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Nonlinear elastic inclusions in isotropic solids

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We introduce a geometric framework to calculate the residual stress fields and deformations of nonlinear solids with inclusions and eigenstrains. Inclusions are regions in a body with different reference configurations from the body itself and can be described by distributed eigenstrains. Geometrically, the eigenstrains define a Riemannian 3-manifold in which the body is stress-free by construction. The problem of residual stress calculation is then reduced to finding a mapping from the Riemannian material manifold to the ambient Euclidean space. Using this construction, we find the residual stress fields of three model systems with spherical and cylindrical symmetries in both incompressible and compressible isotropic elastic solids. In particular, we consider a finite spherical ball with a spherical inclusion with uniform pure dilatational eigenstrain and we show that the stress in the inclusion is uniform and hydrostatic. We also show how singularities in the stress distribution emerge as a consequence of a mismatch between radial and circumferential eigenstrains at the centre of a sphere or the axis of a cylinder.

1. Introduction

Classically, inclusions in an elastic body are pieces of elastic materials that have been inserted into the material. For instance, in the simplest case, a spherical elastic ball is compressed or stretched to fit inside a given spherical shell. The problem is then to find the stress in the new ball and the deformation of both materials. In general, the problem of inclusions is to combine two different stress-free bodies and constrain them geometrically, so that they create a new, possibly residually stressed, body. This process is sometimes called shrink-fit as,

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typically, one of the bodies is compressed to fit in the other one. More generally, we can consider a single body and assume that the body undergoes a local change of volume described by general eigenstrains, the particular shrink-fit problem corresponding to uniform dilatational eigenstrains. Physically, these eigenstrains can be generated by thermal expansions, swelling, shrinking, growth or any other anelastic effects.

In the linearized setting, Eshelby [1] calculated the stress field of an ellipsoidal inclusion with uniform eigenstrains using superposition. For the special class of harmonic materials, there are some recent two-dimensional solutions for inclusions [2–6]. Antman & Shvartsman [7] solved a two-dimensional shrink-fit problem for arbitrary anisotropic nonlinear solids. Basically, a stress-free annulus with inner and outer radii R_{i} and R_{o} is expanded and then left to shrink down upon a stress-free disc of radius $R_{\text{d}} > R_{\text{i}}$. They focused on the question of existence and uniqueness of solutions. However, in the case of isotropic solids, they observed that stress inside the disc is uniform. In terms of eigenstrains, the shrink-fit problem consists of pure dilatational eigenstrain. In the nonlinear case, as far as we know, there are no explicit three-dimensional analytical solutions for inclusions. However, the problem of inclusions with pure dilatational eigenstrains is closely related to the problem of swelling in solids. In some recent works, Pence *et al.* [8–11] presented analytical solutions for swelling in cylindrical and spherical geometries for both incompressible and compressible isotropic solids. The main motivation of these works was cavitation, but one can clearly see a close connection between the swelling models and our geometric formulation. However, we should emphasize that our approach is more general in the sense that it is not restricted to pure dilatational eigenstrains.

Eshelby's problem has been extended to finite bodies in recent years. In particular, Li *et al.* [12] calculated the stress field of a spherical inclusion centred at a finite ball. They observed that, in general, stress inside the inclusion is not uniform. For a recent review of previous works on inclusions in the framework of linearized elasticity, see Zhoua *et al.* [13].

Inclusions in three-dimensional nonlinear solids have been investigated numerically by Diani & Parks [14]. They calculated the stress field of an isotropic inclusion with pure dilatational eigenstrain in an isotropic matrix made of the same material. Their finite-element computations are based on the multiplicative decomposition of deformation gradient into elastic and eigenstrain parts, i.e. $\mathbf{F} = \mathbf{F}^{\text{e}}\mathbf{F}^*$. In their spherical inclusion problem $\mathbf{F}^* = (1+\alpha)\mathbf{I}$, where $\mathbf{I}$ is the identity tensor. Note that this implies that the material metric [15] is $\mathbf{G} = (1+\alpha)^2\mathbf{I}$. In their numerical calculations for $\alpha = 0.1$, Diani & Parks [14] observed that in the inclusion the Cauchy stress is uniform and hydrostatic. In this paper, we revisit this problem for both arbitrary incompressible and certain class of compressible isotropic elastic solids. We will show analytically that for this class of solids the stress in the inclusion is uniform and hydrostatic. Owing to the equivalence between a general theory of growth and the problem of inclusions, the computation of stress fields for some spherical and cylindrical geometries in two and three dimensions has also been investigated [16–20].

In this paper, we develop a geometric framework for the problem of inclusions. The basic idea is that inclusions can be fully described by eigenstrains and that these eigenstrains define a three-dimensional Riemannian manifold in which the body is stress-free. The body being residually stressed means that this 3-manifold, which we call the material manifold, cannot be isometrically embedded in $\mathbb{R}^3$. This geometric framework is identical to that used for calculating residual stresses in the presence of non-uniform temperature distributions [15], bodies with bulk growth [21] and bodies with distributed defects [22–24]. It should also be noted that this approach is general, i.e. it is not restricted to the specific constitutive equations and/or geometry of the problem. However, to be able to find exact solutions, we will present some symmetric problems for isotropic elastic solids.

In this paper, we first find the stress field of a ball made of an arbitrary incompressible isotropic nonlinear solid with a spherically symmetric eigenstrain distribution. In the case of an ellipsoidal inclusion in an infinite linearly isotropic solid, Eshelby observed that stress in the inclusion is uniform. Eshelby's solution is based on superposition, which is not applicable in nonlinear problems. To tackle the nonlinear problem we first find a 3-manifold in which the ball with distributed eigenstrains is stress free. This is a Riemannian manifold with a metric that explicitly depends on the eigenstrain distribution. Once the material manifold is known, our problem is transformed to a classical nonlinear elasticity problem; that is we need to find a mapping from the reference configuration described by the material 3-manifold to the current configuration in the Euclidean ambient space. As a special example, we consider a spherical inclusion at the centre of the ball. We show that for any incompressible isotropic nonlinear elastic solid with pure dilatational eigenstrain, the stress in the inclusion is uniform and hydrostatic. When the radial and circumferential eigenstrains are not equal, we show that stress inside the inclusion is non-uniform and has a logarithmic singularity. We also extend our analysis to compressible solids. We show that the stress inside an inclusion with pure dilatational eigenstrain is uniform and hydrostatic when the ball is made of compressible materials of types I, II and III according to Carroll [25]. We also consider cylindrical inclusions in both finite and infinite circular cylindrical bars made of arbitrary incompressible isotropic solids.

2. Geometric nonlinear elasticity of residually stressed bodies

(a) Kinematics of nonlinear elasticity

We first review some basic concepts of geometric nonlinear elasticity. A body $\mathcal{B}$ is identified with a Riemannian manifold $\mathcal{B}$, and a configuration of $\mathcal{B}$ is a mapping $\varphi:\mathcal{B}\to\mathcal{S}$, where $\mathcal{S}$ is a Riemannian ambient space manifold [26,27] (figure 1*a*). A fundamental assumption in geometric nonlinear elasticity is that the body is stress-free in the material manifold. Any possible residual stresses are described by the geometries of $\mathcal{B}$ and $\mathcal{S}$. The position of a material point $\mathbf{X}\in\mathcal{B}$ in the ambient space $\mathcal{S}$ is given by $\mathbf{x}=\varphi_t(\mathbf{X})=\varphi(\mathbf{X},t)$. The material velocity is defined as $\mathbf{V}_t(\mathbf{X})=\mathbf{V}(\mathbf{X},t)=\partial\varphi(\mathbf{X},t)/\partial t$. The material acceleration is defined by $\mathbf{A}_t(\mathbf{X})=\mathbf{A}(\mathbf{X},t)=\partial\mathbf{V}(\mathbf{X},t)/\partial t$. The deformation gradient—a central object that locally describes deformation—is the tangent map of φ and is denoted by $\mathbf{F}=T\varphi$. Hence, at each point, $\mathbf{X}\in\mathcal{B}$, $\mathbf{F}$ is a linear map $\mathbf{F}(\mathbf{X}):T_{\mathbf{X}}\mathcal{B}\to T_{\varphi(\mathbf{X})}\mathcal{S}$. If $\{x^a\}$ and $\{X^A\}$ are local coordinate charts on $\mathcal{S}$ and $\mathcal{B}$, respectively, then the components of $\mathbf{F}$ read as

$$F^a{}_A(\mathbf{X})=\frac{\partial\varphi^a}{\partial X^A}(\mathbf{X}), \tag{2.1}$$

where $\mathbf{F}$ has the local representation $\mathbf{F}=F^a{}_A\partial_a\otimes \mathrm{d}X^A$. Transpose of $\mathbf{F}$ is a linear map $\mathbf{F}^{\mathsf{T}}:T_{\mathbf{x}}\mathcal{S}\to T_{\mathbf{X}}\mathcal{B}$ and is defined as

$$\langle\!\langle\mathbf{FV},\mathbf{v}\rangle\!\rangle_{\mathbf{g}}=\langle\!\langle\mathbf{V},\mathbf{F}^{\mathsf{T}}\mathbf{v}\rangle\!\rangle_{\mathbf{G}},\quad \forall\,\mathbf{V}\in T_{\mathbf{X}}\mathcal{B},\ \ \mathbf{v}\in T_{\mathbf{x}}\mathcal{S}. \tag{2.2}$$

In components, $(F^{\mathsf{T}}(\mathbf{X}))^A{}_a=g_{ab}(\mathbf{x})F^b{}_B(\mathbf{X})G^{AB}(\mathbf{X})$. The right Cauchy–Green deformation tensor is a linear map $\mathbf{C}(X):T_{\mathbf{X}}\mathcal{B}\to T_{\mathbf{X}}\mathcal{B}$ and is defined by $\mathbf{C}(\mathbf{X})=\mathbf{F}^{\mathsf{T}}(\mathbf{X})\mathbf{F}(\mathbf{X})$. Note that $\mathbf{C}^\flat$ is the pullback of the spatial metric, i.e. $\mathbf{C}^\flat=\varphi^*\mathbf{g}$ or in components $C_{AB}=(g_{ab}\circ\varphi)F^a{}_AF^b{}_B$. Eigenvalues of the right stretch tensor $\mathbf{U}=\sqrt{\mathbf{C}}$ are called principal stretches. For an isotropic body, the strain energy function depends only on the principal stretches, i.e. $W=W(\lambda_1,\lambda_2,\lambda_3)$.

(b) Material manifold of a body with eigenstrains

In classical elasticity, one starts with a stress-free configuration embedded in the ambient space and then makes this embedding time-dependent (a motion) (figure 1*a*). In anelasticity, elastic bodies also have residual stress [28]. These residual stresses can be described geometrically by positing that the stress-free configuration is a Riemannian manifold with a geometry explicitly depending on the anelasticity source(s) (figure 1*b*). The ambient space is a Riemannian manifold $(\mathcal{S},\mathbf{g})$, and hence the computation of stresses requires a Riemannian material manifold $(\mathcal{B},\mathbf{G})$ and a mapping $\varphi:\mathcal{B}\to\mathcal{S}$. For example, in the case of non-uniform temperature changes and bulk growth [15,21], one starts with a material metric $\mathbf{G}$ that specifies the relaxed distances of the material points. In the case of distributed defects, the material metric is calculated indirectly [22–24]. When there are eigenstrains distributed in a body, the material manifold has a metric that explicitly depends on the eigenstrain distribution.

