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Line and point defects in nonlinear anisotropic solids

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Abstract. In this paper, we present some analytical solutions for the stress fields of nonlinear anisotropic solids with distributed line and point defects. In particular, we determine the stress fields of (i) a parallel cylindrically symmetric distribution of screw dislocations in infinite orthotropic and monoclinic media, (ii) a cylindrically symmetric distribution of parallel wedge disclinations in an infinite orthotropic medium, (iii) a distribution of edge dislocations in an orthotropic medium, and (iv) a spherically symmetric distribution of point defects in a transversely isotropic spherical ball.

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1. Introduction

In anelasticity, any measure of strain has both an elastic and a non-elastic part. Given a pair of thermodynamically conjugate stress and strain, locally a nonvanishing strain does not necessarily correspond to a nonvanishing stress. *Elastic* strain refers to the part of strain that is locally related to the corresponding stress. The remaining part is referred to as *eigenstrain*, a term that was first used by Mura [42].[1] Defects are one source of anelasticity. Vito Volterra, in his seminal work [62], pioneered the mathematical study of defects many years before the first experimental observations of defects in solids. He classified line defects into six types, three of which are now called dislocations or translational defects, and the other three are called disclinations or rotational defects. Kondo [30,31] and Bilby et al. [3] independently explored the profound connections between the mechanics of defects and non-Riemannian geometries in the 1950s. Kondo [30,31] discovered that the reference configuration of a solid is not necessarily Euclidean in the presence of defects. He realized that the curvature and the torsion of the reference manifold are measures of incompatibility and the density of dislocations, respectively. Defects due to plastic deformations naturally occur in most of the known problems in mechanics and tribology, e.g., contact mechanics [4,5,25,26], mechanical impact [16], and dislocation-boundary interactions [23,64]. Other examples of anelastic sources include swelling and cavitation [21,41,47], bulk and surface growth [2,53,70], thermal strains [45,51,56], and the presence of inclusions and inhomogeneities [18–20,74]. There have been some theoretical investigations on the effects of eigenstrains in linear anisotropic media, e.g., [17,28,32,66], and references therein.

Very little is known about the effects of material anisotropies on the stress field and energetics of defects in solids. Eshelby [13] investigated infinitely long straight edge dislocations in linear anisotropic solids. He extended Nabarro's calculation of the width of a dislocation to the anisotropic case. His results are limited

[1] One should, however, note that this term had been used by German researchers in its original German version *Eigenspannungen* long before Mura's work. Apparently, Mura decided not to translate *Eigen*; he only translated *Spannung* = Strain. We are grateful to an anonymous reviewer who pointed this out to us.

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to edge dislocations with an axis that is an infinite straight line, but there is no restriction on the type of anisotropy of the medium. Eshelby et al. [14] developed the general solution for the induced displacement fields of dislocations in homogeneous linear anisotropic solids for the special case of the elastic state being independent of one of the three Cartesian coordinates. The dynamical response of uniformly moving dislocations in linear anisotropic media was studied by Teutonico [58]. It was observed that both edge and screw dislocations are prone to exhibiting anomalous dynamical behavior such that the interaction force between two parallel dislocations (on the same slip plane) changes sign when dislocation velocity increases. Head [22] predicted instabilities of dislocations in some anisotropic metallic crystals. It was found that a straight dislocation may decrease its energy if it changes to a zig-zag shape, i.e., a straight dislocation may be unstable. In the setting of the linear theory of elasticity, Willis [67] analyzed dislocations in anisotropic media (see also [68]). Particularly, the displacement fields of infinite straight dislocations and plane curvilinear dislocation loops were obtained. Schaefer and Kronmüller [52] investigated the elastic interaction of point defects in linear isotropic and anisotropic cubic media using Green's function approach. They specifically discussed the differences between the interactions in isotropic and anisotropic materials and the effects of anisotropy on the interaction potential. Some basic developments in the linear theory of dislocations in anisotropic media were presented in [24,34]. Methods for obtaining the induced linear elastic fields of defects in transversely isotropic bimaterials and orthotropic bicrystals (in 2D) were proposed in [78,79], respectively. In particular, some closed-form solutions for the elastic fields of inclusions and dislocations were presented.

A successive approximation method was proposed in [57] to study the nonlinear screw dislocation problem using the linear elasticity solution. Nonetheless, the method fails to find the correct solution near the dislocation axis. Only a handful of exact solutions for defects in nonlinear elastic solids exist in the literature, and they are all restricted to isotropic materials. We should mention [1,15,49,65,72,81] for dislocations, [8,75,81] for disclinations, and [6,73,76] for point defects and discombinations.

To the best of our knowledge, despite the known importance of the anisotropic behavior of solids, especially at finite strains, the study of defects in the setting of nonlinear elasticity has been limited to isotropic solids. In this paper, we study several examples of line and point defects in nonlinear anisotropic solids and present some analytical solutions for their stress fields. We consider an arbitrary cylindrically symmetric distribution of parallel screw dislocations in orthotropic and monoclinic media, along with a parallel cylindrically symmetric distribution of wedge disclinations in an infinite orthotropic medium. As the geometry of the material manifold explicitly depends on the distribution of defects, the material preferred directions (that identify the type of anisotropy) in the reference configuration explicitly depend on the defect distribution as well, and, in general, are different from those of the material in its current configuration. For instance, for the distributed screw dislocations that we consider, the assumption that the dislocated body is orthotropic in the reference (current) configuration implies that the body is monoclinic in the current (reference) configuration.

In this paper, the boundedness of the stress on the dislocation and disclination axes will be discussed. In particular, for an arbitrary cylindrically symmetric distribution of parallel screw dislocations the stress exhibits a logarithmic singularity on the dislocation axis unless the axial deformation is suppressed. Note that these singularities arise due to the presence of anisotropy, e.g., radial fiber reinforcement, and in particular, they do not occur when the material is isotropic. Exploiting the so-called standard reinforcing model (see, e.g., [38]), we obtain conditions under which the energy per unit length and the resultant longitudinal force of a single screw dislocation for a fiber-reinforced material are finite provided that the isotropic base material has a finite axial force and a finite energy per unit length. Employing Cartan's moving frames approach, for a given distribution of edge dislocations we will construct the material manifold and obtain explicit solutions for the stress field when the medium is orthotropic. We will also consider a spherically symmetric distribution of point defects in a finite transversely isotropic spherical ball. We will show that for an arbitrary incompressible transversely isotropic material with the radial material preferred direction a uniform point defect distribution induces a uniform hydrostatic stress inside the region the distribution is supported.

The rest of the paper is structured as follows. In Sect. 2, we tersely review some fundamentals of geometric nonlinear anisotropic elasticity and some related topics on nonlinear defect mechanics. We consider a cylindrically symmetric distribution of parallel screw dislocations in orthotropic and monoclinic media in Sects. 3.1 and 3.2, respectively. A cylindrically symmetric distribution of parallel wedge disclinations in an orthotropic medium is studied in Sect. 3.3. In Sect. 3.4, edge dislocations in an orthotropic medium are considered. In Sect. 3.5, we calculate the residual stresses due to a spherically symmetric distribution of point defects in a transversely isotropic ball. We end the paper with some remarks in Sect. 4.

2. Geometric anelasticity for anisotropic solids

In this section, we briefly review some fundamental elements of the geometric theory of nonlinear elasticity for anisotropic solids. For more detailed discussions, see [36,77].

2.1. Kinematics

A body $\mathcal{B}$ is identified with a Riemannian manifold $(\mathcal{B},\mathbf{G})$, and a configuration of $\mathcal{B}$ is a smooth embedding $\varphi:\mathcal{B}\to\mathcal{S}$, where $(\mathcal{S},\mathbf{g})$ is a Riemannian manifold—the ambient space. An affine connection ∇ on a smooth manifold $\mathcal{B}$ is a linear mapping $\nabla:\mathcal{X}(\mathcal{B})\times\mathcal{X}(\mathcal{B})\to\mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ represents the set of all smooth vector fields on $\mathcal{B}$, such that the following properties are satisfied $\forall\ \mathbf{X},\mathbf{Y},\mathbf{X}_1,\mathbf{X}_2,\mathbf{Y}_1,\mathbf{Y}_2\in\mathcal{X}(\mathcal{B}),\forall\ f,f_1,f_2\in C^\infty(\mathcal{B}),\forall\ a_1,a_2\in\mathbb{R}$ (see [9,48] for more details): (a) $\nabla_{f_1\mathbf{X}_1+f_2\mathbf{X}_2}\mathbf{Y}=f_1\nabla_{\mathbf{X}_1}\mathbf{Y}+f_2\nabla_{\mathbf{X}_2}\mathbf{Y}$, (b) $\nabla_{\mathbf{X}}(a_1\mathbf{Y}_1+a_2\mathbf{Y}_2)=a_1\nabla_{\mathbf{X}}(\mathbf{Y}_1)+a_2\nabla_{\mathbf{X}}(\mathbf{Y}_2)$, (c) $\nabla_{\mathbf{X}}(f\mathbf{Y})=f\nabla_{\mathbf{X}}\mathbf{Y}+(\mathbf{X}f)\mathbf{Y}$. It can be shown that there is a unique torsion-free and compatible affine connection associated with any Riemannian manifold that is called the Levi-Civita connection. Let us denote the Levi-Civita connection associated with the Riemannian manifolds $(\mathcal{B},\mathbf{G})$ and $(\mathcal{S},\mathbf{g})$ by $\nabla^{\mathbf{G}}$ and $\nabla^{\mathbf{g}}$, respectively. We denote the set of all configurations of $\mathcal{B}$ by $\mathcal{C}$. A motion is a curve $c:\mathbb{R}^+\to\varphi_t\in\mathcal{C}$ such that φ_t assigns a spatial point $x=\varphi_t(X)=\varphi(X,t)\in\mathcal{S}$ to every material point $X\in\mathcal{B}$ at any time t. The body is assumed to be stress-free in its reference configuration, which may have a nontrivial geometry, in general, e.g., in the presence of eigenstrains. The deformation gradient $\mathbf{F}$ is the tangent map of φ defined as $\mathbf{F}(X,t)=d\varphi_t(X):T_X\mathcal{B}\to T_{\varphi_t(X)}\mathcal{S}$. The adjoint of $\mathbf{F}$ is defined as $\mathbf{F}^{\mathsf{T}}(X,t):T_{\varphi_t(X)}\mathcal{S}\to T_X\mathcal{B}$, $\mathbf{g}(\mathbf{F}\mathbf{V},\mathbf{v})=\mathbf{G}\left(\mathbf{V},\mathbf{F}^{\mathsf{T}}\mathbf{v}\right)$, $\forall\,\mathbf{V}\in T_X\mathcal{B}$, $\mathbf{v}\in T_{\varphi_t(X)}\mathcal{S}$. The right Cauchy–Green deformation tensor is defined as $\mathbf{C}(X,t)=\mathbf{F}^{\mathsf{T}}(X,t)\mathbf{F}(X,t):T_X\mathcal{B}\to T_X\mathcal{B}$. The Finger deformation tensor is defined as $\mathbf{b}(x,t)=\mathbf{F}(X,t)\mathbf{F}^{\mathsf{T}}(X,t):T_x\varphi(\mathcal{B})\to T_x\varphi(\mathcal{B})$, and in components, $b^{ab}=F^a{}_AF^b{}_BG^{AB}$. Another measure of strain is the Lagrangian strain tensor $\mathbf{E}=\frac{1}{2}\left(\varphi_t^*\mathbf{g}-\mathbf{G}\right)$, where $\varphi_t^*\mathbf{g}$ is the pull-back of the spatial metric (in components $(\varphi_t^*\mathbf{g})_{AB}=F^a{}_AF^b{}_Bg_{ab}$). The Jacobian of deformation J relates the Riemannian volume element of the material manifold $dV(X,\mathbf{G})$ to that of the spatial manifold $dv(\varphi_t(X),\mathbf{g})$, written as

$$J=\sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F}\,,\quad \mathrm{d}v(x,\mathbf{g})=J\,\mathrm{d}V(X,\mathbf{G}). \tag{2.1}$$

2.2. Equilibrium equations

The localized balance of linear momentum in spatial and material forms is written as

$$\operatorname{div}\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\mathbf{a},\quad \operatorname{Div}\mathbf{P}+\rho_0\mathbf{B}=\rho_0\mathbf{A}, \tag{2.2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress and $\mathbf{P}$ is the first Piola–Kirchhoff stress, and ρ_0, $\mathbf{A}$, and $\mathbf{B}$ are the material mass density, material acceleration, and material body force, respectively, and ρ, $\mathbf{a}$, and $\mathbf{b}$ are their corresponding spatial counterparts. Note that the material and spatial divergence operators in components are given as

$$(\mathrm{div}\boldsymbol{\sigma})^a = \sigma^{ab}{}_{|b} = \frac{\partial \sigma^{ab}}{\partial x^b} + \sigma^{ac}\gamma^b{}_{cb} + \sigma^{cb}\gamma^a{}_{cb},$$
$$(\mathrm{Div}\mathbf{P})^a = P^{aA}{}_{|A} = \frac{\partial P^{aA}}{\partial X^A} + P^{aB}\Gamma^A{}_{AB} + P^{cA}F^b{}_A\gamma^a{}_{bc}, \tag{2.3}$$

where $\gamma^a{}_{bc}$ and $\Gamma^A{}_{BC}$ denote the Christoffel symbols of the connections $\nabla^{\mathbf{g}}$ and $\nabla^{\mathbf{G}}$, respectively. Note that in the local coordinate charts $\{x^a\}$ and $\{X^A\}$, one has $\nabla^{\mathbf{g}}{}_{\partial_b}\partial_c = \gamma^a{}_{bc}\partial_a$ and $\nabla^{\mathbf{G}}{}_{\partial_B}\partial_C = \Gamma^A{}_{BC}\partial_A$.

2.3. Material symmetry group

In the case of a *simple* material, the response function at any material point depends only on the first deformation gradient (and its evolution) at that point [43]. Consider an elastic body made of a simple material with the response function W at a material point X. We assume that the response function is the energy function. A response function may be any measure of stress as well. The material symmetry group $\mathcal{G}_X$ associated with the body at the point X with respect to the reference configuration $(\mathcal{B}, \mathbf{G})$ is defined as

$$W(X, \mathbf{FK}, \mathbf{G}, \mathbf{g}) = W(\mathbf{F}, \mathbf{G}, \mathbf{g}), \qquad \forall\, \mathbf{K} \in \mathcal{G}_X, \tag{2.4}$$

for all deformation gradients $\mathbf{F}$, where $\mathbf{K} : T_X\mathcal{B} \to T_X\mathcal{B}$ is an invertible linear transformation. For hyperelastic solids, objectivity implies that the energy function depends on the deformation at a referential point X through the right Cauchy–Green deformation tensor $\mathbf{C}^\flat$, i.e., $W = W(X, \mathbf{C}^\flat, \mathbf{G})$. Thus, the material symmetry group $\mathcal{G}_X$ for a hyperelastic solid is defined to be the subgroup of $\mathbf{G}$-orthogonal transformations Orth($\mathbf{G}$) such that [11]

$$W(X, \mathbf{Q}^{-\star}\mathbf{C}^\flat\mathbf{Q}^{-1}, \mathbf{G}) = W(X, \mathbf{C}^\flat, \mathbf{G}), \qquad \forall\, \mathbf{Q} \in \mathcal{G}_X \leqslant \mathrm{Orth}\,(\mathbf{G})\,, \tag{2.5}$$

where $\mathrm{Orth}(\mathbf{G}) = \left\{\mathbf{Q} : T_X\mathcal{B} \to T_X\mathcal{B} \;\;|\;\; \mathbf{Q}^\top = \mathbf{Q}^{-1}\right\}$, and we use the notation $\mathscr{G} \leqslant \mathscr{H}$ when $\mathscr{G}$ is a subgroup of $\mathscr{H}$. Note that the set of orthogonal transformations is explicitly metric dependent. In other words, if the material metric $\mathbf{G}$ changes, Orth($\mathbf{G}$), and hence, $\mathcal{G}_X$ changes as well. The symmetry group can be equivalently characterized using a finite collection of structural tensors $\boldsymbol{\zeta}_i$ of order μ_i, $i = 1, \ldots, n$, forming a basis for the space of tensors that are invariant under the action of $\mathcal{G}$ as follows (also, see [33,37,54])

$$\mathbf{Q} \in \mathcal{G} \leqslant \mathrm{Orth}\,(\mathbf{G}) \iff \langle\mathbf{Q}\rangle_{\mu_1}\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_1, \ldots, \langle\mathbf{Q}\rangle_{\mu_n}\boldsymbol{\zeta}_n = \boldsymbol{\zeta}_n, \tag{2.6}$$

where $\langle\mathbf{Q}\rangle_\mu$ is the μ-th power Kronecker product of a $\mathbf{G}$-orthogonal transformation $\mathbf{Q}$ defined for any μ-th order tensor $\boldsymbol{\zeta}$ as $(\langle\mathbf{Q}\rangle_\mu\,\zeta)^{\bar{A}_1\cdots\bar{A}_\mu} = Q^{\bar{A}_1}{}_{A_1}\ldots Q^{\bar{A}_\mu}{}_{A_\mu}\zeta^{A_1\cdots A_\mu}$. Note that (2.6) suggests that the material symmetry group $\mathcal{G}$ is the invariance group of the set of the structural tensors $\boldsymbol{\zeta}_i$, $i = 1, \ldots, n$.

