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Universal deformations in anisotropic nonlinear elastic solids

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ABSTRACT

Universal deformations of an elastic solid are deformations that can be achieved for all possible strain–energy density functions and suitable boundary conditions. They play a central role in nonlinear elasticity and their classification has been mostly accomplished for isotropic solids following Ericksen's seminal work. Here, we address the same problem for transversely isotropic, orthotropic, and monoclinic solids. In this case, there are no general solutions unless universal material preferred directions are also specified. First, we show that for compressible transversely isotropic, orthotropic, and monoclinic solids universal deformations are homogeneous and that the material preferred directions are uniform as well. Second, for incompressible transversely isotropic, orthotropic, and monoclinic solids we derive the corresponding *universality constraints*. These are constraints that are imposed by equilibrium equations and the arbitrariness of the energy function. We show that these constraints include those of incompressible isotropic solids. Hence, we consider the known universal deformations for each of the six known families of universal deformations for isotropic solids and find the corresponding universal material preferred directions for transversely isotropic, orthotropic, and monoclinic solids. This work provides a systematic way to study fiber-reinforced elastic solids analytically.

1. Introduction

Universal (controllable) deformations for a given class of materials are those deformations that can be maintained in the absence of body forces by applying only boundary tractions for all strain–energy functions in that class. In the case of (unconstrained) compressible isotropic elastic solids, Ericksen (1955) proved that the only universal deformations are homogeneous deformations. The constrained case is more involved (Saccomandi, 2001). For instance, in the case of incompressible isotropic solids, Ericksen (1954), motivated by the earlier works of Rivlin (1948, 1949a,b), found four families of universal deformations. He conjectured that a deformation with constant principal strain invariants must be homogeneous. Fosdick (1966) found a counter-example, and this led to the discovery of a fifth family of universal deformations independently by Singh and Pipkin (1965) and Klingbeil and Shield (1966). The six known families of universal deformations are:

- Family 0: Homogeneous deformations
- Family 1: Bending, stretching, and shearing of a rectangular block
- Family 2: Straightening, stretching, and shearing of a sector of a cylindrical shell
- Family 3: Inflation, bending, torsion, extension, and shearing of a sector of an annular wedge
- Family 4: Inflation/inversion of a sector of a spherical shell

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- Family 5: Inflation, bending, extension, and azimuthal shearing of an annular wedge

Carroll (1967) and Fosdick (1968) showed that these families are universal dynamically as well for those motions whose acceleration is curl-free, i.e., is gradient of a potential function. Ericksen's problem in the case of incompressible isotropic solids has not been completely solved to this date as the case of deformations with constant principal invariants is still open but the conjecture is that there is no other possible family. In the setting of linear elasticity, Yavari et al. (2020) showed that universal displacements explicitly depend on the material symmetry class; the smaller the symmetry group is the smaller the corresponding space of universal displacements is. Yavari and Goriely (2016) showed that in compressible anelasticity universal deformations are covariantly homogeneous. For the generalization of Ericksen's work to incompressible anelasticity, Goodbrake et al. (2020) showed that a key feature of the analysis is that the extra fields entering the analysis should follow the same symmetry as the deformation.

There has not been any systematic study of universal deformations in anisotropic solids. Ericksen and Rivlin (1954) analyzed a subset of Family 1 for two cases of homogeneous anisotropy. They also analyzed Family 3 for an example of homogeneous anisotropy. See also (Adkins, 1955a,b). Yet, we know plenty of examples of anisotropic fiber-reinforced systems (Spencer, 1982; Qiu and Pence, 1997) with one (i.e. transversely isotropic) or two (i.e. orthotropic) specified material preferred directions that sustain universal deformations either in rectangular (Melnik and Goriely, 2013) or helical geometry (Holzapfel et al., 2000; Demirkoparan and Pence, 2007; Goriely and Tabor, 2013; Demirkoparan and Pence, 2015; Goriely, 2017). The question is then to find all such systems. Here, we do not specify the material preferred directions *a priori*; we find conditions for the existence of universal deformations and then find the universal material preferred directions that satisfy these constraints.