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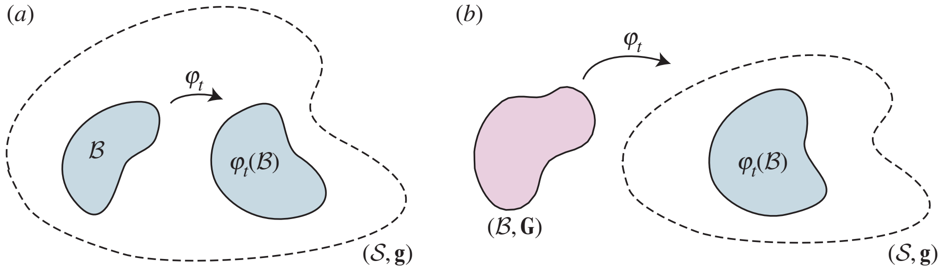

Figure 1. (*a*) In classical nonlinear elasticity, the reference configuration is a submanifold of the ambient space manifold. The material metric is the induced submanifold metric. (*b*) For residually stressed bodies, the material manifold is a Riemannian manifold $(\mathcal{B}, \mathbf{G})$. Motion is defined by a time-dependent mapping from the Riemannian material manifold $(\mathcal{B}, \mathbf{G})$ to the Riemannian ambient space manifold $(\mathcal{S}, \mathbf{g})$. (Online version in colour.)

3. Examples of elastic bodies with distributed eigenstrains

Here, we consider three examples of inclusions in incompressible and compressible isotropic nonlinear elastic solids. The first one is a spherical ball with a spherically symmetric distribution of dilatational eigenstrains. The next two examples are finite and infinite circular cylindrical bars with cylindrically symmetric eigenstrain distributions.

(a) Spherical eigenstrain in a ball

Consider a ball $\mathfrak{B}$ of radius R_o made of a nonlinear elastic solid with a given spherically symmetric distribution of dilatational eigenstrains. Here, we model an eigenstrain by the local natural distances of material points, i.e. a Riemannian metric. For example, a change in temperature changes the natural (relaxed) distances of material points in a solid [15]. In the case of a ball with an inclusion, the natural distances in the inclusion and the matrix are different and this induces a stress field.

A stress-free ball made of a nonlinear elastic solid is isometrically embedded in $\mathbb{R}^3$, and hence natural distances between its material points are measured using the flat metric of $\mathbb{R}^3$, i.e. the material metric has the following representation in the spherical coordinates (R, Θ, Φ): $\mathbf{G}_0(\mathbf{X}) = \mathbf{G}_0(R) = \mathrm{diag}(1, R^2, R^2 \sin^2 \Theta)$. To preserve the spherical symmetry, we can have different radial and circumferential eigenstrains as long as they are functions of R only. Therefore, we consider the following material metric

$$\mathbf{G}(\mathbf{X}) = \mathbf{G}(R) = \begin{pmatrix} e^{2\omega_R(R)} & 0 & 0 \\ 0 & e^{2\omega_\Theta(R)} R^2 & 0 \\ 0 & 0 & e^{2\omega_\Theta(R)} R^2 \sin^2 \Theta \end{pmatrix}, \tag{3.1}$$

where ω_R and ω_Θ are arbitrary functions. For the ball $\mathfrak{B}$ with eigenstrains, given $\omega_R(R)$ and $\omega_\Theta(R)$, we are interested in the resulting residual stress field. We solve the problem for arbitrary $\omega_R(R)$ and $\omega_\Theta(R)$ and then specialize the solution to the case of a spherical inclusion in which radial and circumferential eigenstrains are constants. We use the spherical coordinates (r, θ, ϕ) for the Euclidean ambient space with the flat metric $\mathbf{g} = \mathrm{diag}(1, r^2, r^2 \sin^2 \theta)$. In order to obtain the residual stress field, we embed the material manifold into the ambient space. We look for solutions of the form $(r, \theta, \phi) = (r(R), \Theta, \Phi)$, and hence $\det \mathbf{F} = r'(R)$. We first restrict our attention to incompressible solids for which

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{r^2(R)}{R^2\, e^{\omega_R(R) + 2\omega_\Theta(R)}} r'(R) = 1. \tag{3.2}$$

Assuming that $r(0)=0$ this gives us

$$r(R)=\left(\int_0^R 3\xi^2\,\mathrm{e}^{\omega_R(\xi)+2\omega_\Theta(\xi)}\,\mathrm{d}\xi\right)^{1/3}. \tag{3.3}$$

Physical components of the deformation gradient $\hat{F}^a{}_A$ are related to components of deformation gradient as follows

$$\hat{F}^a{}_A=\sqrt{g_{aa}}\sqrt{G^{AA}}F^a{}_A \quad \text{no summation.} \tag{3.4}$$

Thus

$$\hat{\mathbf{F}}=\begin{pmatrix} \dfrac{R^2}{r^2(R)}\,\mathrm{e}^{2\omega_\Theta(R)} & 0 & 0 \\ 0 & \dfrac{r(R)}{R}\,\mathrm{e}^{-\omega_\Theta(R)} & 0 \\ 0 & 0 & \dfrac{r(R)}{R}\,\mathrm{e}^{-\omega_\Theta(R)} \end{pmatrix}. \tag{3.5}$$

Therefore, the principal stretches are

$$\lambda_1=\frac{R^2}{r^2(R)}\,\mathrm{e}^{2\omega_\Theta(R)},\quad \lambda_2=\lambda_3=\frac{r(R)}{R}\,\mathrm{e}^{-\omega_\Theta(R)}. \tag{3.6}$$

We know that for an isotropic material the strain energy function depends only on the principal stretches, i.e. $W=W(\lambda_1,\lambda_2,\lambda_3)$ [29] (Note that here, for the sake of brevity, we do not write explicitly the dependence of W on R that is required to describe an inhomogeneity. For instance, we will consider an inclusion with different energy–density functions at different locations in the following. For such problems, W explicitly depends on R.). Because of the symmetry of the problem in the spherical coordinates (r,θ,ϕ), the Cauchy stress is diagonal and

$$\sigma^{rr}=\lambda_1 g^{11}\frac{\partial W}{\partial\lambda_1}-p(R)g^{11}=\frac{R^2}{r^2(R)}\,\mathrm{e}^{2\omega_\Theta(R)}\frac{\partial W}{\partial\lambda_1}-p(R), \tag{3.7}$$

$$\sigma^{\theta\theta}=\lambda_2 g^{22}\frac{\partial W}{\partial\lambda_2}-p(R)g^{22}=\frac{\mathrm{e}^{-\omega_\Theta(R)}}{Rr(R)}\frac{\partial W}{\partial\lambda_2}-\frac{p(R)}{r^2(R)} \tag{3.8}$$

and

$$\sigma^{\phi\phi}=\frac{1}{\sin^2\Theta}\sigma^{\theta\theta}. \tag{3.9}$$

In the absence of body forces, the only non-trivial equilibrium equation is $\sigma^{ra}{}_{|a}=0$ ($p=p(R)$ is the consequence of the other two equilibrium equations), where $\sigma^{ra}{}_{|a}$ denotes the r component of the trace of the covariant derivative of the Cauchy stress [26]. This is simplified to read

$$\sigma^{rr}{}_{,r}+\frac{2}{r}\sigma^{rr}-r\sigma^{\theta\theta}-r\sin^2\theta\,\sigma^{\phi\phi}=0. \tag{3.10}$$

Or

$$\frac{1}{r'(R)}\sigma^{rr}{}_{,R}+\frac{2}{r}\sigma^{rr}-2r\sigma^{\theta\theta}=0. \tag{3.11}$$

Therefore

$$\sigma^{rr}{}_{,R}+\frac{R^2\,\mathrm{e}^{\omega_R(R)+2\omega_\Theta(R)}}{r^2(R)}\left(\frac{2}{r}\sigma^{rr}-2r\sigma^{\theta\theta}\right)=0. \tag{3.12}$$

This then gives us $p'(R)=h(R)$, where

$$\begin{aligned} h(R)=&\frac{2R}{r^2(R)}\,\mathrm{e}^{\omega_\Theta(R)}\left(\mathrm{e}^{\omega_\Theta(R)}\frac{\partial W}{\partial\lambda_1}-\mathrm{e}^{\omega_R(R)}\frac{\partial W}{\partial\lambda_2}\right)+\frac{2R^2\omega'_\Theta(R)}{r^2(R)}\,\mathrm{e}^{2\omega_\Theta(R)}\frac{\partial W}{\partial\lambda_1}\\ &+\frac{2R^3}{r^4(R)}\,\mathrm{e}^{4\omega_\Theta(R)}\left[1+R\omega'_\Theta(R)-\frac{R^3}{r^3(R)}\,\mathrm{e}^{\omega_R(R)+2\omega_\Theta(R)}\right]\frac{\partial^2 W}{\partial\lambda_1^2}\\ &-\frac{2\,\mathrm{e}^{\omega_\Theta(R)}}{r(R)}\left[1+R\omega'_\Theta(R)-\frac{R^3}{r^3(R)}\,\mathrm{e}^{\omega_R(R)+2\omega_\Theta(R)}\right]\frac{\partial^2 W}{\partial\lambda_1\partial\lambda_2}. \end{aligned} \tag{3.13}$$

If at the boundary $\sigma^{rr}(R_o) = -p_\infty$, then

$$p(R) = p_\infty + \frac{R_o^2\, e^{2\omega_\Theta(R_o)}}{r^2(R_o)} \frac{\partial W}{\partial \lambda_1}\bigg|_{R=R_o} + \int_R^{R_o} h(\xi)\, d\xi. \tag{3.14}$$

Once the pressure field is known, the stress tensor can be easily calculated.

(i) Spherical inclusion in a ball

Let us consider the following ω_R and ω_Θ distributions

$$\omega_R(R) = \begin{cases} \omega_1, & 0 \le R < R_i, \\ 0, & R_i < R \le R_o, \end{cases} \quad \omega_\Theta(R) = \begin{cases} \omega_2, & 0 \le R < R_i, \\ 0, & R_i < R \le R_o, \end{cases} \tag{3.15}$$

where $R_i < R_o$. Thus

$$0 \le R \le R_i: \quad r(R) = e^{(1/3)\omega_1 + (2/3)\omega_2} R, \tag{3.16}$$

and

$$R_i \le R \le R_o: \quad r(R) = [R^3 + (e^{\omega_1 + 2\omega_2} - 1)R_i^3]^{1/3}. \tag{3.17}$$

This means that for $R \le R_i$

$$\lambda_1 = e^{-(2/3)\omega_1 + (2/3)\omega_2} = \lambda_0^{-2}, \quad \lambda_2 = \lambda_3 = e^{(1/3)\omega_1 - (1/3)\omega_2} = \lambda_0. \tag{3.18}$$

