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GENERALIZATION OF BARENBLATT'S COHESIVE FRACTURE THEORY FOR FRACTAL CRACKS

ARASH YAVARI
Graduate Aeronautical Laboratories, California Institute of Technology
Pasadena, CA 91125, USA
arash@aero.caltech.edu

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Abstract

In this paper, we generalize Barenblatt's cohesive fracture theory for fractal cracks. We discuss the difficulties of generalizing the concept of traction on a fractal surface. Borodich's modification of Griffith's theory for fractal cracks is reviewed. Irwin's driving force is generalized for fractal cracks and a fractal driving force (G_f) is defined. It is shown that to generalize Barenblatt's theory for fractal cracks it is necessary to introduce a new quantity, D-fractal cohesive pseudo-stress. This new quantity is cohesive force per unit of a fractal measure. Fractal modulus of cohesion is seen to be a function of both the material and the fractal dimension of the crack. Equivalence of fractal Barenblatt's and Griffith's theories is discussed. It is seen that the order of stress singularity at the tip of a fractal crack cannot be obtained using modified Barenblatt's theory because this theory is a local theory and assumes the order of stress singularity *a priori*.

Keywords: Fractal Crack; Fractal Fracture; Cohesive Fracture Theory.

1. INTRODUCTION

Fractal geometry is a somewhat new branch of mathematics that studies infinitely irregular sets. Fractal geometry was introduced by Mandelbrot[1,2] less than three decades ago. Fractals are sets that seem to be random and infinitely irregular but indeed are orderly. There are many applications of fractals in different fields of science and engineering. Looking at the large number of experimental studies that confirm the existence of fractals in different natural phenomena, now we can say fractal geometry could be a nice model for nature. The idea of fractal geometry is to model natural phenomena by fractals instead of smooth Euclidean sets.

There is a recent interest in applications of fractal geometry in the solid mechanics community. There are a large number of experimental and a few number of theoretical investigations. To date, fractal models have been used mostly in fracture mechanics and contact mechanics. From experiments, we know that fracture surfaces are fractals in a wide range of length scales. Therefore, fractals might be more suitable models for cracks than smooth curves or surfaces. Some researchers of solid mechanics and materials science have tried to find a relation between fractal dimensions of cracks and toughness. However, so far there is no convincing evidence for existence of such an interesting and useful relationship.

Application of fractals to continuum mechanics seems to be very difficult. The most important theorem of continuum mechanics, i.e. Cauchy's stress theorem, fails to be valid on a fractal surface. There is no generalization of the concept of stress tensor for material points on a fractal surface. There are some recent investigations into generalization of Stokes' theorem for differential forms on domains with fractal boundaries.[3–6] These researches show that under some conditions, Stokes' theorem holds for a domain with a fractal boundary. This amazing result gives us the hope of future progresses in fractal analysis.

The first investigations into theoretical fractal fracture mechanics are due to Mosolov,[7–9] Borodich,[10] Mosolov and Borodich,[11] and Goldshtein and Mosolov.[12,13] These pioneering researchers investigated various problems such as the order of stress singularity at the tip of a fractal crack, path dependence of J-integral for fractal cracks, fracture in compression, etc. Other interesting theoretical results can be seen in the works of Balankin,[14–16] Borodich,[11,17–19] Cherepanov et al.,[20] Xie,[21] Xie and Sanderson,[22] and Yavari et al.[23–25] For a more complete review of fractal fracture mechanics, the reader may refer to Cherepanov et al.,[20] Borodich,[26] and Yavari et al.[23,24]

In this paper, we generalize Barenblatt's fracture theory for fractal cracks. To our best knowledge, there is no discussion of this theory for fractal cracks in the literature. The concept of traction on fractal surfaces is discussed. A critical review of Griffith and Barenblatt theories is presented and the equivalence of the two theories is discussed. D-fractal cohesive pseudo-stress is defined and Barenblatt's theory is generalized for fractal cracks. The relation between fractal Barenblatt's theory and fractal Griffith's theory is investigated. This paper is organized as follows. Section 2 discusses the concept of traction on fractal surfaces and difficulties in generalizing the concepts of classical continuum mechanics for bodies with fractal boundaries. In Sec. 3, Griffith's fracture theory, its history, and its generalization for fractal cracks is reviewed. Section 4 reviews Barenblatt's fracture theory and its equivalence to Griffith's theory. In Sec. 5, Barenblatt's theory is generalized for fractal cracks. The equivalence of fractal Griffith's and Barenblatt's theories is discussed. Conclusions are given in Sec. 6.