We consider the following six classes of anisotropic materials: (i) compressible transversely isotropic, (ii) compressible orthotropic, (iii) compressible monoclinic, (iv) incompressible transversely isotropic, (v) incompressible orthotropic, and (vi) incompressible monoclinic solids. Using the representation of Cauchy stress for each class we find the universality constraints imposed by both the equilibrium equations in the absence of body forces, and the arbitrariness of the energy function. Perhaps unsurprisingly, our analysis shows that the set of universality constraints for each class includes those of isotropic solids. In the case of compressible solids it implies that universal deformations must be homogeneous and we show that the extra universality constraints force the universal material preferred directions to be uniform for non-isochoric deformations. In the case of incompressible solids we find, for each of the six known families of universal deformations, the corresponding universal material preferred directions assuming that they respect the symmetry of the universal deformations encoded in the right Cauchy–Green tensor.

This paper is organized as follows. In Section 2 we briefly review nonlinear anisotropic elasticity. In Section 3, we consider compressible transversely isotropic, orthotropic, and monoclinic solids. The universal deformations and universal material preferred directions of incompressible transversely isotropic solids are analyzed for each of the known six families in Section 4. In Sections 5 and 6 similar analyses are presented for incompressible orthotropic and incompressible monoclinic solids. Conclusions are given in Section 7.

2. Nonlinear anisotropic elasticity

Kinematics. In nonlinear anelasticity a body $\mathcal{B}$ is identified with a Riemannian manifold $(\mathcal{B},\mathbf{G})$, where $\mathbf{G}$ is the material metric that characterizes the natural distances in the body. In nonlinear elasticity, which is the focus of this paper $(\mathcal{B},\mathbf{G})$ is a submanifold of the Euclidean 3-space. A deformation of the body is a mapping $\varphi:\mathcal{B}\to\mathcal{S}$, where $(\mathcal{S},\mathbf{g})$ is another Riemannian manifold — the ambient space, which is assumed to be the Euclidean 3-space. The material velocity $\mathbf{V}_t:\mathcal{B}\to T_{\varphi_t(\mathbf{X})}\mathcal{S}$ is defined as $\mathbf{V}_t(\mathbf{X})=\mathbf{V}(\mathbf{X},t)=\frac{\partial\varphi(\mathbf{X},t)}{\partial t}$. The spatial velocity is defined as $\mathbf{v}=\mathbf{V}\circ\varphi_t^{-1}$. The primary object to study deformations in nonlinear elasticity is the deformation gradient, which is the tangent map (or derivative) of φ and is denoted by $\mathbf{F}=T\varphi$. At each material point $\mathbf{X}\in\mathcal{B}$, deformation gradient is a linear map $\mathbf{F}(\mathbf{X}):T_{\mathbf{X}}\mathcal{B}\to T_{\varphi(\mathbf{X})}\mathcal{S}$. With respect to local coordinate charts $\{x^a\}$ and $\{X^A\}$ on $\mathcal{S}$ and $\mathcal{B}$, respectively, the deformation gradient has components

$$F^a{}_A(\mathbf{X})=\frac{\partial\varphi^a}{\partial X^A}(\mathbf{X})\,. \tag{2.1}$$

The transpose of deformation gradient is defined as

$$\mathbf{F}^{\mathsf{T}}:T_{\mathbf{x}}\mathcal{S}\to T_{\mathbf{X}}\mathcal{B},\quad \langle\!\langle\mathbf{F}\mathbf{V},\mathbf{v}\rangle\!\rangle_{\mathbf{g}}=\langle\!\langle\mathbf{V},\mathbf{F}^{\mathsf{T}}\mathbf{v}\rangle\!\rangle_{\mathbf{G}},\quad \forall\mathbf{V}\in T_{\mathbf{X}}\mathcal{B},\ \mathbf{v}\in T_{\mathbf{x}}\mathcal{S}\,, \tag{2.2}$$

and has components

$$(F^{\mathsf{T}}(\mathbf{X}))^A{}_a=g_{ab}(\mathbf{x})F^b{}_B(\mathbf{X})G^{AB}(\mathbf{X})\,. \tag{2.3}$$

