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On Hashin's Hollow Cylinder and Sphere Assemblages in Anisotropic Nonlinear Elasticity

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Abstract
We generalize Hashin's nonlinear isotropic hollow cylinder and sphere assemblages to nonlinear anisotropic solids. More specifically, we find the effective hydrostatic constitutive equation of nonlinear transversely isotropic hollow sphere assemblages with radial material preferred directions. We also derive the effective constitutive equations of finite and infinitely-long hollow cylinder assemblages made of incompressible orthotropic solids with axial, radial, and circumferential material preferred directions. In both sphere and cylinder assemblages the spherical and cylindrical shells can be radially inhomogeneous as long as Hashin's definition of *similar* shells is properly generalized.

Keywords Neutral inclusions · Nonlinear composites · Porous materials · Composite assemblage · Nonlinear elasticity · Anisotropic solids

Mathematics Subject Classification 74B20 · 70G45 · 74E10 · 15A72 · 74Fxx

1 Introduction

The idea of neutral holes in elastic sheets was first introduced by Gurney [11], Reissner and Morduchow [27], and Mansfield [21] in the setting of linear elasticity. Neutral holes under finite radial deformations were studied in [36]. Neutral inhomogeneities when inserted in an elastic matrix do not perturb the stress and deformation fields outside the inclusions [1, 13–16]. Sphere assemblages were introduced by Hashin [12, 15, 16]. See also [26]. Hashin [13] analyzed the hollow sphere assemblages under large dilatational deformations and calculated their exact effective hydrostatic constitutive equations. Note that in the case of Hashin's hollow cylinder and sphere assemblages all the inclusions are neutral when the body is under a pure dilatational finite deformation. Lopez-Pamies et al. [19] showed that there exists an isotropic porous material consisting of mesoscopic and microscopic pores

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that is stiffer than Hashin's hollow cylinder assemblage under hydrostatic loading. In this paper we construct anisotropic analogues of Hashin's isotropic composite hollow cylinder and sphere assemblages and find their effective hydrostatic constitutive equations.

This paper is organized as follows. In §2 we briefly review nonlinear anisotropic elasticity. In §3, we analyze transversely isotropic hollow sphere assemblages and calculate their effective hydrostatic constitutive equations. In §4 the same problem is studied for orthotropic hollow cylinder assemblages. Conclusions are given in §5.

2 Anisotropic Nonlinear Elasticity

Kinematics In nonlinear elasticity, motion is a time-dependent mapping between a reference configuration (or natural configuration) and the ambient space, i.e., $\varphi_t : \mathcal{B} \to \mathcal{S}$, where $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ are the material and the ambient space Riemannian manifolds, respectively [22]. Here, $\mathbf{G}$ is the material metric (that allows one to measure distances in a natural stress-free configuration) and $\mathbf{g}$ is the background metric of the ambient space. The deformation gradient $\mathbf{F}$ is the tangent map of φ_t, which is defined as $\mathbf{F}(X,t) = T\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{S}$. The transpose of $\mathbf{F}$ is denoted by $\mathbf{F}^\mathsf{T}$, where

$$\mathbf{F}^\mathsf{T}(X,t) : T_{\varphi_t(X)}\mathcal{S} \to T_X\mathcal{B}, \quad \langle\!\langle \mathbf{W}, \mathbf{F}^\mathsf{T}\mathbf{w} \rangle\!\rangle_{\mathbf{G}} = \langle\!\langle \mathbf{F}\mathbf{W}, \mathbf{w} \rangle\!\rangle_{\mathbf{g}}, \quad \forall\, \mathbf{W} \in T_X\mathcal{B},\ \mathbf{w} \in T_{\varphi_t(X)}\mathcal{S}\,. \tag{2.1}$$

In components, $(F^\mathsf{T})^A{}_a = G^{AB} F^b{}_B\, g_{ab}$. The right Cauchy-Green deformation tensor is defined as $\mathbf{C} = \mathbf{F}^\mathsf{T}\mathbf{F} : T_X\mathcal{B} \to T_X\mathcal{B}$, which in components reads $C^A{}_B = F^a{}_M F^b{}_B\, g_{ab}\, G^{AM}$. Note that $\mathbf{C}^\flat$ agrees with the pull-back of the ambient space metric by φ_t, i.e., $\mathbf{C}^\flat = \varphi_t^* \mathbf{g}$.

Balance Laws The balance of linear momentum in spatial form reads

$$\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}, \tag{2.2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress. ρ, $\mathbf{b}$, and $\mathbf{a}$ are the mass density, body force, and acceleration, respectively.

The Jacobian of deformation relates the deformed and undeformed Riemannian volume elements as $dv(x,\mathbf{g}) = J\,dV(X,\mathbf{G})$, and is defined as $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}}\det \mathbf{F}$.

Material Symmetry Consider an elastic body $\mathcal{B}$ made of a *simple* material with the response function $\mathscr{R}$.[1] The material symmetry group $\mathcal{G}_X$ associated with the body at a point X with respect to the reference configuration $(\mathcal{B}, \mathbf{G})$ is defined as

$$\mathscr{R}(\mathbf{F}\mathbf{K}) = \mathscr{R}(\mathbf{F})\,, \quad \forall\, \mathbf{K} \in \mathcal{G}_X\,, \tag{2.3}$$

for all deformation gradients $\mathbf{F}$, where $\mathbf{K} : T_X\mathcal{B} \to T_X\mathcal{B}$ is an invertible linear transformation. Objectivity requires that the energy function of a hyperelastic solid depend on the deformation through the right Cauchy-Green deformation tensor $\mathbf{C}^\flat$, i.e., $W = W(X, \mathbf{C}^\flat, \mathbf{G})$ at a referential point X. Therefore, for a hyperelastic solid the material symmetry group $\mathcal{G}_X$ is defined to be the subgroup of $\mathbf{G}$-orthogonal transformations Orth $(\mathbf{G})$ such that[2] [5]

$$W(X, \boldsymbol{Q}^{-\star}\mathbf{C}^\flat \boldsymbol{Q}^{-1}, \mathbf{G}) = W(X, \mathbf{C}^\flat, \mathbf{G})\,, \quad \forall\, \boldsymbol{Q} \in \mathcal{G}_X \leq \mathrm{Orth}(\mathbf{G})\,. \tag{2.4}$$

[1] Here we assume that $\mathscr{R}$ is the energy function. Response function may be any measure of stress as well.

[2] Note that Orth $(\mathbf{G}) = \left\{\mathbf{Q} : T_X\mathcal{B} \to T_X\mathcal{B} \;\middle|\; \mathbf{Q}^\top = \mathbf{Q}^{-1}\right\}$. We use the notation $\mathscr{G} \leqslant \mathscr{H}$ when $\mathscr{G}$ is a subgroup of $\mathscr{H}$.

Constitutive Equations The energy function (per unit undeformed volume) of an inhomogeneous anisotropic hyperelastic material at a material point X is written in the following form

$$W = \hat{W}(X, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_n)\,, \tag{2.5}$$

where $\boldsymbol{\zeta}_i, i = 1, \dots, n$ are a collection of the so called *structural tensors* characterizing the material symmetry group at the point X (see also [2, 18, 20, 30, 31, 39]) such that

$$\bar{\boldsymbol{\zeta}}_j(X) = \boldsymbol{\zeta}_j(X)\,, \quad j = 1, \dots, n \iff \boldsymbol{Q} \in \mathcal{G}_X\,, \tag{2.6}$$

where $\bar{\boldsymbol{\zeta}}_j$ is the $\boldsymbol{Q}$-transformed $\boldsymbol{\zeta}_j$. Using the Doyle-Ericksen formula [3, 22, 38], the second Piola-Kirchhoff stress and the Cauchy stress tensors are expressed as

$$\mathbf{S} = 2\frac{\partial \hat{W}}{\partial \mathbf{C}^\flat}\,, \qquad \boldsymbol{\sigma} = \frac{2}{J}\frac{\partial \hat{W}}{\partial \mathbf{g}}\,, \tag{2.7}$$

where, with a slight abuse of notation, we may write

$$\hat{W}\big(X, \mathbf{G}, \mathbf{C}^\flat, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_n\big) = \hat{W}\big(x, \mathbf{G} \circ \varphi^{-1}, \mathbf{g}, \mathbf{F}, \boldsymbol{\zeta}_1 \circ \varphi^{-1}, \dots, \boldsymbol{\zeta}_n \circ \varphi^{-1}\big)\,. \tag{2.8}$$

Thus, using (2.5) and (2.7), one can write

$$\mathbf{S} = \hat{\mathbf{S}}(X, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_n)\,. \tag{2.9}$$