2. TRACTION ON FRACTAL SURFACES

Consider a solid body $\mathcal{B}$ occupying a spatial domain Ω with boundary $\partial\Omega$ under some applied forces. Suppose that a fractal surface Γ divides the domain Ω into subdomains Ω_1 and Ω_2 i.e.

$$\Omega = \Omega_1 \cup \Omega_2 \,. \tag{1}$$

There are internal forces between $\mathcal{B}_1$ and $\mathcal{B}_2$. The body $\mathcal{B}_1$ exerts some forces on $\mathcal{B}_2$, and $\mathcal{B}_2$ exerts some forces on $\mathcal{B}_1$. The bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ occupy the special domains Ω_1 and Ω_2, respectively (as shown in Fig. 1). All the mechanical effects of $\mathcal{B}_2$ on $\mathcal{B}_1$ is represented by a force $\mathbf{F}_{1,2}$ and a moment $\mathbf{M}_{1,2}$. Similarly, all the mechanical effects of $\mathcal{B}_1$ on $\mathcal{B}_2$ is represented by a force $\mathbf{F}_{2,1}$ and a moment $\mathbf{M}_{2,1}$. Now consider a spatial point $\mathbf{x}$ on Γ and a differential element in Ω_1 containing $\mathbf{x}$. Body $\mathcal{B}_2$ exerts a force $\Delta\mathbf{F}$ on this differential element (effect of the moment $\Delta\mathbf{M}$ is assumed to be of higher order and hence negligible, i.e. the material is nonpolar). The true area of the surface of action of $\Delta\mathbf{F}$ is infinity no matter how small the volume of the differential element is. Therefore, stress vector does not exist in its classical definition. For smooth surfaces, according to Cauchy's hypothesis stress vector at time t is a function of $\mathbf{x}$, t and the unit normal vector to the surface $\mathbf{n}$.[a] In other words, for two different smooth surfaces with the same unit normal vector traction is the same. Fractal surfaces can be distinguished from each other by their dimension. However, there

[a]Noll[27] showed that this dependence on the unit normal vector is a consequence of blance of linear momentum and not an assumption. In other word, stress vector cannot be a function of curvature or any other characteristic of the surface at the point $\mathbf{x}$, i.e. $\mathbf{t} = \mathbf{t}(\mathbf{x}, \mathbf{n}, t)$.

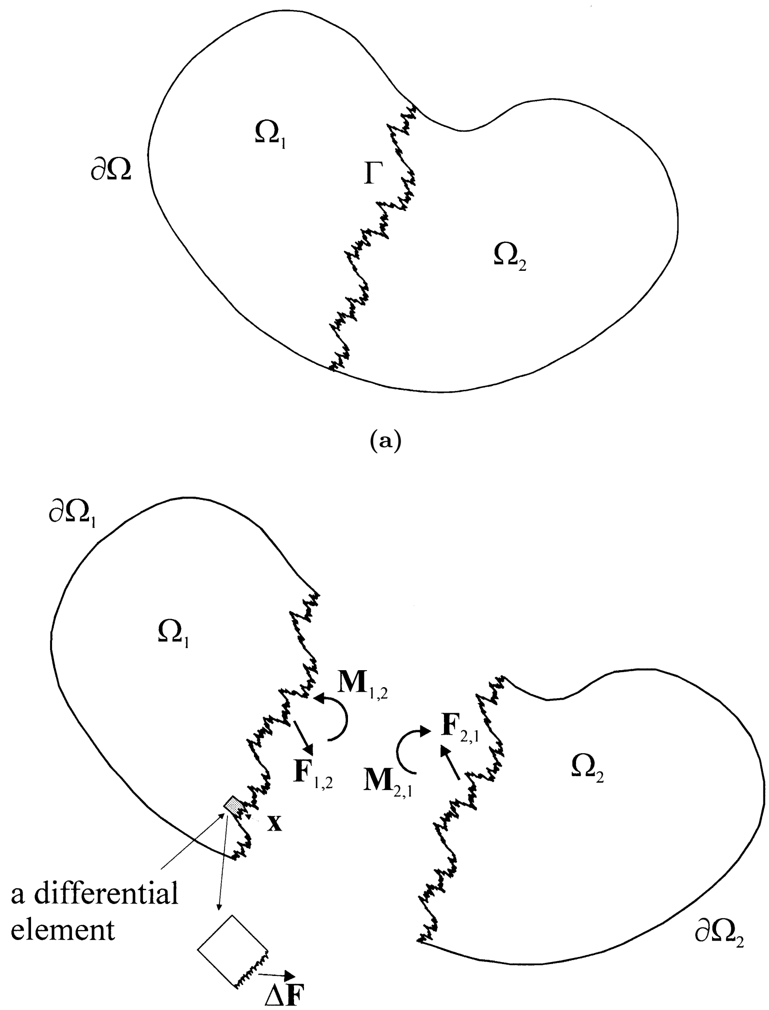

Fig. 1 **(a)** A solid body and a fractal curve that partitions it into two sub-bodies. **(b)** The two sub-bodies and the internal system of loads between them. A differential element on the fractal surface is shown.

is no simple way of classifying fractal surfaces with the same fractal dimension because unit normal vector is not defined on a fractal surface. Therefore, it is not easy to generalize the concept of traction on a fractal surface. The following naive generalization of traction might seem to be reasonable at first glance.