The right Cauchy–Green deformation tensor is defined as $\mathbf{C}(X)=\mathbf{F}(\mathbf{X})^{\mathsf{T}}\mathbf{F}(\mathbf{X}):T_{\mathbf{X}}\mathcal{B}\to T_{\mathbf{X}}\mathcal{B}$ and has components $C^A{}_B=(F^{\mathsf{T}})^A{}_aF^a{}_B$. Note that $C_{AB}=(g_{ab}\circ\varphi)F^a{}_AF^b{}_B$, which means that $\mathbf{C}^\flat=\varphi^*(\mathbf{g})$, where $\flat$ is the flat operator induced by the metric $\mathbf{g}$. The left Cauchy–Green deformation tensor is defined as $\mathbf{B}^\sharp=\varphi^*(\mathbf{g}^\sharp)$, which has components $B^{AB}=(F^{-1})^A{}_a(F^{-1})^B{}_b\,g^{ab}$. The spatial analogues of $\mathbf{C}^\flat$ and $\mathbf{B}^\sharp$ are $\mathbf{c}^\flat$ and $\mathbf{b}^\sharp$, respectively, and are defined as

$$\begin{aligned}\mathbf{c}^\flat&=\varphi_*(\mathbf{G}), & c_{ab}&=\left(F^{-1}\right)^A{}_a\left(F^{-1}\right)^B{}_b\,G_{AB},\\ \mathbf{b}^\sharp&=\varphi_*(\mathbf{G}^\sharp), & b^{ab}&=F^a{}_AF^b{}_BG^{AB}.\end{aligned} \tag{2.4}$$

$\mathbf{b}^\sharp$ is called the Finger deformation tensor. The tensors $\mathbf{C}$ and $\mathbf{b}$ have the same principal invariants I_1, I_2, and I_3 (Ogden, 1984), which are defined as

$$\begin{aligned} I_1 &= \operatorname{tr}\mathbf{b} = b^a{}_a = b^{ab}\,g_{ab},\\ I_2 &= \frac{1}{2}\left(I_1^2 - \operatorname{tr}\mathbf{b}^2\right) = \frac{1}{2}\left(I_1^2 - b^a{}_b\,b^b{}_a\right) = \frac{1}{2}\left(I_1^2 - b^{ab}b^{cd}\,g_{ac}\,g_{bd}\right),\\ I_3 &= \det\mathbf{b}. \end{aligned} \tag{2.5}$$

Balance laws. Conservation of mass and the balance of linear and angular momenta in material form read

$$\frac{\partial \rho_0}{\partial t} = 0, \tag{2.6}$$

$$\operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \rho_0\mathbf{A}, \tag{2.7}$$

$$\mathbf{P}\mathbf{F}^{\mathsf{T}} = \mathbf{F}\mathbf{P}^{\mathsf{T}}, \tag{2.8}$$

where ρ_0 is the material mass density, $\mathbf{B}$ is body force per unit undeformed volume, $\mathbf{A}$ is the material acceleration, and $\mathbf{P}$ is the first Piola–Kirchhoff stress. The relation between $\mathbf{P}$ and the Cauchy stress $\boldsymbol{\sigma}$ is $J\sigma^{ab} = P^{aA}F^b{}_A$, where J is the Jacobian of deformation that relates the material (dV) and spatial (dv) Riemannian volume forms as $dv = J\,dV$, and is defined as

$$J = \sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\,\det\mathbf{F}. \tag{2.9}$$

The balance equations in terms of the spatial mass density ρ and the Cauchy stress $\boldsymbol{\sigma}$ read

$$\mathbf{L}_{\mathbf{v}}\rho = 0, \tag{2.10}$$

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a}, \tag{2.11}$$

$$\boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma}, \tag{2.12}$$

where $\mathbf{b} = \mathbf{B}\circ\varphi_t^{-1}$, $\mathbf{a}$ is the spatial acceleration, and $\mathbf{L}_{\mathbf{v}}\rho$ is the Lie derivative of the spatial mass density with respect to the spatial velocity.