Using structural tensors makes the energy function and the stress tensor isotropic functions of their arguments, i.e.

$$\forall \boldsymbol{Q} \in \mathrm{Orth}(\mathbf{G}): \quad \bar{\mathbf{S}}(X, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_n) = \mathbf{S}(X, \bar{\mathbf{C}}^\flat, \bar{\mathbf{G}}, \bar{\boldsymbol{\zeta}}_1, \dots, \bar{\boldsymbol{\zeta}}_n)\,. \tag{2.10}$$

It is also noted that $\mathbf{S}$ (and $\hat{W}$) is an anisotropic function of $\mathbf{C}^\flat$ and $\mathbf{G}$ alone, with the type of anisotropy given by the symmetry group $\mathcal{G}_X$. To see this, using (2.6) and (2.10), one has

$$\bar{\mathbf{S}}(X, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_n) = \mathbf{S}(X, \bar{\mathbf{C}}^\flat, \bar{\mathbf{G}}, \boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_n)\,, \quad \forall \boldsymbol{Q} \in \mathcal{G}_X \leq \mathrm{Orth}(\mathbf{G})\,. \tag{2.11}$$

According to Hilbert's theorem, for any finite number of tensors, there exist a finite number of isotropic invariants forming a basis called *integrity basis* for the space of isotropic invariants of the collection of tensors. Thus, if I_j, $j = 1, \dots, m$, form an integrity basis for the set of tensors in (2.5), we have $W = W(X, I_1, \dots, I_m)$. Hence, using (2.7), one obtains

$$\mathbf{S} = \sum_{j=1}^{m} 2W_j \frac{\partial I_j}{\partial \mathbf{C}^\flat}\,, \quad W_j := \frac{\partial W}{\partial I_j}\,, \quad j = 1, \dots, m\,. \tag{2.12}$$

Isotropic Solids In the case of isotropic materials, the energy function is expressed as $W = W(X, I_1, I_2, I_3)$, where $I_1 = \operatorname{tr}\mathbf{C}$, $I_2 = \det\mathbf{C}\operatorname{tr}\mathbf{C}^{-1}$, and $I_3 = \det\mathbf{C}$ are the principal invariants of the right Cauchy-Green deformation tensor. It follows from (2.12) that

$$\mathbf{S} = 2\left[W_1\mathbf{G}^\sharp + W_2(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}) + W_3 I_3\mathbf{C}^{-1}\right]. \tag{2.13}$$

If the material is incompressible, i.e., $I_3 = 1$, one writes

$$\mathbf{S} = -p\mathbf{C}^{-1} + 2\left[W_1\mathbf{G}^\sharp - W_2\mathbf{C}^{-2}\right], \tag{2.14}$$

where p is the Lagrange multiplier associated with the incompressibility constraint $J = \sqrt{I_3} = 1$. The Cauchy stress $\sigma^{ab} = \frac{1}{J} F^a{}_A F^b{}_B S^{AB}$ similarly reads

$$\boldsymbol{\sigma} = -p\mathbf{g}^\sharp + \frac{2}{J}\frac{\partial \hat{W}}{\partial \mathbf{g}}\,. \tag{2.15}$$

In components

$$\sigma^{ab} = \frac{2}{\sqrt{I_3}}\left[W_1 b^{ab} + (I_2 W_2 + I_3 W_3) g^{ab} - I_3 W_2\, c^{ab}\right], \tag{2.16}$$

where

$$b^{ab} = F^a{}_A F^b{}_B\, G^{AB}, \quad c^{ab} = (F^{-1})^M{}_m (F^{-1})^N{}_n\, G_{MN}\, g^{am} g^{bn}\,. \tag{2.17}$$

In the case of incompressible solids

$$\sigma^{ab} = -pg^{ab} + 2\left(W_1 b^{ab} - W_2\, c^{ab}\right). \tag{2.18}$$

Transversely Isotropic Solids Let us assume a compressible transversely isotropic material such that the unit vector $\mathbf{N}(\mathrm{X})$ identifies the material preferred direction at a point X in the reference configuration. The strain energy density per unit volume of the reference configuration is given as (see, e.g., [3, 20, 31]) $W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A})$, where $\mathbf{A} = \mathbf{N} \otimes \mathbf{N}$ is a structural tensor representing the transverse isotropy of the material symmetry group. The energy function W depends on the following five independent invariants defined as

$$I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \det\mathbf{C}\operatorname{tr}\mathbf{C}^{-1}, \quad I_3 = \det\mathbf{C}, \quad I_4 = \mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}, \quad I_5 = \mathbf{N}\cdot\mathbf{C}^2\cdot\mathbf{N}. \tag{2.19}$$

In components they read

$$\begin{aligned} &I_1 = C^A{}_A\,, \quad I_2 = \det[C^A{}_B](C^{-1})^D{}_D\,, \quad I_3 = \det[C^A{}_B]\,,\\ &I_4 = N^A N^B C_{AB}\,, \quad I_5 = N^A N^B C_{BM} C^M{}_A\,. \end{aligned} \tag{2.20}$$

Thus, one obtains

$$\mathbf{S} = \sum_{j=1}^{5} 2W_j \frac{\partial I_j}{\partial \mathbf{C}^\flat}\,, \qquad W_j := \frac{\partial W}{\partial I_j}\,,\ j = 1,\ldots,5\,. \tag{2.21}$$

Note that

$$\begin{aligned} &\frac{\partial I_1}{\partial \mathbf{C}^\flat} = \mathbf{G}^\sharp, \quad \frac{\partial I_2}{\partial \mathbf{C}^\flat} = I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}, \quad \frac{\partial I_3}{\partial \mathbf{C}^\flat} = I_3\mathbf{C}^{-1},\\ &\frac{\partial I_4}{\partial \mathbf{C}^\flat} = \mathbf{N}\otimes\mathbf{N}, \quad \frac{\partial I_5}{\partial \mathbf{C}^\flat} = \mathbf{N}\otimes(\mathbf{C}\cdot\mathbf{N}) + (\mathbf{C}\cdot\mathbf{N})\otimes\mathbf{N}\,. \end{aligned} \tag{2.22}$$

Therefore, using (2.22), we obtain the following representation for the second Piola-Kirchhoff stress tensor

$$\begin{aligned} \mathbf{S} = 2\Big\{&W_1\mathbf{G}^\sharp + W_2\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + W_3 I_3\mathbf{C}^{-1}\\ &+ W_4\,(\mathbf{N}\otimes\mathbf{N}) + W_5\,[\mathbf{N}\otimes(\mathbf{C}\cdot\mathbf{N}) + (\mathbf{C}\cdot\mathbf{N})\otimes\mathbf{N}]\Big\}\,. \end{aligned} \tag{2.23}$$

The Cauchy stress tensor is represented in component form as

$$\sigma^{ab} = \frac{2}{\sqrt{I_3}}\Big[W_1 b^{ab} + (I_2 W_2 + I_3 W_3) g^{ab} - I_3 W_2\, c^{ab} + W_4\, n^a n^b + W_5 (n^a b^{bc} n_c + n^b b^{ac} n_c)\Big], \tag{2.24}$$

where $n^a = F^a{}_A N^A$. If the material is incompressible, then $I_3 = 1$, and hence, $W = W(\mathrm{X}, I_1, I_2, I_4, I_5)$. Thus, from (2.23), $\mathbf{S}$ is expressed as

$$\mathbf{S} = -p\mathbf{C}^{-1} + 2\Big\{W_1 \mathbf{G}^\sharp + W_2\left(I_2 \mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + W_4\,(\mathbf{N}\otimes\mathbf{N}) + W_5\,[\mathbf{N}\otimes(\mathbf{C}\cdot\mathbf{N}) + (\mathbf{C}\cdot\mathbf{N})\otimes\mathbf{N}]\Big\}. \tag{2.25}$$

The Cauchy stress tensor is represented in component form as [7–9, 32]

$$\sigma^{ab} = -p g^{ab} + 2\left[W_1 b^{ab} - W_2\, c^{ab} + W_4\, n^a n^b + W_5 (n^a b^{bc} n^d g_{cd} + n^b b^{ac} n^d g_{cd})\right]. \tag{2.26}$$