The mutual mechanical effects of $\mathcal{B}_1$ and $\mathcal{B}_2$ separated by a fractal surface of dimension D at a point $\mathbf{x}$ is completely described by a D-fractal stress vector $\mathbf{t}^D$, which is defined as

$$\mathbf{t}^D(\mathbf{x}, D) = \lim_{\Delta m_D \to 0} \frac{\Delta \mathbf{F}}{\Delta m_D} \tag{2}$$

where m_D is a fractal measure, say Hausdorff D-measure.

According to this definition, fractal traction depends only on the position and the dimension of the fractal surface, which is not necessarily true. It is known that some very different surfaces might have the same fractal dimension and this definition overlooks this fact. Obviously, a mathematically sound definition of traction on a fractal surface should recover the special case of $D = 1$. Definition (2) lacks this essential requirement because for $D = 1$, it implies that traction is not a function of unit normal vector $\mathbf{n}$, which is not consistent with classical continuum mechanics. We call $\mathbf{t}^D$, a D-fractal pseudo-stress vector. Now we can have a more precise definition of a fractal crack.

Fractal Crack: In classical fracture mechanics, a crack is a free surface inside a solid body. Here, "free" means "free of stress." In other words, on a crack surface, stress vector is identically zero. Similarly, a fractal crack is defined as a fractal surface free of fractal pseudo-stress inside a solid body. For a fractal crack in a three-dimensional solid, the crack edge is the boundary of the fractal surface and for a fractal crack in a two-dimensional solid, crack tips are the two end points of the fractal curve.

3. GRIFFITH'S FRACTURE THEORY FOR SMOOTH AND FRACTAL CRACKS

Griffith[28] proposed a fracture theory, which is now known as Griffith's theory or criterion. Griffith introduced a surface energy U_s, which is a measure of the resistance of the material to crack extension. This surface energy is expressed as

$$U_s = 2A\gamma \tag{3}$$

where A is the area of one crack face and γ is the specific surface energy, which is assumed to be a material property. Griffith used this surface energy in a modified energy equation to find the critical stress in an infinite solid with a single finite crack under uniform all around tension at infinity. Later Griffith[29] noticed that the critical stress that he found in his previous paper was erroneous and gave the correct critical stress. But he did not explain the method he used to arrive at the correct formula. Later, some other researchers worked on this problem.[30–32] Griffith stated his theory in a global form. The local form of the theory was recognized by Irwin.[33] According to the local form of Griffith's theory, a crack of length $2a$ extends by an amount Δa if

$$\Delta U_e = \Delta U_s \tag{4}$$

where ΔU_e is the strain energy release due to the crack growth Δa and ΔU_s is the required surface energy for this crack growth. The surface energy ΔU_s is spent on breaking the bonds that oppose crack propagation. It should be noted that Griffith's criterion is a necessary condition for brittle fracture. It is also worth mentioning that γ is a macroscopic quantity and it is the work done to create a unit of (macroscopic) crack extension.[34]

Irwin[33] introduced the concept of driving force for a smooth crack. Driving force G is the generalized force corresponding to the generalized displacement Δa. Thus,

$$\Delta W = G\Delta a\,. \tag{5}$$

The driving force G can be viewed as the force tending to cause crack propagation. For a mode I crack, the driving force has the following relation with crack tip parameters

$$\begin{aligned} G &= \frac{K_I^2}{E} \quad \text{(plane stress)} \\ G &= \frac{K_I^2}{(1-\nu^2)E} \quad \text{(plane strain)}\,. \end{aligned} \tag{6}$$

There are similar relations for modes II and III cracks. Driving force for fractal cracks will be discussed in sequel. It is to be noted that Griffith's criterion and the concept of surface energy are applicable only to brittle fracture. Orowan[35] modified the theory for ductile fracture. It is known that in brittle fracture of ductile metals there is always some amount of plastic deformation, which is concentrated in a thin layer at crack surfaces. Orowan modified Griffith's criterion for cases in which plastic deformation may be assumed to be concentrated in such thin layers. He modified the theory by adding a plastic specific surface energy γ_p to γ_0, i.e. $\gamma = \gamma_0 + \gamma_p$, where γ_0 is the brittle specific surface energy. Usually plastic surface energy is much higher ($\gamma_0 \ll \gamma_p$) and hence $\gamma_0 + \gamma_p \approx \gamma_p$. It should be noted that γ_p is proportional to γ_0.[36,37]

Rice,[38] considering the possibility of crack healing, proposed the following more generalized form of Griffith's criterion

$$(G - 2\gamma)\dot{a} \geq 0 \tag{7}$$

where G is Irwin energy release rate (driving force), γ is the specific surface energy, and $\dot{a}$ is the crack speed.