Constitutive equations. For an anisotropic hyperelastic solid the energy function (per unit undeformed volume) is written as

$$W = \hat{W}(\mathbf{C}^\flat, \mathbf{G}, \zeta_1, \dots, \zeta_n), \tag{2.13}$$

where $\zeta_i, i = 1, \dots, n$ are the *structural tensors* that characterize the material symmetry group of the solid. Structural tensors make the energy function an isotropic function of its arguments. Hilbert's theorem tells us that for any finite number of tensors there is a finite number of isotropic invariants that form an *integrity basis* for the space of isotropic invariants of the collection of tensors. Therefore, if $I_j, j = 1, \dots, m$, form an integrity basis for the set of tensors in (2.13), one has $W = W(X, I_1, \dots, I_m)$. Using the Doyle–Ericksen formula (Doyle and Ericksen, 1956; Marsden and Hughes, 1994; Yavari et al., 2006), one obtains the following representation for the second Piola–Kirchhoff stress tensor

$$\mathbf{S} = 2\frac{\partial \hat{W}}{\partial \mathbf{C}^\flat} = \sum_{j=1}^{m} 2W_j \frac{\partial I_j}{\partial \mathbf{C}^\flat}, \qquad W_j := \frac{\partial W}{\partial I_j}, \quad j = 1, \dots, m. \tag{2.14}$$

Note that $S^{AB} = (F^{-1})^A{}_a P^{aB} = J(F^{-1})^A{}_a (F^{-1})^B{}_b\,\sigma^{ab}$.

Isotropic solids. For an isotropic solid, the energy function has the form $W = W(I_1, I_2, I_3)$, where I_1, I_2, and I_3 are the principal invariants of the right Cauchy–Green deformation tensor given in (2.5). From (2.14) we have

$$\mathbf{S} = 2W_1\mathbf{G}^\sharp + 2W_2\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + 2W_3 I_3\mathbf{C}^{-1}. \tag{2.15}$$

Similarly, the Cauchy stress has the representation

$$\sigma^{ab} = \frac{2}{\sqrt{I_3}}\left[W_1 b^{ab} + (I_2W_2 + I_3W_3)g^{ab} - I_3W_2\,c^{ab}\right], \tag{2.16}$$

where $c^{ab} = (F^{-1})^M{}_m (F^{-1})^N{}_n G_{MN}\, g^{am} g^{bn}$. For incompressible isotropic solids $I_3 = 1$, and one writes

$$\mathbf{S} = -p\mathbf{C}^{-1} + 2W_1\mathbf{G}^\sharp - 2W_2\mathbf{C}^{-2}, \tag{2.17}$$

where p is the Lagrange multiplier associated with the incompressibility constraint $J = \sqrt{I_3} = 1$. The Cauchy stress similarly reads

$$\sigma^{ab} = -pg^{ab} + 2W_1 b^{ab} - 2W_2\,c^{ab}. \tag{2.18}$$

Transversely isotropic solids. A transversely isotropic solid has a single material preferred direction at every point that is normal to the plane of isotropy at that point. Let us assume that the unit vector $\mathbf{N}(\mathbf{X})$ identifies the material preferred direction at a point $\mathbf{X} \in \mathcal{B}$. The energy function has the form $W = W(\mathbf{G}, \mathbf{C}^\flat, \mathbf{A})$, where $\mathbf{A} = \mathbf{N}\otimes\mathbf{N}$ is a structural tensor (Doyle and Ericksen, 1956;

Spencer, 1982; Lu and Papadopoulos, 2000). The energy function W depends on the following five independent invariants

$$I_1 = \operatorname{tr}\mathbf{C}\,, \quad I_2 = \det\mathbf{C}\ \operatorname{tr}\ \mathbf{C}^{-1}\,, \quad I_3 = \det\mathbf{C}\,, \quad I_4 = \mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}\,, \quad I_5 = \mathbf{N}\cdot\mathbf{C}^2\cdot\mathbf{N}\,. \tag{2.19}$$