Note that

$$\omega'_\Theta(R) = -\omega_2 \delta(R - R_i). \tag{3.19}$$

Multiplication of a distribution and a smooth function is well defined. However, here $e^{\omega(R)}$, for example, is not smooth, and the product $e^{\omega(R)}\delta(R - R_i)$ is indeterminate. We denote the sum of products of all the terms with $\omega'_\Theta(R)$ by $A\delta(R - R_i)$. The unknown constant A will be determined after enforcing continuity of traction vector on the boundary of the inclusion. That is, we have

$$h(R) = A\delta(R - R_i) + \hat{h}(R), \tag{3.20}$$

where

$$\begin{aligned} \hat{h}(R) = {} & \frac{2R}{r^2(R)} e^{\omega_\Theta(R)} \left(e^{\omega_\Theta(R)} \frac{\partial W}{\partial \lambda_1} - e^{\omega_R(R)} \frac{\partial W}{\partial \lambda_2} \right) \\ & + \frac{2R^3}{r^4(R)} e^{4\omega_\Theta(R)} \left[1 - \frac{R^3}{r^3(R)} e^{\omega_R(R) + 2\omega_\Theta(R)} \right] \frac{\partial^2 W}{\partial \lambda_1^2} \\ & - \frac{2\, e^{\omega_\Theta(R)}}{r(R)} \left[1 - \frac{R^3}{r^3(R)} e^{\omega_R(R) + 2\omega_\Theta(R)} \right] \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}. \end{aligned} \tag{3.21}$$

Note that for $R < R_i$

$$\hat{h}(R) = \frac{h_0}{R}, \tag{3.22}$$

where

$$h_0 = 2e^{-(2/3)(\omega_1 + 2\omega_2)} \left(e^{\omega_2} \frac{\partial W}{\partial \lambda_1} - e^{\omega_1} \frac{\partial W}{\partial \lambda_2} \right)\bigg|_{\lambda_1 = \lambda_0^{-2},\, \lambda_2 = \lambda_3 = \lambda_0}. \tag{3.23}$$

Note also that when $\omega_1 \ne \omega_2$, $h_0 \ne 0$. However, when $\omega_1 = \omega_2 = \omega_0$, $\lambda_0 = 1$ and hence

$$h_0 = 2e^{-\omega_0} \left(\frac{\partial W}{\partial \lambda_1} - \frac{\partial W}{\partial \lambda_2} \right)\bigg|_{\lambda_1 = \lambda_2 = \lambda_3 = 1}. \tag{3.24}$$

We know that [29], $W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_2, \lambda_1, \lambda_3)$, and in particular $W(x, 1, 1) = W(1, x, 1)$. Taking derivative with respect of x and evaluating at $x = 1$, we obtain

$$\frac{\partial W}{\partial \lambda_1}\bigg|_{\lambda_1 = \lambda_2 = \lambda_3 = 1} = \frac{\partial W}{\partial \lambda_2}\bigg|_{\lambda_1 = \lambda_2 = \lambda_3 = 1}. \tag{3.25}$$

Therefore, in this case $h_0=0$. Note that

$$0\le R<R_{\mathrm{i}}:\quad \int_R^{R_{\mathrm{o}}} h(\xi)\,\mathrm{d}\xi=\int_R^{R_{\mathrm{i}}}\hat{h}(\xi)\,\mathrm{d}\xi+A+\int_{R_{\mathrm{i}}}^{R_{\mathrm{o}}}\hat{h}(\xi)\,\mathrm{d}\xi$$
$$=h_0\ln\left(\frac{R_{\mathrm{i}}}{R}\right)+A+\int_{R_{\mathrm{i}}}^{R_{\mathrm{o}}}\bar{h}(\xi)\,\mathrm{d}\xi, \tag{3.26}$$

and

$$R_{\mathrm{i}}<R\le R_{\mathrm{o}}:\quad \int_R^{R_{\mathrm{o}}} h(\xi)\,\mathrm{d}\xi=\int_R^{R_{\mathrm{o}}}\bar{h}(\xi)\,\mathrm{d}\xi, \tag{3.27}$$

where

$$\bar{h}(R)=\frac{2R}{r^2(R)}\left(\frac{\partial W}{\partial\lambda_1}-\frac{\partial W}{\partial\lambda_2}\right)+\frac{2R^3}{r^4(R)}\left(1-\frac{R^3}{r^3(R)}\right)\frac{\partial^2 W}{\partial\lambda_1^2}$$
$$-\frac{1}{r(R)}\left(1-\frac{R^3}{r^3(R)}\right)\frac{\partial^2 W}{\partial\lambda_1\partial\lambda_2}. \tag{3.28}$$

Therefore

$$0\le R<R_{\mathrm{i}}:\quad p(R)=p_\infty+\frac{R_{\mathrm{o}}^2}{r^2(R_{\mathrm{o}})}\frac{\partial W}{\partial\lambda_1}\bigg|_{R=R_{\mathrm{o}}}-h_0\ln\left(\frac{R_{\mathrm{i}}}{R}\right)-A-\int_{R_{\mathrm{i}}}^{R_{\mathrm{o}}}\bar{h}(\xi)\,\mathrm{d}\xi, \tag{3.29}$$

and

$$R_{\mathrm{i}}<R\le R_{\mathrm{o}}:\quad p(R)=p_\infty+\frac{R_{\mathrm{o}}^2}{r^2(R_{\mathrm{o}})}\frac{\partial W}{\partial\lambda_1}\bigg|_{R=R_{\mathrm{o}}}-\int_R^{R_{\mathrm{o}}}\bar{h}(\xi)\,\mathrm{d}\xi. \tag{3.30}$$

Now the radial stress inside and outside of the inclusion has the following distributions

$$0\le R<R_{\mathrm{i}}:\quad \sigma^{rr}(R)=\mathrm{e}^{-(2/3)(\omega_1-\omega_2)}\frac{\partial W}{\partial\lambda_1}\bigg|_{\lambda_1=\lambda_0^{-2},\lambda_2=\lambda_3=\lambda_0}-p_\infty-\frac{R_{\mathrm{o}}^2}{r^2(R_{\mathrm{o}})}\frac{\partial W}{\partial\lambda_1}\bigg|_{R=R_{\mathrm{o}}}$$
$$+h_0\ln\left(\frac{R_{\mathrm{i}}}{R}\right)+A+\int_{R_{\mathrm{i}}}^{R_{\mathrm{o}}}\bar{h}(\xi)\,\mathrm{d}\xi, \tag{3.31}$$

and

$$R_{\mathrm{i}}<R\le R_{\mathrm{o}}:\quad \sigma^{rr}(R)=\frac{R^2}{r^2(R)}\frac{\partial W}{\partial\lambda_1}-p_\infty-\frac{R_{\mathrm{o}}^2}{r^2(R_{\mathrm{o}})}\frac{\partial W}{\partial\lambda_1}\bigg|_{R=R_{\mathrm{o}}}+\int_R^{R_{\mathrm{o}}}\bar{h}(\xi)\,\mathrm{d}\xi. \tag{3.32}$$

Continuity of the traction vector on the boundary of the inclusion implies that σ^{rr} must be continuous at $R=R_{\mathrm{i}}$. Thus

$$A=\mathrm{e}^{-(2/3)(\omega_1+2\omega_2)}\frac{\partial W}{\partial\lambda_1}\bigg|_{R=R_{\mathrm{i}}^+}-\mathrm{e}^{-(2/3)(\omega_1-\omega_2)}\frac{\partial W}{\partial\lambda_1}\bigg|_{\lambda_1=\lambda_0^{-2},\,\lambda_2=\lambda_3=\lambda_0}. \tag{3.33}$$

Therefore, inside the inclusion, the radial stress has the following distribution

$$\sigma^{rr}(R)=h_0\ln\left(\frac{R_{\mathrm{i}}}{R}\right)+\sigma_i, \tag{3.34}$$

where

$$\sigma_i=\mathrm{e}^{-(2/3)(\omega_1+2\omega_2)}\frac{\partial W}{\partial\lambda_1}\bigg|_{R=R_{\mathrm{i}}^+}-p_\infty-\frac{R_{\mathrm{o}}^2}{r^2(R_{\mathrm{o}})}\frac{\partial W}{\partial\lambda_1}\bigg|_{R=R_{\mathrm{o}}}+\int_{R_{\mathrm{i}}}^{R_{\mathrm{o}}}\bar{h}(\xi)\,\mathrm{d}\xi. \tag{3.35}$$

It is seen that unless $\omega_1=\omega_2$, there is a logarithmic singularity. Note that when W explicitly depends on R, $\partial W/\partial\lambda_1|_{R=R_{\mathrm{i}}^+}$ and $\partial W/\partial\lambda_1|_{R=R_{\mathrm{i}}^-}$ may not be equal.

In curvilinear coordinates, the components of a tensor may have different physical dimensions. The following relation holds between the Cauchy stress components (unbarred) and its physical components (barred) [30]

$$\bar{\sigma}^{ab} = \sigma^{ab}\sqrt{g_{aa}g_{bb}} \quad \text{no summation on } a \text{ or } b. \tag{3.36}$$

The non-zero physical Cauchy stress components read

$$\left.\begin{aligned} \bar{\sigma}^{rr} &= \sigma^{rr}, \\ \bar{\sigma}^{\theta\theta} &= r^2\sigma^{\theta\theta} = \frac{r(R)\,\mathrm{e}^{-\omega_\Theta(R)}}{R}\frac{\partial W}{\partial\lambda_2} - p(R) \\ \text{and}\qquad \bar{\sigma}^{\phi\phi} &= r^2\sin^2\theta\,\sigma^{\phi\phi} = \bar{\sigma}^{\theta\theta}. \end{aligned}\right\} \tag{3.37}$$

Thus, for $R < R_\mathrm{i}$

$$\begin{aligned} \bar{\sigma}^{\theta\theta}(R) = \bar{\sigma}^{\phi\phi}(R) = h_0 \ln\left(\frac{R_\mathrm{i}}{R}\right) + \mathrm{e}^{(1/3)(\omega_1-\omega_2)}\left.\frac{\partial W}{\partial\lambda_2}\right|_{\lambda_1=\lambda_0^{-2},\lambda_2=\lambda_3=\lambda_0} \\ - \mathrm{e}^{-(2/3)(\omega_1-\omega_2)}\left.\frac{\partial W}{\partial\lambda_1}\right|_{\lambda_1=\lambda_0^{-2},\lambda_2=\lambda_3=\lambda_0} + \sigma_i. \end{aligned} \tag{3.38}$$

Again, note that if $\omega_1 \neq \omega_2$, then the stress in the inclusion is not hydrostatic and exhibits a logarithmic singularity.

Proposition 3.1. *Consider an isotropic and incompressible elastic ball subject to a uniform pressure on its boundary sphere. Assume that there is a spherical inclusion at the centre of the ball with uniform radial and circumferential eigenstrains. Then, unless the radial and circumferential eigenstrains are equal, the inclusion exhibits a logarithmic singularity. If the eigenstrains are equal, then the stress in the inclusion is uniform and hydrostatic.*

Remark 3.2. Note that when the radial and circumferential eigenstrains are not equal, the linearized eigenstrains would not be homogeneous. Hence, it is not surprising that the stress inside the inclusion is not uniform. Logarithmic singularities for the stress have also been observed in an isotropic linear elastic perfectly plastic body with a growing inclusion using the classical equations in [31]. In this case, the stress inside the inclusion is not uniform and has a logarithmic singularity [32]. Note that using either Tresca or von Mises criterion everywhere inside the inclusion $\bar{\sigma}^{rr} - \bar{\sigma}^{\theta\theta}$ is constant. In our inclusion with anisotropic eigenstrains, we see that this quantity is also constant inside the inclusion.

Goriely *et al.* [19] observed that when a solid sphere undergoes anisotropic bulk growth, stress becomes singular at the centre of the ball and causes cavitation. In another study, Moulton & Goriely [20] showed that a thick spherical shell undergoing anisotropic growth builds up residual stresses that resist anti-cavitation. In both these works, there is a clear connection between anisotropy of growth and unboundedness of stress at the centre of the growing ball. Our calculations determine the nature of this singularity.