3.1 Griffith's Theory for Fractal Cracks

In Griffith's theory, it is implicitly assumed that the crack (in a two-dimensional body) is a rectifiable curve. Therefore, it is not possible to apply this criterion to non-rectifiable cracks. Because fractal curves are non-rectifiable, Griffith's criterion cannot be used for them. Borodich[10,17–19] noticed that using the classical form of Griffith's criterion for a fractal crack leads to the conclusion that fractal cracking is impossible because the required surface energy for creating an increment of fractal crack extension is infinity no matter how small the nominal crack growth length is. Borodich modified Griffith's criterion for fractal cracks by defining a specific surface energy per unit of a fractal measure. Generalized Griffith's criterion may be stated in a local form as

$$\Delta U_e = \Delta U_s = 2\gamma_f(D)\Delta m_D \tag{8}$$

where $\gamma_f(D)$ is a D-fractal specific surface energy and Δm_D is a fractal measure of the crack extension. It should be noted that there are different definitions for measure and all are acceptable for defining γ_f. As was mentioned in Yavari, et al.,[24] this fractal specific surface energy is not a material property; it is a function of both the material and the fractal dimension of the fractal crack.

Yavari, et al.[24,25] utilized fractal Griffith's theory for finding the order of stress (and couple-stress) singularity at the tip of a fractal crack using dimensional analysis considerations. The power of Griffith's criterion lies in the fact that Griffith's theory is a global (macroscopic) theory and does not make use of the form of stress distribution around the crack tip. This is why it is possible to find the form of the radial variation of stresses around the tip of a fractal crack using dimensional analysis.

To find the stress distribution around the tip of a fractal crack, an elasticity problem with fractal boundaries should be solved. For solving this problem, we need to have an analytic representation for a fractal crack with fractal dimension D (or Hurst exponent H in the case of a self-affine fractal crack). Fractal dimension is a complexity index and tells us about the degree of irregularity of a fractal set but does not specify the fractal set uniquely. In other words, fractal dimension provides us with a limited amount of information. Sets with different topological properties could have equal fractal dimensions. Therefore, for solving an elasticity problem with fractal boundaries knowing the fractal dimension of the boundary is not enough. There is no general analytic representation for a fractal curve with fractal dimension D (or Hurst exponent H). We can consider a specific fractal curve, for example Weierstrass-Mandelbrot function, which is a self-affine fractal. However, the results obtained for such a specific fractal cannot be generalized to all fractal curves with the same fractal dimension. In other words, Weierstrass-Mandelbrot function is not a representative for all fractal single-valued functions.

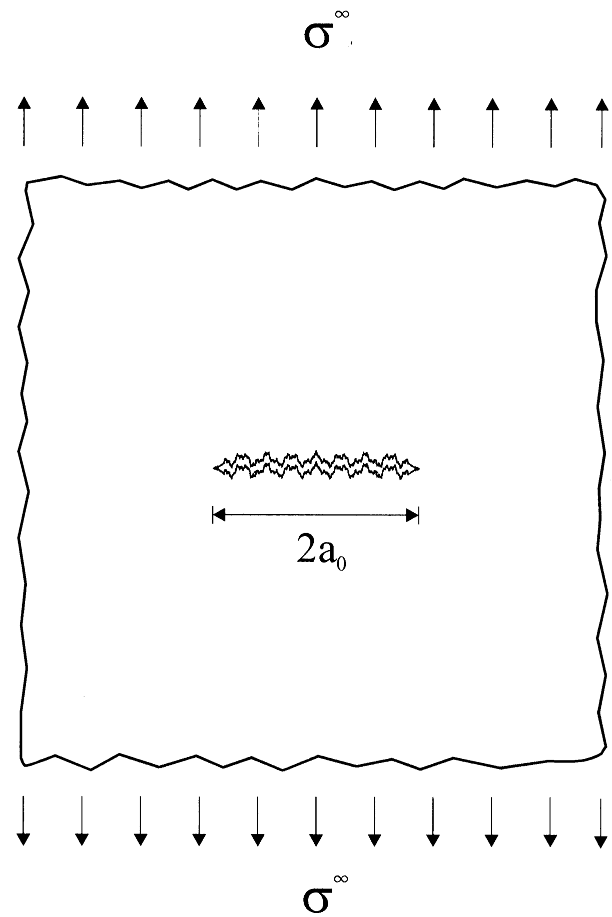

Fig. 2 A mode I fractal crack with nominal length of $2a_0$.