In components

$$I_1 = C^A{}_A\,, \quad I_2 = \det(C^A{}_B)(C^{-1})^D{}_D\,, \quad I_3 = \det(C^A{}_B)\,, \quad I_4 = N^A N^B C_{AB}\,, \quad I_5 = N^A N^B C_{BM} C^M{}_A\,. \tag{2.20}$$

The second Piola–Kirchhoff stress tensor is written as

$$\mathbf{S} = \sum_{j=1}^{5} 2W_j \frac{\partial I_j}{\partial \mathbf{C}^\flat}\,, \qquad W_j := \frac{\partial W}{\partial I_j}\,,\ j = 1,\dots,5\,, \tag{2.21}$$

where

$$\frac{\partial I_1}{\partial \mathbf{C}^\flat} = \mathbf{G}^\sharp\,, \quad \frac{\partial I_2}{\partial \mathbf{C}^\flat} = I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\,, \quad \frac{\partial I_3}{\partial \mathbf{C}^\flat} = I_3\mathbf{C}^{-1}\,, \quad \frac{\partial I_4}{\partial \mathbf{C}^\flat} = \mathbf{N}\otimes\mathbf{N}\,, \quad \frac{\partial I_5}{\partial \mathbf{C}^\flat} = \mathbf{N}\otimes(\mathbf{C}\cdot\mathbf{N}) + (\mathbf{C}\cdot\mathbf{N})\otimes\mathbf{N}\,. \tag{2.22}$$

From (2.22), the second Piola–Kirchhoff stress tensor has the following representation

$$\mathbf{S} = 2W_1\mathbf{G}^\sharp + 2W_2\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + 2W_3 I_3\mathbf{C}^{-1} + 2W_4\,(\mathbf{N}\otimes\mathbf{N}) + 2W_5\,[\mathbf{N}\otimes(\mathbf{C}\cdot\mathbf{N}) + (\mathbf{C}\cdot\mathbf{N})\otimes\mathbf{N}]\,. \tag{2.23}$$

The Cauchy stress tensor has the following component representation (Ericksen and Rivlin, 1954; Golgoon and Yavari, 2018a,b)

$$\sigma^{ab} = \frac{2}{\sqrt{I_3}}\left[W_1 b^{ab} + (I_2W_2 + I_3W_3)g^{ab} - I_3W_2\,c^{ab} + W_4\,n^a n^b + W_5\,\ell^{ab}\right]\,, \tag{2.24}$$

where $n^a = F^a{}_A N^A$, and $\ell^{ab} = n^a b^{bc}\, n_c + n^b b^{ac}\, n_c$. For an incompressible transversely isotropic solid ($I_3 = 1$), $W = W\left(I_1, I_2, I_4, I_5\right)$. Thus

$$\mathbf{S} = -p\mathbf{C}^{-1} + 2W_1\mathbf{G}^\sharp + 2W_2\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + 2W_4\,(\mathbf{N}\otimes\mathbf{N}) + 2W_5\,[\mathbf{N}\otimes(\mathbf{C}\cdot\mathbf{N}) + (\mathbf{C}\cdot\mathbf{N})\otimes\mathbf{N}]\,. \tag{2.25}$$

The Cauchy stress tensor is represented in components as (Ericksen and Rivlin, 1954; Spencer, 1986; Golgoon and Yavari, 2018a,b)

$$\sigma^{ab} = -pg^{ab} + 2W_1 b^{ab} - 2W_2\,c^{ab} + 2W_4\,n^a n^b + 2W_5(n^a b^{bc} n^d g_{cd} + n^b b^{ac} n^d g_{cd})\,. \tag{2.26}$$