Remark 2.1. We define the material symmetry group at a material point in its natural (stress-free) state. This has the following physical interpretation. One is given a body with a distribution of defects that is residually stressed in its current configuration. Now imagine that the body is partitioned into a large number of small elements and each is allowed to relax. The symmetry group of a material point in this locally relaxed configuration is the same as its symmetry group in the Riemannian material manifold $(\mathcal{B}, \mathbf{G})$.

Remark 2.2. In the so-called theory of "material uniformity" of Noll [44] and Wang [63], one characterizes uniformity of the mechanical response of a body that may be residually stressed, although Noll and Wang did not explicitly mention residual stresses (see also [12]). Their developments are essentially based on the multiplicative decomposition of the deformation gradient into elastic and plastic parts: $\mathbf{F} = \overset{\mathrm{e}}{\mathbf{F}}\overset{\mathrm{p}}{\mathbf{F}}$. They call $\overset{\mathrm{p}}{\mathbf{F}}$ and $\overset{\mathrm{e}}{\mathbf{F}}$ a "local configuration," and a "local deformation," respectively. A body is "materially uniform" if its energy function (or any response function) depends only on $\overset{\mathrm{e}}{\mathbf{F}}$ and not on the material point X. In our formulation of anelasticity, the deformation gradient is purely elastic; the plastic (defect) part is buried into the material metric. Therefore, our symmetry group $\mathcal{G}_X$ is what Noll and Wang call the "isotropy group relative" to $\overset{\mathrm{p}}{\mathbf{F}}$. Their "isotropy group relative" to $\overset{\mathrm{p}}{\mathbf{F}}$ explicitly depends on $\overset{\mathrm{p}}{\mathbf{F}}$, while our material symmetry group explicitly depends on the material metric $\mathbf{G}$. What Wang [63] calls an "intrinsic" Riemannian metric is our material metric $\mathbf{G} = \overset{\mathrm{p}}{\mathbf{F}}{}^{\mathsf{T}} \overset{\mathrm{p}}{\mathbf{F}}$. However, Noll and Wang did not use the concept of natural distances and a stress-free reference configuration; their main interest was the material symmetry group. In particular, for isotropic solids the symmetry group is preserved under a uniform scaling of the "intrinsic" Riemannian metric, i.e., this metric is unique for isotropic solids up to a constant factor [63]. However, note that a scaling of $\mathbf{G}$ changes the natural distances. In other words, two material metrics related by a uniform scaling are not equivalent in our geometric theory of anelasticity as they do not correspond to equivalent reference configurations. More specifically, if the body is stress-free in $(\mathcal{B}, \mathbf{G})$, in general, it is not stress-free in $(\mathcal{B}, \alpha^2\mathbf{G})$, if for example, the boundary has prescribed displacements.

2.4. Constitutive equations

In this paper, our calculations are restricted to incompressible transversely isotropic, orthotropic, and monoclinic solids. To establish a materially covariant strain energy density function, *structural tensors* corresponding to the symmetry group of the material are used. For detailed discussions on structural tensors and the determination of the integrity basis and the corresponding invariants of a set of tensors, see [33,35,54,55,80].

2.5. Transverse isotropy

Let us assume a compressible transversely isotropic material such that the unit vector $\mathbf{N}(\mathrm{X})$ identifies the material preferred direction at a point X in the reference configuration. The strain energy density per unit volume of the reference configuration is given as (see, e.g., [10,35,55]) $W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A})$, where $\mathbf{A} = \mathbf{N} \otimes \mathbf{N}$ is a structural tensor representing the transverse isotropy of the material symmetry group, and $(.)^\flat$ denotes the flat operator for lowering tensor indices. The second Piola–Kirchhoff stress tensor is given by

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}^\flat}. \tag{2.7}$$

The energy function W depends on the following five independent invariants defined as

$$I_1 = \mathrm{tr}\,\mathbf{C}, \quad I_2 = \det \mathbf{C}\, \mathrm{tr}\ \mathbf{C}^{-1}, \quad I_3 = \det \mathbf{C}, \quad I_4 = \mathbf{N} \cdot \mathbf{C} \cdot \mathbf{N}, \quad I_5 = \mathbf{N} \cdot \mathbf{C}^2 \cdot \mathbf{N}. \tag{2.8}$$

In components they read

$$I_1 = C^A{}_A,\ I_2 = \det(C^A{}_B)(C^{-1})^D{}_D,\ I_3 = \det(C^A{}_B),\ I_4 = N^A N^B C_{AB}\,,\ I_5 = N^A N^B C_{BQ} C^Q{}_A. \tag{2.9}$$

Using (2.7), one obtains[2]

$$\mathbf{S} = \sum_{n=1}^{5} 2W_{I_n} \frac{\partial I_n}{\partial \mathbf{C}^\flat}\,, \qquad W_{I_n} := \frac{\partial W}{\partial I_n}\,,\ \ n = 1, \ldots, 5. \tag{2.10}$$

[2] For the sake of brevity, we do not assume an explicit dependence of W on X, which in the case of inhomogeneous bodies is needed. We suppose instead that the material is piece-wise homogeneous and model an inhomogeneity using different energy functions in different regions of the body.

Note that

$$\frac{\partial I_1}{\partial \mathbf{C}^\flat} = \mathbf{G}^\sharp\,, \ \frac{\partial I_2}{\partial \mathbf{C}^\flat} = I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\,, \ \frac{\partial I_3}{\partial \mathbf{C}^\flat} = I_3\mathbf{C}^{-1}\,, \ \frac{\partial I_4}{\partial \mathbf{C}^\flat} = \mathbf{N}\otimes\mathbf{N}\,, \ \frac{\partial I_5}{\partial \mathbf{C}^\flat} = \mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N} + \mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}, \tag{2.11}$$

where $(.)^\sharp$ is the sharp operator for raising tensor indices. Thus, from (2.10) and (2.11), one obtains the following representation for the second Piola–Kirchhoff stress tensor

$$\mathbf{S} = 2\left\{W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + W_{I_3}I_3\mathbf{C}^{-1} + W_{I_4}\left(\mathbf{N}\otimes\mathbf{N}\right) + W_{I_5}\left(\mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N} + \mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\right)\right\}. \tag{2.12}$$

If the material is incompressible, then $I_3 = 1$, and thus, $W = W(\mathrm{X}, I_1, I_2, I_4, I_5)$. Therefore, from (2.12), $\mathbf{S}$ is expressed as

$$\mathbf{S} = 2\left\{W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + W_{I_4}\left(\mathbf{N}\otimes\mathbf{N}\right) + W_{I_5}\left(\mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N} + \mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\right)\right\} - p\mathbf{C}^{-1}, \tag{2.13}$$

in which p is the Lagrange multiplier associated with the incompressibility condition $J = 1$. The Cauchy stress tensor $\sigma^{ab} = \frac{1}{J}F^a{}_A F^b{}_B S^{AB}$ is represented in component form as[3]

$$\sigma^{ab} = 2F^a{}_A F^b{}_B\left[(W_{I_1} + I_1 W_{I_2})G^{AB} - W_{I_2}C^{AB} + W_{I_4}N^A N^B + W_{I_5}\left(N^Q N^A C^B{}_Q + N^P N^B C_P{}^A\right)\right] - pg^{ab}. \tag{2.14}$$

2.6. Orthotropy

Next, we consider a compressible orthotropic material with three $\mathbf{G}$-orthonormal vectors $\mathbf{N}_1(\mathrm{X})$, $\mathbf{N}_2(\mathrm{X})$, and $\mathbf{N}_3(\mathrm{X})$ specifying the orthotropic axes in the reference configuration at a point X. A choice of structural tensors is given by $\mathbf{A}_1 = \mathbf{N}_1\otimes\mathbf{N}_1$, $\mathbf{A}_2 = \mathbf{N}_2\otimes\mathbf{N}_2$, and $\mathbf{A}_3 = \mathbf{N}_3\otimes\mathbf{N}_3$, where only two of which are independent as $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I}$. Hence, the energy function is given as [10,35,55]

$$W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A}_1, \mathbf{A}_2). \tag{2.15}$$

The energy function W is represented in terms of the following seven independent invariants

$$\begin{aligned} &I_1 = \operatorname{tr}\mathbf{C}\,, \quad I_2 = \det\mathbf{C}\operatorname{tr}\,\mathbf{C}^{-1}\,, \quad I_3 = \det\mathbf{C}\,, \quad I_4 = \mathbf{N}_1\cdot\mathbf{C}\cdot\mathbf{N}_1\,,\\ &I_5 = \mathbf{N}_1\cdot\mathbf{C}^2\cdot\mathbf{N}_1\,, \quad I_6 = \mathbf{N}_2\cdot\mathbf{C}\cdot\mathbf{N}_2\,, \quad I_7 = \mathbf{N}_2\cdot\mathbf{C}^2\cdot\mathbf{N}_2. \end{aligned} \tag{2.16}$$

Using (2.7), one writes

$$\mathbf{S} = \sum_{n=1}^{7} 2W_{I_n}\frac{\partial I_n}{\partial \mathbf{C}^\flat}\,, \qquad W_{I_n} := \frac{\partial W}{\partial I_n}\,, \ n = 1, \ldots, 7. \tag{2.17}$$

Substituting (2.11) into (2.17), the second Piola–Kirchhoff stress tensor is given by

$$\begin{aligned} \mathbf{S} = 2\Big\{&W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right)\\ &+ W_{I_3}I_3\mathbf{C}^{-1} + W_{I_4}\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + W_{I_5}\left(\mathbf{N}_1\otimes\mathbf{C}\cdot\mathbf{N}_1 + \mathbf{N}_1\cdot\mathbf{C}\otimes\mathbf{N}_1\right)\\ &+ W_{I_6}\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + W_{I_7}\left(\mathbf{N}_2\otimes\mathbf{C}\cdot\mathbf{N}_2 + \mathbf{N}_2\cdot\mathbf{C}\otimes\mathbf{N}_2\right)\Big\}. \end{aligned} \tag{2.18}$$

[3] Note that one can use the Cayley–Hamilton theorem to obtain $\frac{\partial I_2}{\partial \mathbf{C}^\flat} = I_2(\mathbf{C}^{-1})^\sharp - I_3(\mathbf{C}^{-2})^\sharp = I_1\mathbf{G}^\sharp - \mathbf{C}^\sharp$.

In the case of incompressible solids $I_3 = 1$ and $W = W(\mathrm{X}, I_1, I_2, I_4, I_5, I_6, I_7)$. Therefore, using (2.18), one obtains the following representation for the second Piola–Kirchhoff stress tensor

$$\mathbf{S} = 2\Big\{W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + W_{I_4}\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + W_{I_5}\left(\mathbf{N}_1\otimes\mathbf{C}\cdot\mathbf{N}_1 + \mathbf{N}_1\cdot\mathbf{C}\otimes\mathbf{N}_1\right) + W_{I_6}\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + W_{I_7}\left(\mathbf{N}_2\otimes\mathbf{C}\cdot\mathbf{N}_2 + \mathbf{N}_2\cdot\mathbf{C}\otimes\mathbf{N}_2\right)\Big\} - p\mathbf{C}^{-1}. \tag{2.19}$$

In components, the Cauchy stress tensor is given as

$$\sigma^{ab} = 2F^a{}_AF^b{}_B\Big[(W_{I_1} + I_1W_{I_2})G^{AB} - W_{I_2}C^{AB} + W_{I_4}N_1{}^AN_1{}^B + W_{I_5}\left(N_1{}^QN_1{}^AC^B{}_Q + N_1{}^PN_1{}^BC_P{}^A\right) + W_{I_6}N_2{}^AN_2{}^B + W_{I_7}\left(N_2{}^SN_2{}^AC^B{}_S + N_2{}^KN_2{}^BC_K{}^A\right)\Big] - pg^{ab}. \tag{2.20}$$

2.7. Monoclinic symmetry

One of the preferred directions of a material with a monoclinic symmetry (say $\mathbf{N}_3(\mathrm{X})$) is perpendicular to the plane of the other two (denoted by $\mathbf{N}_1(\mathrm{X})$ and $\mathbf{N}_2(\mathrm{X})$), which are not orthogonal. As an example one can consider an isotropic base material reinforced with two families of fibers such that the fibers are not at right angles, nor are they mechanically equivalent. In this case, the energy function is similar to that of orthotropic materials given by (2.15), where $\mathbf{A}_1 = \mathbf{N}_1\otimes\mathbf{N}_1$ and $\mathbf{A}_2 = \mathbf{N}_2\otimes\mathbf{N}_2$. Nonetheless, an extra invariant $I_8 = (\mathbf{N}_1\cdot\mathbf{N}_2)\mathbf{N}_1\cdot\mathbf{C}\cdot\mathbf{N}_2$ that models the coupling between the fibers (in $\mathbf{N}_1$ and $\mathbf{N}_2$ directions) is needed to express the energy function for monoclinic materials as $\mathbf{N}_1$ and $\mathbf{N}_2$ are not perpendicular (see [7,40,61]). Therefore,

$$\mathbf{S} = \sum_{n=1}^{8} 2W_{I_n}\frac{\partial I_n}{\partial \mathbf{C}^\flat}, \qquad W_{I_n} := \frac{\partial W}{\partial I_n}, \;\; n = 1,\ldots,8. \tag{2.21}$$

Hence[4]

$$\mathbf{S} = 2\Big\{W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + W_{I_3}I_3\mathbf{C}^{-1} + W_{I_4}\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + W_{I_5}\left(\mathbf{N}_1\otimes\mathbf{C}\cdot\mathbf{N}_1 + \mathbf{N}_1\cdot\mathbf{C}\otimes\mathbf{N}_1\right) + W_{I_6}\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + W_{I_7}\left(\mathbf{N}_2\otimes\mathbf{C}\cdot\mathbf{N}_2 + \mathbf{N}_2\cdot\mathbf{C}\otimes\mathbf{N}_2\right) + \frac{W_{I_8}}{2}\left(\mathbf{N}_1\otimes\mathbf{N}_2 + \mathbf{N}_2\otimes\mathbf{N}_1\right)\Big\}, \tag{2.22}$$

and for incompressible solids

$$\mathbf{S} = 2\Big\{W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + W_{I_4}\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + W_{I_5}\left(\mathbf{N}_1\otimes\mathbf{C}\cdot\mathbf{N}_1 + \mathbf{N}_1\cdot\mathbf{C}\otimes\mathbf{N}_1\right) + W_{I_6}\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + W_{I_7}\left(\mathbf{N}_2\otimes\mathbf{C}\cdot\mathbf{N}_2 + \mathbf{N}_2\cdot\mathbf{C}\otimes\mathbf{N}_2\right) + \frac{W_{I_8}}{2}\left(\mathbf{N}_1\otimes\mathbf{N}_2 + \mathbf{N}_2\otimes\mathbf{N}_1\right)\Big\} - p\mathbf{C}^{-1}. \tag{2.23}$$

The Cauchy stress is given in components as

$$\begin{aligned}\sigma^{ab} = 2F^a{}_AF^b{}_B\Big[&(W_{I_1} + I_1W_{I_2})G^{AB} - W_{I_2}C^{AB} + W_{I_4}N_1{}^AN_1{}^B \\ &+ W_{I_5}\left(N_1{}^QN_1{}^AC^B{}_Q + N_1{}^PN_1{}^BC_P{}^A\right) \\ &+ W_{I_6}N_2{}^AN_2{}^B + W_{I_7}\left(N_2{}^SN_2{}^AC^B{}_S + N_2{}^KN_2{}^BC_K{}^A\right) \\ &+ \frac{W_{I_8}}{2}\left(N_1{}^AN_2{}^B + N_2{}^AN_1{}^B\right)\Big] - pg^{ab}.\end{aligned} \tag{2.24}$$

[4] Note that $\frac{\partial I_8}{\partial \mathbf{C}^\flat} = \mathbf{N}_1\otimes\mathbf{N}_2 + \mathbf{N}_2\otimes\mathbf{N}_1$.