3.2 Driving Force for a Fractal Crack

For a fractal crack with dimension D, we define the fractal driving force G_f as the generalized force corresponding to the generalized displacement Δm_D, i.e.

$$\Delta U_e = G_f \Delta m_D \quad \text{or} \quad G_f = \frac{\Delta U_e}{\Delta m_D}. \tag{9}$$

Consider the mode I fractal crack shown in Fig .2. Stress distribution around the crack tip has the following form

$$\sigma_{ij}^f(r, \theta) = K_I^f r^{-\alpha} f_{ij}(\theta, D) + \text{higher-order terms} \tag{10a}$$

where

$$\alpha = \frac{2-D}{2}, \; K_I^f = \phi(d)\sqrt{\pi a^{2-D}}\,\sigma^\infty. \tag{10b}$$

Here K_I^f is the fractal stress intensity factor.[23,24] The fractal driving force G_f is a function of K_I^f, D,

E and ν. Here, the effects of "a" and σ^∞ are hidden in K_I^f. Thus,

$$G_f = \Phi(K_I^f, E, D, \nu). \tag{11}$$

In Eq. (11), independent variables are K_I^f and E. According to Buckingham's Π-theorem, we must have

$$\frac{G_f}{(K_I^f)^2 E^{-1}} = \Psi(D, \nu) \quad \text{or} \quad G_f = \Psi(D, \nu)\frac{(K_I^f)^2}{E} \tag{12}$$

which is similar to Eq. (6). Obviously, we recover the classical relation for $D = 1$ if $\Psi(1, \nu) = 1$.

4. BARENBLATT'S COHESIVE FRACTURE THEORY FOR SMOOTH CRACKS AND ITS EQUIVALENCE TO GRIFFITH'S THEORY

In this section, Barenblatt's fracture theory is reviewed and compared to Griffith's theory. Griffith's theory, as a global (macroscopic) fracture theory, does not make any direct use of the form of stress distribution around the crack tip. In the framework of classical (local) elasticity, stresses at the tip of a smooth crack in a homogeneous solid are unbounded and asymptotically behave as $O(r^{-\frac{1}{2}})$, where r is the distance from the crack tip. However, for such an elastic system, strain energy in a finite region containing the crack tip is bounded. There is a strain energy release due to a crack propagation Δa. What is important in Griffith's theory is the amount of the strain energy release, not the form of the stresses that contribute to this energy. As was shown by Eringen et al.,[39] for a smooth crack in a nonlocal linear elastic medium, stresses are finite at the crack tip, and hence the maximum stress criterion may be utilized. However, Griffith's theory can be used as well. When a crack in a non-local solid propagates, there is a strain energy release ΔU_e, which should be equal to ΔU_s. As it is seen, Griffith's theory can be applied to both local and nonlocal cracked systems and its form is the same in both cases.

Barenblatt[40] proposed a local (microscopic) fracture theory that is now known as Barenblatt's cohesive fracture theory or cohesive fracture theory. In reality, there is no stress singularity at the crack tip. Barenblatt resolved this paradox by introducing a more physically sound model for a crack. He considered a nonlinear region of length "d" in which cohesive stresses are active. We refer to this region as the "end-region" [see Fig. 3(a)]. Barenblatt developed a theory based on the idea that cohesive forces must distribute in such a way as to be able to close the crack faces smoothly and remove the stress singularity at the crack tip. Barenblatt made the following two postulates: (i) postulate of smallness of the end-region, and (ii) postulate of autonomy.

In the postulate of smallness of the end-region, it is assumed that $d \gg a$ ($2a$ is the crack length) because cohesive stresses act in a very localized region close to the crack tip. Most cracks in practice satisfy this requirement. According to the postulate of autonomy, for a crack that is about to propagate the shape of the crack profile (and consequently, the distribution of cohesive stresses) is a material property, i.e. it does not depend on crack length, applied loads, or the geometry of the cracked structure. Because the distribution of the cohesive forces is unknown, a modulus of cohesion is defined. Modulus of cohesion K_{coh} represents the effect of cohesive stresses and is assumed to be a material property