Orthotropic Solids Next, we consider a compressible orthotropic material with three $\mathbf{G}$-orthonormal vectors $\mathbf{N}_1(\mathrm{X})$, $\mathbf{N}_2(\mathrm{X})$, and $\mathbf{N}_3(\mathrm{X})$ specifying the orthotropic axes in the reference configuration at a point X. A choice of structural tensors is given by $\mathbf{A}_1 = \mathbf{N}_1 \otimes \mathbf{N}_1$, $\mathbf{A}_2 = \mathbf{N}_2 \otimes \mathbf{N}_2$, and $\mathbf{A}_3 = \mathbf{N}_3 \otimes \mathbf{N}_3$, where only two of which are independent as $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I}$. Hence, the energy function is given as $W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A}_1, \mathbf{A}_2)$ [3, 20, 31]. The energy function W is represented in terms of the following seven independent invariants

$$\begin{aligned} &I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \det\mathbf{C}\operatorname{tr}\mathbf{C}^{-1}, \quad I_3 = \det\mathbf{C}, \quad I_4 = \mathbf{N}_1\cdot\mathbf{C}\cdot\mathbf{N}_1, \\ &I_5 = \mathbf{N}_1\cdot\mathbf{C}^2\cdot\mathbf{N}_1, \quad I_6 = \mathbf{N}_2\cdot\mathbf{C}\cdot\mathbf{N}_2, \quad I_7 = \mathbf{N}_2\cdot\mathbf{C}^2\cdot\mathbf{N}_2. \end{aligned} \tag{2.27}$$

Thus

$$\mathbf{S} = \sum_{j=1}^{7} 2W_j \frac{\partial I_j}{\partial \mathbf{C}^\flat}, \qquad W_j := \frac{\partial W}{\partial I_j},\ j = 1,\ldots,7. \tag{2.28}$$

Hence, the second Piola-Kirchhoff stress tensor is given by

$$\begin{aligned} \mathbf{S} = 2\Big\{&W_1\mathbf{G}^\sharp + W_2\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + W_3 I_3 \mathbf{C}^{-1} \\ &+ W_4\,(\mathbf{N}_1\otimes\mathbf{N}_1) + W_5\,[\mathbf{N}_1\otimes(\mathbf{C}\cdot\mathbf{N}_1) + (\mathbf{C}\cdot\mathbf{N}_1)\otimes\mathbf{N}_1] \\ &+ W_6\,(\mathbf{N}_2\otimes\mathbf{N}_2) + W_7\,[\mathbf{N}_2\otimes(\mathbf{C}\cdot\mathbf{N}_2) + (\mathbf{C}\cdot\mathbf{N}_2)\otimes\mathbf{N}_2]\Big\}. \end{aligned} \tag{2.29}$$

The Cauchy stress tensor is represented in component form as

$$\begin{aligned} \sigma^{ab} = \frac{2}{\sqrt{I_3}}\Big[&W_1 b^{ab} + (I_2 W_2 + I_3 W_3) g^{ab} - I_3 W_2\, c^{ab} \\ &+ W_4\, n_1^a n_1^b + W_5\left(n_1^a\, b^{bc}\, n_1^d\, g_{cd} + n_1^b\, b^{ac}\, n_1^d\, g_{cd}\right) \\ &+ W_6\, n_2^a n_2^b + W_7\left(n_2^a\, b^{bc}\, n_2^d\, g_{cd} + n_2^b\, b^{ac}\, n_2^d\, g_{cd}\right)\Big], \end{aligned} \tag{2.30}$$

where $n_1^a = F^a{}_A N_1^A$, and $n_2^a = F^a{}_A N_2^A$. In the case of incompressible solids one has the following representation for the second Piola-Kirchhoff stress tensor

$$\begin{aligned}\mathbf{S} = & -p\mathbf{C}^{-1} + 2\Big\{ W_1 \mathbf{G}^\sharp + W_2 \left(I_2 \mathbf{C}^{-1} - \mathbf{C}^{-2} \right) \\ & + W_4 \left(\mathbf{N}_1 \otimes \mathbf{N}_1 \right) + W_5 \left[\mathbf{N}_1 \otimes (\mathbf{C} \cdot \mathbf{N}_1) + (\mathbf{C} \cdot \mathbf{N}_1) \otimes \mathbf{N}_1 \right] \\ & + W_6 \left(\mathbf{N}_2 \otimes \mathbf{N}_2 \right) + W_7 \left[\mathbf{N}_2 \otimes (\mathbf{C} \cdot \mathbf{N}_2) + (\mathbf{C} \cdot \mathbf{N}_2) \otimes \mathbf{N}_2 \right] \Big\} .\end{aligned} \tag{2.31}$$

In components, the Cauchy stress tensor is given as

$$\begin{aligned}\sigma^{ab} = & -pg^{ab} + 2F^a{}_A F^b{}_B \Big[(W_1 + I_1 W2) G^{AB} - W_2 C^{AB} \\ & + W_4 N_1^A N_1^B + W_5 \left(N_1^Q N_1^A C^B{}_Q + N_1^P N_1^B C_P{}^A \right) \\ & + W_6 N_2^A N_2^B + W_7 \left(N_2^S N_2^A C^B{}_S + N_2^K N_2^B C_K{}^A \right) \Big] .\end{aligned} \tag{2.32}$$

Or equivalently [8, 9, 29, 32]

$$\begin{aligned}\sigma^{ab} = & -pg^{ab} + 2\Big[W_1 b^{ab} - I_3 W_2 c^{ab} \\ & + W_4 n_1^a n_1^b + W_5 \left(n_1^a b^{bc} n_1^d g_{cd} + n_1^b b^{ac} n_1^d g_{cd} \right) \\ & + W_6 n_2^a n_2^b + W_7 \left(n_2^a b^{bc} n_2^d g_{cd} + n_2^b b^{ac} n_2^d g_{cd} \right) \Big] .\end{aligned} \tag{2.33}$$

3 Hollow Transversely Isotropic Sphere Assemblages

Let us consider a spherical shell of inner radius R_i and outer radius R_o in its undeformed configuration made of a nonlinear incompressible transversely isotropic material with the strain energy function $W = W(I_1, I_2, I_4, I_5)$. We assume that the material preferred direction is radial, i.e., $\mathbf{N} = \hat{\mathbf{R}}$, where $\hat{\mathbf{R}}$ is a unit vector in the radial direction.[3] More specifically, with respect to the spherical coordinates (R, Θ, Φ) the material metric and the material preferred unit vector have the following representations

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 \sin^2 \Theta \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{3.1}$$

We consider radially-symmetric deformations such that in the spherical coordinates $(r, \theta, \phi) = (r(R), \Theta, \Phi)$.[4] The radial stretch is denoted by $\lambda(R) = r(R)/R$. Assume that the shell deforms such that in the deformed configuration $\lambda(R_o) = \lambda_0$, where λ_0 is a positive constant, and the hole surface is traction-free. The incompressibility constraint implies that

$$\lambda(R) = \left[1 + \frac{R_o^3}{R^3} \left(\lambda_0^3 - 1 \right) \right]^{\frac{1}{3}} . \tag{3.2}$$

[3]This can be thought of as a model for a polymer with spherulitic microstructure [4]. See also [17, 28].

[4]We assume that λ_0 is small enough such that radial deformations are the only possible deformations.

The right Cauchy-Green deformation tensor reads $\mathbf{C} = \mathrm{diag}\left(\lambda^{-4}(R), \lambda^{2}(R), \lambda^{2}(R)\right)$. Using (2.22) and (2.26) the Cauchy stress tensor has the following non-zero physical components

$$\begin{aligned}
\hat{\sigma}^{rr}(R) &= -p(R) + 2\lambda^{-4}(R)\left[W_1(R) + W_4(R)\right] + 4\lambda^{-2}(R)W_2(R) + 4\lambda^{-8}(R)W_5(R)\,,\\
\hat{\sigma}^{\theta\theta}(R) &= \hat{\sigma}^{\phi\phi}(R) = -p(R) + 2\lambda^{2}(R)W_1(R) + 2\left[\lambda^{-2}(R) + \lambda^{4}(R)\right]W_2(R)\,.
\end{aligned} \tag{3.3}$$

The invariants of the energy function read (note that $I_3 = 1$ due to incompressibility)

$$\begin{aligned}
I_1(R) &= 2\lambda^{2}(R) + \lambda^{-4}(R)\,, \qquad I_2(R) = 2\lambda^{-2}(R) + \lambda^{4}(R)\,,\\
I_4(R) &= \lambda^{-4}(R)\,, \qquad I_5(R) = \lambda^{-8}(R)\,.
\end{aligned} \tag{3.4}$$