(ii) Comparison with the linear solution

It is of interest to compare our results with the classical linear solution. To do this, we assume that $p_\infty = 0$, $\omega_1 = \omega_2 = \omega_0$ and consider a neo-Hookean solid. We compare the residual stress field with the solution predicted by the theory of linear elasticity when $\nu = \frac{1}{2}$. For a neo-Hookean material, we have $W = (\mu/2)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$, and hence

$$\sigma^{rr} = \frac{\mu R^4}{r^4(R)}\,\mathrm{e}^{4\omega_\Theta(R)} - p(R), \tag{3.39}$$

$$\sigma^{\theta\theta} = \frac{\mu}{R^2}\,\mathrm{e}^{-2\omega_\Theta(R)} - \frac{p(R)}{r^2(R)}, \tag{3.40}$$

and

$$\sigma^{\phi\phi} = \frac{1}{\sin^2\Theta}\sigma^{\theta\theta}. \tag{3.41}$$

Therefore, $A = (\mathrm{e}^{-4\omega_0} - 1)\mu$, and we have

$$0 \le R \le R_\mathrm{i}: \quad \sigma^{rr}(R) = \mu \mathrm{e}^{-4\omega_0} - \mu \frac{R_\mathrm{o}^4}{r^4(R_\mathrm{o})} + \int_{R_\mathrm{i}}^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi, \tag{3.42}$$

and

$$R_\mathrm{i} \le R \le R_\mathrm{o}: \quad \sigma^{rr}(R) = \mu \frac{R^4}{r^4(R)} - \mu \frac{R_\mathrm{o}^4}{r^4(R_\mathrm{o})} + \int_{R_\mathrm{i}}^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi. \tag{3.43}$$

Note that

$$\bar{h}(R) = -\frac{2\mu}{r(R)}\left[1 - \frac{R^3}{r^3(R)}\right]^2 = O(\omega_0^2). \tag{3.44}$$

Thus

$$\int_{R_\mathrm{i}}^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi = \int_{R_\mathrm{i}}^{R_\mathrm{o}} O(\omega_0^2)\,\mathrm{d}\xi = O\left(\int_{R_\mathrm{i}}^{R_\mathrm{o}} \omega_0^2\,\mathrm{d}\xi\right) = O(\omega_0^2). \tag{3.45}$$

We now have the following asymptotic expansion (for small ω_0) for the radial stress inside and outside the inclusion.

$$0 \le R \le R_\mathrm{i}: \quad \sigma^{rr}(R) = -4\mu\omega_0\left[1 - \left(\frac{R_\mathrm{i}}{R_\mathrm{o}}\right)^3\right] + O(\omega_0^2), \tag{3.46}$$

and

$$R_\mathrm{i} \le R \le R_\mathrm{o}: \quad \sigma^{rr}(R) = -4\mu\omega_0\left[\left(\frac{R_\mathrm{i}}{R}\right)^3 - \left(\frac{R_\mathrm{i}}{R_\mathrm{o}}\right)^3\right] + O(\omega_0^2). \tag{3.47}$$

Because the leading terms are identical to the linear elasticity solution [14,33], we recover the classical result in this limit.

(iii) Limit of a single point defect

For a ball of radius R_o made of an incompressible linear elastic solid with a single point defect at the origin [34], the radial stress is

$$\sigma^{rr} = -\frac{4\mu C}{R^3}\left(1 - \frac{R^3}{R_\mathrm{o}^3}\right), \tag{3.48}$$

where $C = \delta v/4\pi$ and δv is the volume change due to the point defect. In our example, the change in volume owing to the inclusion of radius R_i is

$$\delta v = \frac{4\pi}{3} R_\mathrm{i}^3(\mathrm{e}^{3\omega_0} - 1). \tag{3.49}$$

Next, we keep δv constant while shrinking the inclusion. Note that for $R > R_\mathrm{i}$

$$r(R) = [R^3 + (\mathrm{e}^{3\omega_0} - 1)R_\mathrm{i}^3]^{1/3} = \left[R^3 + \frac{3}{4\pi}\delta v\right]^{1/3}. \tag{3.50}$$

Therefore, we have

$$\sigma^{rr}(R) = \mu \frac{R^4}{r^4(R)} - \mu \frac{R_\mathrm{o}^4}{r^4(R_\mathrm{o})} + \int_R^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi = -\frac{4\mu C}{R^3}\left(1 - \frac{R^3}{R_\mathrm{o}^3}\right) + O(\delta v^2). \tag{3.51}$$

That is, we recover the stress field of a single point defect with strength δv.

(iv) Zero-stress spherically symmetric eigenstrain distributions

Next, we determine those spherically symmetric eigenstrain distributions that leave a stress-free ball. Here, we are interested in the local nature of incompatibilities; other stresses may arise due to global constraints. In the spherical coordinates (R, Θ, Φ), $R \ge 0$, $0 \le \Theta \le \pi$, $0 \le \Phi < 2\pi$, having the material metric (3.1) is equivalent to having the following orthonormal coframe field (see [35] and [22] for an introduction to Cartan's moving frames and applications to residual stress calculations.)

$$\vartheta^1 = e^{\omega_R(R)}\,dR, \quad \vartheta^2 = e^{\omega_\Theta(R)} R\,d\Theta \quad \text{and} \quad \vartheta^3 = e^{\omega_\Theta(R)} R \sin\Theta\,d\Phi. \tag{3.52}$$

We now calculate the Levi–Civita connection 1-forms. The material connection being metric compatible means that the matrix of connection 1-forms is antisymmetric, i.e.

$$\boldsymbol{\omega} = [\omega^\alpha{}_\beta] = \begin{pmatrix} 0 & \omega^1{}_2 & -\omega^3{}_1 \\ -\omega^1{}_2 & 0 & \omega^2{}_3 \\ \omega^3{}_1 & -\omega^2{}_3 & 0 \end{pmatrix}. \tag{3.53}$$

Using Cartan's first structural equations $\mathcal{T}^\alpha = d\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta$ and knowing that the Levi–Civita connection is torsion free, we have the following set of equations for the three unknown connection 1-forms $\omega^1{}_2, \omega^2{}_3$ and $\omega^3{}_1$.

$$d\vartheta^1 + \omega^1{}_2 \wedge \vartheta^2 - \omega^3{}_1 \wedge \vartheta^3 = 0, \tag{3.54}$$

$$d\vartheta^2 - \omega^1{}_2 \wedge \vartheta^1 + \omega^2{}_3 \wedge \vartheta^3 = 0 \tag{3.55}$$

and

$$d\vartheta^3 + \omega^3{}_1 \wedge \vartheta^1 - \omega^2{}_3 \wedge \vartheta^2 = 0. \tag{3.56}$$

Hence, after a simple calculation

$$\omega^1{}_2 = -e^{-\omega_R(R)} \frac{1 + R\omega'_\Theta(R)}{R} \vartheta^2, \quad \omega^2{}_3 = -e^{-\omega_\Theta(R)} \frac{\cot\Theta}{R} \vartheta^3 \quad \text{and} \quad \omega^3{}_1 = e^{-\omega_R(R)} \frac{1 + R\omega'_\Theta(R)}{R} \vartheta^3. \tag{3.57}$$

Using Cartan's second structural equations $\mathcal{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$, we obtain the following Levi–Civita curvature 2-forms

$$\mathcal{R}^1{}_2 = -e^{-2\omega_R} \left[\omega''_\Theta + \frac{1}{R}(2\omega'_\Theta - \omega'_R) + \omega'_\Theta(\omega'_\Theta - \omega'_R) \right] \vartheta^1 \wedge \vartheta^2, \tag{3.58}$$

$$\mathcal{R}^2{}_3 = -e^{-2\omega_R} \left[\frac{1}{R^2}(1 - e^{-2\omega_\Theta + 2\omega_R}) + \omega'_\Theta \left(\omega'_\Theta + \frac{2}{R} \right) \right] \vartheta^2 \wedge \vartheta^3 \tag{3.59}$$

and

$$\mathcal{R}^3{}_1 = -e^{-2\omega_R} \left[\omega''_\Theta + \frac{1}{R}(2\omega'_\Theta - \omega'_R) + \omega'_\Theta(\omega'_\Theta - \omega'_R) \right] \vartheta^3 \wedge \vartheta^1. \tag{3.60}$$

The spherically symmetric dilatational eigenstrain distribution is locally impotent (zero stress) if and only if the Riemannian material manifold is flat (this condition is sufficient only for simply connected bodies). This means that for ω_R and ω_Θ to be stress-free they must satisfy the following system of nonlinear ODEs.

$$\omega''_\Theta(R) + \frac{1}{R}(2\omega'_\Theta(R) - \omega'_R(R)) + \omega'_\Theta(R)(\omega'_\Theta(R) - \omega'_R(R)) = 0, \tag{3.61}$$

and

$$\frac{1}{R^2}(1 - e^{-2\omega_\Theta(R) + 2\omega_R(R)}) + \omega'_\Theta(R) \left(\omega'_\Theta(R) + \frac{2}{R} \right) = 0. \tag{3.62}$$

Surprisingly, these two equations are compatible, in the sense that the former is the derivative of the latter. Therefore, a general solution for the stress-free problem is provided by an arbitrary $\omega_R(R)$ and

$$\omega_\Theta(R) = \ln \left[\frac{K \pm \int_0^R e^{\omega_R(\rho)}\,d\rho}{R} \right]. \tag{3.63}$$

As long as ω_R is analytic at $R = 0$, there is a well-behaved solution for the solid ball with $K = 0$, and $\omega_\Theta \neq \omega_R$ given by

$$\omega_\Theta(R) = \ln \left[\frac{\int_0^R e^{\omega_R(\rho)}\,d\rho}{R} \right], \tag{3.64}$$

otherwise, the solution is only well-defined on a spherical shell. In the special case of $\omega_R(R)=\omega_\Theta(R)=\omega(R)$, we have

$$\omega(R)=\omega(R_o)+\ln\left(\frac{R_o}{R}\right)^2. \tag{3.65}$$

For a solid ball, this solution is unbounded.