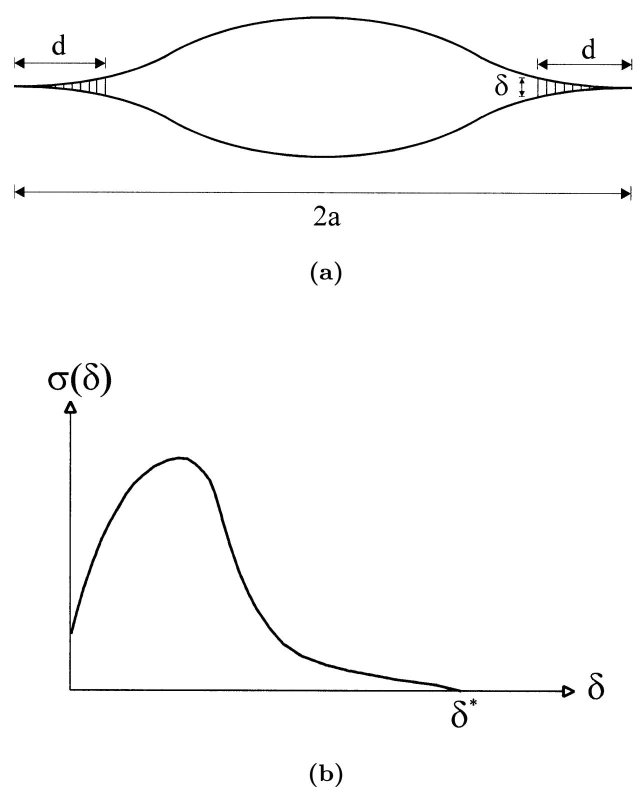

Fig. 3 **(a)** A crack with its two end-regions. **(b)** Cohesive stress-displacement relation.

defined as

$$K_{\text{coh}} = \int_0^d \frac{G(\xi)}{\sqrt{\xi}}\, d\xi \qquad (13)$$

where $G(\xi)$ is the cohesive stress at point ξ in the end region. Note that K_{coh} has been defined based on the knowledge that stress has an $r^{-1/2}$ singularity at the crack tip. For calculating the critical length of a crack, two loading conditions are considered: (a) applied external loads without cohesive stresses, and (b) cohesive stresses without applied external loads. In the loading condition (a), stresses have the following distribution

$$\sigma_{ij}^{\text{applied}}(r,\theta) = K_I r^{-\frac{1}{2}} f_{ij}(\theta, D) + \text{non-singular terms}. \qquad (14)$$

Similarly, stresses have the following distribution in the loading condition (b)

$$\sigma_{ij}^{\text{cohesive}}(r,\theta) = -\frac{K_{\text{coh}}}{\pi}\, r^{-\frac{1}{2}} f_{ij}(\theta, D) + \text{non-singular terms}. \qquad (15)$$

Therefore, for having non-singular stresses at the crack tip, we must have

$$K_I = \frac{K_{\text{coh}}}{\pi}. \qquad (16)$$

We call Eq. (16) "finiteness condition."

At first glance, Griffith and Barenblatt theories seem to be very different. But they are indeed equivalent. When a cracked structure is under some external loads the crack surfaces are subjected to the cohesive forces, which restrain the crack faces to separate from each other. The cohesive stress σ at a point x in the end-region is a function of the relative displacement of crack faces δ, i.e. $\sigma = \sigma(\delta)$. A typical cohesive stress-displacement diagram is shown in Fig. 3(b). Due to applied external loads, δ increases from zero until it reaches δ^*. At this moment, the bond between crack faces breaks and new free surfaces are created. In this process, cohesive stresses do some amount of work (per unit of crack length), which can be written as

$$W = \int_0^{\delta^*} \sigma(\delta) d\delta\,. \qquad (17)$$

The surface energy needed for a crack propagation of amount Δa is

$$\Delta U_s = \int_0^{\Delta a}\int_0^{\delta^*} \sigma(\delta)d\delta\; dx = \Delta a \int_0^{\delta^*} \sigma(\delta)d\delta\,. \qquad (18)$$

Therefore,

$$\int_0^{\delta^*} \sigma(\delta)d\delta = 2\gamma\,. \qquad (19)$$

This is actually a link between Griffith and Barenblatt theories. Rice[41,42] showed that for a Barenblatt-type crack J-integral has the following value

$$J = \int_0^{\delta^*} \sigma(\delta)d\delta \qquad (20)$$

and hence he was able to prove the equivalence of Griffith and Barenblatt theories for a smooth (straight) crack in a general nonlinear material. The equivalence of the two theories had previously been proven for a smooth crack in a linear elastic material by Willis.[43]