The equilibrium equation, i.e., $\hat{\sigma}^{rr}{}_{,r} + \frac{2}{r}\left(\hat{\sigma}^{rr} - \hat{\sigma}^{\theta\theta}\right) = 0$, and the boundary condition $\hat{\sigma}^{rr}(R_i) = 0$ imply that

$$\begin{aligned}
\hat{\sigma}^{rr}(R) &= \int_{R_i}^{R} \frac{4}{\xi\lambda(\xi)}\Big\{W_1(\xi)\left[1 - \lambda^{-6}(\xi)\right] + W_2(\xi)\left[\lambda^{2}(\xi) - \lambda^{-4}(\xi)\right]\\
&\qquad - W_4(\xi)\lambda^{-6}(\xi) - 2W_5(\xi)\lambda^{-10}(\xi)\Big\}d\xi\,,\\
\hat{\sigma}^{\theta\theta}(R) &= \hat{\sigma}^{\phi\phi}(R) = 2W_1(R)\left[\lambda^{2}(R) - \lambda^{-4}(R)\right] + 2W_2(R)\left[\lambda^{4}(R) - \lambda^{-2}(R)\right]\\
&\qquad - 2W_4(R)\lambda^{-4}(R) - 4W_5(R)\lambda^{-8}(R)\\
&\qquad + \int_{R_i}^{R} \frac{4}{\xi\lambda(\xi)}\Big\{W_1(\xi)\left[1 - \lambda^{-6}(\xi)\right] + W_2(\xi)\left[\lambda^{2}(\xi) - \lambda^{-4}(\xi)\right]\\
&\qquad - W_4(\xi)\lambda^{-6}(\xi) - 2W_5(\xi)\lambda^{-10}(\xi)\Big\}d\xi\,,
\end{aligned} \tag{3.5}$$

where $W_j(\xi) = W_j(I_1(\xi), I_2(\xi), I_4(\xi), I_5(\xi)) = \bar{W}_j(\lambda(\xi))$, $j = 1, 2, 4, 5$. Note that the stress components depend on the coordinate R only through the radial stretch $\lambda(R)$, and thus, as long as the ratio R_i/R_o is fixed for the spherical shells with different radii R_o (*similar* spherical shells), they will have the same stress distribution and will require the same boundary traction to maintain the deformation.[5]

Remark 3.1 Suppose that the spherical shell is inhomogeneous but still radially symmetric, i.e., $W = W(R, I_1, I_2, I_4, I_5)$. In this case stresses are still given by (3.5) but with $W_j(\xi) = W_j(\xi, I_1(\xi), I_2(\xi), I_4(\xi), I_5(\xi)) = \bar{W}_j(\xi, \lambda(\xi))$, $j = 1, 2, 4, 5$. In this case, two radially inhomogeneous spherical shells with inner radii R_i, $\tilde{R}_i$, and outer radii R_o, $\tilde{R}_o$, are

[5]For the same λ_0 consider another hollow spherical shell with inner and outer radii $\tilde{R}_i$ and $\tilde{R}_o$, respectively, such that $\tilde{R}_i = kR_i$, and $\tilde{R}_o = kR_o$, $k > 0$. For the second spherical shell $\tilde{R}_i < \tilde{R} < \tilde{R}_o$, where $\tilde{R} = kR$. Note that

$$\tilde{\lambda}(\tilde{R}) = \left[1 + \frac{\tilde{R}_o^3}{\tilde{R}^3}\left(\lambda_0^3 - 1\right)\right]^{\frac{1}{3}} = \left[1 + \frac{R_o^3}{R^3}\left(\lambda_0^3 - 1\right)\right]^{\frac{1}{3}} = \lambda(R)\,.$$

Also $\tilde{I}_1(\tilde{R}) = I_1(R)$, $\tilde{I}_2(\tilde{R}) = I_2(R)$, $\tilde{I}_4(\tilde{R}) = I_4(R)$, $\tilde{I}_5(\tilde{R}) = I_5(R)$, and hence $W_1(\tilde{R}) = W_1(R)$, $W_2(\tilde{R}) = W_2(R)$, $W_4(\tilde{R}) = W_4(R)$, and $W_5(\tilde{R}) = W_5(R)$. Therefore, $\tilde{\hat{\sigma}}^{rr}(\tilde{R}) = \hat{\sigma}^{rr}(R)$, and similarly for the other components of the Cauchy stress. This was Hashin's observation in the case of isotropic composite spheres.

called similar if:[6]

$$\frac{\tilde{R}_i}{\tilde{R}_o} = \frac{R_i}{R_o}\,, \text{ and } \quad \tilde{W}(\tilde{R}, I_1, I_2, I_4, I_5) = W(R, I_1, I_2, I_4, I_5)\,. \tag{3.6}$$

It is straightforward to show that for inhomogeneous similar spherical shells $\hat{\tilde{\sigma}}^{rr}(\tilde{R}) = \hat{\sigma}^{rr}(R)$, and $\hat{\tilde{\sigma}}^{\theta\theta}(\tilde{R}) = \hat{\sigma}^{\theta\theta}(R)$. In other words the hollow sphere assemblage can be constructed for radially inhomogeneous inclusions as well.

Remark 3.2 It is known that radial deformations of a spherical shell are universal deformations for incompressible isotropic solids [6]. More specifically, the class of deformations considered here are a subset of Family 4 of universal deformations for incompressible isotropic solids. Note that the full Family 4 includes inversions too. Recently, Yavari and Goriely [37] showed that for incompressible transversely isotropic solids Family 4 deformations are universal and the only universal material preferred directions consistent with Family 4 deformations are radial. Also, Yavari [35] has shown that for inhomogeneous incompressible isotropic solids Family 4 is universal as long as the energy function has the form $W = W(R, I_1, I_2,)$. Here we observe that for radially inhomogeneous transversely isotropic spherical shells with radial material preferred direction radial deformations are still universal. See also [10] for a recent generalization of Ericksen's work to anelasticity.

Now consider a finite compressible homogeneous and isotropic elastic body subjected to a pure dilatational deformation such that $\mathbf{F} = \lambda_0 \mathbf{I}$, where $\mathbf{I}$ is the identity (in components $F^a{}_A = \lambda_0\, \delta^a_A$). The strain energy function of the (matrix) material is denoted by $W^M = W^M(I_1, I_2, I_3)$, where $I_1 = 3\lambda_0^2$, $I_2 = 3\lambda_0^4$, $I_3 = \lambda_0^6$ are the principal invariants of the right Cauchy-Green strain tensor. It immediately follows that the stress is hydrostatic in the isotropic matrix and is written as $\boldsymbol{\sigma} = \sigma_0 \mathbf{g}$, where σ_0 is a constant given by

$$\sigma_0 = \sigma_0(\lambda_0) = \frac{2}{\lambda_0}\left(W^M_{I_1} + 2\lambda_0^2 W^M_{I_2} + \lambda_0^4 W^M_{I_3}\right)\,. \tag{3.7}$$

When $\lambda_0 > 1$ ($\lambda_0 < 1$) one expects $\sigma_0 > 0$ ($\sigma_0 < 0$). These are the pressure-compression (P-C) inequalities [25, 34].

Following Hashin's construction of a composite sphere assemblage, any solid sphere of radius R_o in the isotropic compressible homogeneous matrix can be replaced by an incompressible transversely isotropic spherical shell with inner and outer radii R_i and R_o without perturbing the stress field in the remaining part of the body as long as

$$\begin{aligned}&\int_{\lambda_i\left(\lambda_0, \frac{R_i}{R_o}\right)}^{\lambda_0} \frac{4\eta}{1-\eta^3}\left[\bar{W}_1(\eta)\left(1-\eta^{-6}\right) + \bar{W}_2(\eta)\left(\eta^2-\eta^{-4}\right) - 2\bar{W}_5(\eta)\eta^{-10} - \bar{W}_4(\eta)\eta^{-6}\right] d\eta \\ &\quad = \sigma_o\,,\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}&\lambda_i = \lambda(R_i) = \left[1 + \frac{R_o^3}{R_i^3}\left(\lambda_0^3 - 1\right)\right]^{1/3}, \\ &I_1(\eta) = 2\eta^2 + \eta^{-4}, \quad I_2(\eta) = 2\eta^{-2} + \eta^4, \quad I_4(\eta) = \eta^{-4}, \quad I_5(\eta) = \eta^{-8}\,,\end{aligned} \tag{3.9}$$

[6] Similar radially inhomogeneous cylindrical shells are defined analogously using (3.6) if instead of spherical coordinates cylindrical coordinates are used.