(b) Spherical eigenstrain in a compressible ball

Here, we release the incompressibility constraint for the problem of a ball with a spherically symmetric eigenstrain distribution. For an isotropic solid instead of considering the strain energy density as a function of the principal invariants of $\mathbf{C}$, one can assume that W explicitly depends on the principal invariants of $\mathbf{U}$, i.e. $W=\hat{W}(i_1,i_2,i_3)$, where

$$i_1=\lambda_1+\lambda_2+\lambda_3,\quad i_2=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1\quad\text{and}\quad i_3=\lambda_1\lambda_2\lambda_3. \tag{3.66}$$

We use Carroll's representation of the Cauchy stress for an isotropic elastic solid in terms of the left stretch tensor [25]. In our geometric framework, it takes the form

$$\boldsymbol{\sigma}=\left(\frac{i_2}{i_3}\frac{\partial\hat{W}}{\partial i_2}+\frac{\partial\hat{W}}{\partial i_3}\right)\mathbf{g}^\sharp+\frac{1}{i_3}\frac{\partial\hat{W}}{\partial i_1}\mathbf{V}^\sharp-\frac{\partial\hat{W}}{\partial i_2}\mathbf{V}^{-1}. \tag{3.67}$$

Or in components

$$\sigma^{ab}=\left(\frac{i_2}{i_3}\frac{\partial\hat{W}}{\partial i_2}+\frac{\partial\hat{W}}{\partial i_3}\right)g^{ab}+\frac{1}{i_3}\frac{\partial\hat{W}}{\partial i_1}V^{ab}-\frac{\partial\hat{W}}{\partial i_2}(V^{-1})^{ab}. \tag{3.68}$$

Note that in components

$$b^{ab}=V^{am}V^{bn}g_{mn},\quad c^{ab}=(V^{-1})^{am}(V^{-1})^{bn}g_{mn}, \tag{3.69}$$

where $\mathbf{b}^\sharp=\varphi_*(\mathbf{G}^\sharp)$ and $\mathbf{c}^\flat=\varphi_*(\mathbf{G})$ and in components $b^{ab}=F^a{}_AF^b{}_BG^{AB}$ and $c_{ab}=(F^{-1})^A{}_a(F^{-1})^B{}_b\,G_{AB}$ [26]. Note that $\mathbf{V}^\sharp=\sqrt{\mathbf{b}^\sharp}$ and $\mathbf{V}^{-1}=\sqrt{\mathbf{c}^\sharp}$. Carroll [25] considers a special class of compressible materials for which $\hat{W}(i_1,i_2,i_3)=u(i_1)+v(i_2)+w(i_3)$, where u, v and w are arbitrary C^2 functions. For this class of materials

$$\boldsymbol{\sigma}=\left(\frac{i_2}{i_3}v'(i_2)+w'(i_3)\right)\mathbf{g}^\sharp+\frac{u'(i_1)}{i_3}\mathbf{V}^\sharp-v'(i_2)\mathbf{V}^{-1}. \tag{3.70}$$

In the case of a ball with a spherically symmetric and isotropic eigenstrain distribution $\omega_R(R)=\omega_\Theta(R)=\omega(R)$, we have

$$\lambda_1=r'(R)\,\mathrm{e}^{-\omega(R)},\quad \lambda_2=\lambda_3=\frac{r(R)\,\mathrm{e}^{-\omega(R)}}{R}. \tag{3.71}$$

Thus

$$i_1=\mathrm{e}^{-\omega(R)}\left(r'(R)+\frac{2r(R)}{R}\right),\quad i_2=\mathrm{e}^{-2\omega(R)}\left(\frac{2r(R)r'(R)}{R}+\frac{r^2(R)}{R^2}\right),\quad i_3=\frac{r'(R)r^2(R)\,\mathrm{e}^{-3\omega(R)}}{R^2}. \tag{3.72}$$

A simple calculation gives us

$$\mathbf{V}^{\sharp} = \begin{pmatrix} r'(R)\,\mathrm{e}^{-\omega(R)} & 0 & 0 \\ 0 & \dfrac{\mathrm{e}^{-\omega(R)}}{Rr(R)} & 0 \\ 0 & 0 & \dfrac{\mathrm{e}^{-\omega(R)}}{Rr(R)\sin^2\Theta} \end{pmatrix},$$

$$\mathbf{V}^{-1} = \begin{pmatrix} \dfrac{\mathrm{e}^{\omega(R)}}{r'(R)} & 0 & 0 \\ 0 & \dfrac{R\,\mathrm{e}^{\omega(R)}}{r^3(R)} & 0 \\ 0 & 0 & \dfrac{R\,\mathrm{e}^{\omega(R)}}{r^3(R)\sin^2\Theta} \end{pmatrix}. \tag{3.73}$$

Hence, the non-zero stress components read

$$\sigma^{rr}(R) = \frac{R^2 u'(i_1)}{r^2(R)}\,\mathrm{e}^{2\omega(R)} + \frac{2Rv'(i_2)}{r(R)}\,\mathrm{e}^{\omega(R)} + w'(i_3), \tag{3.74}$$

$$\sigma^{\theta\theta}(R) = \frac{Ru'(i_1)}{r^3(R)r'(R)}\,\mathrm{e}^{2\omega(R)} + v'(i_2)\left(\frac{R}{r^3(R)} + \frac{1}{r^2(R)r'(R)}\right)\mathrm{e}^{\omega(R)} + \frac{w'(i_3)}{r^2(R)} \tag{3.75}$$

and

$$\sigma^{\phi\phi}(R) = \frac{1}{\sin^2\Theta}\sigma^{\theta\theta}(R). \tag{3.76}$$

The equilibrium equation (3.10) is simplified to read

$$\frac{R^2}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(u'\,\mathrm{e}^{2\omega}) + \frac{2R}{r}\frac{\mathrm{d}}{\mathrm{d}r}(v'\,\mathrm{e}^{\omega}) + \frac{\mathrm{d}w'}{\mathrm{d}r} = 0. \tag{3.77}$$

We first work with a harmonic material [36] for which $v(i_2) = c_2(i_2 - 3)$ and $w(i_3) = c_3(i_3 - 1)$, where c_2 and c_3 are constants (class I materials according to Carroll [25]). In this case, the above ODE is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}r}(u'\,\mathrm{e}^{2\omega}) + \frac{2c_2 r}{R}\frac{\mathrm{d}}{\mathrm{d}r}(\mathrm{e}^{\omega}) = 0. \tag{3.78}$$

Equation (3.78) can be solved for the eigenstrain distribution (3.15) when $\omega_1 = \omega_2 = \omega_0$. For both $R < R_\mathrm{i}$ and $R > R_\mathrm{i}$, we have $(\mathrm{d}/\mathrm{d}r)u'(i_1) = 0$, which implies that i_1 is constant in each interval. Note that for $R < R_\mathrm{i}$, $i_1 = \mathrm{e}^{-\omega_0}(r'(R) + 2r(R)/R)$ and for $R > R_\mathrm{i}$, $i_1 = r'(R) + 2r(R)/R$. Thus, we have

$$0 \le R \le R_\mathrm{i}: \quad r(R) = C_1 R + \frac{C_2}{R^2} \tag{3.79}$$

and

$$R_\mathrm{i} \le R \le R_\mathrm{o}: \quad r(R) = C_3 R + \frac{C_4}{R^2}, \tag{3.80}$$

which is identical to Carroll's [25] solution in each interval. For $r(R)$ to be bounded at $R = 0$, we must have $C_2 = 0$ and continuity of $r(R)$ at R_i implies that $C_4 = (C_1 - C_3)R_\mathrm{i}^3$. Thus, the only remaining unknowns are C_1 and C_3. These will be determined using continuity of traction at R_i and the boundary condition $\sigma^{rr}(R_\mathrm{o}) = -p_\infty$, namely

$$\frac{\mathrm{e}^{2\omega_0}u'(3\,\mathrm{e}^{-\omega_0}C_1)}{C_1^2} + \frac{2c_2\,\mathrm{e}^{\omega_0}}{C_1} = \frac{u'(3C_3)}{C_1^2} + \frac{2c_2}{C_1} \tag{3.81}$$

and

$$u'(3C_3)\frac{R_\mathrm{o}^2}{r^2(R_\mathrm{o})} + \frac{2c_2 R_\mathrm{o}}{r(R_\mathrm{o})} + c_3 = -p_\infty, \tag{3.82}$$

where $r(R_\mathrm{o}) = C_1 R_\mathrm{i}(R_\mathrm{i}/R_\mathrm{o})^2 + C_3 R_\mathrm{o}[1 - (R_\mathrm{i}/R_\mathrm{o})^3]$. Note that the radial stress inside the inclusion has the following value

$$\sigma^{rr}(R) = \frac{u'(3C_3\,\mathrm{e}^{2\omega_0})}{C_1^2} + \frac{2c_2\,\mathrm{e}^{\omega_0}}{C_1} + c_3 = \sigma_i, \tag{3.83}$$

i.e. the radial stress is uniform inside the inclusion. It is straightforward to show that the physical components of $\sigma^{\theta\theta}$ and $\sigma^{\phi\phi}$ are uniform inside the inclusion as well and are equal to σ_i.

Proposition 3.3. *Consider a spherical ball made of a harmonic solid, subject to a uniform pressure on its boundary sphere. Assume that there is a spherical inclusion at the centre of the ball with pure dilatational eigenstrain. Then, the stress inside the inclusion is uniform and hydrostatic.*

Remark 3.4. For class II and III materials according to Carroll [25], $u(i_1)=c_1(i_1-3), w(i_3)=c_3(i_3-1)$ and $u(i_1)=c_1(i_1-3), v(i_2)=c_2(i_2-3)$, respectively. For class II materials, we have

$$0 \le R \le R_{\mathrm{i}}: \quad r^2(R)=C_1R^2+\frac{C_2}{R} \tag{3.84}$$

and

$$R_{\mathrm{i}} \le R \le R_{\mathrm{o}}: \quad r^2(R)=C_3R^2+\frac{C_4}{R}. \tag{3.85}$$

For class III materials, we have

$$0 \le R \le R_{\mathrm{i}}: \quad r^3(R)=C_1R^3+C_2 \tag{3.86}$$

and

$$R_{\mathrm{i}} \le R \le R_{\mathrm{o}}: \quad r^3(R)=C_3R^3+C_4. \tag{3.87}$$

Imposing $r(0)=0$ in both cases $C_2=0$ and hence inside the inclusion $r(R)=\alpha R$, where α is a constant. This is identical to what we saw for harmonic materials. Therefore, proposition (3.2) also holds for materials of types II and III.

(c) Cylindrical eigenstrain in a finite cylindrical bar

Let us consider a circular cylindrical bar of initial length L and radius R_{o}. We assume a cylindrically symmetric distribution of eigenstrains. In the cylindrical coordinates (R,Θ,Z), we assume that the eigenstrains in the R, Θ and Z directions are different, in general. The metric of the material manifold is assumed to have the following form

$$\mathbf{G}=\begin{pmatrix} e^{2\omega_R(R)} & 0 & 0 \\ 0 & R^2\, e^{2\omega_\Theta(R)} & 0 \\ 0 & 0 & e^{2\omega_Z(R)} \end{pmatrix}, \tag{3.88}$$

where $\omega_R(R)$, $\omega_\Theta(R)$ and $\omega_Z(R)$ are arbitrary functions. We use the cylindrical coordinates (r,θ,z) for the Euclidean ambient space with the metric $\mathbf{g}=\mathrm{diag}(1,r^2,1)$. In order to obtain the residual stress field, we embed the material manifold into the ambient space and look for solutions of the form $(r,\theta,z)=(r(R),\Theta,\beta Z)$, where β is a constant to be determined. The deformation gradient reads $\mathbf{F}=\mathrm{diag}(r'(R),1,\beta)$ and hence $\det\mathbf{F}=\beta r'(R)$. For an incompressible solid, we have

$$J=\sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F}=\frac{\beta r(R)}{R\, e^{\omega_R(R)+\omega_\Theta(R)+\omega_Z(R)}}r'(R)=1. \tag{3.89}$$

Assuming that $r(0)=0$ this gives us

$$r(R)=\left(\int_0^R \frac{2\xi}{\beta}\, e^{\omega_R(\xi)+\omega_\Theta(\xi)+\omega_Z(\xi)}\, d\xi\right)^{1/2}. \tag{3.90}$$

The *physical* deformation gradient reads

$$\hat{\mathbf{F}}=\begin{pmatrix} \dfrac{R}{\beta r(R)}\, e^{\omega_\Theta(R)+\omega_Z(R)} & 0 & 0 \\ 0 & \dfrac{r(R)}{R}\, e^{-\omega_\Theta(R)} & 0 \\ 0 & 0 & \beta\, e^{-\omega_Z(R)} \end{pmatrix}. \tag{3.91}$$