5. GENERALIZATION OF BARENBLATT'S THEORY FOR FRACTAL CRACKS

This section generalizes Barenblatt's theory for fractal cracks. Griffith and Barenblatt theories have a fundamental difference. Griffith does not make any direct use of the stress distribution around the crack tip. Therefore, Griffith's theory can be easily generalized for fractal cracks. The theory is modified by defining a specific surface energy per unit of a fractal measure. Barenblatt's theory makes use of the fact that at the crack tip, stresses have an $r^{-\frac{1}{2}}$ singularity and based on this knowledge a modulus of cohesion is defined. In generalizing Barenblatt's theory for fractal cracks, first the following question must be answered. Are the end-regions smooth or fractal? Because the end regions are continuations of the crack trajectory, for a fractal crack they have to be fractal curves with the same fractal dimension

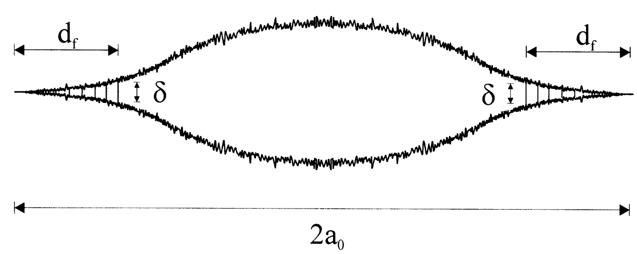

Fig. 4 A fractal crack with its two end-regions.

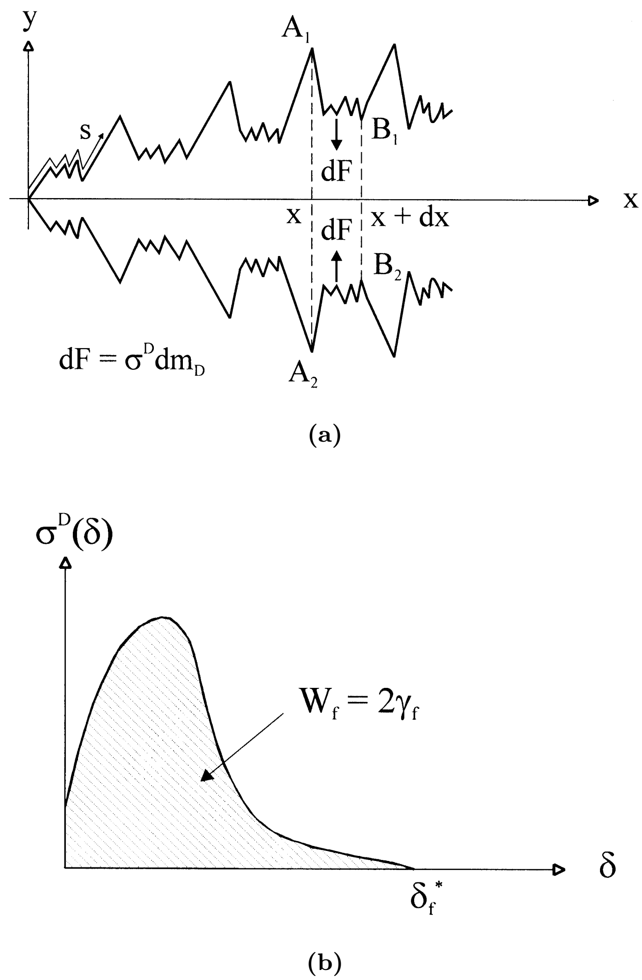

Fig. 5 **(a)** End-region of a fractal crack. The true length between any two points on the crack surfaces is infinity. There is a cohesive force dF in the interval $[x, x+dx]$. **(b)** D-fractal cohesive pseudo-stress-displacement relation.

D. A fractal crack with cohesive zones is shown in Fig. 4.

End-regions are fractal curves and do not have normal vectors at all points in the interval $[0, d_f]$, where d_f is the nominal length of the fractal end-region. Obviously, cohesive stress does not exist in its classical definition. However, we can define fractal cohesive pseudo-stresses. For a smooth crack, cohesive stresses are defined as cohesive force per unit of crack length. But this definition is not acceptable for a fractal crack. In Fig. 5(a), a fractal crack is shown (actually what we can show here is a pre-fractal curve). For any two points A and B on the crack surfaces, we have

$$s(B) - s(A) = ds = +\infty \tag{21}$$

where s is the true length of the crack. We know that a cohesive force dF acts in the interval $[x, x+dx]$. This cohesive force restrains crack faces from separation by relative displacement δ. Because dF is a cohesive force, it must be perpendicular to the x-axis. In the interval $[x, x+dx]$ the fractal curve has a finite D-measure, i.e.

$$m_D(B) - m_D(A) = dm_D < \infty\,. \tag{22}$$

Now a D-fractal cohesive pseudo-stress is defined as

$$dF = \sigma^D dm_D\,. \tag{23}$$

We now make the following two postulates:

(i) $d_f \ll a_o$, where d_f and $2a_0$ are nominal lengths of the end-region and the crack, respectively.
(ii) For all fractal cracks with fractal dimension D (or Hurst exponent H in the case of self-affine fractal cracks) that are about to propagate, the distribution of D-fractal cohesive pseudo-stresses is a material property. In other words, the distribution of σ^D does not depend on applied external loads, nominal crack length, or geometry of the cracked structure.