2.8. Cartan's moving frame

At a point X of a manifold $\mathcal{B}$ consider an orthonormal frame field $\{\mathbf{e}_\alpha\}_{\alpha=1}^N$ forming a basis for $T_X\mathcal{B}$. This frame field is not necessarily a coordinate basis for the tangent space. However, given a coordinate basis $\{\frac{\partial}{\partial X^A}\}$, one can obtain an arbitrary frame field $\{\mathbf{e}_\alpha\}$ using an $SO(N,\mathbb{R})$-rotation of the coordinate basis such that $\mathbf{e}_\alpha = \mathsf{F}^A{}_\alpha \frac{\partial}{\partial X^A}$. For a coordinate frame $\left[\frac{\partial}{\partial X^A}, \frac{\partial}{\partial X^B}\right] = 0$,[5] whereas for the non-coordinate frame, $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = -c^\gamma{}_{\alpha\beta}\mathbf{e}_\gamma$, where $c^\gamma{}_{\alpha\beta}$ are the components of the object of anholonomy. One can show that $c^\gamma{}_{\alpha\beta} = \mathsf{F}^A{}_\alpha \mathsf{F}^B{}_\beta \left(\partial_A \mathsf{F}^\gamma{}_B - \partial_B \mathsf{F}^\gamma{}_A\right)$, where $\mathsf{F}^\gamma{}_A$ is the inverse of $\mathsf{F}^A{}_\gamma$. Connection 1-forms are defined by $\nabla \mathbf{e}_\alpha = \mathbf{e}_\gamma \otimes \omega^\gamma{}_\alpha$, and in components, $\nabla_{\mathbf{e}_\beta}\mathbf{e}_\alpha = \langle \omega^\gamma{}_\alpha, \mathbf{e}_\beta\rangle\, \mathbf{e}_\gamma = \omega^\gamma{}_{\beta\alpha}\mathbf{e}_\gamma$. In terms of the co-frame field $\{\vartheta^\alpha\}_{\alpha=1}^N$ corresponding to $\{\mathbf{e}_\alpha\}$, one has $\omega^\gamma{}_\alpha = \omega^\gamma{}_{\beta\alpha}\vartheta^\beta$. Similarly, one obtains $\nabla\vartheta^\alpha = -\omega^\alpha{}_\gamma\vartheta^\gamma$ and $\nabla_{\mathbf{e}_\beta}\vartheta^\alpha = -\omega^\alpha{}_{\beta\gamma}\vartheta^\gamma$. The metric tensor is represented as $\mathbf{G} = \delta_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$. Metric compatibility of ∇ gives the following constraints on the connection 1-forms: $\delta_{\alpha\gamma}\omega^\gamma{}_\beta + \delta_{\beta\gamma}\omega^\gamma{}_\alpha = 0$. In a non-coordinate basis, the torsion and curvature have the following components: $T^\alpha{}_{\beta\gamma} = \omega^\alpha{}_{\beta\gamma} - \omega^\alpha{}_{\gamma\beta} + c^\alpha{}_{\beta\gamma}$ and $\mathcal{R}^\alpha{}_{\beta\lambda\mu} = \partial_\beta\omega^\alpha{}_{\lambda\mu} - \partial_\lambda\omega^\alpha{}_{\beta\mu} + \omega^\alpha{}_{\beta\xi}\omega^\xi{}_{\lambda\mu} - \omega^\alpha{}_{\lambda\xi}\omega^\xi{}_{\beta\mu} + \omega^\alpha{}_{\xi\mu}c^\xi{}_{\beta\lambda}$, respectively. Torsion and curvature 2-forms are, respectively, given by $\mathcal{T}^\alpha = d\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta$ and $\mathcal{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$. These are Cartan's first and second structural equations. The density of Burgers' vector $\mathbf{b}$ at a point X of $\mathcal{B}$ is related to torsion 2-from as follows

$$b^\alpha(X; C_s) = \int_{\Omega_s} \mathsf{P}^\alpha{}_\beta \mathcal{T}^\beta, \tag{2.25}$$

where $\Omega_s \in \mathcal{B}$ is a smooth surface with a boundary given by the curve C_s, and $\mathsf{P}(C_s)^t_\tau : T_{C_s(\tau)}\mathcal{B} \to T_{C_s(t)}\mathcal{B}$ parallel transports vectors tangent to the manifold at $C_s(\tau)$ to $C_s(t)$ (see [27,46] for more details).

3. Examples of anisotropic bodies with distributed defects

In this section, we consider several examples of distributed defects in cylindrical bars made of orthotropic and monoclinic solids as well as distributed defects in spherical balls made of transversely isotropic solids. Particularly, we consider cylindrically symmetric distributions of parallel screw dislocations and disclinations in an orthotropic medium, a spherically symmetric distribution of point defects in a transversely isotropic spherical ball, and a cylindrically symmetric distribution of screw dislocations in a monoclinic medium. We also discuss the effects of the constitutive parameters on the induced stress fields for different types of defects.

3.1. A cylindrically symmetric distribution of parallel screw dislocations in an orthotropic medium

Let us consider a cylindrically symmetric distribution of screw dislocations parallel to the Z-axis with a radially symmetric Burgers' vector density $b(R)$ (in a cylindrical coordinate system (R, Θ, Z)) in an infinite orthotropic medium. We assume that in the reference configuration the dislocated body is orthotropic. The material preferred directions at a material point X are denoted by $\mathbf{N}_1(\mathrm{X})$, $\mathbf{N}_2(\mathrm{X})$, and $\mathbf{N}_3(\mathrm{X})$ in the reference configuration. In the current configuration, the preferred directions are given by $\mathbf{n}_1(\mathrm{x})$, $\mathbf{n}_2(\mathrm{x})$, and $\mathbf{n}_3(\mathrm{x})$ at the point x corresponding to the material point X. We assume that $\mathbf{N}_1$ and $\mathbf{N}_2$ are in the radial and axial directions, respectively. Note that $\mathbf{N}_3$, which is perpendicular to $\mathbf{N}_1$ and $\mathbf{N}_2$, explicitly depends on the distribution of screw dislocations as will be seen in the following. This is because the geometry of the material manifold has an explicit nontrivial dependence on the dislocation distribution (see (3.1)). In the current configuration, the body will have monoclinic anisotropy as $\mathbf{n}_1$ will

[5] Note that for any pair of vector fields $\mathbf{U}$ and $\mathbf{V}$ on $\mathcal{B}$, one can define a new vector field—the commutator—given by $[\mathbf{U}, \mathbf{V}]_X f := \mathbf{U}_X(\mathbf{V}f) - \mathbf{V}_X(\mathbf{V}f)$, for any smooth function at X on $\mathcal{B}$.

be perpendicular to the plane of $\mathbf{n}_2$ and $\mathbf{n}_3$, which will not be orthogonal in the ambient space. It is known that the material manifold for a nonlinear solid with distributed dislocations is a Weitzenböck manifold, i.e., a manifold with torsion having a flat connection and vanishing non-metricity (see [46,72] for more details). Therefore, the material metric for the dislocated body is written as

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 + f(R)^2 & f(R) \\ 0 & f(R) & 1 \end{pmatrix}, \tag{3.1}$$

where $f(R)$ is related to the Burgers' vector density $b(R)$ such that $f'(R) = \frac{R}{2\pi} b(R)$. Let us endow the ambient space with the Euclidean metric $\mathbf{g} = \operatorname{diag}\{1, r^2, 1\}$. We then assume an embedding of the material manifold into the ambient space of the form $(r, \theta, z) = (r(R), \Theta, \alpha Z)$, where α is a positive constant denoting the longitudinal stretch. Hence, $\mathbf{F} = \operatorname{diag}\{r'(R), 1, \alpha\}$. Assuming incompressibility, i.e., $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = 1$, one obtains $\frac{r(R)}{R} r'(R) \alpha = 1$. Eliminating the rigid body translation by setting $r(0) = 0$, one obtains $r(R) = \frac{1}{\sqrt{\alpha}} R$. Therefore, the right Cauchy–Green deformation tensor is written as[6]

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & -\frac{\alpha^2 f(R)}{R^2} \\ 0 & -\frac{f(R)}{\alpha} & \frac{\alpha^2}{R^2}(R^2 + f(R)^2) \end{pmatrix}. \tag{3.2}$$

Note that $\mathbf{N}_1 = \mathbf{E}_R$, $\mathbf{N}_2 = \mathbf{E}_Z$, and $\mathbf{N}_3 = \frac{1}{R}\mathbf{E}_\Theta - \frac{f(R)}{R}\mathbf{E}_Z$. Note also that $\mathbf{N}_3$ is obtained using the orthonormality of the material preferred directions, and $\mathbf{E}_R = \partial/\partial R$, $\mathbf{E}_Z = \partial/\partial Z$, and $\mathbf{E}_\Theta = \partial/\partial\Theta$ form a basis for $T_X\mathcal{B}$. Using (2.16), the invariants of the strain energy function are simplified and are written as

$$\begin{aligned} &I_1 = \operatorname{tr}\mathbf{C} = \frac{2}{\alpha} + \frac{\alpha^2}{R^2}(R^2 + f(R)^2), \quad I_2 = \frac{1}{2}\left[\operatorname{tr}(\mathbf{C}^2) - (\operatorname{tr}\mathbf{C})^2\right] = \frac{1}{\alpha^2} + 2\alpha + \alpha\frac{f(R)^2}{R^2}, \\ &I_4 = \frac{1}{\alpha}, \qquad I_5 = \frac{1}{\alpha^2}, \qquad I_6 = \alpha^2, \quad I_7 = \frac{\alpha^4}{R^2}(R^2 + f(R)^2). \end{aligned} \tag{3.3}$$

The nonzero components of the Cauchy stress tensor following (2.20) read

$$\sigma^{rr} = \frac{2}{\alpha^2}\left[W_{I_2}\left(\alpha^3 + \frac{\alpha^3 f(R)^2}{R^2} + 1\right) + \alpha W_{I_4} + 2W_{I_5}\right] + \frac{2W_{I_1}}{\alpha} - p(R), \tag{3.4}$$

$$\sigma^{\theta\theta} = \frac{2\alpha W_{I_1} + 2\left(\alpha^3 + 1\right) W_{I_2} - \alpha^2 p(R)}{\alpha R^2}, \tag{3.5}$$

$$\sigma^{zz} = \frac{2\alpha}{R^2}\Big[\left(f(R)^2 + R^2\right)\left(\alpha W_{I_1} + W_{I_2} + 2\alpha^3 W_{I_7}\right) + R^2(W_{I_2} + \alpha W_{I_6})\Big] - p(R), \tag{3.6}$$

$$\sigma^{\theta z} = -\frac{2f(R)}{R^2}\left(\alpha W_{I_1} + W_{I_2} + \alpha^3 W_{I_7}\right). \tag{3.7}$$

We assume that the stress vanishes when the body is dislocation-free and the longitudinal stretch $\alpha = 1$ (see also [19,38,61]). Thus

$$\left(W_{I_4} + 2W_{I_5}\right)|_{I_1=I_2=3, I_4=I_5=I_6=I_7=1} = 0, \quad \text{and} \quad \left(W_{I_6} + 2W_{I_7}\right)|_{I_1=I_2=3, I_4=I_5=I_6=I_7=1} = 0. \tag{3.8}$$

[6] The symbolic computations in this paper were performed using Mathematica [69].

In the absence of body and inertial forces, the only nontrivial equilibrium equation is ${\sigma^{rb}}_{|b} = 0$, implying[7] that (cf. (2.3)) ${\sigma^{rr}}_{,r} + \frac{\sigma^{rr}}{r} - r\sigma^{\theta\theta} = 0$. Therefore, $p'(R) = h(R)$, where

$$\begin{aligned} h(R) = \frac{2}{\alpha R^5} \Big[& 2R^3 f(R) f'(R) \Big(\alpha^2 W_{I_2} + \alpha^2 W_{I_1 I_1} + \alpha(2+\alpha^3) W_{I_1 I_2} + \alpha^2 W_{I_1 I_4} + 2\alpha W_{I_1 I_5} + \alpha^4 W_{I_1 I_7} \\ & + (\alpha^3+1) W_{I_2 I_2} + \alpha W_{I_2 I_4} + 2 W_{I_2 I_5} + \alpha^3(\alpha^3+1) W_{I_2 I_7} + \alpha^4 W_{I_4 I_7} + 2\alpha^3 W_{I_5 I_7} \Big) \\ & + 2\alpha^3 R f(R)^3 f'(R) \left(\alpha W_{I_1 I_2} + W_{I_2 I_2} + \alpha^3 W_{I_2 I_7} \right) - R^2 f(R)^2 \Big\{ 2\alpha W_{I_1 I_2} (2+\alpha^3) \\ & + 2\alpha^2 W_{I_1 I_4} + 4\alpha W_{I_1 I_5} + 2\alpha^4 W_{I_1 I_7} + \alpha^2 W_{I_2} + 2(\alpha^3+1) W_{I_2 I_2} + 2\alpha W_{I_2 I_4} + 4 W_{I_2 I_5} \\ & + 2\alpha^3(\alpha^3+1) W_{I_2 I_7} + 2\alpha^4 W_{I_4 I_7} + 4\alpha^3 W_{I_5 I_7} + 2\alpha^2 W_{I_1 I_1} \Big\} \\ & - 2\alpha^3 f(R)^4 \left(\alpha W_{I_1 I_2} + W_{I_2 I_2} + \alpha^3 W_{I_2 I_7} \right) + R^4 W_{I_4} \Big] + \frac{4 W_{I_5}}{\alpha^2 R}. \end{aligned} \tag{3.9}$$

If one assumes that the medium is a cylindrical bar with a finite radius R_o and the surface $R = R_o$ is traction-free, one obtains

$$p(R) = \int_{R_o}^{R} h(\zeta) \mathrm{d}\zeta + \frac{2}{\alpha^2} \left[\left(\alpha^3 + \frac{\alpha^3 f(R_o)^2}{R_o^2} + 1 \right) W_{I_2}|_{R=R_o} + \alpha W_{I_4}|_{R=R_o} + 2 W_{I_5}|_{R=R_o} \right] + \frac{2}{\alpha} W_{I_1}|_{R=R_o}. \tag{3.10}$$

Let us employ the so-called *standard reinforcing model* for compressible materials, which is defined as [38,39,59]

$$W = W\left(I_1, I_2, I_4, I_5, I_6, I_7\right) = W_{\text{iso}}\left(I_1, I_2\right) + W_{\text{fib}}^{R}\left(I_4, I_5\right) + W_{\text{fib}}^{Z}\left(I_6, I_7\right), \tag{3.11}$$

where W_{iso} denotes the strain energy function for the isotropic base material, whereas W_{fib}^{R} and W_{fib}^{Z} represent the anisotropy effects due to the fiber reinforcement in the radial and longitudinal directions, respectively. Consider as an example a cylindrical body made of a Mooney–Rivlin solid reinforced with fibers in the radial and longitudinal directions such that

$$\begin{aligned} W(I_1, I_2, I_4, I_5, I_6, I_7) = & \frac{\mu_1}{2}\left(I_1 - 3\right) + \frac{\mu_2}{2}\left(I_2 - 3\right) + \frac{\gamma_1}{2}\left(I_4 - 1\right)^2 \\ & + \frac{\gamma_2}{2}\left(I_5 - 1\right)^2 + \frac{\xi_1}{2}\left(I_6 - 1\right)^2 + \frac{\xi_2}{2}\left(I_7 - 1\right)^2. \end{aligned} \tag{3.12}$$

Using (3.9), we have

$$h(R) = \alpha\mu_2 \frac{f(R)}{R^3} \left[2R f'(R) - f(R) \right] + \frac{2}{R} \frac{1}{\alpha^2} \left(\frac{1}{\alpha} - 1 \right) \left[\alpha\gamma_1 + 2\gamma_2 \left(1 + \frac{1}{\alpha} \right) \right]. \tag{3.13}$$

Thus, from (3.10)

$$\begin{aligned} p(R) = & \alpha\mu_2 \int_{R_o}^{R} \frac{f(\zeta)}{\zeta^3} \left[2\zeta f'(\zeta) - f(\zeta) \right] \mathrm{d}\zeta + \frac{2}{\alpha^2} \left(\frac{1}{\alpha} - 1 \right) \left[\alpha\gamma_1 + 2\gamma_2 \left(1 + \frac{1}{\alpha} \right) \right] \left[1 + \ln \frac{R}{R_o} \right] \\ & + \frac{\mu_1}{\alpha} + \frac{\mu_2}{\alpha^2} \left(\alpha^3 + \frac{\alpha^3 f(R_o)^2}{R_o^2} + 1 \right). \end{aligned} \tag{3.14}$$

[7] Note that $p = p(R)$ is implied from the other equilibrium equations.