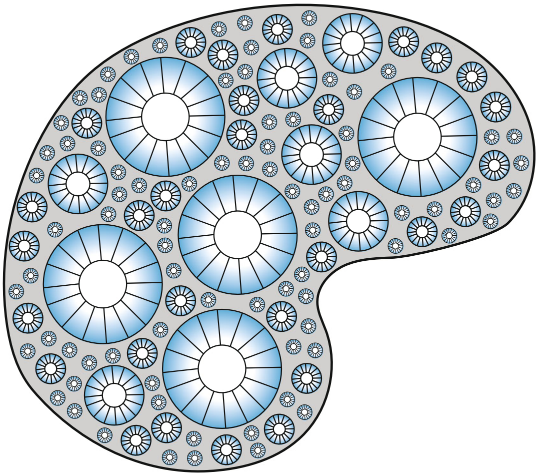

Fig. 1 An anisotropic hollow sphere/cylinder assemblage with inhomogeneous spherical/cylindrical shells. An assemblage with homogeneous spherical/cylindrical shells would be a special case

and the traction boundary condition $\hat{\sigma}^{rr}(R_i) = 0$ was used. For a fixed ratio R_i/R_o, and a given energy function W, (3.9) gives a relation $\sigma_0 = \sigma(\lambda_0)$. Hashin [13] does not consider an arbitrary compressible isotropic matrix; he assumes that the compressible isotropic matrix has an energy function that gives the same hydrostatic constitutive equation $\sigma_0 = \sigma(\lambda_0)$. Replacing the remaining part of the body by transversely isotropic spheres of diminishing sizes with the same ratio R_i/R_o one reaches the so called *sphere assemblage* geometrical arrangement. In the limit of the hollow sphere assemblage one obtains a porous material with initial porosity $c_0 = R_i^3/R_o^3$ that has the same hydrostatic constitutive equation $\sigma = \sigma(\lambda)$ as each of its constituent hollow spherical shells does. In other words, $\sigma = \sigma(\lambda, c_0)$ is the effective hydrostatic constitutive equation of the assemblage. In Fig. 1 a body with a transversely isotropic hollow sphere assemblage is shown (more precisely this is an element of a sequence that in the limit is a hollow sphere assemblage). The pure dilatational response of this composite is identical to that of any of its transversely isotropic hollow spherical shells. Note that the effective hydrostatic constitutive equations of assemblages made of similar homogenous and radially inhomogeneous spherical shells both have the form $\sigma = \sigma(\lambda, c_0)$. However, the class of sphere assemblages made of radially inhomogeneous spherical shells is much richer.

Example 3.3 Let us assume that the hollow spherical shells are made of an incompressible Mooney-Rivlin reinforced model (I_4 reinforcement) with energy function [23, 24, 33]

$$W(I_1, I_2, I_4) = C_1(I_1 - 3) + C_2(I_2 - 3) + \frac{\mu_1}{2}(I_4 - 1)^2, \tag{3.10}$$

where C_1, C_2, $\mu_1 > 0$, and $\mu_2 > 0$ are constants. Equation (3.8) is simplified to read

$$\begin{aligned}\sigma(\lambda_0, c_0) = {} & C_1\left(\frac{1}{\lambda_o^4} + \frac{4}{\lambda_o} - \frac{1}{\lambda_i^4} - \frac{4}{\lambda_i}\right) + 2C_2\left(\frac{1}{2\lambda_o^2} - \lambda_0 - \frac{1}{2\lambda_i^2} + \lambda_i\right) \\ & + \mu_1\left[\left(\frac{1}{2\lambda_0^8} + \frac{4}{5\lambda_0^5} - \frac{1}{\lambda_0^4} + \frac{2}{\lambda_0^2} - \frac{4}{\lambda_0}\right) - \left(\frac{1}{2\lambda_i^8} + \frac{4}{5\lambda_i^5} - \frac{1}{\lambda_i^4} + \frac{2}{\lambda_i^2} - \frac{4}{\lambda_i}\right)\right] \\ & + \frac{8\mu_1}{\sqrt{3}}\left\{\arctan\left[\frac{1+2\lambda_i}{\sqrt{3}}\right] - \arctan\left[\frac{1+2\lambda_o}{\sqrt{3}}\right]\right\},\end{aligned} \tag{3.11}$$

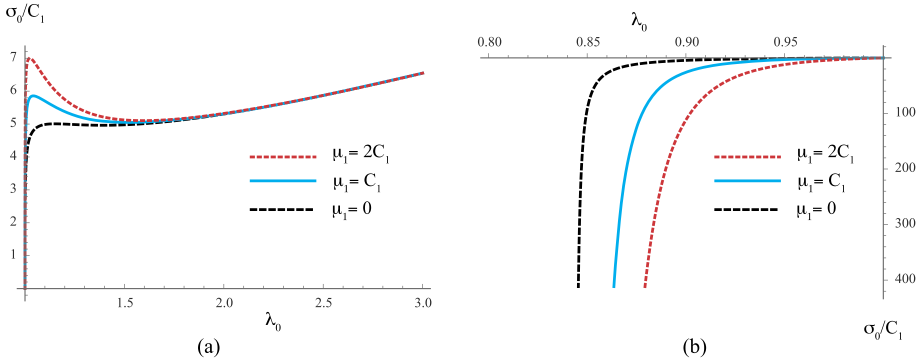

Fig. 2 (**a**) The constitutive equation (3.11) for $C_1 = 0.5$ MPa, $C_2 = 0.05$ MPa, $c_0 = 0.001$, and $\lambda_0 > 1$. (**b**) The constitutive equation (3.11) for $C_1 = 0.5$ MPa, $C_2 = 0.05$ MPa, $c_0 = 0.4$, and $\lambda_0 < 1$

where $\lambda_i = [1 + c_0^{-1}(\lambda_0^3 - 1)]^{\frac{1}{3}}$. When $\mu_1 = 0$, the hydrostatic constitutive equation (3.11) is identical to what Hashin [13] obtained for isotropic spherical shells. Following [13], let us assume that $C_1 = 0.5$ MPa, and $C_2 = 0.05$ MPa. Figure 2 shows the effective isotropic stress-strain relations for spherical assemblages for three different values of μ_1. We observe that even with I_4 reinforcement the responses of the assemblages in tension and compression are quite different. Figure 2(a) shows the tensile response of three assemblages that all have porosity $c_0 = 0.001$. It is seen that initially the assemblages with larger values of μ_1 are stiffer but for $\lambda_0 > 2$ the responses of the three assemblages are almost identical. The effect of I_4 reinforcement is more pronounced in compression as can be seen in Fig. 2(b).

Example 3.4 Let us assume that the hollow spherical shells are made of an incompressible Mooney-Rivlin reinforced model with I_5 reinforcement with energy function

$$W(I_1, I_2, I_5) = C_1(I_1 - 3) + C_2(I_2 - 3) + \frac{\mu_2}{2}(I_5 - 1)^2 . \tag{3.12}$$

In this case (3.8) is simplified to read

$$\begin{aligned}\sigma(\lambda_0, c_0) = {} & C_1\left(\frac{1}{\lambda_o^4} + \frac{4}{\lambda_o} - \frac{1}{\lambda_i^4} - \frac{4}{\lambda_i}\right) + 2C_2\left(\frac{1}{2\lambda_o^2} - \lambda_0 - \frac{1}{2\lambda_i^2} + \lambda_i\right) \\ & + \mu_2\Bigg[\left(\frac{1}{2\lambda_0^{16}} + \frac{8}{13\lambda_0^{13}} + \frac{4}{5\lambda_0^{10}} - \frac{1}{\lambda_0^8} + \frac{8}{7\lambda_0^7} - \frac{8}{5\lambda_0^5} + \frac{2}{\lambda_0^4} - \frac{4}{\lambda_0^2} + \frac{8}{\lambda_0}\right) \\ & - \left(\frac{1}{2\lambda_i^{16}} + \frac{8}{13\lambda_i^{13}} + \frac{4}{5\lambda_i^{10}} - \frac{1}{\lambda_i^8} + \frac{8}{7\lambda_i^7} - \frac{8}{5\lambda_i^5} + \frac{2}{\lambda_i^4} - \frac{4}{\lambda_i^2} + \frac{8}{\lambda_i}\right)\Bigg] \\ & + \frac{16\mu_2}{\sqrt{3}}\left\{\arctan\left[\frac{1+2\lambda_o}{\sqrt{3}}\right] - \arctan\left[\frac{1+2\lambda_i}{\sqrt{3}}\right]\right\},\end{aligned} \tag{3.13}$$

where $\lambda_i = [1 + c_0^{-1}(\lambda_0^3 - 1)]^{\frac{1}{3}}$. Figure 3 shows the effective isotropic stress-strain relations for spherical assemblages for three different values of μ_2. We again observe that even with I_5 reinforcement the responses of the assemblages in tension and compression are very different. The tensile responses of three assemblages that all have porosity $c_0 = 0.001$ are shown Fig. 3(a). We observe that initially the assemblages with larger values of μ_2 are stiffer