Thus, the principal stretches are

$$\lambda_1 = \frac{R}{\beta r(R)}\, e^{\omega_\Theta(R)+\omega_Z(R)}, \quad \lambda_2 = \frac{r(R)}{R}\, e^{-\omega_\Theta(R)}, \quad \lambda_3 = \beta\, e^{-\omega_Z(R)}. \tag{3.92}$$

The non-zero Cauchy stress components read

$$\sigma^{rr} = \frac{R}{\beta r(R)}\, e^{\omega_\Theta(R)+\omega_Z(R)} \frac{\partial W}{\partial \lambda_1} - p(R), \tag{3.93}$$

$$\sigma^{\theta\theta} = \frac{e^{-\omega_\Theta(R)}}{R r(R)} \frac{\partial W}{\partial \lambda_2} - \frac{p(R)}{r^2(R)} \tag{3.94}$$

and

$$\sigma^{zz} = \beta\, e^{-\omega_Z(R)} \frac{\partial W}{\partial \lambda_3} - p(R). \tag{3.95}$$

In the absence of body forces, the only non-trivial equilibrium equation reads $\sigma^{ra}{}_{|a} = 0$ ($p = p(R)$ is the consequence of the other two equilibrium equations), which is simplified to read

$$\sigma^{rr}{}_{,r} + \frac{1}{r}\sigma^{rr} - r\sigma^{\theta\theta} = 0. \tag{3.96}$$

Or

$$\sigma^{rr}{}_{,R} + \frac{R\, e^{\omega_R(R)+\omega_\Theta(R)+\omega_Z(R)}}{\beta r(R)} \left(\frac{1}{r}\sigma^{rr} - r\sigma^{\theta\theta}\right) = 0. \tag{3.97}$$

This then gives us $p'(R) = k(R)$, where

$$\begin{aligned} k(R) = {} & \frac{e^{\omega_Z(R)}}{\beta r(R)} \left(e^{\omega_\Theta(R)} \frac{\partial W}{\partial \lambda_1} - e^{\omega_R(R)} \frac{\partial W}{\partial \lambda_2} \right) + \frac{R(\omega'_\Theta(R) + \omega'_Z(R))\, e^{\omega_\Theta(R)+\omega_Z(R)}}{\beta r(R)} \frac{\partial W}{\partial \lambda_1} \\ & + \frac{R\, e^{2\omega_\Theta(R)+2\omega_Z(R)}}{\beta^2 r^2(R)} \left[1 + R(\omega'_\Theta(R) + \omega'_Z(R)) - \frac{R^2\, e^{\omega_R(R)+\omega_\Theta(R)+\omega_Z(R)}}{\beta r^2(R)} \right] \frac{\partial^2 W}{\partial \lambda_1^2} \\ & + \frac{e^{\omega_Z(R)}}{\beta R} \left[\frac{R^2\, e^{\omega_R(R)+\omega_\Theta(R)+\omega_Z(R)}}{\beta r^2(R)} - (1 + R\omega'_\Theta(R)) \right] \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \\ & - \frac{R\omega'_Z(R)\, e^{\omega_\Theta(R)}}{r(R)} \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_3}. \end{aligned} \tag{3.98}$$

Assuming that at the boundary $\sigma^{rr}(R_0) = -p_\infty$, we have

$$p(R) = \frac{R_o}{\beta r(R_o)}\, e^{\omega_\Theta(R_o)+\omega_Z(R_o)} \left.\frac{\partial W}{\partial \lambda_1}\right|_{R=R_o} + p_\infty - \int_R^{R_o} k(\xi)\, d\xi. \tag{3.99}$$

The radial stress is now written as

$$\sigma^{rr}(R) = \frac{R}{\beta r(R)}\, e^{\omega_\Theta(R)+\omega_Z(R)} \frac{\partial W}{\partial \lambda_1} - \frac{R_o}{\beta r(R_o)}\, e^{\omega_\Theta(R_o)+\omega_Z(R_o)} \left.\frac{\partial W}{\partial \lambda_1}\right|_{R=R_o} - p_\infty + \int_R^{R_o} k(\xi)\, d\xi. \tag{3.100}$$

Note that β is not known and will be determined using boundary conditions. Next, we consider a cylindrical inclusion in the bar and calculate its stress field.

(i) A cylindrical inclusion in a finite cylindrical bar

For the cylindrical bar, we consider the following ω_R, ω_Θ and ω_Z distributions

$$\omega_R(R) = \begin{cases} \omega_1, & 0 \le R < R_i, \\ 0, & R_i < R \le R_o, \end{cases} \quad \omega_\Theta(R) = \begin{cases} \omega_2, & 0 \le R < R_i, \\ 0, & R_i < R \le R_o, \end{cases} \quad \omega_Z(R) = \begin{cases} \omega_3, & 0 \le R < R_i, \\ 0, & R_i < R \le R_o, \end{cases} \tag{3.101}$$

where $R_i < R_o$. Therefore

$$0 \le R \le R_i: \quad r(R) = \frac{1}{\sqrt{\beta}}\, e^{(1/2)(\omega_1+\omega_2+\omega_3)} R, \tag{3.102}$$

and

$$R_{\rm i}\le R\le R_{\rm o}:\quad r(R)=\frac{1}{\sqrt{\beta}}[R^2+(e^{\omega_1+\omega_2+\omega_3}-1)R_{\rm i}^2]^{1/2}. \tag{3.103}$$

Note that for $R<R_{\rm i}$

$$\lambda_1(R)=\bar{\lambda}_1=\frac{1}{\sqrt{\beta}}\,e^{(1/2)(-\omega_1+\omega_2+\omega_3)},\quad \lambda_2(R)=\bar{\lambda}_2=\frac{1}{\sqrt{\beta}}\,e^{(1/2)(\omega_1-\omega_2+\omega_3)},\quad \lambda_3(R)=\bar{\lambda}_3=\beta\,e^{-\omega_3}. \tag{3.104}$$

Note also that $\omega'_{\Theta}(R)=-\omega_2\delta(R-R_{\rm i})$, $\omega'_Z(R)=-\omega_3\delta(R-R_{\rm i})$ and hence $k(R)=B\delta(R-R_{\rm i})+\hat{k}(R)$, where

$$\hat{k}(R)=\frac{e^{\omega_Z(R)}}{\beta r(R)}\left(e^{\omega_\Theta(R)}\frac{\partial W}{\partial\lambda_1}-e^{\omega_R(R)}\frac{\partial W}{\partial\lambda_2}\right)+\frac{R\,e^{2\omega_\Theta(R)+2\omega_Z(R)}}{\beta^2r^2(R)}\left[1-\frac{R^2\,e^{\omega_R(R)+\omega_\Theta(R)+\omega_Z(R)}}{\beta r^2(R)}\right]\frac{\partial^2W}{\partial\lambda_1^2}$$
$$+\frac{e^{\omega_Z(R)}}{\beta R}\left[\frac{R^2\,e^{\omega_R(R)+\omega_\Theta(R)+\omega_Z(R)}}{\beta r^2(R)}-1\right]\frac{\partial^2W}{\partial\lambda_1\partial\lambda_2}, \tag{3.105}$$

and B is an unknown constant that will be determined after enforcing the continuity of the traction vector on the boundary of the inclusion. For $R<R_{\rm i}$, $\hat{k}(R)=h_0/R$, where

$$h_0=\frac{e^{(1/2)(-\omega_1-\omega_2+\omega_3)}}{\sqrt{\beta}}\left(e^{\omega_2}\frac{\partial W}{\partial\lambda_1}-e^{\omega_1}\frac{\partial W}{\partial\lambda_2}\right)\Bigg|_{\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3}. \tag{3.106}$$

Note that $h_0=0$ only if $\omega_1=\omega_2$. Now, we have

$$0\le R<R_{\rm i}:\quad \int_R^{R_{\rm o}}k(\xi)\,d\xi=\int_R^{R_{\rm i}}\hat{k}(\xi)\,d\xi+B+\int_{R_{\rm i}}^{R_{\rm o}}\hat{k}(\xi)\,d\xi$$
$$=h_0\ln\left(\frac{R_{\rm i}}{R}\right)+B+\int_{R_{\rm i}}^{R_{\rm o}}\bar{k}(\xi)\,d\xi \tag{3.107}$$

and

$$R_{\rm i}<R\le R_{\rm o}:\quad \int_R^{R_{\rm o}}k(\xi)\,d\xi=\int_R^{R_{\rm o}}\bar{k}(\xi)\,d\xi, \tag{3.108}$$

where

$$\bar{k}(R)=\frac{1}{\beta r(R)}\left(\frac{\partial W}{\partial\lambda_1}-\frac{\partial W}{\partial\lambda_2}\right)+\frac{R}{\beta^2r^2(R)}\left[1-\frac{R^2}{\beta r^2(R)}\right]\frac{\partial^2W}{\partial\lambda_1^2}+\frac{1}{\beta R}\left[\frac{R^2}{\beta r^2(R)}-1\right]\frac{\partial^2W}{\partial\lambda_1\partial\lambda_2}. \tag{3.109}$$

Therefore

$$0\le R<R_{\rm i}:\quad p(R)=p_\infty+\frac{R_{\rm o}}{\beta r(R_{\rm o})}\frac{\partial W}{\partial\lambda_1}\bigg|_{R_{\rm o}}-h_0\ln\left(\frac{R_{\rm i}}{R}\right)-B-\int_{R_{\rm i}}^{R_{\rm o}}\bar{k}(\xi)\,d\xi \tag{3.110}$$

and

$$R_{\rm i}<R\le R_{\rm o}:\quad p(R)=p_\infty+\frac{R_{\rm o}}{\beta r(R_{\rm o})}\frac{\partial W}{\partial\lambda_1}\bigg|_{R_{\rm o}}-\int_R^{R_{\rm o}}\bar{k}(\xi)\,d\xi. \tag{3.111}$$

Now, the radial stress inside and outside the inclusion has the following distributions

$$0\le R<R_{\rm i}:\quad \sigma^{rr}(R)=\frac{e^{(1/2)(-\omega_1+\omega_2+\omega_3)}}{\sqrt{\beta}}\frac{\partial W}{\partial\lambda_1}\bigg|_{\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3}-p(R) \tag{3.112}$$

and

$$R_{\rm i}<R\le R_{\rm o}:\quad \sigma^{rr}(R)=\frac{R}{\beta r(R)}\frac{\partial W}{\partial\lambda_1}-p(R). \tag{3.113}$$

Continuity of the traction vector on the boundary of the inclusion implies that σ^{rr} must be continuous at $R=R_{\rm i}$. Thus

$$B=\frac{e^{(1/2)(-\omega_1+\omega_2+\omega_3)}}{\sqrt{\beta}}\left(e^{-\omega_2-\omega_3}\frac{\partial W}{\partial\lambda_1}\bigg|_{R=R_{\rm i}^+}-\frac{\partial W}{\partial\lambda_1}\bigg|_{\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3}\right). \tag{3.114}$$

The physical component $\bar{\sigma}^{\theta\theta}$ reads

$$0 \le R < R_{\rm i}: \quad \bar{\sigma}^{\theta\theta}(R) = \frac{e^{(1/2)(\omega_1-\omega_2+\omega_3)}}{\sqrt{\beta}} \left.\frac{\partial W}{\partial \lambda_2}\right|_{\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3} - p(R) \tag{3.115}$$

and

$$R_{\rm i} < R \le R_{\rm o}: \quad \bar{\sigma}^{\theta\theta}(R) = \frac{r(R)}{R}\frac{\partial W}{\partial \lambda_2} - p(R). \tag{3.116}$$

The axial stress has the following expressions inside and outside the inclusion.

$$0 \le R < R_{\rm i}: \quad \sigma^{zz}(R) = \beta\, e^{-\omega_3} \left.\frac{\partial W}{\partial \lambda_3}\right|_{\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3} - p(R) \tag{3.117}$$

and

$$R_{\rm i} < R \le R_{\rm o}: \quad \sigma^{zz}(R) = \beta \frac{\partial W}{\partial \lambda_3} - p(R). \tag{3.118}$$

We observe that all the non-zero components of Cauchy stress have a logarithmic singularity. Note that the constant β depends on the axial boundary conditions.