The physical components of the Cauchy stress read[8]

$$\hat{\sigma}^{rr} = \alpha\mu_2\left(\frac{f(R)^2}{R^2} - \frac{f(R_o)^2}{R_o^2}\right) + \alpha\mu_2\int_R^{R_o}\frac{f(\zeta)}{\zeta^3}\left[2\zeta f'(\zeta) - f(\zeta)\right]\mathrm{d}\zeta - \frac{2}{\alpha^2}\left(\frac{1}{\alpha}-1\right)\left\{\alpha\gamma_1 + 2\gamma_2\left(1+\frac{1}{\alpha}\right)\right\}\ln\frac{R}{R_o}, \tag{3.15}$$

$$\hat{\sigma}^{\theta\theta} = \alpha\mu_2\int_R^{R_o}\frac{f(\zeta)}{\zeta^3}\left[2\zeta f'(\zeta) - f(\zeta)\right]\mathrm{d}\zeta - \alpha\mu_2\frac{f(R_o)^2}{R_o^2} - \frac{2}{\alpha^2}\left(\frac{1}{\alpha}-1\right)\left\{\alpha\gamma_1 + 2\gamma_2\left(1+\frac{1}{\alpha}\right)\right\}\left[1+\ln\frac{R}{R_o}\right], \tag{3.16}$$

$$\hat{\sigma}^{zz} = 2\alpha^2(\alpha^2-1)\xi_1 + \frac{4\alpha^4\xi_2}{R^2}\left(f(R)^2+R^2\right)\left[\frac{\alpha^4}{R^2}\left(f(R)^2+R^2\right)-1\right] + \alpha\mu_2\left(\frac{f(R)^2}{R^2} - \frac{f(R_o)^2}{R_o^2}\right) - \frac{2}{\alpha^2}\left(\frac{1}{\alpha}-1\right)\left\{\alpha\gamma_1 + 2\gamma_2\left(1+\frac{1}{\alpha}\right)\right\}\left[1+\ln\frac{R}{R_o}\right] + \alpha\mu_2\int_R^{R_o}\frac{f(\zeta)}{\zeta^3}\left[2\zeta f'(\zeta) - f(\zeta)\right]\mathrm{d}\zeta + (\alpha\mu_1+\mu_2)\left(\alpha - \frac{1}{\alpha^2}\right) + \alpha^2\mu_1\frac{f(R)^2}{R^2}, \tag{3.17}$$

$$\hat{\sigma}^{\theta z} = -\frac{f(R)}{\alpha^{\frac{1}{2}}R}\left(\alpha\mu_1 + \mu_2 + 2\alpha^3\xi_2\left[\frac{\alpha^4}{R^2}\left(R^2+f(R)^2\right)-1\right]\right). \tag{3.18}$$

Remark 3.1. From (3.14), for an arbitrary cylindrically symmetric distribution of parallel screw dislocations, the pressure $p(R)$, and hence, $\hat{\sigma}^{rr}$, $\hat{\sigma}^{\theta\theta}$, and $\hat{\sigma}^{zz}$ exhibit a logarithmic singularity on the dislocation axis ($R=0$) unless $\alpha=1$. Note that this singularity is inherently due to the anisotropy effects, i.e., the presence of the reinforcement in the radial direction. In particular, the singularity does not occur when $\gamma_1=\gamma_2=0$, e.g., when the material is isotropic. Note also that as $R\to 0$, $f(R)=\frac{b(0)}{4\pi}R^2+\mathcal{O}(R^3)$, and thus, $\frac{f(R)}{R}$ is finite at $R=0$. This implies that unlike the other stress components, $\hat{\sigma}^{\theta z}$ is nonsingular.

Remark 3.2. Note that in the case of fiber-reinforced neo-Hookean materials ($\mu_2=0$) and a given arbitrary cylindrically symmetric distribution of screw dislocations supported on a cylinder of radius R_i, the stress field for $R>R_i$ is independent of $b(R)$ and is identical to that of a single screw dislocation with Burgers vector $b_0=\int_0^{R_i}\eta b(\eta)\mathrm{d}\eta$. Acharya [1] and Yavari and Goriely [72] had observed this for isotropic neo-Hookean solids.

As an example, let us assume the following Burgers' vector density distribution:

$$b(R) = \begin{cases} b_0 & 0<R\leq R_i, \\ 0 & R_i<R\leq R_o, \end{cases} \tag{3.19}$$

where $R_i\leq R_o$. Thus,

$$f(R) = \frac{1}{2\pi}\int_0^R \eta b(\eta)\mathrm{d}\eta = \frac{b_0}{4\pi}\begin{cases} R^2 & 0<R\leq R_i, \\ R_i^2 & R_i<R\leq R_o. \end{cases} \tag{3.20}$$

Figure 1 depicts the variation of the different components of the Cauchy stress for the Burgers' vector density distribution (3.19) such that $R_i/R_o=0.5$ and $b_0R_o=20$. Notice that the $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$ vanish for a neo-Hookean solid.

[8] The physical components of the Cauchy stress tensor, i.e., $\hat{\sigma}^{ab}=\sigma^{ab}\sqrt{g_{aa}g_{bb}}$ (no summation) [60] are given as $\hat{\sigma}^{rr}=\sigma^{rr}$, $\hat{\sigma}^{\theta\theta}=r^2(R)\sigma^{\theta\theta}$, $\hat{\sigma}^{zz}=\sigma^{zz}$, and $\hat{\sigma}^{\theta z}=r(R)\sigma^{\theta z}$.

Remark 3.3. As noted by Zubov [81], the energy per unit length (along the dislocation line) of a single screw dislocation in a Mooney–Rivlin solid is unbounded.[9] This is also the case for a fiber-reinforced Mooney–Rivlin material due to the standard reinforcing model considered here (cf. (3.11)). Let us consider incompressible isotropic base materials, for which the energy per unit length of a single screw dislocation remains bounded, i.e., $2\pi \int_0^{R_o} W_{\text{iso}}(I_1(\xi), I_2(\xi))\xi d\xi < \infty$, for finite R_o (examples include Varga [81], incompressible power-law [29,49], generalized incompressible neo-Hookean materials [72], and Hencky material [71]). Exploiting the standard reinforcing model, the energy function for the fiber-reinforced material with the isotropic base with the energy function $W_{\text{iso}}(I_1, I_2)$ is assumed to be given as

$$W = W_{\text{iso}}(I_1, I_2) + \frac{\gamma_1}{2}(I_4 - 1)^2 + \frac{\gamma_2}{2}(I_5 - 1)^2 + \frac{\xi_1}{2}(I_6 - 1)^2 + \frac{\xi_2}{2}(I_7 - 1)^\beta. \tag{3.21}$$

Then, the energy per unit length along a single screw dislocation line is finite if $\beta < 1$. To see this, we need to show that $2\pi \int_0^{R_o} \frac{\xi_2}{2}(I_7(\zeta) - 1)^\beta \zeta d\zeta < \infty$ as the finiteness of the contribution of the other terms in the energy per unit length is trivial (cf. (3.3)). Noting that for a single screw dislocation with Burgers vector b_i, $b(R) = 2\pi b_i \delta^2(R)$, and hence, $f(R) = \frac{b_i}{2\pi} H(R)$, we have

$$\pi\xi_2 \int_0^{R_o} (I_7(\zeta) - 1)^\beta \zeta d\zeta = \pi\xi_2 \int_0^{R_o} \left[\frac{\alpha^4}{\zeta^2}(\zeta^2 + f(\zeta)^2) - 1\right]^\beta \zeta d\zeta = \pi\xi_2 \int_0^{R_o} \left[\alpha^4 - 1 + \frac{\alpha^4 b_i^2}{4\pi^2\zeta^2}\right]^\beta \zeta d\zeta < \infty, \tag{3.22}$$

provided that $\beta < 1$. Similarly, one can show that if the resultant longitudinal force, i.e., $F_Z = 2\pi \int_0^{R_o} \hat{\sigma}^{zz}(\zeta)\zeta d\zeta$, induced by a single screw dislocation is finite for the isotropic base material with the energy function $W_{\text{fib}}(I_1, I_2)$, so is the axial force for the fiber-reinforced material with the energy function (3.21) when $\beta < 1$.

As the underlying geometry of the material manifold explicitly depends on the distribution of defects, so are the material preferred directions (and thus, the material symmetry group). One of the consequences of this is that the class of anisotropy of the defective body is, in general, different in the reference and current configurations given that the reference configuration has a nontrivial geometry, whereas the geometry of the current configuration is trivial. The next section is aimed at illustrating that depending on whether the dislocated body is orthotropic in its reference configuration or in its current configuration, the induced residual stresses are different.

3.2. A cylindrically symmetric distribution of parallel screw dislocations in a monoclinic medium

In the previous section, we assumed that the dislocated body is orthotropic in the reference configuration. Instead, let us assume that the medium with the cylindrically symmetric distribution of parallel screw dislocations is orthotropic in its current configuration such that the orthotropic axes are in the radial, circumferential, and axial directions in the ambient space. In the reference configuration, the material will be monoclinic such that $\mathbf{N}_3 = \hat{\mathbf{R}}$ is perpendicular to the plane of $\mathbf{N}_1 = \hat{\mathbf{\Theta}}$ and $\mathbf{N}_2 = \hat{\mathbf{Z}}$.[10] We assume the same class of deformations as was assumed in the previous section, and thus, $r(R) = \frac{1}{\sqrt{\alpha}} R$. Hence, the right Cauchy–Green deformation tensor is given by (3.2). The invariants of the strain energy function

[9] Note, however, that the energy of distributed screw dislocations is not necessarily unbounded (see also [50]). In particular, a Mooney–Rivlin reinforced material with the energy function (3.12) and the Burgers' vector distribution (3.19) has a finite energy per unit length.

[10] Note that $\mathbf{N}_1 = \mathbf{E}_\Theta/(R^2 + f(R)^2)^{1/2}$ and $\mathbf{N}_2 = \mathbf{E}_Z$ are not orthogonal in the nontrivial geometry of the reference configuration (cf. (3.1)).

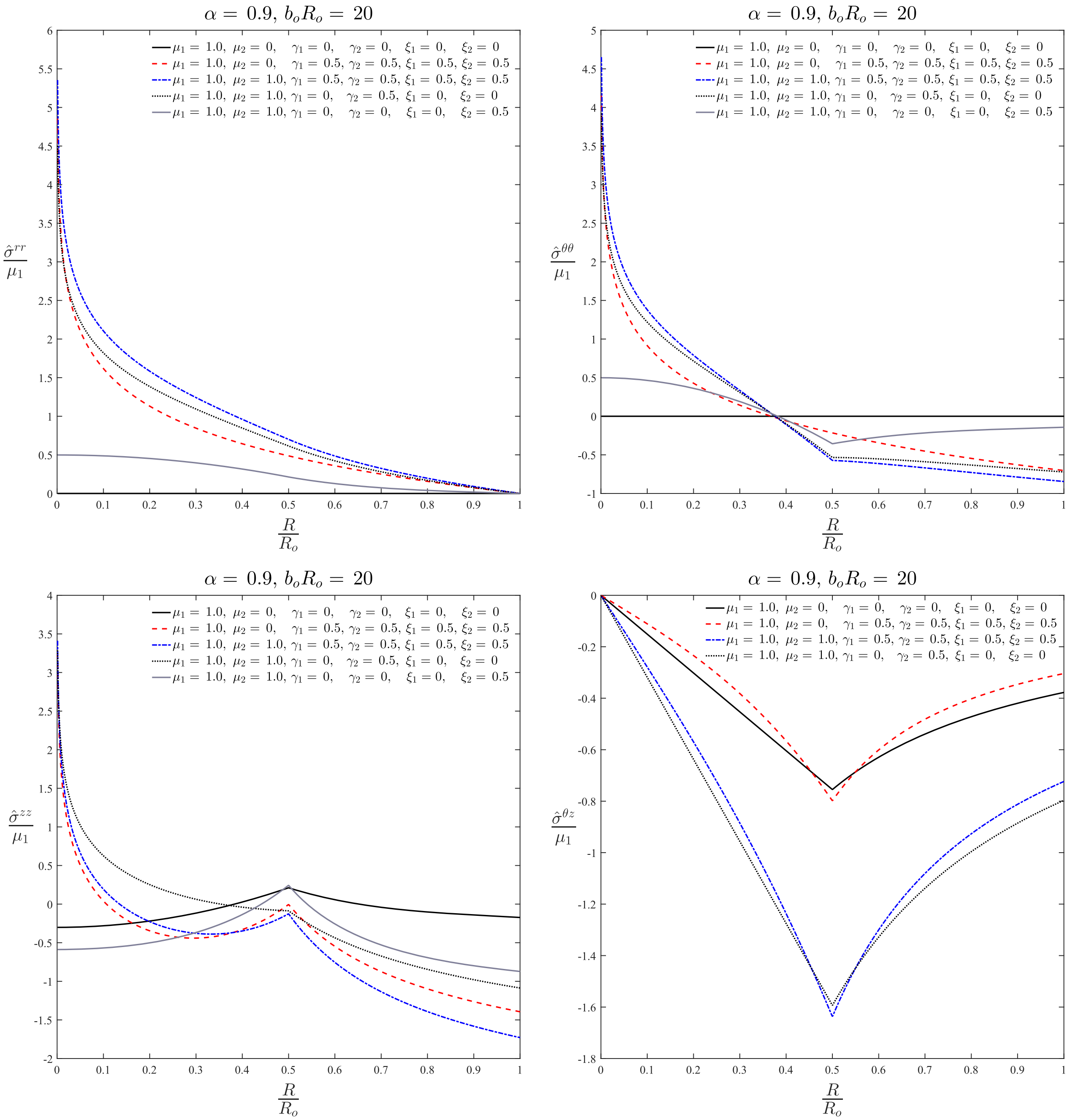

FIG. 1. Stress distribution in a medium with the constitutive equation (3.12) and the dislocation distribution (3.19) such that $R_i/R_o = 0.5$, $b_0R_o = 20$, and $\alpha = 0.9$ for different values of the constitutive parameters

for the monoclinic material are given as

$$I_1 = \mathrm{tr}(\mathbf{C}) = \frac{2}{\alpha} + \frac{\alpha^2}{R^2}(R^2 + f(R)^2), \quad I_2 = \frac{1}{2}\left[\mathrm{tr}(\mathbf{C}^2) - (\mathrm{tr}\,\mathbf{C})^2\right] = \frac{1}{\alpha^2} + 2\alpha + \alpha\frac{f(R)^2}{R^2},$$
$$I_4 = \frac{R^2}{\alpha(R^2 + f(R)^2)},\ I_5 = \frac{R^2}{\alpha^2(R^2 + f(R)^2)},\ I_6 = \alpha^2,\ I_7 = \frac{\alpha^4}{R^2}(R^2 + f(R)^2),\ I_8 = 0. \tag{3.23}$$

