, 1984

104. SIMO, J.C., MARSDEN, J.E., KRISHNAPRASAD, P.S.: The Hamiltonian structure of nonlinear elasticity: the material and convective representations of solids, rods, and plates. *Arch. Ration. Mech. Anal.* **104**(2), 125–183, 1988
105. SKLAN, S.R., PAK, R.Y.S., LI, B.: Seismic invisibility: elastic wave cloaking via symmetrized transformation media. *N. J. Phys.* **20**(6), 063013, 2018
106. SOZIO, F., YAVARI, A.: Nonlinear mechanics of surface growth for cylindrical and spherical elastic bodies. *J. Mech. Phys. Solids* **98**, 12–48, 2017
107. SOZIO, F., YAVARI, A.: Nonlinear mechanics of accretion. *J. Nonlinear Sci.* 1–51, 2019
108. SPENCER, A.J.M.: *The Formulation of Constitutive Equation for Anisotropic Solids*, pp. 3–26. Springer, Dordrecht 1982
109. SPENCER, A.J.M.: Part III. Theory of invariants. *Contin. Phys.* **1**, 239–353, 1971
110. STEIGMANN, D.J.: On the frame invariance of linear elasticity theory. *Zeitschrift für Angewandte Mathematik und Physik* **58**(1), 121–136, 2007
111. STOJANOVIĆ, R.: *Recent Developments in the Theory of Polar Continua*. Springer, New York 1970
112. TOUPIN, R.A.: Elastic materials with couple-stresses. *Arch. Ration. Mech. Anal.* **11**(1), 385–414, 1962
113. TOUPIN, R.A.: Theories of elasticity with couple-stress. *Arch. Ration. Mech. Anal.* **17**(2), 85–112, 1964
114. TRUESDELL, C.: The physical components of vectors and tensors. *Zeitschrift für Angewandte Mathematik und Mechanik* **33**(10–11), 345–356, 1953
115. YANO, K.L.: *The Theory of Lie Derivatives and its Applications*. North-Holland, Amsterdam, 1957
116. YAVARI, A.: A geometric theory of growth mechanics. *J. Nonlinear Sci.* **20**(6), 781–830, 2010
117. YAVARI, A., GORIELY, A.: Riemann–Cartan geometry of nonlinear dislocation mechanics. *Arch. Ration. Mech. Anal.* **205**(1), 59–118, 2012a
118. YAVARI, A., GORIELY, A.: Weyl geometry and the nonlinear mechanics of distributed point defects. *Proc. R. Soc. A* **468**(2148), 3902–3922, 2012b
119. YAVARI, A., GORIELY, A.: The geometry of discombinations and its applications to semi-inverse problems in anelasticity. *Proc. R. Soc. A* **470**(2169), 20140403, 2014
120. YAVARI, A., GORIELY, A.: The anelastic Ericksen problem: universal eigenstrains and deformations in compressible isotropic elastic solids. *Proc. R. Soc. A* **472**(2196), 20160690, 2016
121. YAVARI, A.: On geometric discretization of elasticity. *J. Math. Phys.* **49**(2), 022901, 2008
122. YAVARI, A.: Compatibility equations of nonlinear elasticity for non-simply-connected bodies. *Arch. Ration. Mech. Anal.* **209**(1), 237–253, 2013
123. YAVARI, A., GORIELY, A.: Riemann–Cartan geometry of nonlinear disclination mechanics. *Math. Mech. Solids* **18**(1), 91–102, 2012c
124. YAVARI, A., MARSDEN, J.E.: Covariant balance laws in continua with microstructure. *Rep. Math. Phys.* **63**(1), 1–42, 2009a
125. YAVARI, A., MARSDEN, J.E.: Energy balance invariance for interacting particle systems. *Zeitschrift für Angewandte Mathematik und Physik* **60**(4), 723–738, 2009b
126. YAVARI, A., OZAKIN, A.: Covariance in linearized elasticity. *Zeitschrift für Angewandte Mathematik und Physik* **59**(6), 1081–1110, 2008
127. YAVARI, A., MARSDEN, J.E., ORTIZ, M.: On spatial and material covariant balance laws in elasticity. *J. Math. Phys.* **47**(4), 042903, 2006
128. ZHANG, J.M., RYCHLEWSKI, J.: Structural tensors for anisotropic solids. *Arch. Mech.* **42**(3), 267–277, 1990
129. ZHANG, S., GENOV, D.A., SUN, C., ZHANG, X.: Cloaking of matter waves. *Phys. Rev. Lett.* **100**(12), 123002, 2008
130. ZHENG, Q.S.: Theory of representations for tensor functions. *Appl. Mech. Rev.* **47**(11), 545–587, 1994

131. ZHENG, Q.-S., SPENCER, A.J.M.: Tensors which characterize anisotropies. *Int. J. Eng. Sci.* **31**(5), 679–693, 1993a
132. ZHENG, Q.-S., SPENCER, A.J.M.: On the canonical representations for Kronecker powers of orthogonal tensors with application to material symmetry problems. *Int. J. Eng. Sci.* **31**(4), 617–635, 1993b
133. ZHOU, X., GENGKAI, H., TIANJIAN, L.: Elastic wave transparency of a solid sphere coated with metamaterials. *Phys. Rev. B* **77**(2), 024101, 2008

ARASH YAVARI

School of Civil and Environmental Engineering &
The George W. Woodruff School of Mechanical Engineering,
Georgia Institute of Technology,
Atlanta, GA 30332, USA.
e-mail: arash.yavari@ce.gatech.edu

and

ASHKAN GOLGOON

School of Civil and Environmental Engineering,
Georgia Institute of Technology,
Atlanta, GA 30332, USA.
e-mail: agolgoon3@gatech.edu

(Received July 26, 2018 / Accepted April 15, 2019)

Published online May 16, 2019

© Springer-Verlag GmbH Germany, part of Springer Nature (2019)