Proposition 3.5. *Consider a finite isotropic and incompressible elastic cylinder subject to a uniform pressure on its boundary cylinder. Assume that there is a cylindrical inclusion at the centre of the bar with uniform radial and circumferential eigenstrains. Unless these strains are equal, the radial stress field exhibits a logarithmic singularity. If the eigenstrains are equal, then the stress inside the inclusion is uniform but not necessarily hydrostatic.*

Remark 3.6. Note that similar to the spherical inclusion, when the eigenstrain inside the inclusion is pure dilatational, the stress inside the inclusion would be uniform even if the ball is made of compressible materials of types I, II or III.

(d) Cylindrical inclusions in an infinite cylindrical bar

We now consider an infinite circular cylindrical bar with a cylindrically symmetric distribution of eigenstrains, i.e. a plane strain problem. The material metric in the cylindrical coordinates (R, Θ, Z) reads

$$\mathbf{G} = \begin{pmatrix} e^{2\omega_R(R)} & 0 & 0 \\ 0 & R^2\, e^{2\omega_\Theta(R)} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.119}$$

We assume that the bar is made of an arbitrary incompressible, isotropic solid. We know that for an isotropic solid strain energy density is only a function of the principal stretches, i.e. $W = W(\lambda_1, \lambda_2, \lambda_3 = 1)$. Note that the deformation gradient is given by $\mathbf{F} = \mathrm{diag}(r'(R), 1, 1)$. Incompressibility implies that

$$r'(R) = \frac{R}{r(R)}\, e^{\omega_R(R)+\omega_\Theta(R)}. \tag{3.120}$$

Assuming that $r(0) = 0$ we obtain

$$r(R) = \left(\int_0^R 2\xi\, e^{\omega_R(\xi)+\omega_\Theta(\xi)}\, d\xi\right)^{1/2}. \tag{3.121}$$

The principal stretches read

$$\lambda_1 = r'(R)\, e^{-\omega_R(R)} = \frac{R}{r(R)}\, e^{\omega_\Theta(R)}, \quad \lambda_2 = \frac{r(R)}{R}\, e^{-\omega_\Theta(R)}, \quad \lambda_3 = 1. \tag{3.122}$$

The symmetry of the problem dictates that in the spherical coordinates (r, θ, z) the Cauchy stress is diagonal. Hence, we have [29]

$$\sigma^{aa} = \lambda_a \frac{\partial W}{\partial \lambda_a} g^{aa} - p g^{aa}, \text{ no summation on } a. \tag{3.123}$$

Therefore, the non-zero stress components read

$$\sigma^{rr} = \frac{R\,\mathrm{e}^{\omega_\Theta(R)}}{r(R)}\frac{\partial W}{\partial \lambda_1} - p(R), \tag{3.124}$$

$$\sigma^{\theta\theta} = \frac{\mathrm{e}^{-\omega_\Theta(R)}}{Rr(R)}\frac{\partial W}{\partial \lambda_2} - \frac{p(R)}{r^2(R)} \tag{3.125}$$

and

$$\sigma^{zz} = \frac{\partial W}{\partial \lambda_3} - p(R). \tag{3.126}$$

The equilibrium equation (3.96) gives the ODE $p'(R) = h(R)$ for $p = p(R)$, where

$$\begin{aligned} h(R) = {} & \frac{1}{r(R)}\left(\mathrm{e}^{\omega_\Theta(R)}\frac{\partial W}{\partial \lambda_1} - \mathrm{e}^{\omega_R(R)}\frac{\partial W}{\partial \lambda_2}\right) + \frac{R\omega'_\Theta(R)\,\mathrm{e}^{\omega_\Theta(R)}}{r(R)}\frac{\partial W}{\partial \lambda_1} \\ & + \frac{R\,\mathrm{e}^{2\omega_\Theta(R)}}{r^2(R)}\left[1 + R\omega'_\Theta(R) - \frac{R^2}{r^2(R)}\,\mathrm{e}^{\omega_R(R)+\omega_\Theta(R)}\right]\frac{\partial^2 W}{\partial \lambda_1^2} \\ & - \frac{1}{R}\left[1 + R\omega'_\Theta(R) - \frac{R^2}{r^2(R)}\,\mathrm{e}^{\omega_R(R)+\omega_\Theta(R)}\right]\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}. \end{aligned} \tag{3.127}$$

If at the boundary $\sigma^{rr}(R_\mathrm{o}) = -p_\infty$, then

$$p(R) = p_\infty - \frac{R_\mathrm{o}\,\mathrm{e}^{\omega_\Theta(R_\mathrm{o})}}{r(R_\mathrm{o})}\left.\frac{\partial W}{\partial \lambda_1}\right|_{R=R_\mathrm{o}} - \int_R^{R_\mathrm{o}} h(\xi)\,\mathrm{d}\xi. \tag{3.128}$$

Once the pressure field is known, all the stress components can be easily calculated.

(i) A cylindrical inclusion in an infinite cylindrical bar

We consider the following ω_R and ω_Θ distributions

$$\omega_R(R) = \begin{cases} \omega_1, & 0 \le R < R_\mathrm{i}, \\ 0, & R_\mathrm{i} < R \le R_\mathrm{o}, \end{cases} \quad \omega_\Theta(R) = \begin{cases} \omega_2, & 0 \le R < R_\mathrm{i}, \\ 0, & R_\mathrm{i} < R \le R_\mathrm{o}, \end{cases} \tag{3.129}$$

where $R_\mathrm{i} < R_\mathrm{o}$. Thus

$$0 \le R \le R_\mathrm{i}: \quad r(R) = \mathrm{e}^{(1/2)(\omega_1+\omega_2)}R, \tag{3.130}$$

and

$$R_\mathrm{i} \le R \le R_\mathrm{o}: \quad r(R) = [R^2 + (\mathrm{e}^{\omega_1+\omega_2} - 1)R_\mathrm{i}^2]^{(1/2)}. \tag{3.131}$$

This means that for $R \le R_\mathrm{i}$, $\lambda_1 = \mathrm{e}^{(1/2)(\omega_2-\omega_1)} = \lambda_0$, $\lambda_2 = \mathrm{e}^{(1/2)(\omega_1-\omega_2)} = \lambda_0^{-1}$, $\lambda_3 = 1$. Note that $\omega'_\Theta(R) = -\omega_2\delta(R - R_\mathrm{i})$ and hence $h(R) = C\delta(R - R_\mathrm{i}) + \hat{h}(R)$, where

$$\begin{aligned} \hat{h}(R) = {} & \frac{1}{r(R)}\left(\mathrm{e}^{\omega_\Theta(R)}\frac{\partial W}{\partial \lambda_1} - \mathrm{e}^{\omega_R(R)}\frac{\partial W}{\partial \lambda_2}\right) + \frac{R\,\mathrm{e}^{2\omega_\Theta(R)}}{r^2(R)}\left[1 - \frac{R^2}{r^2(R)}\,\mathrm{e}^{\omega_R(R)+\omega_\Theta(R)}\right]\frac{\partial^2 W}{\partial \lambda_1^2} \\ & - \frac{1}{R}\left[1 - \frac{R^2}{r^2(R)}\,\mathrm{e}^{\omega_R(R)+\omega_\Theta(R)}\right]\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}, \end{aligned} \tag{3.132}$$

and C is an unknown constant that will be determined after enforcing the continuity of the traction vector on the boundary of the inclusion. Note that for $R < R_\mathrm{i}$, $\hat{h}(R) = h_0/R$, where

$$h_0 = \mathrm{e}^{-(1/2)(\omega_1+\omega_2)}\left.\left(\mathrm{e}^{\omega_2}\frac{\partial W}{\partial \lambda_1} - \mathrm{e}^{\omega_1}\frac{\partial W}{\partial \lambda_2}\right)\right|_{\lambda_1=\lambda_0,\lambda_2=\lambda_0^{-1},\lambda_3=1}. \tag{3.133}$$

Thus

$$0 \le R < R_\mathrm{i}: \quad \int_R^{R_\mathrm{o}} h(\xi)\,\mathrm{d}\xi = \int_R^{R_\mathrm{i}} \hat{h}(\xi)\,\mathrm{d}\xi + C + \int_{R_\mathrm{i}}^{R_\mathrm{o}} \hat{h}(\xi)\,\mathrm{d}\xi$$

$$= h_0 \ln\left(\frac{R_\mathrm{i}}{R}\right) + C + \int_{R_\mathrm{i}}^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi \tag{3.134}$$

and

$$R_\mathrm{i} < R \le R_\mathrm{o}: \quad \int_R^{R_\mathrm{o}} h(\xi)\,\mathrm{d}\xi = \int_R^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi, \tag{3.135}$$

where

$$\bar{h}(R) = \frac{1}{r(R)}\left(\frac{\partial W}{\partial \lambda_1} - \frac{\partial W}{\partial \lambda_2}\right) + \frac{R}{r^2(R)}\left[1 - \frac{R^2}{r^2(R)}\right]\frac{\partial^2 W}{\partial \lambda_1^2} + \frac{R}{r^2(R)}\left[1 - \frac{r^2(R)}{R^2}\right]\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}. \tag{3.136}$$

Therefore

$$0 \le R < R_\mathrm{i}: \quad p(R) = p_\infty + \frac{R_\mathrm{o}}{r(R_\mathrm{o})}\left.\frac{\partial W}{\partial \lambda_1}\right|_{R=R_\mathrm{o}} - h_0 \ln\left(\frac{R_\mathrm{i}}{R}\right) - C - \int_{R_\mathrm{i}}^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi \tag{3.137}$$

and

$$R_\mathrm{i} < R \le R_\mathrm{o}: \quad p(R) = p_\infty + \frac{R_\mathrm{o}}{r(R_\mathrm{o})}\left.\frac{\partial W}{\partial \lambda_1}\right|_{R=R_\mathrm{o}} - \int_R^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi. \tag{3.138}$$

Now the radial stress inside and outside of the inclusion has the following distributions

$$0 \le R < R_\mathrm{i}: \quad \sigma^{rr}(R) = \mathrm{e}^{(1/2)(\omega_2 - \omega_1)}\left.\frac{\partial W}{\partial \lambda_1}\right|_{\lambda_1=\lambda_0, \lambda_2=\lambda_0^{-1}, \lambda_3=1} - p_\infty \tag{3.139}$$

$$- \frac{R_\mathrm{o}}{r(R_\mathrm{o})}\left.\frac{\partial W}{\partial \lambda_1}\right|_{R=R_\mathrm{o}} + h_0 \ln\left(\frac{R_\mathrm{i}}{R}\right) + C + \int_{R_\mathrm{i}}^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi, \tag{3.140}$$

and

$$R_\mathrm{i} < R \le R_\mathrm{o}: \quad \sigma^{rr}(R) = \frac{R}{r(R)}\frac{\partial W}{\partial \lambda_1} - p_\infty - \frac{R_\mathrm{o}}{r(R_\mathrm{o})}\left.\frac{\partial W}{\partial \lambda_1}\right|_{R=R_\mathrm{o}} + \int_R^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi. \tag{3.141}$$