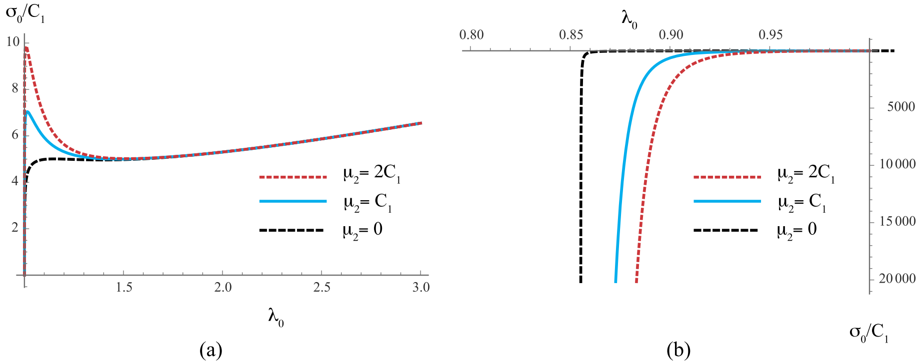

Fig. 3 (**a**) The constitutive equation (3.13) for $C_1 = 0.5$ MPa, $C_2 = 0.05$ MPa, $c_0 = 0.001$, and $\lambda_0 > 1$. (**b**) The constitutive equation (3.11) for $C_1 = 0.5$ MPa, $C_2 = 0.05$ MPa, $c_0 = 0.4$, and $\lambda_0 < 1$

but for $\lambda_0 > 1.5$ the responses of the three assemblages are almost identical. The effect of I_5 reinforcement is more pronounced in compression as can be seen in Fig. 3(b).

Remark 3.5 It is straightforward to show that one can alternatively use compressible transversely isotropic shells (with the radial material preferred direction) instead of incompressible ones provided that in lieu of (3.8) the following condition holds ($\lambda_0 = r(R_o)/R_o$)

$$2r'(R_o)\Big[\lambda_0^{-2}\left\{\left(W_{I_1(R_o)} + W_{I_4(R_o)}\right) + 2W_{I_5}r'(R_o)^2\right\} + 2W_{I_2(R_o)} + \lambda_0^2 W_{I_3(R_o)}\Big] = \sigma_0\,, \quad (3.14)$$

where

$$\begin{aligned} I_1 &= r'(R)^2 + 2\frac{r^2(R)}{R^2}\,, \quad I_2 = \frac{r^4(R)}{R^4} + 2\frac{r(R)^2}{R^2}r'(R)^2\,, \\ I_3 &= \frac{r^4(R)}{R^4}r'(R)^2\,, \quad I_4 = r'(R)^2\,, \quad I_5 = r'(R)^4\,, \end{aligned} \quad (3.15)$$

and $r(R)$ satisfies a second-order ODE dictated by the equilibrium equation in the radial direction. Similar to the incompressible case, it can be shown that the dependence of the stress and the energy function invariants on R is only through $\lambda(R)$. This is done by showing that the ODE governing $r(R)$ can be rewritten as a second-order ODE for $\lambda(R)$.

Remark 3.6 (Neutral hollow spherical inclusions) Consider a homogeneous body made of an isotropic compressible solid with energy function $W^M(I_1, I_2, I_3)$. Let us assume that the matrix is under a pure dilatational deformation λ_0. As was discussed earlier the body would be in a state of hydrostatic stress $\sigma_0 = \frac{2}{\lambda_0}(W^M_{I_1} + 2\lambda_0^2 W^M_{I_2} + \lambda_0^4 W^M_{I_3})$. Now consider the same body but with a hole of radius R_i in the undeformed configuration. Under the same deformation the state of stress will be perturbed by the hole. We would like to cloak the hole by a spherical shell with outer radius R_o such that outside the cloak, i.e., for $R \geq R_o$ the state of stress is the hydrostatic stress σ_0. The design parameters are the elastic properties of the cloaking shell. Let us assume that the cloak is made of an incompressible neo-Hookean solid with energy function $\frac{\mu}{2}(I_1 - 3)$. From (3.8) we have

$$\mu = \frac{4}{\lambda_0}\left[\frac{1+4\lambda_0^2}{\lambda_0^4} - \frac{1+4\lambda_i^2}{\lambda_i^4}\right]^{-1}\left(W^M_{I_1} + 2\lambda_0^2 W^M_{I_2} + \lambda_0^4 W^M_{I_3}\right)\,, \quad (3.16)$$

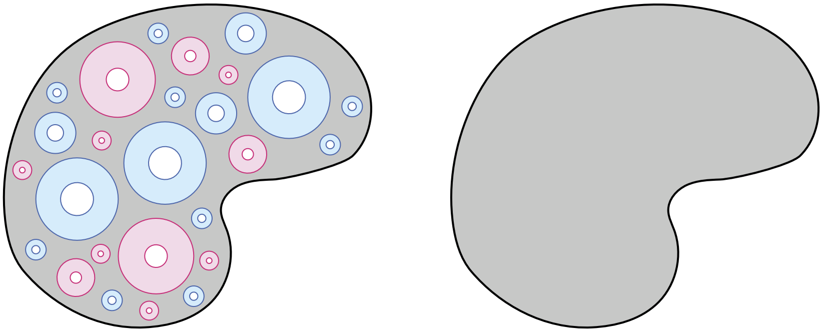

Fig. 4 Left: A compressible isotropic body with holes. Two different incompressible neo-Hookean spherical shells are used to cloak the holes. Right: The same body without holes. Under a specific pure dilatational deformation $\varphi(X) = \lambda_0 X$ the matrix of the body with holes has the same uniform hydrostatic stress as the homogeneous body does

where $\lambda_i = [1 + c_0^{-1}(\lambda_0^3 - 1)]^{\frac{1}{3}}$, and $c_0 = R_i^3/R_o^3$. Note that

$$\begin{cases} \lambda_i > \lambda_0, & \lambda_0 > 1, \\ \lambda_i < \lambda_0, & \lambda_0 < 1\,. \end{cases} \tag{3.17}$$

Thus

$$\begin{cases} \frac{1+4\lambda_0^2}{\lambda_0^4} - \frac{1+4\lambda_i^2}{\lambda_i^4} > 0, & \lambda_0 > 1, \\ \frac{1+4\lambda_0^2}{\lambda_0^4} - \frac{1+4\lambda_i^2}{\lambda_i^4} < 0, & \lambda_0 < 1\,. \end{cases} \tag{3.18}$$

Therefore, we conclude that $\mu > 0$ for any compressible isotropic matrix that satisfies the P-C inequalities [25, 34]. In other words, for a given λ_0, cloaking is always possible using an incompressible spherical shell made of a homogeneous neo-Hookean solid.

Note that μ explicitly depends on λ_0. Also, for a given W^M and λ_0, μ depends on R_i/R_o. This means that the same incompressible neo-Hookean material can be used for cloaking spherical cavities of different sizes as long as the appropriate size R_o for the cloak is chosen. One should also note that μ is a strictly increasing function of R_i/R_o, and $\mu \to \infty$, as $R_i/R_o \to 1$. Figure 4 shows a body with spherical cavities that are cloaked using two different incompressible neo-Hookean solids. In summary, for a given compressible isotropic matrix with energy function $W^M = W^M(I_1, I_2, I_3)$, and under a given pure dilatational deformation λ_0, one can cloak a spherical hole (or a set of spherical holes) such that outside the cloak(s) the hydrostatic response of the body is identical to that of the same body made of the homogeneous and isotropic matrix. Note that a cloak designed for λ_0 would not work for other values of stretch, in general. This is a nonlinear analogue of Mansfield [21]'s neutral holes. See also, [36].