Our first postulate is very similar to that of Barenblatt. The only difference is that instead of using true lengths, nominal lengths of the end-regions and the crack are used. The second postulate has a fundamental difference with Barenblatt's second postulate. Here, the form of cohesive pseudo-stresses is not a material property; it is a function of both the material and the fractal dimension of the crack. The reason is that it is possible to have two cracks with different fractal dimensions in the same material. This is similar to D-dependence of the specific surface energy in fractal Griffith's theory.[24]

In the generalization of Barenblatt's theory, fractal cohesive pseudo-stresses distribute in such a way as to be able to remove the stress singularity at the crack tip. Consider two systems of loading: (a) there are no fractal cohesive pseudo-stresses in the end-regions and the cracked structure is under applied external loads, and (b) there are no applied external loads, and the crack is under fractal cohesive pseudo-stresses in the end-regions. In the loading condition (a), stresses at the crack tip have the following distribution

$$\sigma_{ij}^{f,\text{applied}}(r, \theta) = K_I^f r^{-\alpha} f_{ij}(\theta, D) + \text{non-singular terms} \tag{24}$$

where $\alpha = \alpha(D)$ is the unknown order of stress singularity. Note that the fractal crack has two-measure (area) zero and hence stress tensor can be defined far from the fractal crack. Similarly, in the loading condition (b), stresses have the following distribution

$$\sigma_{ij}^{f,\text{cohesive}}(r, \theta) = -\frac{K_{\text{coh}}^f}{\pi} r^{-\alpha} f_{ij}(\theta, D) + \text{ non-singular terms} \qquad (25)$$

where K_{coh}^f is the fractal modulus of cohesion and is a function of both the material and the fractal dimension of the crack. For stresses to be non-singular when $r \to 0^+$, we must have

$$K_I^f = \frac{K_{\text{coh}}^f}{\pi} \qquad (26)$$

which is a generalization of the finiteness condition (16). Because this theory assumes the form of the stress distribution at the crack tip *a priori*, it is not possible to find the order of stress singularity α.

We now demonstrate the equivalence of Griffith's and Barenblat's theories for fractal cracks. Before applying external loads, crack opening displacement δ is zero. Due to external loads, δ increases from zero until it reaches the value δ_f^*. At this moment, the bond between crack faces at this point breaks and new fractal free surfaces are created. In this process, D-fractal cohesive pseudo-stress σ^D is a function of the relative displacement of crack faces δ, i.e. $\sigma^D = \sigma^D(\delta)$. We assume that the δ-dependence of σ^D is of the form shown in Fig. 5(b). In the process of increasing δ from zero to δ_f^*, the cohesive pseudo-stress does some amount of fractal work (work per unit of a fractal measure), which can be expressed as

$$W_f = \int_0^{\delta^*} \sigma^D(\delta) d\delta\,. \qquad (27)$$

Hence, the surface energy required for propagation of the fractal crack by an amount Δm_D is defined by the following Lebesque integral

$$\Delta U_s = \int_0^{\Delta m_D} \int_0^{\delta^*} \sigma^D(\delta) d\delta\, dm_D = \Delta m_D \int_0^{\delta^*} \sigma^D(\delta) d\delta\,. \qquad (28)$$

But we know that

$$\Delta U_s = 2\Delta m_D\, \gamma_f(D)\,. \qquad (29)$$

Therefore, we find the following interesting relation

$$\int_0^{\delta^*} \sigma^D(\delta) d\delta = 2\gamma_f(D) \qquad (30)$$

which shows the equivalence of the two theories for fractal cracks.

6. CONCLUSIONS

In this paper, a D-fractal pseudo-stress vector is defined. A critical review of Griffith and Barenblatt fracture theories is presented. Differences between these theories and their equivalence are discussed. Borodich's modification of Griffith's theory for fractal cracks is reviewed. The concept of driving force is generalized for fractal cracks and its dependence on crack tip parameters and fractal dimension of the crack is discussed. Barenblatt's theory is generalized for fractal cracks. Fractal cohesive pseudo-stress is defined and the equivalence of Griffith's and Barenblatt's theories for fractal cracks is demonstrated. It is seen that Griffith's theory has the superiority that it does not rely on our knowledge of the form of stress distribution around the crack tip.

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