From (2.24), the nonzero components of the Cauchy stress tensor read

$$\sigma^{rr} = \frac{2W_{I_2}}{\alpha^2}\left(\alpha^3 + \frac{\alpha^3 f(R)^2}{R^2} + 1\right) + \frac{2W_{I_1}}{\alpha} - p(R), \tag{3.24}$$

$$\sigma^{\theta\theta} = \frac{2\alpha W_{I_1} + 2\left(\alpha^3+1\right)W_{I_2} - \alpha^2 p(R)}{\alpha R^2} + \frac{2(\alpha W_{I_4} + 2W_{I_5})}{\alpha(R^2+f(R)^2)}, \tag{3.25}$$

$$\sigma^{zz} = \frac{2\alpha}{R^2}\Big[\left(f(R)^2+R^2\right)\left(\alpha W_{I_1} + W_{I_2} + 2\alpha^3 W_{I_7}\right) + R^2(W_{I_2} + \alpha W_{I_6})\Big] - p(R), \tag{3.26}$$

$$\sigma^{\theta z} = -\frac{2f(R)}{R^2}\left(\alpha W_{I_1} + W_{I_2} + \alpha^3 W_{I_7}\right) - \frac{2f(R)}{R^2+f(R)^2}W_{I_5} + \frac{\alpha}{(R^2+f(R)^2)^{\frac{1}{2}}}W_{I_8}. \tag{3.27}$$

Note that for the stress to vanish when $\alpha = 1$ and the body is dislocation-free, i.e., $f(R) = 0$ (identically), one needs to have $(W_{I_4} + 2W_{I_5}) = (W_{I_6} + 2W_{I_7}) = W_{I_8} = 0$, evaluated at $I_1 = I_2 = 3$, $I_4 = I_5 = I_6 = I_7 = 1$, $I_8 = 0$. The equilibrium equation implies that $p'(R) = S(R)$, where

$$\begin{aligned} S(R) = &-\frac{1}{\alpha^4 R^3}\Bigg[-4\alpha f(R)\left(Rf'(R) - f(R)\right)\left(\alpha^4 W_{I_1I_1} + \alpha^3 W_{I_1I_2} + \alpha^6 W_{I_1I_7} - \frac{R^4(\alpha W_{I_1I_4} + W_{I_1I_5})}{\left(f(R)^2+R^2\right)^2}\right) \\ &- 4f(R)\Big(\alpha^3 + \frac{\alpha^3 f(R)^2}{R^2} + 1\Big)\left(Rf'(R) - f(R)\right)\Big\{\alpha^4 W_{I_1I_2} + \alpha^3 W_{I_2I_2} + \alpha^6 W_{I_2I_7} \\ &- \frac{R^4(\alpha W_{I_2I_4} + W_{I_2I_5})}{\left(f(R)^2+R^2\right)^2}\Big\} + 4\alpha^5 f(R) W_{I_2}\left(f(R) - Rf'(R)\right) + \frac{2\alpha^2 R^4(\alpha W_{I_4} + 2W_{I_5})}{f(R)^2+R^2} \\ &- 2\alpha^5 f(R)^2 W_{I_2}\Bigg]. \end{aligned} \tag{3.28}$$

Assuming that the surface $R = R_o$ is traction-free the pressure field is obtained as

$$p(R) = \int_{R_o}^{R} S(\zeta)\mathrm{d}\zeta + \frac{2}{\alpha^2}\left(\alpha^3 + \frac{\alpha^3 f(R_o)^2}{R_o^2} + 1\right)W_{I_2}|_{R=R_o} + \frac{2}{\alpha}W_{I_1}|_{R=R_o}. \tag{3.29}$$

Let us consider the following model for the strain energy function

$$W = W\left(I_1, I_2, I_4, I_5, I_6, I_7, I_8\right) = W_{\text{iso}}\left(I_1, I_2\right) + W_{\text{fib}}^{\Theta}\left(I_4, I_5\right) + W_{\text{fib}}^{Z}\left(I_6, I_7\right) + W_{\text{fib}}^{Z\Theta}(I_8), \tag{3.30}$$

where W_{iso} describes that part of the energy function pertaining to the isotropic base material, while W_{fib}^{Θ} and W_{fib}^{Z} represent the reinforcement effects in the circumferential and axial directions. $W_{\text{fib}}^{Z\Theta}(I_8)$ models the coupling between the axial and circumferential fibers. Note, however, that $I_8 = 0$, and for the stress to vanish for the dislocation-free body, one needs $W_{I_8} = 0$ at $I_8 = 0$, which implies that $W_{\text{fib}}^{Z\Theta}(I_8) = 0$, i.e., the coupling term must vanish. A way out would be to require that the coupling term depends on some other invariants as well, e.g., one can define $W_{\text{fib}}^{Z\Theta}(I_1, I_8) = \eta I_8(I_1 - 3)$ for some positive constant η. For the sake of simplicity, as an example, we consider a fiber-reinforced Mooney–Rivlin material with the following energy function

$$\begin{aligned} W(I_1, I_2, I_4, I_5, I_6, I_7) = &\frac{\mu_1}{2}\left(I_1 - 3\right) + \frac{\mu_2}{2}\left(I_2 - 3\right) + \frac{\lambda_1}{2}\left(I_4 - 1\right)^2 \\ &+ \frac{\lambda_2}{2}\left(I_5 - 1\right)^2 + \frac{\xi_1}{2}\left(I_6 - 1\right)^2 + \frac{\xi_2}{2}\left(I_7 - 1\right)^2. \end{aligned} \tag{3.31}$$

Therefore, one obtains

$$S(R) = \frac{1}{\alpha^2 R^3\left(f(R)^2+R^2\right)}\Bigg[\alpha^3\mu_2 f(R)(R^2+f(R)^2)(2Rf'(R)-f(R)) \\ +2R^4\left\{\alpha\lambda_1 - \frac{R^2}{f(R)^2+R^2}(\frac{2\lambda_2}{\alpha^2}+\lambda_1)+2\lambda_2\right\}\Bigg]. \tag{3.32}$$

Thus, the physical components of the stress are given as

$$\hat{\sigma}^{rr} = \alpha\mu_2\left(\frac{f(R)^2}{R^2}-\frac{f(R_o)^2}{R_o^2}\right)+\int_R^{R_o} S(\zeta)\mathrm{d}\zeta, \tag{3.33}$$

$$\hat{\sigma}^{\theta\theta} = \int_R^{R_o} S(\zeta)\mathrm{d}\zeta - \alpha\mu_2\frac{f(R_o)^2}{R_o^2}+\frac{2R^2}{\alpha^2(R^2+f(R)^2)}\Bigg[\alpha\lambda_1\left(\frac{R^2}{\alpha(R^2+f(R)^2)}-1\right) \\ +2\lambda_2\left(\frac{R^2}{\alpha^2(R^2+f(R)^2)}-1\right)\Bigg], \tag{3.34}$$

$$\hat{\sigma}^{zz} = 2\alpha^2\xi_1(\alpha^2-1)+4\alpha^4\xi_2\left(1+\frac{f(R)^2}{R^2}\right)\left[\alpha^4\left(1+\frac{f(R)^2}{R^2}\right)-1\right]+\int_R^{R_o} S(\zeta)\mathrm{d}\zeta \\ +\alpha\mu_2\left(\frac{f(R)^2}{R^2}-\frac{f(R_o)^2}{R_o^2}\right)+(\alpha\mu_1+\mu_2)(\alpha-\frac{1}{\alpha^2})+\alpha^2\mu_1\frac{f(R)^2}{R^2}, \tag{3.35}$$

$$\hat{\sigma}^{\theta z} = -\frac{1}{\sqrt{\alpha}}\frac{f(R)}{R}(\alpha\mu_1+\mu_2)-2\xi_2\alpha^{\frac{5}{2}}\frac{f(R)}{R}\left[\alpha^4\left(1+\frac{f(R)^2}{R^2}\right)-1\right] \\ -\frac{2\lambda_2}{\sqrt{\alpha}}\frac{Rf(R)}{R^2+f(R)^2}\left(\frac{R^2}{\alpha^2(R^2+f(R)^2)}-1\right). \tag{3.36}$$

Remark 3.4. For an arbitrary cylindrically symmetric distribution of parallel screw dislocations with a smooth Burgers' vector density $b(R)$ in a monoclinic material, the pressure, and hence, $\hat{\sigma}^{rr}$, $\hat{\sigma}^{\theta\theta}$, and $\hat{\sigma}^{zz}$ have a logarithmic singularity on the dislocation axis unless $\alpha=1$. Nevertheless, the shear component $\hat{\sigma}^{\theta z}$ is finite and vanishes at $R=0$. This is because as $R\to 0$, $f(R)=\frac{b(0)}{4\pi}R^2+\mathcal{O}(R^3)$, and thus, from (3.32)

$$S(R) = \frac{2\left(\alpha-1\right)}{\alpha^2}\left[\lambda_1+\frac{2\lambda_2}{\alpha}\left(1+\frac{1}{\alpha}\right)\right]\frac{1}{R}+\mathcal{O}(R). \tag{3.37}$$

Therefore, $p(R)=C-\frac{2(\alpha-1)}{\alpha^2}\left[\lambda_1+\frac{2\lambda_2}{\alpha}\left(1+\frac{1}{\alpha}\right)\right]\ln\frac{R}{R_o}+\mathcal{O}(R^2)$ as $R\to 0$, where C is a constant. It is straightforward to see that when $\alpha=1$, the stress is finite and $\hat{\sigma}^{rr}=\hat{\sigma}^{\theta\theta}=\hat{\sigma}^{zz}$ at $R=0$.

In Fig. 2, the stress field is shown for the dislocation distribution (3.19), where $R_i/R_o=0.5$ and $b_0R_o=20$ for different values of the constitutive parameters given by (3.31).

3.3. A parallel cylindrically symmetric distribution of wedge disclinations in an orthotropic medium

Let us consider a parallel cylindrically symmetric distribution of wedge disclinations in an infinite orthotropic medium in the reference configuration. In the cylindrical coordinates (R,Θ,Z), assume that the material orthotropic axes are in the R, Θ, and Z directions. The radial density of the wedge disclinations is denoted by $w(R)$. The material manifold for a body having a distribution of wedge disclinations is a Riemannian manifold with a nonvanishing curvature. The material metric for the disclinated body is

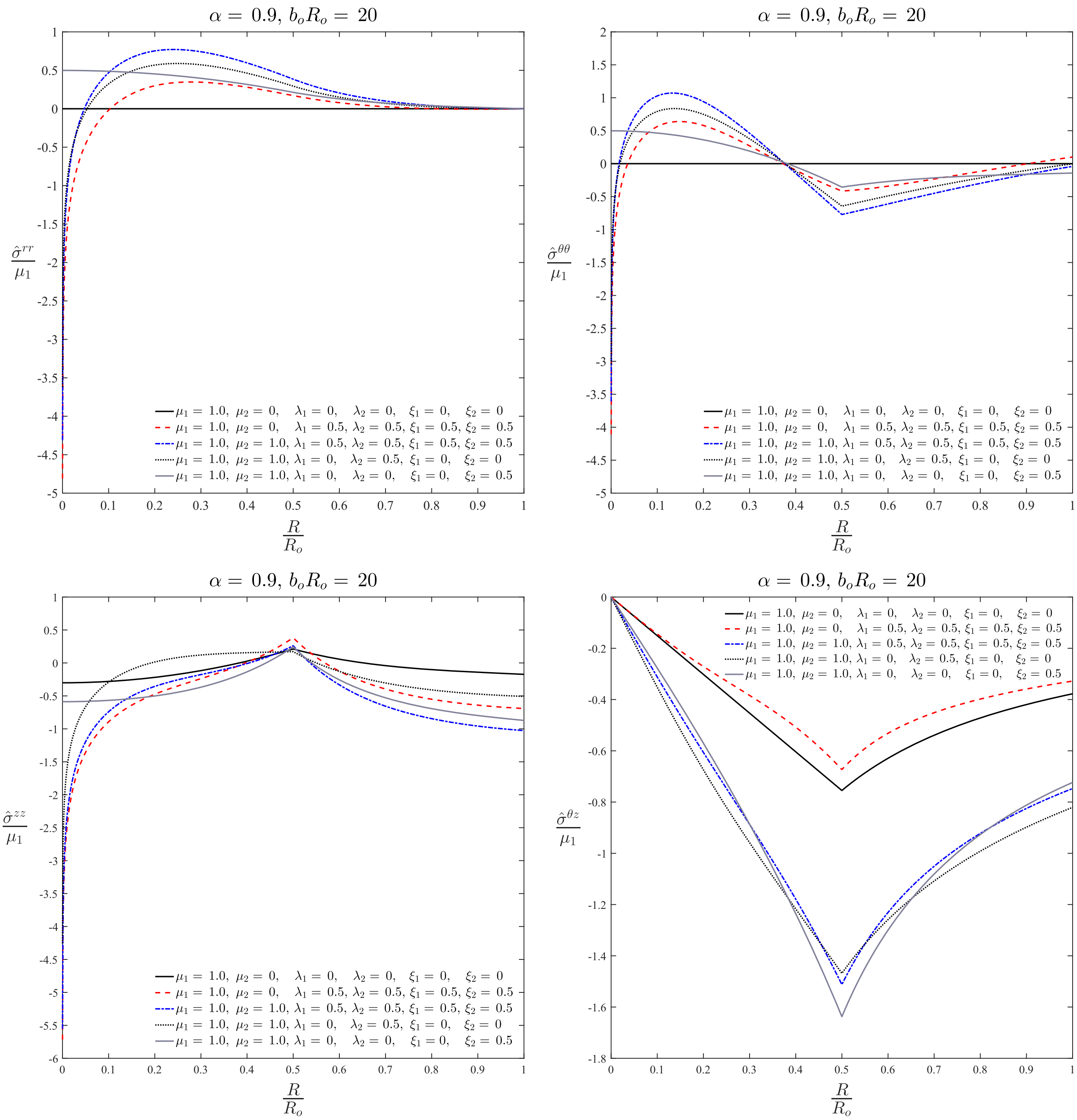

Fig. 2. Stress distribution in a medium with the constitutive equation (3.31) and the dislocation distribution (3.19) such that $R_i/R_o = 0.5$, $b_o R_o = 20$, and $\alpha = 0.9$ for different values of the constitutive parameters

given by [75]