4 Hollow Orthotropic Cylinder Assemblages

4.1 Finite Hollow Cylinders

Let us next consider a finite incompressible orthotropic cylindrical shell of length L and inner and outer radii R_i and R_o, respectively, in its undeformed configuration. Assume that the material orthotropic axes are in the radial, circumferential, and axial directions in the cylindrical coordinates (R, Θ, Z), i.e., $\mathbf{N}_1 = \hat{\mathbf{R}}$, $\mathbf{N}_2 = \hat{\boldsymbol{\Theta}}$, and $\mathbf{N}_3 = \hat{\mathbf{Z}}$, where $\hat{\mathbf{R}}$, $\hat{\boldsymbol{\Theta}}$, and $\hat{\mathbf{Z}}$ denote the unit vectors in the radial, circumferential, and longitudinal directions, respectively. More specifically, with respect to the cylindrical coordinates (R, Θ, Z) the material metric and the three material preferred unit vectors have the following representations

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N}_1 = \begin{bmatrix} 0 \\ \frac{1}{R} \\ 0 \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.1}$$

The strain energy function of the shell is denoted by $W = W(I_1, I_2, I_4, I_5, I_6, I_7)$. Let us consider deformations of the form $(r, \theta, z) = (r(R), \Theta, \alpha Z)$, where $\alpha > 0$ is the axial stretch. The radial stretch is denoted by $\lambda(R) = r(R)/R$. Suppose that the shell deforms such that the inner surface of the cylindrical shell remains traction-free, the radial stretch at $R = R_o$ is $\lambda(R_o) = \lambda_0$, and the axial stretch is α. The right Cauchy-Green deformation tensor reads $\mathbf{C} = \mathrm{diag}(\alpha^{-2}\lambda^{-2}(R), \lambda^2(R), \alpha^2)$. Incompressibility constraint dictates that $r(R)r'(R) = R/\alpha$, and hence

$$\lambda(R) = \left[\frac{1}{\alpha} + \left(\lambda_0^2 - \frac{1}{\alpha}\right)\frac{R_o^2}{R^2}\right]^{1/2}. \tag{4.2}$$

Employing (2.27) and (2.32), the non-zero components of the stress read

$$\begin{aligned}
\hat{\sigma}^{rr}(R) &= -p(R) + 2\alpha^{-2}\lambda^{-2}(R)\left[W_1(R) + W_4(R)\right] + 2\left[\lambda^{-2}(R) + \alpha^{-2}\right] W_2(R) \\
&\quad + 4\alpha^{-4}\lambda^{-4}(R) W_5(R)\,, \\
\hat{\sigma}^{\theta\theta}(R) &= -p(R) + 2\lambda^2(R)\left[W_1(R) + \alpha^2 W_2(R)\right] + 2\alpha^{-2} W_2(R)\,, \\
\hat{\sigma}^{zz}(R) &= -p(R) + 2\alpha^2\left[W_1(R) + W_6(R) + 2\alpha^2 W_7(R)\right] + 2\left[\lambda^{-2}(R) + \alpha^2\lambda^2(R)\right] W_2(R)\,.
\end{aligned} \tag{4.3}$$

The energy function has the following invariants

$$\begin{aligned}
&I_1 = \lambda^2(R) + \alpha^{-2}\lambda^{-2}(R) + \alpha^2\,, \quad I_2 = \lambda^{-2}(R) + \alpha^2\lambda^2(R) + \alpha^{-2}\,, \\
&I_4 = \alpha^{-2}\lambda^{-2}(R)\,, \quad I_5 = \alpha^{-4}\lambda^{-4}(R)\,, \quad I_6 = \alpha^2\,, \quad I_7 = \alpha^4\,.
\end{aligned} \tag{4.4}$$

The only non-trivial equilibrium equation, $\hat{\sigma}^{rr}{}_{,r} + \frac{1}{r}\left(\hat{\sigma}^{rr} - \hat{\sigma}^{\theta\theta}\right) = 0$, implies that

$$\begin{aligned}
\hat{\sigma}^{rr}(R) = \int_{R_i}^{R} \frac{2}{\alpha\xi}\Big\{&\left[1 - \alpha^{-2}\lambda^{-4}(\xi)\right]\left[W_1(\xi) + \alpha^2 W_2(\xi)\right] \\
&- \alpha^{-2}\lambda^{-4}(\xi) W_4(\xi) - 2\alpha^{-4}\lambda^{-6}(\xi) W_5(\xi)\Big\} d\xi\,,
\end{aligned}$$

$$
\begin{aligned}
\hat{\sigma}^{\theta\theta}(R) &= 2\left[\lambda^2(R) - \alpha^{-2}\lambda^{-2}(R)\right]\left[W_1(R) + \alpha^2 W_2(R)\right] \\
&\quad - 2\alpha^{-2}\lambda^{-2}(R)\left[W_4(R) + 2\alpha^{-2}\lambda^{-2}(R) W_5(R)\right] \\
&\quad + \int_{R_i}^{R} \frac{2}{\alpha\xi}\Big\{\left[1 - \alpha^{-2}\lambda^{-4}(\xi)\right]\left[W_1(\xi) + \alpha^2 W_2(\xi)\right] \\
&\qquad\qquad - \alpha^{-2}\lambda^{-4}(\xi) W_4(\xi) - 2\alpha^{-4}\lambda^{-6}(\xi) W_5(\xi)\Big\} d\xi\,, \\
\hat{\sigma}^{zz}(R) &= 2\left[\alpha^2 - \alpha^{-2}\lambda^{-2}(R)\right]\left[W_1(R) + \lambda^2(R) W_2(R)\right] \\
&\quad - 2\alpha^{-2}\lambda^{-2}(R)\left[W_4(R) + 2\alpha^{-2}\lambda^{-2}(R) W_5(R)\right] + 2\alpha^2\left[W_6(R) + 2\alpha^2 W_7(R)\right] \\
&\quad + \int_{R_i}^{R} \frac{2}{\alpha\xi}\Big\{\left[1 - \alpha^{-2}\lambda^{-4}(\xi)\right]\left[W_1(\xi) + \alpha^2 W_2(\xi)\right] \\
&\qquad\qquad - \alpha^{-2}\lambda^{-4}(\xi) W_4(\xi) - 2\alpha^{-4}\lambda^{-6}(\xi) W_5(\xi)\Big\} d\xi\,.
\end{aligned}
\tag{4.5}
$$

Similar to the case of anisotropic spherical shells, the stress and the invariants of the energy function depend on the radial parameter R though the radial stretch $\lambda(R)$. In particular, orthotropic cylindrical shells with the same R_i/R_o ratio (and different radii R_o) will have the same stress field (and boundary tractions). Note also that similar to spherical shells, the above formulas and the following calculations are still valid for radially inhomogeneous cylindrical shells if the more general definition of similar cylindrical shells (3.6) is adopted.

Remark 4.1 It is known that inflation and extension of cylindrical shells are universal deformations for incompressible isotropic solids [6]. More specifically, the deformations considered here are a subset of Family 3 of universal deformations for incompressible isotropic solids. Yavari and Goriely [37] showed that for incompressible orthotropic isotropic solids Family 3 deformations are universal and there are two classes of universal material preferred directions consistent with Family 3 deformations: (i) radial, circumferential, and axial, and (ii) radial and two orthogonal families of circular helices. Here we have considered class (i) of universal material preferred directions.

Let us consider an elastic compressible, homogeneous, isotropic body with an energy function $W^M = W^M(I_1, I_2, I_3)$ such that it has a finite thickness L in the Z-direction in the Cartesian coordinate system (X, Y, Z). Assume that $\varphi(X, Y, Z) = (\lambda_0 X, \lambda_0 Y, \alpha Z)$, i.e., $\mathbf{F} = \mathrm{diag}(\lambda_0, \lambda_0, \alpha)$. The non-zero stress components are

$$
\begin{aligned}
\hat{\sigma}^{xx} = \hat{\sigma}^{yy} &= 2\alpha^{-1}\left(W^M_{I_1} + \left(\alpha^2 + \lambda_0^2\right) W^M_{I_2} + \alpha^2\lambda_0^2 W^M_{I_3}\right) = \sigma_0\,, \\
\hat{\sigma}^{zz} &= 2\alpha\left(\lambda_0^{-2} W^M_{I_1} + 2W^M_{I_2} + \lambda_0^2 W^M_{I_3}\right)\,,
\end{aligned}
\tag{4.6}
$$

where $I_1 = 2\lambda_0^2 + \alpha^2$, $I_2 = \lambda_0^2\left(\lambda_0^2 + 2\alpha^2\right)$, and $I_3 = \alpha^2\lambda_0^4$. It is apparent that the stress in the $X - Y$ plane is everywhere the same and is equal to σ_0. It is now possible to replace any isotropic cylindrical part of the compressible homogeneous body by an incompressible orthotropic cylindrical shell without perturbing the outside stress field provided that

$$
\begin{aligned}
&\int_{\lambda_i(\lambda_0, \frac{R_i}{R_o}, \alpha)}^{\lambda_0} \frac{2\eta}{1 - \alpha\eta^2}\Big[\left(W_1(\eta) + \alpha^2 W_2(\eta)\right)\left(1 - \alpha^{-2}\eta^{-4}\right) \\
&\qquad\qquad - \alpha^{-2}\eta^{-4} W_4(\eta) - 2\alpha^{-4}\eta^{-6} W_5(\eta)\Big] d\eta = \sigma_0\,,
\end{aligned}
\tag{4.7}
$$

where

$$\lambda_i = \lambda(R_i) = \left[\frac{1}{\alpha} + \frac{R_o^2}{R_i^2}\left(\lambda_0^2 - \frac{1}{\alpha}\right)\right]^{1/2}, \tag{4.8}$$

and

$$\begin{aligned} I_1(\eta) &= \eta^2 + \alpha^{-2}\eta^{-2} + \alpha^2, \qquad I_2(\eta) = \eta^{-2} + \alpha^2\eta^2 + \alpha^{-2}, \\ I_4(\eta) &= \alpha^{-2}\eta^{-2}, \qquad I_5(\eta) = \alpha^{-4}\eta^{-4}. \end{aligned} \tag{4.9}$$

We can then continue replacing the remaining part of the body by the hollow shells with as small radii as we desire and reach the *cylinder assemblage* geometrical arrangement. Equation (4.7) gives the effective constitutive equation of the assemblage in the form $\sigma = \sigma(\lambda, \alpha, c_0)$, where $c_0 = R_i^2/R_o^2$.