The continuity of traction vector on the boundary of the inclusion dictates that σ^{rr} must be continuous at $R = R_\mathrm{i}$ and hence

$$C = \mathrm{e}^{-(1/2)(\omega_1 + \omega_2)}\left.\frac{\partial W}{\partial \lambda_1}\right|_{R=R_\mathrm{i}^+} - \mathrm{e}^{(1/2)(\omega_2 - \omega_1)}\left.\frac{\partial W}{\partial \lambda_1}\right|_{\lambda_1=\lambda_0, \lambda_2=\lambda_0^{-1}, \lambda_3=1}. \tag{3.142}$$

Therefore, the radial stress inside the inclusion has the following distribution

$$\sigma^{rr}(R) = h_0 \ln\left(\frac{R_\mathrm{i}}{R}\right) + \sigma_i, \tag{3.143}$$

where

$$\sigma_i = \mathrm{e}^{-(1/2)(\omega_1 + \omega_2)}\left.\frac{\partial W}{\partial \lambda_1}\right|_{R=R_\mathrm{i}^+} - p_\infty - \frac{R_\mathrm{o}}{r(R_\mathrm{o})}\left.\frac{\partial W}{\partial \lambda_1}\right|_{R=R_\mathrm{o}} + \int_{R_\mathrm{i}}^{R_\mathrm{o}} \bar{h}(\xi)\,\mathrm{d}\xi. \tag{3.144}$$

We conclude that the radial stress has a logarithmic singularity unless $\omega_1 = \omega_2$. Inside the inclusion, the other two stress components have the following distributions

$$\bar{\sigma}^{\theta\theta} = h_0 \ln\left(\frac{R_\mathrm{i}}{R}\right) + \sigma_i + \left.\left[\mathrm{e}^{(1/2)(\omega_1 - \omega_2)}\frac{\partial W}{\partial \lambda_2} - \mathrm{e}^{(1/2)(\omega_2 - \omega_1)}\frac{\partial W}{\partial \lambda_1}\right]\right|_{\lambda_1=\lambda_0, \lambda_2=\lambda_0^{-1}, \lambda_3=1} \tag{3.145}$$

and

$$\sigma^{zz} = h_0 \ln\left(\frac{R_\mathrm{i}}{R}\right) + \sigma_i + \left.\left[\frac{\partial W}{\partial \lambda_3} - \mathrm{e}^{(1/2)(\omega_2 - \omega_1)}\frac{\partial W}{\partial \lambda_1}\right]\right|_{\lambda_1=\lambda_0, \lambda_2=\lambda_0^{-1}, \lambda_3=1}. \tag{3.146}$$

Proposition 3.7. *Consider an infinite isotropic and incompressible elastic circular cylinder subject to uniform pressure on its boundary cylinder. Assume that there is a cylindrical inclusion at the centre of the bar with uniform radial and circumferential eigenstrains. Unless these strains are equal, the stress field has* *a logarithmic singularity. If the eigenstrains are equal, then the stress inside the inclusion is uniform and hydrostatic.*

Remark 3.8. Note that when the radial and circumferential eigenstrains are equal the stress inside the inclusion would be uniform hydrostatic even if the bar is made of compressible materials of types I, II or III.

(e) A multi-layered cylindrical inclusion

We can generalize the previous example to the case when the eigenstrain is piecewise constant in the cylindrical bar. Let $R_0 = R_\mathrm{i} < R_1 < \cdots < R_n = R_\mathrm{o}$ and assume that

$$\omega_R(R) = \omega_\Theta(R) = \omega_0 + \sum_{k=0}^{n-1} (\omega_{k+1} - \omega_k) H(R - R_k). \tag{3.147}$$

Thus

$$\omega'_\Theta(R) = \sum_{k=0}^{n-1} (\omega_{k+1} - \omega_k) \delta(R - R_k). \tag{3.148}$$

Using incompressibility, for $R_k \leq R \leq R_{k+1}$, we have

$$r(R) = \left[\mathrm{e}^{2\omega_k} R^2 + \sum_{j=1}^{k} \left(\mathrm{e}^{2\omega_{j-1}} - \mathrm{e}^{2\omega_j} \right) R_{j-1}^2 \right]^{1/2}. \tag{3.149}$$

In the same interval

$$\lambda_1 = \frac{R}{r(R)}\, \mathrm{e}^{\omega_k}, \quad \lambda_2 = \frac{r(R)}{R}\, \mathrm{e}^{-\omega_k}, \quad \lambda_3 = 1. \tag{3.150}$$

Note that in $p'(R) = h(R)$ there are n delta distributions on the right-hand side with indeterminate coefficients, i.e.

$$h(R) = \hat{h}(R) + \sum_{k=0}^{n-1} A_k \delta(R - R_k). \tag{3.151}$$

Knowing that $\sigma^{rr}(R_n) = -p_\infty$, we have

$$p(R_n) = p_\infty + \frac{R_n\, \mathrm{e}^{\omega_{n-1}}}{r(R_n)} \left. \frac{\partial W}{\partial \lambda_1} \right|_{R_n}. \tag{3.152}$$

Denoting the jump in a quantity by $[\![\cdot]\!]_R = (\cdot)_{R^+} - (\cdot)_{R^-}$, it can be shown that

$$[\![\sigma^{rr}]\!]_{R_k} = \frac{R_k}{r(R_k)} \left(\mathrm{e}^{\omega_k} \left. \frac{\partial W}{\partial \lambda_1} \right|_{R_k^+} - \mathrm{e}^{\omega_{k-1}} \left. \frac{\partial W}{\partial \lambda_1} \right|_{R_k^-} \right) - [\![p]\!]_{R_k} = 0. \tag{3.153}$$

It is straightforward to show that $A_k = [\![p]\!]_{R_k}$ and hence

$$A_k = \frac{R_k}{r(R_k)} \left(\mathrm{e}^{\omega_k} \left. \frac{\partial W}{\partial \lambda_1} \right|_{R_k^+} - \mathrm{e}^{\omega_{k-1}} \left. \frac{\partial W}{\partial \lambda_1} \right|_{R_k^-} \right). \tag{3.154}$$

Therefore

$$R_k < R < R_{k+1}: \quad p(R) = p_\infty + \frac{R_n\, \mathrm{e}^{\omega_{n-1}}}{r(R_n)} \left. \frac{\partial W}{\partial \lambda_1} \right|_{R_n} - \sum_{j=k+1}^{n-1} A_j - \int_R^{R_n} \hat{h}(\xi)\, \mathrm{d}\xi. \tag{3.155}$$

Having the pressure field, all stress components are easily calculated.

(i) Zero-stress cylindrically symmetric eigenstrain distributions

Next, we find the locally impotent cylindrically symmetric eigenstrain distributions. In the cylindrical coordinates (R, Θ, Z), having the material metric (3.119) is equivalent to having the following orthonormal coframe field

$$\vartheta^1 = e^{\omega_R(R)}\,dR, \quad \vartheta^2 = e^{\omega_\Theta(R)} R\,d\Theta, \quad \vartheta^3 = dZ. \tag{3.156}$$

To calculate the Levi–Civita connection 1-forms, we use Cartan's first structural equations. Knowing that the Levi–Civita connection is torsion free, we find the three unknown connection 1-forms $\omega^1{}_2, \omega^2{}_3$ and $\omega^3{}_1$ as

$$\omega^1{}_2 = -e^{-\omega_R(R)} \frac{1 + R\omega'_\Theta(R)}{R} \vartheta^2, \quad \omega^2{}_3 = \omega^3{}_1 = 0. \tag{3.157}$$

Using Cartan's second structural equations, we obtain the following Riemann curvature 2-forms

$$\mathcal{R}^1{}_2 = -e^{-2\omega_R}\left[\omega''_\Theta + \frac{1}{R}(2\omega'_\Theta - \omega'_R) + \omega'_\Theta(\omega'_\Theta - \omega'_R)\right]\vartheta^1 \wedge \vartheta^2, \quad \mathcal{R}^2{}_3 = \mathcal{R}^3{}_1 = 0. \tag{3.158}$$

The Riemannian material manifold is flat if and only if

$$\omega''_\Theta(R) + \frac{1}{R}(2\omega'_\Theta(R) - \omega'_R(R)) + \omega'_\Theta(R)(\omega'_\Theta(R) - \omega'_R(R)) = 0. \tag{3.159}$$

Given ω_Θ, ω_R must satisfy the following linear ODE for the eigenstrain distribution to be zero stress.

$$\left[\omega'_\Theta(R) + \frac{1}{R}\right]\omega'_R(R) = \omega''_\Theta(R) + \omega'_\Theta(R)^2 + \frac{2}{R}\omega'_\Theta(R). \tag{3.160}$$

The general solution of this equation for a given $\omega_R(R)$ is

$$\omega_\Theta(R) = C_1 - \int_0^R \left(\frac{1}{\zeta} - \frac{e^{\omega_R(\zeta)}}{C_2 + \int_0^\zeta e^{\omega_R(\rho)}\,d\rho}\right) d\zeta. \tag{3.161}$$

When $\omega_\Theta = \omega_R = \omega$, the solution is simply

$$\omega(R) = C_1 + C_2 \ln R. \tag{3.162}$$

For a solid bar to have an invertible material metric at $R = 0$, we must have $C_2 = 0$, i.e. a uniform eigenstrain is the only zero-stress cylindrically symmetric eigenstrain distribution.

4. Conclusion

In this paper, we developed a general framework to compute residual stress fields induced by general eigenstrains and inclusions. As examples, we first computed the stress field of a ball with a spherically symmetric distribution of dilatational eigenstrains. We assumed that the ball is made of an arbitrary incompressible isotropic solid. As a particular example, we looked at a spherical inclusion at the centre of a finite ball. Assuming that the ball is made of an incompressible solid and that the eigenstrain in the inclusion is purely dilatational, we showed that stress in the inclusion is hydrostatic. When the radial and circumferential eigenstrains are not equal, we showed that the stress inside the inclusion is non-homogeneous and has a logarithmic singularity. We also looked at a special class of compressible materials (materials of types I, II and III according to Carroll [25]). We showed that when the eigenstrain in a spherical inclusion is purely dilatational, the stress in the inclusion is again uniform and hydrostatic. We also considered finite and infinite circular cylindrical bars with cylindrically symmetric distributions of eigenstrains and obtained similar results for their residual stress fields.

The geometric framework used for the inclusion problem is rigorous, general and flexible. On the formal side, it provides a direct geometric interpretation for the notion of incompatibility and the generation of residual stress. On the practical side, geometric tools provide a systematic method to compute residual stress and deformations created by eigenstrains. It is clear from the procedure described here that all semi-inverse problems can be treated along these lines as long as the eigenstrains respect the underlying symmetry of the deformations (in our case, spherical and cylindrical shells are locally deformed into spherical or cylindrical shells). The analysis presented here also provides the starting point for a generalized homogenization computation for a material with a dilute distribution of inclusions with given eigenstrains in order to compute effective material properties.

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