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f(R)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.38}$$

where $f''(R) = -\frac{R}{2\pi}w(R)$. The ambient space is endowed with the Euclidean metric $\mathbf{g} = \mathrm{diag}\{1, r^2, 1\}$. We embed the material manifold into the ambient space by looking for mappings[11] of the form $(r, \theta, z) = (r(R), \Theta, \alpha Z)$, where α is a constant representing the axial stretch of the bar that depends on the axial boundary conditions. Therefore, the deformation gradient reads $\mathbf{F} = \mathrm{diag}\,(r'(R), 1, \alpha)$. Incompressibility constraint dictates that $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}}\det \mathbf{F} = \alpha \frac{r(R)}{f(R)} r'(R) = 1$. Thus, imposing $r(0) = 0$, we have $r(R) = \left(\frac{2}{\alpha}\int_0^R f(\xi)\mathrm{d}\xi\right)^{\frac{1}{2}}$. The right Cauchy–Green deformation tensor reads $\mathbf{C} = \mathrm{diag}\left\{\frac{1}{\alpha^2}\frac{f(R)^2}{r(R)^2}, \frac{r(R)^2}{f(R)^2}, \alpha^2\right\}$. From (2.16), the invariants of the strain energy function are simplified to read

$$
\begin{aligned}
I_1 =& \mathrm{tr}(\mathbf{C}) = \alpha^2 + \frac{1}{\alpha^2}\frac{f(R)^2}{r(R)^2} + \frac{r(R)^2}{f(R)^2}, \quad I_2 = \frac{1}{2}(\mathrm{tr}(\mathbf{C}^2) - \mathrm{tr}(\mathbf{C})^2) = \frac{1}{\alpha^2} + \alpha^2\frac{r(R)^2}{f(R)^2} + \frac{f(R)^2}{r(R)^2},\\
I_4 =& \frac{1}{\alpha^2}\frac{f(R)^2}{r(R)^2}, \qquad I_5 = \frac{1}{\alpha^4}\frac{f(R)^4}{r(R)^4}, \qquad I_6 = \alpha^2, \quad I_7 = \alpha^4.
\end{aligned}
\tag{3.39}
$$

The nonzero physical components of the Cauchy stress are as follows[12]

$$
\hat{\sigma}^{rr} = \frac{2}{\alpha^2}\frac{f(R)^2}{r(R)^2}\left(W_{I_1} + \alpha^2 W_{I_2} + W_{I_4}\right) + \frac{4}{\alpha^4}\frac{f(R)^4}{r(R)^4}W_{I_5} + \frac{2}{\alpha^2}W_{I_2} - p(R), \tag{3.40}
$$

$$
\hat{\sigma}^{\theta\theta} = 2\frac{r(R)^2}{f(R)^2}\left(W_{I_1} + \alpha^2 W_{I_2}\right) + \frac{2}{\alpha^2}W_{I_2} - p(R), \tag{3.41}
$$

$$
\hat{\sigma}^{zz} = 2\alpha^2\left(W_{I_1} + W_{I_6} + 2\alpha^2 W_{I_7}\right) + 2W_{I_2}\left(\frac{f(R)^2}{r(R)^2} + \alpha^2\frac{r(R)^2}{f(R)^2}\right) - p(R). \tag{3.42}
$$

The equilibrium equation implies that $p'(R) = k(R)$, where

$$
\begin{aligned}
k(R) = \frac{2}{\alpha^8 f^3 r^9}\Big[& 2\alpha^4 f^4 r^7 f'\left(\alpha^2 W_{I_1} + W_{I_1 I_2} - 2W_{I_1 I_5} + \alpha^4 W_{I_2} + \alpha^2 W_{I_2 I_2} + W_{I_2 I_4} - 2\alpha^2 W_{I_2 I_5} + \alpha^2 W_{I_4}\right)\\
&+ 2\alpha^2 f^6 r^5 f'\left(\alpha^2 W_{I_1 I_1} + 2\alpha^4 W_{I_1 I_2} + 2\alpha^2 W_{I_1 I_4} + \alpha^6 W_{I_2 I_2} + 2\alpha^4 W_{I_2 I_4} + 2W_{I_2 I_5} + \alpha^2 W_{I_4 I_4} + 4\alpha^2 W_{I_5}\right)\\
&- 2\alpha^6 f^2 r^9 f'\left(W_{I_1 I_1} + 2\alpha^2 W_{I_1 I_2} + W_{I_1 I_4} + \alpha^4 W_{I_2 I_2} + \alpha^2 W_{I_2 I_4}\right)\\
&+ 8\alpha^2 f^8 r^3 f'\left(W_{I_1 I_5} + \alpha^2 W_{I_2 I_5} + W_{I_4 I_5}\right) + 8f^{10} W_{I_5 I_5} r f' - 2\alpha^6 r^{11} f'\left(W_{I_1 I_2} + \alpha^2 W_{I_2 I_2}\right)\\
&- \alpha^3 f^6 r^5\left(\alpha^2 W_{I_1} + 2W_{I_1 I_2} - 4W_{I_1 I_5} + \alpha^4 W_{I_2} + 2\alpha^2 W_{I_2 I_2} + 2W_{I_2 I_4} - 4\alpha^2 W_{I_2 I_5} + \alpha^2 W_{I_4}\right)\\
&- \alpha^5 f^2 r^9\left(\alpha^2 W_{I_1} - 2W_{I_1 I_2} + \alpha^4 W_{I_2} - 2\alpha^2 W_{I_2 I_2}\right) - 8\alpha f^{10} r\left(W_{I_1 I_5} + \alpha^2 W_{I_2 I_5} + W_{I_4 I_5}\right)\\
&- 2\alpha f^8 r^3\left(\alpha^2 W_{I_1 I_1} + 2\alpha^4 W_{I_1 I_2} + 2\alpha^2 W_{I_1 I_4} + \alpha^6 W_{I_2 I_2} + 2\alpha^4 W_{I_2 I_4} + 2W_{I_2 I_5} + \alpha^2 W_{I_4 I_4} + 3\alpha^2 W_{I_5}\right)\\
&+ 2\alpha^5 f^4 r^7\left(W_{I_1 I_1} + 2\alpha^2 W_{I_1 I_2} + W_{I_1 I_4} + \alpha^4 W_{I_2 I_2} + \alpha^2 W_{I_2 I_4}\right) - \frac{8f^{12} W_{I_5 I_5}}{\alpha r}\Big].
\end{aligned}
\tag{3.43}
$$

Assuming (3.12) for the energy function, one obtains

$$
\begin{aligned}
p'(R) = -\frac{1}{\alpha^9 f r^{10}}\Big[& -2\alpha^7 f^2 r^8 f'\left(\alpha^2\mu_2 - 2\gamma_1 + \mu_1\right) + \alpha^5 f^4 r^6\left\{\alpha\left(\alpha^2\mu_2 - 2\gamma_1 + \mu_1\right) - 8(\gamma_1 - 2\gamma_2)f'\right\}\\
&- 32\alpha\gamma_2 f^8 r^2 f' + 6\alpha^4(\gamma_1 - 2\gamma_2) f^6 r^4 + 28\gamma_2 f^{10} + \alpha^8 r^{10}\left(\alpha^2\mu_2 + \mu_1\right)\Big].
\end{aligned}
\tag{3.44}
$$

[11] Note that for the class of deformations that is considered, the material will be orthotropic in its current configuration as well. The orthotropic axes in the current configuration will be in the radial, circumferential, and axial directions (similar to those in the reference configuration).

[12] When the body is disclination-free $f(R) = R$, and the stress vanishes if the energy function satisfies (3.8).

Knowing that the traction vanishes on the outer boundary $R = R_o$, one finds

$$p(R_o) = \frac{1}{\alpha^2}\frac{f(R_o)^2}{r(R_o)^2}\left\{\mu_1 + \alpha^2\mu_2 + 2\gamma_1\left[\frac{1}{\alpha^2}\frac{f(R_o)^2}{r(R_o)^2} - 1\right]\right\} + \frac{4\gamma_2}{\alpha^4}\frac{f(R_o)^4}{r(R_o)^4}\left[\frac{1}{\alpha^4}\frac{f(R_o)^4}{r(R_o)^4} - 1\right] + \frac{\mu_2}{\alpha^2}. \quad (3.45)$$

Therefore, $p(R) = \int_{R_o}^{R} p'(\xi)\mathrm{d}\xi + p(R_o)$. The stress components are simplified and read

$$\hat{\sigma}^{rr} = \frac{1}{\alpha^2}\frac{f(R)^2}{r(R)^2}\left\{\mu_1 + \alpha^2\mu_2 + 2\gamma_1\left[\frac{1}{\alpha^2}\frac{f(R)^2}{r(R)^2} - 1\right]\right\} + \frac{4\gamma_2}{\alpha^4}\frac{f(R)^4}{r(R)^4}\left[\frac{1}{\alpha^4}\frac{f(R)^4}{r(R)^4} - 1\right] + \frac{\mu_2}{\alpha^2} - p(R), \quad (3.46)$$

$$\hat{\sigma}^{\theta\theta} = \frac{r(R)^2}{f(R)^2}\left(\mu_1 + \alpha^2\mu_2\right) + \frac{\mu_2}{\alpha^2} - p(R), \quad (3.47)$$

$$\hat{\sigma}^{zz} = \alpha^2\left[\mu_1 + 2\xi_1(\alpha^2 - 1) + 4\alpha^2\xi_2(\alpha^4 - 1)\right] + \mu_2\left[\frac{f(R)^2}{r(R)^2} + \alpha^2\frac{r(R)^2}{f(R)^2}\right] - p(R). \quad (3.48)$$

Example 3.5. For a uniform disclination distribution $w(R) = w_o$, one has $f''(R) = -\frac{R}{2\pi}w_o$, and thus, $f(R) = R - \frac{w_o}{12\pi}R^3$. Therefore

$$r(R) = \frac{R}{\alpha^{\frac{1}{2}}}\left(1 - \frac{w_o}{24\pi}R^2\right)^{\frac{1}{2}}, \quad (3.49)$$

provided that $w_o < 24\pi/R_o^2$.

Remark 3.6. For the uniform disclination distribution, the stress field exhibits a logarithmic singularity on the disclinations axis unless the axial stretch $\alpha = 1$. Moreover, when $\alpha = 1$, the stress is finite and hydrostatic at $R = 0$. To see this, as $R \to 0$, $r(R) = \frac{R}{\alpha^{\frac{1}{2}}} + \mathcal{O}(R^3)$, and $f(R) = R + \mathcal{O}(R^3)$. From (3.44), therefore

$$p(R) = C + \frac{2(1-\alpha)}{\alpha^4}\left[\alpha^2\gamma_1 + 2\gamma_2(1+\alpha)\right]\ln R + \mathcal{O}(R^2), \quad (3.50)$$

where C is a constant. Hence, the stress is logarithmically unbounded at $R = 0$ unless $\alpha = 1$. Similar to the case of parallel screw dislocations in an orthotropic medium (cf. Remark. 3.1), the singularity arises as a result of radial reinforcement effects, and does not, in particular, occur in isotropic materials. Note that for $\alpha = 1$, at $R = 0$, one has $\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{zz} = \mu_1 + 2\mu_2 + \int_0^{R_o} p'(\xi)\mathrm{d}\xi - p(R_o)$.

In Fig. 3, we show the variation of the stress components for the uniform disclination distribution with $w_o = 8\pi/R_o^2$ and for some different values of the constitutive parameters.[13]

Example 3.7. For a single wedge disclination $\omega(R) = 2\pi\Theta_o\delta^2(R)$, where Θ_o is the angle of the wedge shape region that is removed in Volterra's cut-and-weld operation (see [75] for more details). Therefore, $f''(R) = -\frac{\Theta_o}{2\pi}\delta(R)$, which implies that $f(R) = R(1 - \frac{\Theta_o}{2\pi})$, and thus, $r(R) = \frac{R}{\alpha^{\frac{1}{2}}}(1 - \frac{\Theta_o}{2\pi})^{\frac{1}{2}}$. Figure 4 illustrates the stress distribution for different values of the reinforcement and the base material parameters in the case of a single wedge disclination of positive sign with $\Theta_o = \frac{\pi}{2}$.

[13] Note that the numerical values shown in [75]'s Fig. 4 are not correct. This was caused by a typo in the sign of the integral term in the numerical evaluation of the pressure function from Eq. (4.23). In other words, the numerical values in that figure correspond to the following (incorrect) relation for the pressure with a positive sign for the integral term

$$p(R) = \mu\frac{f^2(R_o)}{r^2(R_o)} + \mu\int_R^{R_o}\left[\frac{f(\eta)f'(\eta)}{\int_0^{\eta} f(\xi)\mathrm{d}\xi} - \frac{f^3(\eta)}{4(\int_0^{\eta} f(\xi)\mathrm{d}\xi)^2} - \frac{1}{f(\eta)}\right]\mathrm{d}\eta.$$

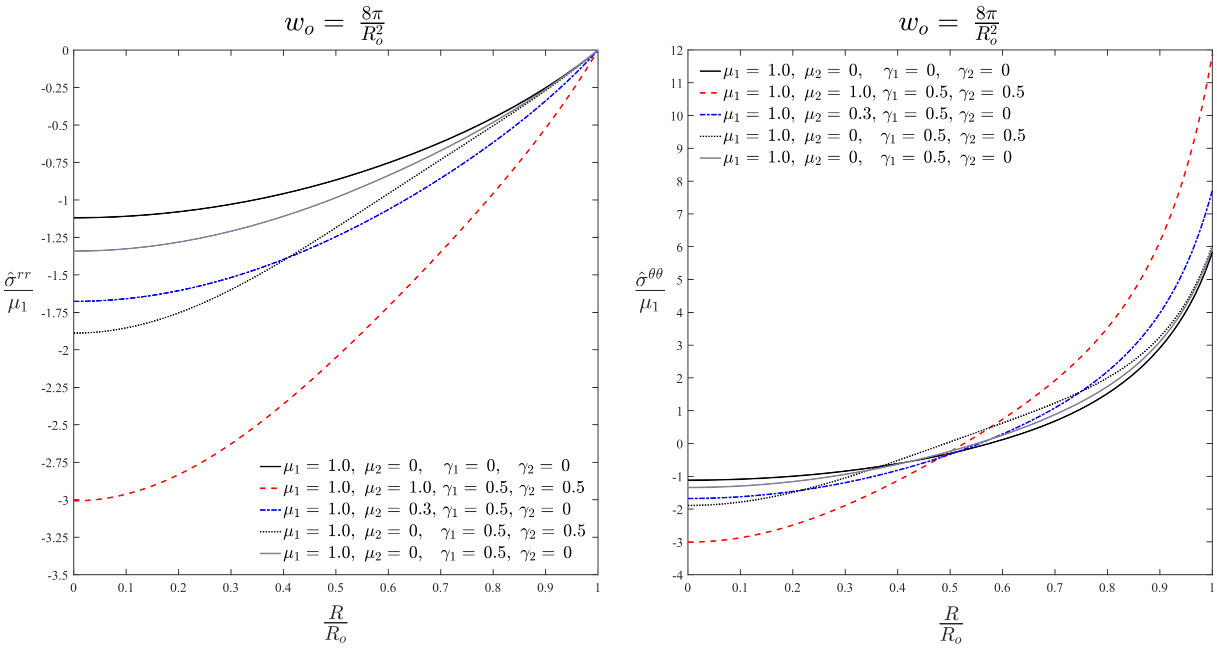

FIG. 3. $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$ distributions for different values of the constitutive parameters for a uniform disclination distribution with $\omega_o = 8\pi/R_o^2$ such that $\alpha = 1$