Remark 4.2 (The effective plane-stress constitutive equation) The axial force required to maintain the deformation for a pair of radial and axial stretches (λ_0, α) is calculated as

$$F_z = 2\pi \int_0^{r_o} \hat{\sigma}^{zz}(r)\, r dr = \frac{2\pi}{\alpha}\int_0^{R_o} \hat{\sigma}^{zz}(R) R dR. \tag{4.10}$$

If there is no applied axial force, i.e., $F_z = 0$, from the relation

$$\int_0^{R_o} R\,\hat{\sigma}^{zz}(R)\, dR = 0, \tag{4.11}$$

one obtains $\alpha = \alpha(\lambda_0)$. Then, $\sigma = \bar{\sigma}(\lambda, c_0) = \sigma(\lambda, \alpha(\lambda), c_0)$ would be the effective plane-stress constitutive equation of the assemblage. Note that even for a neo-Hookean solid the relation $\alpha = \alpha(\lambda_0)$ would need to be calculated numerically.

4.2 Infinitely-Long Hollow Cylinders

Let us next consider an infinitely-long incompressible orthotropic cylindrical shell of inner and outer radii R_i and R_o, respectively, in its undeformed configuration. Again, assume that the material orthotropic axes are in the radial, circumferential, and axial directions in the cylindrical coordinates (R, Θ, Z). For deformations of the form $(r, \theta, z) = (r(R), \Theta, Z)$, the right Cauchy-Green deformation tensor reads $\mathbf{C} = \mathrm{diag}(\lambda^{-2}(R), \lambda^2(R), 1)$. The incompressibility constraint is written as $r(R)r'(R) = R$, and hence

$$\lambda(R) = \left[1 + \left(\lambda_0^2 - 1\right)\frac{R_o^2}{R^2}\right]^{1/2}. \tag{4.12}$$

From (2.27) and (2.32), the non-zero components of the stress read

$$\begin{aligned} \hat{\sigma}^{rr} &= -p(R) + 2\lambda^{-2}(R)\left[W_1(R) + W_4(R)\right] + 2\left[\lambda^{-2}(R) + 1\right]W_2(R) + 4\lambda^{-4}(R)W_5(R), \\ \hat{\sigma}^{\theta\theta} &= -p(R) + 2\lambda^2(R)\left[W_1(R) + W_2(R)\right] + 2W_2(R), \\ \hat{\sigma}^{zz} &= -p(R) + 2\left[W_1(R) + W_6(R) + 2W_7(R)\right] + 2\left[\lambda^{-2}(R) + \lambda^2(R)\right]W_2(R). \end{aligned} \tag{4.13}$$

The energy function has the following invariants

$$\begin{aligned} I_1 &= \lambda^2(R) + \lambda^{-2}(R) + 1\,, \quad I_2 = \lambda^{-2}(R) + \lambda^2(R) + 1\,, \\ I_4 &= \lambda^{-2}(R)\,, \quad I_5 = \lambda^{-4}(R)\,, \quad I_6 = I_7 = 1\,. \end{aligned} \tag{4.14}$$

The equilibrium equation implies that

$$\begin{aligned} \hat{\sigma}^{rr}(R) &= \int_{R_i}^{R} \frac{2}{\xi}\Big\{ \big[1-\lambda^{-4}(\xi)\big]\left[W_1(\xi)+W_2(\xi)\right] - \lambda^{-4}(\xi)W_4(\xi) - 2\lambda^{-6}(\xi)W_5(\xi)\Big\} d\xi\,, \\ \hat{\sigma}^{\theta\theta}(R) &= 2\big[\lambda^2(R)-\lambda^{-2}(R)\big]\left[W_1(R)+W_2(R)\right] - 2\lambda^{-2}(R)\big[W_4(R)+2\lambda^{-2}(R)W_5(R)\big] \\ &\quad + \int_{R_i}^{R} \frac{2}{\xi}\Big\{ \big[1-\lambda^{-4}(\xi)\big]\left[W_1(\xi)+W_2(\xi)\right] - \lambda^{-4}(\xi)W_4(\xi) - 2\lambda^{-6}(\xi)W_5(\xi)\Big\} d\xi\,. \end{aligned} \tag{4.15}$$

Again, the stress and the invariants of the energy function depend on the radial parameter R through the radial stretch $\lambda(R)$. In particular, orthotropic cylindrical shells with the same R_i/R_o ratio (and different radii R_o) will have the same stress field (and boundary tractions).

Let us consider an elastic compressible, homogeneous, isotropic body with the energy function $W^M = W^M(I_1, I_2, I_3)$ such that it is infinitely extended in the Z-direction in the Cartesian coordinate system (X, Y, Z). For deformations of the form $\varphi(X,Y,Z) = (\lambda_0 X, \lambda_0 Y, Z)$, i.e., $\mathbf{F} = \operatorname{diag}(\lambda_0, \lambda_0, 1)$, the non-zero stress components are

$$\hat{\sigma}^{xx} = \hat{\sigma}^{yy} = 2\left[W^M_{I_1} + \left(1+\lambda_0^2\right) W^M_{I_2} + \lambda_0^2 W^M_{I_3}\right] = \sigma_0\,, \tag{4.16}$$

where $I_1 = 2\lambda_0^2 + 1$, $I_2 = \lambda_0^2\left(\lambda_0^2+2\right)$, and $I_3 = \lambda_0^4$. Note that the isotropic body is under the plane strain condition, and the stress in the $X-Y$ plane is everywhere the same and is equal to σ_0. One can replace any isotropic cylindrical part of the compressible homogeneous body by an incompressible orthotropic cylindrical shell without perturbing the outside stress field if

$$\begin{aligned} &\int_{\lambda_i(\lambda_0, \frac{R_i}{R_o})}^{\lambda_0} \frac{2\eta}{1-\eta^2}\Big\{ \left(1-\eta^{-4}\right)\left[W_1(\eta)+W_2(\eta)\right] - \eta^{-4}W_4(\eta) - 2\eta^{-6}W_5(\eta)\Big\} d\eta \\ &= \sigma_0\,, \end{aligned} \tag{4.17}$$

where

$$\lambda_i = \lambda(R_i) = \left[1 + \frac{R_o^2}{R_i^2}\left(\lambda_0^2 - 1\right)\right]^{1/2}, \tag{4.18}$$

and

$$I_1(\eta) = \eta^2 + \eta^{-2} + 1, \quad I_2(\eta) = \eta^{-2} + \eta^2 + 1, \quad I_4(\eta) = \eta^{-2}, \quad I_5(\eta) = \eta^{-4}\,. \tag{4.19}$$

We can then continue replacing the remaining part of the body by the hollow shells with as small radii as we desire and reach the *cylinder assemblage* geometrical arrangement. Equation (4.17) gives the effective plane-strain constitutive equation of the assemblage in the form $\sigma = \sigma(\lambda, c_0)$, where $c_0 = R_i^2/R_o^2$.

5 Concluding Remarks

In this paper, we generalized Hashin's hollow sphere assemblage to nonlinear transversely isotropic solids with radial material preferred direction. Each incompressible shell can be radially inhomogeneous as long as *similar* spherical shells are defined properly. We made a connection with cloaking spherical cavities in a given compressible homogeneous isotropic solid. In particular, it is noted that mechanical properties of cloaks explicitly depend on the applied pure dilatational deformation. We analyzed both finite and infinitely-long hollow cylinder assemblages made of orthotropic incompressible solids with axial, radial, and circumferential material preferred directions. In the case of finite cylinders the effective constitutive equation of the assemblage is a function of both the axial and the radial stretches. The effective plane-stress constitutive equation was derived as well. In the case of infinitely-long hollow cylinders the effective plane-strain constitutive equation of the assemblage was derived. The cylindrical shells can, in general, be radially inhomogeneous as long as a proper definition of *similar* cylindrical shells is used.

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