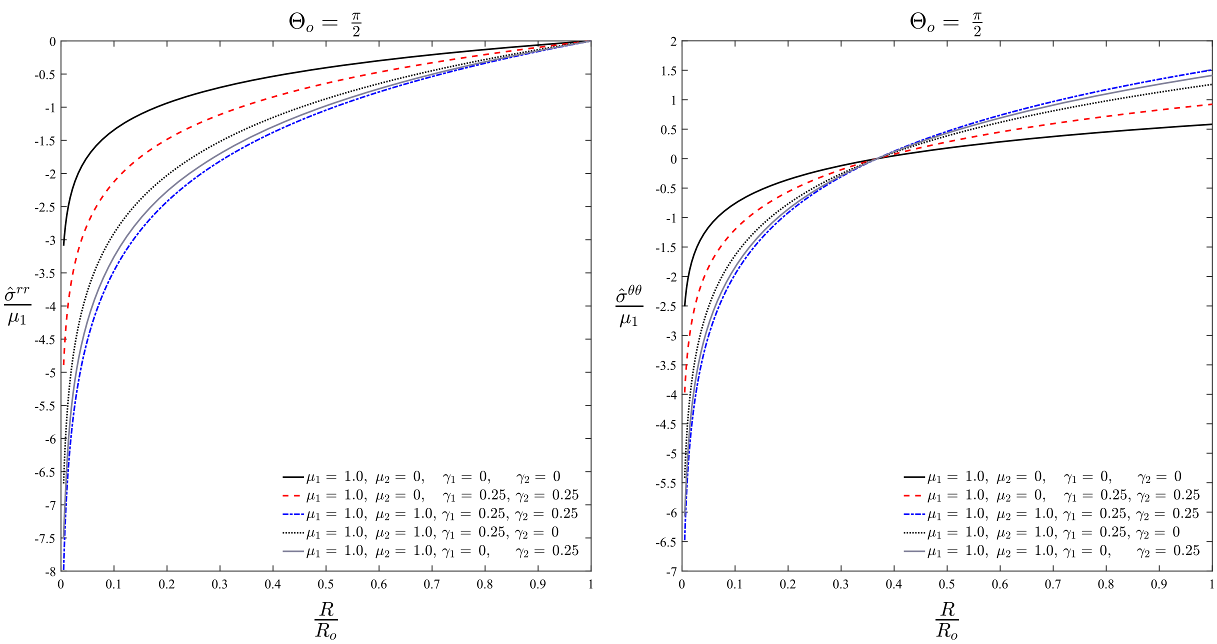

FIG. 4. $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$ distributions for different values of the constitutive parameters for a single positive wedge disclination with $\Theta_o = \pi/2$ such that $\alpha = 1$

3.4. Distributed edge dislocations in an orthotropic medium

Next, we consider a distribution of edge dislocations in an orthotropic medium such that the material preferred directions are parallel to the Cartesian axes in the Cartesian coordinates (X,Y,Z). Let us consider the orthonormal frame field $\{\mathbf{e}_\alpha(X,Y,Z)\}_{\alpha=1}^3$, where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are in the X, Y, and Z-directions, respectively. We assume that the edge dislocation distribution consists of dislocations with (i) the dislocation line parallel to the Z-axis such that the Burgers' vector density is given by $b_1(Z)\mathbf{e}_1 + c_1(Z)\mathbf{e}_2$, (ii) X-oriented Burgers' vector density $b_2(X,Y,Z)\mathbf{e}_1$ such that the dislocation line is parallel to the Y-axis, (iii) Y-oriented Burgers' vector $c_2(X,Y,Z)\mathbf{e}_2$ with the dislocation line parallel to the X-axis. Let us consider the following co-frame field (see also [72])

$$\vartheta^1 = e^{\xi(Z)+\gamma(Y)}\mathrm{d}X, \quad \vartheta^2 = e^{\eta(Z)+\lambda(X)}\mathrm{d}Y, \quad \vartheta^3 = e^{\psi(Z)}\mathrm{d}Z, \tag{3.51}$$

where $\xi(Z)$, $\gamma(Y)$, $\eta(Z)$, $\lambda(X)$, and $\psi(Z)$ are scalar functions to be determined. The corresponding frame field reads

$$\mathbf{e}_1 = e^{-\xi(Z)-\gamma(Y)}\partial_X, \quad \mathbf{e}_2 = e^{-\eta(Z)-\lambda(X)}\partial_Y, \quad \mathbf{e}_3 = e^{-\psi(Z)}\partial_Z. \tag{3.52}$$

Note that $\mathbf{G} = \delta_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$, and thus

$$\mathbf{G} = \mathrm{diag}\left\{ e^{2\left(\xi(Z)+\gamma(Y)\right)}, e^{2\left(\eta(Z)+\lambda(X)\right)}, e^{2\psi(Z)} \right\}. \tag{3.53}$$

The above dislocation distribution corresponds to the following torsion 2-forms (cf. (2.25))

$$\mathcal{T}^1 = b_1(Z)\vartheta^3\wedge\vartheta^1 + b_2(X,Y,Z)\vartheta^1\wedge\vartheta^2, \quad \mathcal{T}^2 = c_1(Z)\vartheta^2\wedge\vartheta^3 + c_2(X,Y,Z)\vartheta^1\wedge\vartheta^2, \quad \mathcal{T}^3 = 0. \tag{3.54}$$

This represents a distribution of edge dislocations with the following total Burgers' vector density

$$\begin{aligned} \mathbf{b}(X,Y,Z) &= \left(b_1(Z) + b_2(X,Y,Z)\right)\mathbf{e}_1 + \left(c_1(Z) + c_2(X,Y,Z)\right)\mathbf{e}_2 \\ &= e^{-\xi(Z)-\gamma(Y)}\left[b_1(Z) + b_2(X,Y,Z)\right]\partial_X + e^{-\eta(Z)-\lambda(X)}\left[c_1(Z) + c_2(X,Y,Z)\right]\partial_Y. \end{aligned} \tag{3.55}$$

From (3.51), one obtains

$$\begin{aligned} &\mathrm{d}\vartheta^1 = e^{-\psi(Z)}\xi'(Z)\vartheta^3\wedge\vartheta^1 + e^{-\eta(Z)-\lambda(X)}\gamma'(Y)\vartheta^2\wedge\vartheta^1, \\ &\mathrm{d}\vartheta^2 = e^{-\psi(Z)}\eta'(Z)\vartheta^3\wedge\vartheta^2 + e^{-\xi(Z)-\gamma(Y)}\lambda'(X)\vartheta^1\wedge\vartheta^2, \qquad \mathrm{d}\vartheta^3 = 0. \end{aligned} \tag{3.56}$$

Metric compatibility implies the following connection 1-forms matrix

$$\boldsymbol{\omega} = [\omega^\alpha{}_\beta] = \begin{pmatrix} 0 & \omega^1{}_2 & -\omega^3{}_1 \\ -\omega^1{}_2 & 0 & \omega^2{}_3 \\ \omega^3{}_1 & -\omega^2{}_3 & 0 \end{pmatrix}. \tag{3.57}$$

Cartan's first structural equations give the following connection 1-forms

$$\begin{aligned} &\omega^1{}_2 = \left(b_2(X,Y,Z) + \gamma'(Y)e^{-\eta(Z)-\lambda(X)}\right)\vartheta^1 + \left(c_2(X,Y,Z) - \lambda'(X)e^{-\xi(Z)-\gamma(Y)}\right)\vartheta^2, \\ &\omega^2{}_3 = \left(c_1(Z) + \eta'(Z)e^{-\psi(Z)}\right)\vartheta^2, \qquad \omega^3{}_1 = \left(b_1(Z) - \xi'(Z)e^{-\psi(Z)}\right)\vartheta^1. \end{aligned} \tag{3.58}$$

The second structural equations, i.e., $\mathcal{R}^\alpha{}_\beta = 0$ are trivially satisfied if one assumes that

$$\begin{gathered} \xi'(Z) = b_1(Z)e^{\psi(Z)}, \quad \gamma'(Y) = -b_2(X,Y,Z)e^{\eta(Z)+\lambda(X)}, \quad \eta'(Z) = -c_1(Z)e^{\psi(Z)}, \\ \lambda'(X) = c_2(X,Y,Z)e^{\xi(Z)+\gamma(Y)}. \end{gathered} \tag{3.59}$$

Thus,

$$\eta'(Z) = -\frac{1}{b_2}\frac{\partial b_2}{\partial Z}, \ \lambda'(X) = -\frac{1}{b_2}\frac{\partial b_2}{\partial X}, \ \gamma'(Y) = -\frac{1}{c_2}\frac{\partial c_2}{\partial Y}, \ \xi'(Z) = -\frac{1}{c_2}\frac{\partial c_2}{\partial Z}, \ \psi(Z) = \ln\left[\frac{-1}{b_1 c_2}\frac{\partial c_2}{\partial Z}\right], \tag{3.60}$$

where one needs to have $\frac{\partial b_2}{\partial Z} = -\frac{c_1 b_2}{b_1 c_2}\frac{\partial c_2}{\partial Z}$ and $-\frac{1}{b_1 c_2}\frac{\partial c_2}{\partial Z} > 0$. If we assume that b_2 and c_2 are separable in X, Y, and Z, i.e., $b_2 = b_{2X}(X)b_{2Y}(Y)b_{2Z}(Z)$ and $c_2 = c_{2X}(X)c_{2Y}(Y)c_{2Z}(Z)$, then

$$e^{\eta(Z)} = \frac{C_1}{b_{2Z}(Z)}, \quad e^{\lambda(X)} = \frac{C_2}{b_{2X}(X)}, \quad e^{\gamma(Y)} = \frac{C_3}{c_{2Y}(Y)}, \quad e^{\xi(Z)} = \frac{C_4}{c_{2Z}(Z)}, \quad e^{\psi(Z)} = -\frac{c_{2Z}(Z)'}{b_1(Z)c_{2Z}(Z)}, \tag{3.61}$$

where C_i, $i = 1,\ldots,4$ are constants of integration. The compatibility conditions are written as

$$C_1C_2\, b_{2Y}(Y) = \frac{c'_{2Y}(Y)}{c_{2Y}(Y)}, \quad C_3C_4\, c_{2X}(X) = -\frac{b'_{2X}(X)}{b_{2X}(X)}, \quad \frac{b'_{2Z}(Z)}{b_{2Z}(Z)} = -\frac{c_1(Z)}{b_1(Z)}\frac{c'_{2Z}(Z)}{c_{2Z}(Z)}. \tag{3.62}$$

Therefore, we have the material manifold (3.53) for the edge dislocation distributions with the Burgers' vector density (3.55). For the sake of simplicity of calculations, in the remaining of this section we consider two simplified cases of the distribution (3.55): (i) $b_2(X,Y,Z) = c_2(X,Y,Z) = 0$, $\gamma(Y) = 0$, $\lambda(X) = 0$, $\psi(Z) = 0$, and (ii) $b_2(X,Y,Z) = c_2(X,Y,Z) = 0$, $\gamma(Y) = 0$, $\lambda(X) = 0$, $c_1(Z) = 0$, $\eta(Z) = 0$.

Case (i). From (3.55), the Burgers' vector density reads $\mathbf{b} = \mathbf{b}(Z) = b_1(Z)\mathbf{e}_1 + c_1(Z)\mathbf{e}_2 = b_1(Z)e^{-\xi(Z)}\partial_X + e^{-\eta(Z)}c_1(Z)\partial_Y$, where, using (3.59), $\xi'(Z) = b_1(Z)$ and $\eta'(Z) = -c_1(Z)$. The material metric (3.53) is simplified as $\mathbf{G} = \operatorname{diag}\left\{e^{2\xi(Z)}, e^{2\eta(Z)}, 1\right\}$. Looking for solutions of the form $(x,y,z) = (X,Y,\alpha Z)$, the incompressibility constraint implies that $J = \frac{\alpha}{e^{\xi(Z)+\eta(Z)}} = 1$, and thus, $\xi(Z) + \eta(Z) = \ln\alpha$. This means that $\xi'(Z) + \eta'(Z) = 0$, and hence, $c_1(Z) = b_1(Z)$. Choosing orthonormal vectors $\mathbf{N}_1 = e^{-\xi(Z)}\partial_X$, $\mathbf{N}_2 = e^{-\eta(Z)}\partial_Y$, and $\mathbf{N}_3 = \partial_Z$ as the orthotropic axes, the invariants of the energy function are obtained from (2.16) as follows

$$\begin{aligned} I_1 &= \alpha^2 + e^{-2\xi(Z)} + \frac{1}{\alpha^2}e^{2\xi(Z)}, \quad I_2 = \frac{1}{\alpha^2} + e^{2\xi(Z)} + \alpha^2 e^{-2\xi(Z)}, \\ I_5 &= I_4^2 = e^{-4\xi(Z)}, \quad I_7 = I_6^2 = \frac{e^{4\xi(Z)}}{\alpha^4}. \end{aligned} \tag{3.63}$$

Therefore, the nonzero components of the Cauchy stress tensor read

$$\hat{\sigma}^{xx} = 2e^{-2\xi(Z)}\left[W_{I_1} + \left(\frac{e^{2\xi(Z)}}{\alpha^2} + \alpha^2\right)W_{I_2} + W_{I_4}\right] + 4e^{-4\xi(Z)}W_{I_5} - p(Z), \tag{3.64}$$

$$\hat{\sigma}^{yy} = \frac{2}{\alpha^2}e^{2\xi(Z)}\left[W_{I_1} + \left(e^{-2\xi(Z)} + \alpha^2\right)W_{I_2} + W_{I_6}\right] + \frac{4}{\alpha^4}e^{4\xi(Z)}W_{I_7} - p(Z), \tag{3.65}$$

$$\hat{\sigma}^{zz} = 2\alpha^2\left[W_{I_1} + \left(e^{-2\xi(Z)} + \frac{e^{2\xi(Z)}}{\alpha^2}\right)W_{I_2}\right] - p(Z). \tag{3.66}$$

Equilibrium equations imply that $\hat{\sigma}^{zz} = C$, where C is a constant. Vanishing of the traction vector on surfaces parallel to the $\mathrm{X}-\mathrm{Y}$ plane gives the pressure field as

$$p(Z) = 2\alpha^2\left[W_{I_1} + \left(e^{-2\xi(Z)} + \frac{e^{2\xi(Z)}}{\alpha^2}\right)W_{I_2}\right]. \tag{3.67}$$

Case (ii). The Burgers' vector density is given by $\mathbf{b} = \mathbf{b}(Z) = b_1(Z)\mathbf{e}_1 = b_1(Z)e^{-\xi(Z)}\partial_X$, where $\xi'(Z) = b_1(Z)$. From (3.53), the material metric reads $\mathbf{G} = \operatorname{diag}\left\{e^{2\xi(Z)}, 1, e^{2\psi(Z)}\right\}$. We then look for solutions of the form $(x,y,z) = (X,Y,\alpha Z)$. Incompressibility implies that $J = \frac{\alpha}{e^{\xi(Z)+\psi(Z)}} = 1$, and hence, $\xi(Z) + \psi(Z) = \ln\alpha$. The orthotropic axes are $\mathbf{N}_1 = e^{-\xi(Z)}\partial_X$, $\mathbf{N}_2 = \partial_Y$, and $\mathbf{N}_3 = e^{-\psi(Z)}\partial_Z$. The invariants of the strain energy function read

$$I_1 = 1 + e^{-2\xi(Z)} + e^{2\xi(Z)}, \quad I_2 = 1 + e^{2\xi(Z)} + e^{-2\xi(Z)}, \quad I_5 = I_4^2 = e^{-4\xi(Z)}, \quad I_6 = I_7 = 1. \tag{3.68}$$

The nonzero components of the Cauchy stress are given as

$$\hat{\sigma}^{xx} = 2W_{I_2} + 2e^{-2\xi(Z)}\left(W_{I_1} + W_{I_2} + W_{I_4}\right) + 4e^{-4\xi(Z)}W_{I_5} - p(Z), \tag{3.69}$$

$$\hat{\sigma}^{yy} = 2W_{I_1} + 2W_{I_2}\left(e^{-2\xi(Z)} + e^{2\xi(Z)}\right) + 2W_{I_6} + 4W_{I_7} - p(Z), \tag{3.70}$$

$$\hat{\sigma}^{zz} = 2\left(W_{I_2} + e^{2\xi(Z)}(W_{I_1} + W_{I_2})\right) - p(Z). \tag{3.71}$$

The equilibrium equation and the vanishing of traction vector on surfaces parallel to X − Y plane give

$$p(Z) = 2\left(W_{I_2} + e^{2\xi(Z)}(W_{I_1} + W_{I_2})\right). \tag{3.72}$$

3.5. A spherically symmetric distribution of point defects in a transversely isotropic ball

In this section, we calculate the stress field of a spherically symmetric distribution of point defects with the volume density $\mathsf{n}(R)$[14] in a transversely isotropic ball of radius R_o. The material manifold of a medium with distributed point defects is a flat Weyl manifold [73]. Let us assume that the material preferred direction is radial, i.e., $\mathbf{N} = \hat{\mathbf{R}}$,[15] where $\hat{\mathbf{R}}$ is a unit vector in the radial direction. The material metric for the body with a radial distribution of point defects in the spherical coordinates (R, Θ, Φ) reads $\mathbf{G} = \mathrm{diag}\left\{f^2(R), R^2, R^2\sin^2\Theta\right\}$, where

$$f(R) = \frac{1 - \mathsf{n}(R)}{1 - \frac{1}{R^3}\int_0^R 3y^2\mathsf{n}(y)\mathrm{d}y}. \tag{3.73}$$

We endow the ambient space with the flat Euclidean metric $\mathbf{g} = \mathrm{diag}\left\{1, r^2, r^2\sin^2\theta\right\}$ in the spherical coordinates (r, θ, ϕ). Given an embedding of the form $(r, \theta, \phi) = (r(R), \Theta, \Phi)$, the deformation gradient is written as $\mathbf{F} = \mathrm{diag}\{r'(R), 1, 1\}$. The right Cauchy–Green deformation tensor reads $\mathbf{C} = \mathrm{diag}\left\{\frac{r'^2(R)}{f^2(R)}, \frac{r^2(R)}{R^2}, \frac{r^2(R)}{R^2}\right\}$. Assuming incompressibility, the Jacobean is expressed as

$$J = \sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F} = \frac{r^2(R)r'(R)}{R^2 f(R)} = 1. \tag{3.74}$$

This gives $r(R) = \left(\int_0^R 3\xi^2 f(\xi)\mathrm{d}\xi\right)^{\frac{1}{3}}$. Using (2.8), the invariants are written as

$$I_1 = \mathrm{tr}(\mathbf{C}) = \frac{R^4}{r^4(R)} + 2\frac{r^2(R)}{R^2}, \quad I_2 = \frac{1}{2}(\mathrm{tr}(\mathbf{C}^2) - \mathrm{tr}(\mathbf{C})^2) = \frac{r^4(R)}{R^4} + \frac{2R^2}{r(R)^2},$$
$$I_5 = I_4^2 = \frac{R^8}{r^8(R)}. \tag{3.75}$$

[14] Note that $\mathsf{n}(R) < 0$ for a distribution of vacancies, and $\mathsf{n}(R) > 0$ for a distribution of interstitials.

[15] Note that $\hat{\mathbf{R}} = \frac{1}{f(R)}\mathbf{E}_R$ is the unit vector identifying the material preferred direction, where $\mathbf{E}_R = \frac{\partial}{\partial R}$ such that $\langle\!\langle \mathbf{E}_R, \mathbf{E}_R \rangle\!\rangle_{\mathbf{G}} = G_{RR}$.

Using (3.74), the nonzero stress components read[16]

$$\hat{\sigma}^{rr} = 2\frac{R^4}{r^4(R)}\left(W_{I_1} + W_{I_4}\right) + 4\frac{R^2}{r^2(R)}W_{I_2} + 4\frac{R^8}{r^8(R)}W_{I_5} - p(R),$$
$$\hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = 2\frac{r^2(R)}{R^2}W_{I_1} + 2\frac{R^2}{r^2(R)}W_{I_2} + 2\frac{r^4(R)}{R^4}W_{I_2} - p(R). \tag{3.77}$$

The nontrivial equilibrium equation is simplified to read $\frac{1}{r'(R)}\sigma^{rr}{}_{,R} + \frac{2}{r}\sigma^{rr} - 2r\sigma^{\theta\theta} = 0$. This gives $p'(R) = q(R)$, where

$$\begin{aligned} q(R) = &-\frac{4}{R^3 r^{19}}\Big[f\Big\{R^9 r^{12}\left(W_{I_1} + 4W_{I_2I_2} - 4W_{I_2I_5} + W_{I_4}\right) + R^3 r^{18}\left(W_{I_1} - 4W_{I_2I_2}\right) \\ &+ R^7 r^{14}\left[W_{I_2} - 2\left(W_{I_1I_1} + W_{I_1I_4}\right)\right] + 2R^{13}r^8\left(W_{I_1I_1} + 2W_{I_1I_4} + W_{I_4I_4} + 3W_{I_5}\right) \\ &+ 2R^{11}r^{10}\left(3W_{I_1I_2} - 2W_{I_1I_5} + 3W_{I_2I_4}\right) - 2R^5 r^{16}\left(3W_{I_1I_2} + W_{I_2I_4}\right) + 8R^{17}r^4\left(W_{I_1I_5} + W_{I_4I_5}\right) \\ &+ RW_{I_2}r^{20} + 12R^{15}W_{I_2I_5}r^6 + 8R^{21}W_{I_5I_5}\Big\} - 2r^3\Big\{R^6 r^{12}\left(W_{I_1} + 2W_{I_2I_2} - 2W_{I_2I_5} + W_{I_4}\right) \\ &+ R^4 r^{14}\left(W_{I_2} - W_{I_1I_1} - W_{I_1I_4}\right) + R^{10}r^8\left(W_{I_1I_1} + 2W_{I_1I_4} + W_{I_4I_4} + 4W_{I_5}\right) \\ &+ R^8 r^{10}\left(3W_{I_1I_2} - 2W_{I_1I_5} + 3W_{I_2I_4}\right) - R^2 r^{16}\left(3W_{I_1I_2} + W_{I_2I_4}\right) + 4R^{14}r^4\left(W_{I_1I_5} + W_{I_4I_5}\right) \\ &- 2W_{I_2I_2}r^{18} + 6R^{12}W_{I_2I_5}r^6 + 4R^{18}W_{I_5I_5}\Big\}\Big]. \end{aligned} \tag{3.78}$$

Next, we assume an energy function corresponding to a radially reinforced Mooney–Rivlin spherical ball of the following form

$$W(I_1, I_2, I_4, I_5) = \frac{\mu_1}{2}\left(I_1 - 3\right) + \frac{\mu_2}{2}\left(I_2 - 3\right) + \frac{\gamma_1}{2}\left(I_4 - 1\right)^2 + \frac{\gamma_2}{2}\left(I_5 - 1\right)^2, \tag{3.79}$$

where μ_1 and μ_2 are constants of the Mooney–Rivlin base material, while γ_1 and γ_2 are nonnegative material constants pertaining to the reinforcement strength in the radial direction. Thus, (3.78) is simplified to read

$$\begin{aligned} p'(R) = &-\frac{2}{R^2 r^{19}}\Big[f\left\{6R^{12}(\gamma_1 - 2\gamma_2)r^8 + R^8(\mu_1 - 2\gamma_1)r^{12} + \mu_2 R^6 r^{14} + \mu_1 R^2 r^{18} + \mu_2 r^{20} + 28\gamma_2 R^{20}\right\} \\ &- 2R^3 r^3\left(4R^6(\gamma_1 - 2\gamma_2)r^8 + R^2(\mu_1 - 2\gamma_1)r^{12} + \mu_2 r^{14} + 16\gamma_2 R^{14}\right)\Big]. \end{aligned} \tag{3.80}$$

The stress components are also simplified and read

$$\hat{\sigma}^{rr} = \frac{R^4}{r^4}\left[\mu_1 + 2\gamma_1\left(\frac{R^4}{r^4} - 1\right)\right] + 2\mu_2\frac{R^2}{r^2} + 4\gamma_2\frac{R^8}{r^8}\left(\frac{R^8}{r^8} - 1\right) - p,$$
$$\hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = \mu_1\frac{r^2}{R^2} + \mu_2\frac{R^2}{r^2} + \mu_2\frac{r^4}{R^4} - p. \tag{3.81}$$

Assuming that the boundary of the ball is traction-free, one obtains

$$p(R_o) = \frac{R_o^4}{r^4(R_o)}\left\{\mu_1 + 2\gamma_1\left[\frac{R_o^4}{r^4(R_o)} - 1\right]\right\} + 2\mu_2\frac{R_o^2}{r^2(R_o)} + 4\gamma_2\frac{R_o^8}{r^8(R_o)}\left[\frac{R_o^8}{r^8(R_o)} - 1\right]. \tag{3.82}$$

[16] When the body is defect-free, $f(R) = 1$, and thus, $I_1 = I_2 = 3$ and $I_4 = I_5 = 1$. If one assumes that the stress vanishes in this case, one has (see [38,61] for similar conditions)

$$\left(2W_{I_5} + W_{I_4}\right)|_{I_1=I_2=3, I_4=I_5=1} = 0. \tag{3.76}$$

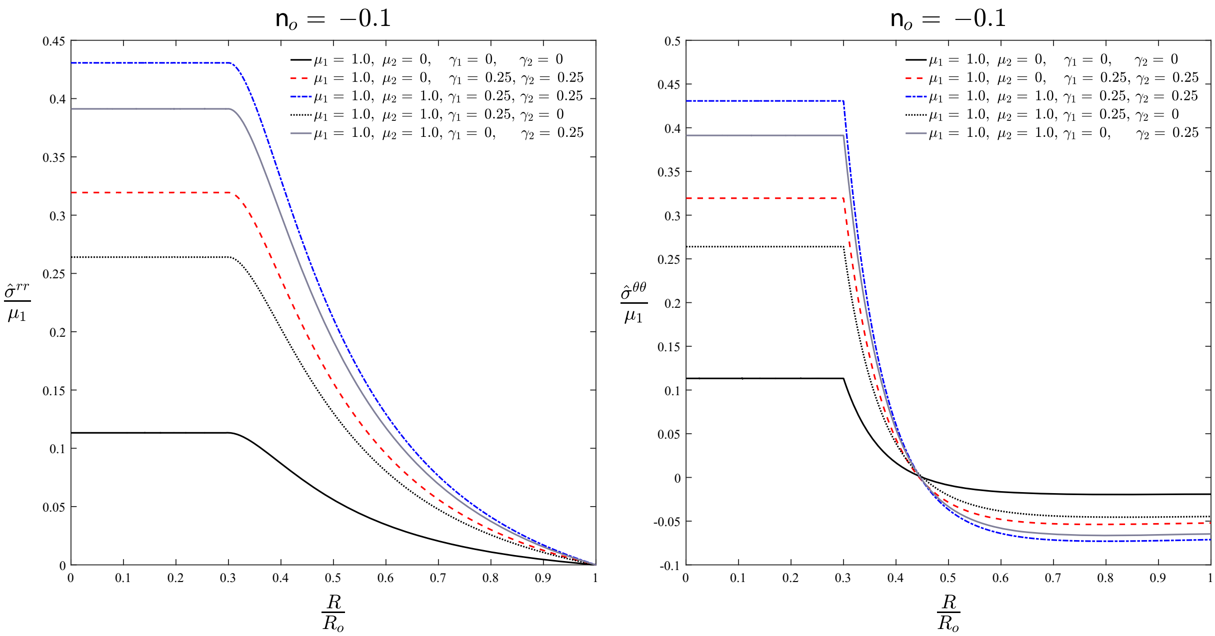

FIG. 5. $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$ distributions for different values of the constitutive parameters for the point defect distribution (3.84) with $R_i/R_o = 0.3$ and $\mathsf{n}_o = -0.1$

Thus,

$$
\begin{aligned}
p(R) = 2\int_R^{R_o} \frac{1}{\xi^2 r^{19}(\xi)} \Bigg\{ & f(\xi)\Big[6\xi^{12}(\gamma_1-2\gamma_2)r^8(\xi)+\xi^8(\mu_1-2\gamma_1)r^{12}(\xi)+\mu_2\xi^6 r^{14}(\xi)+\mu_1\xi^2 r^{18}(\xi) \\
& +\mu_2 r^{20}+28\gamma_2\xi^{20}\Big]-2\xi^3 r^3(\xi)\Big[4\xi^6(\gamma_1-2\gamma_2)r^8(\xi)+\xi^2(\mu_1-2\gamma_1)r^{12}+\mu_2 r^{14}(\xi)+16\gamma_2\xi^{14}\Big]\Bigg\}\mathrm{d}\xi \\
& + p(R_o).
\end{aligned}
\tag{3.83}
$$

Let us consider the following distribution of point defects in the ball

$$
\mathsf{n}(R) = \begin{cases} \mathsf{n}_o & 0 \le R \le R_i, \\ 0 & R_i < R \le R_o. \end{cases} \tag{3.84}
$$

Therefore, from (3.73)

$$
f(R) = \begin{cases} 1, & 0 \le R \le R_i, \\ (1-\mathsf{n}_o(R_i/R)^3)^{-1}, & R_i < R \le R_o, \end{cases} \tag{3.85}
$$

and hence,

$$
r(R) = \begin{cases} R, & 0 \le R \le R_i, \\ \left[R^3 + \mathsf{n}_o R_i^3 \ln \frac{(R/R_i)^3 - \mathsf{n}_o}{1-\mathsf{n}_o}\right]^{1/3}, & R_i < R \le R_o. \end{cases} \tag{3.86}
$$

Figure 5 shows the stress field variation for the point defect distribution (3.84), when $R_i/R_o = 0.3$ and $\mathsf{n}_o = -0.1$ for different values of the reinforcement and base material constants in (3.79).

Remark 3.8. Consider an arbitrary nonlinear incompressible transversely isotropic spherical ball of radius R_o such that the material preferred direction is radial. Suppose that the ball is subject to a uniform pressure on its boundary and has the point defect distribution (3.84). Then, in the ball $R \leq R_i$, the stress is uniform and hydrostatic. Interestingly, the value of the hydrostatic stress inside the ball $R \leq R_i$ has an explicit dependence on the reinforcement parameters (see Fig. 5). To show this, for $R \leq R_i$, $f(R) = 1$ and $r(R) = R$, following (3.85) and (3.86), respectively. Therefore, after some simplification, (3.78) implies that $p'(R) = q(R) = \frac{4}{R}\left(W_{I_4} + 2W_{I_5}\right)|_{I_1=I_2=3, I_4=I_5=1} = 0$ using the relation (3.76). Hence, for $R \leq R_i$, $p(R) = C$, where C is a constant depending on the reinforcement and the base material parameters. From (3.77), $\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = 2\left(W_{I_1} + 2W_{I_2}\right)|_{I_1=I_2=3, I_4=I_5=1} - C$ for $R \leq R_i$.

4. Concluding remarks

Despite the crucial role that anisotropy plays in the overall response of materials in the presence of large strains, the study of defects in nonlinear solids has been overwhelmingly restricted to isotropic materials to this date. In this paper, we presented a few analytical solutions for the stress fields induced by distributed line and point defects in nonlinear anisotropic solids. We considered a parallel cylindrically symmetric distribution of screw dislocations in infinite orthotropic and monoclinic media, and also, a cylindrically symmetric distribution of parallel wedge disclinations in an orthotropic medium. Because the material manifold is endowed with a nontrivial Riemannian metric that explicitly depends on the defect distribution, the material preferred directions, and hence, the class of anisotropy of the defective body is, in general, different in the reference and current configurations. We observed, in particular, that for a cylindrically symmetric distribution of screw dislocations, assuming that the body is orthotropic in its reference (current) configuration, it is monoclinic in its current (reference) configuration. We found that for an arbitrary cylindrically symmetric distribution of parallel screw dislocations, and for a uniform wedge disclination distribution, the stress field is logarithmically singular on the dislocation and the disclination axes unless the axial deformation is suppressed. These stress singularities are inherently due to the anisotropy effects, e.g., the radial fiber reinforcement, and do not, in particular, arise in isotropic materials. This observation demonstrates the significance of taking material anisotropy into consideration in the analysis of solids with distributed defects. For a single screw dislocation, we employed the standard reinforcing model and discussed the conditions that guarantee that the energy per unit length and the resultant axial force are finite for a fiber-reinforced material as long as the isotropic base material has a finite energy per unit length and a finite axial force. For a distribution of edge dislocations, the resulting stresses are calculated when the medium is orthotropic. Finally, we studied a spherically symmetric distribution of point defects in a transversely isotropic spherical ball. We showed that for an incompressible transversely isotropic ball with the radial material preferred direction, a uniform point defect distribution results in a uniform hydrostatic stress field inside the spherical region the distribution is supported in. The dynamics, stability, and interactions of defects in anisotropic solids, at finite strains are exciting problems that will be the subjects of future communications.

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