Orthotropic solids. Orthotropic solids at every point have reflection symmetry with respect to three mutually perpendicular planes. Suppose that three $\mathbf{G}$-orthonormal vectors $\mathbf{N}_1(\mathbf{X})$, $\mathbf{N}_2(\mathbf{X})$, and $\mathbf{N}_3(\mathbf{X})$ specify the orthotropic axes in the reference configuration at a point $\mathbf{X}$. One choice of structural tensors is $\mathbf{A}_1 = \mathbf{N}_1\otimes\mathbf{N}_1$, $\mathbf{A}_2 = \mathbf{N}_2\otimes\mathbf{N}_2$, and $\mathbf{A}_3 = \mathbf{N}_3\otimes\mathbf{N}_3$. However, only two of them are independent as $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I}$. Thus, the energy function has the functional form $W = W(\mathbf{G}, \mathbf{C}^\flat, \mathbf{A}_1, \mathbf{A}_2)$ (Doyle and Ericksen, 1956; Spencer, 1982; Lu and Papadopoulos, 2000) and is represented in terms of the following seven independent invariants

$$I_1 = \operatorname{tr}\mathbf{C}\,, \quad I_2 = \det\mathbf{C}\operatorname{tr}\ \mathbf{C}^{-1}\,, \quad I_3 = \det\mathbf{C}\,, \quad I_4 = \mathbf{N}_1\cdot\mathbf{C}\cdot\mathbf{N}_1\,, \quad I_5 = \mathbf{N}_1\cdot\mathbf{C}^2\cdot\mathbf{N}_1\,, \quad I_6 = \mathbf{N}_2\cdot\mathbf{C}\cdot\mathbf{N}_2\,, \quad I_7 = \mathbf{N}_2\cdot\mathbf{C}^2\cdot\mathbf{N}_2\,. \tag{2.27}$$

Thus

$$\mathbf{S} = \sum_{j=1}^{7} 2W_j \frac{\partial I_j}{\partial \mathbf{C}^\flat}\,, \qquad W_j := \frac{\partial W}{\partial I_j}\,, \quad j = 1,\dots,7\,. \tag{2.28}$$

The second Piola–Kirchhoff stress tensor has the following representation

$$\begin{aligned}\mathbf{S} =& 2W_1\mathbf{G}^\sharp + 2W_2\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + 2W_3 I_3\mathbf{C}^{-1} + 2W_4\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + 2W_5\left[\mathbf{N}_1\otimes(\mathbf{C}\cdot\mathbf{N}_1) + (\mathbf{C}\cdot\mathbf{N}_1)\otimes\mathbf{N}_1\right]\\ &+ 2W_6\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + 2W_7\left[\mathbf{N}_2\otimes(\mathbf{C}\cdot\mathbf{N}_2) + (\mathbf{C}\cdot\mathbf{N}_2)\otimes\mathbf{N}_2\right]\,.\end{aligned} \tag{2.29}$$

The Cauchy stress tensor is represented in component form as (Smith and Rivlin, 1958; Spencer, 1986; Golgoon and Yavari, 2018a,b)

$$\begin{aligned}\sigma^{ab} = \frac{2}{\sqrt{I_3}}\Big[\ & W_1 b^{ab} + (I_2W_2 + I_3W_3)g^{ab} - I_3W_2\,c^{ab} + W_4\,n_1^a n_1^b + W_5\left(n_1^a b^{bc} n_1^d g_{cd} + n_1^b b^{ac} n_1^d g_{cd}\right)\\ &+ W_6\,n_2^a n_2^b + W_7\left(n_2^a b^{bc} n_2^d g_{cd} + n_2^b b^{ac} n_2^d g_{cd}\right)\Big]\,,\end{aligned} \tag{2.30}$$

where $n_1^a = F^a{}_A N_1^A$, and $n_2^a = F^a{}_A N_2^A$.

For incompressible orthotropic solids ($I_3 = 1$), $W = W\left(I_1, I_2, I_4, I_5, I_6, I_7\right)$. Therefore, using (2.29), one obtains the following representation

$$\begin{aligned}\mathbf{S} = -p\mathbf{C}^{-1} + 2W_1\mathbf{G}^\sharp + 2W_2\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + 2W_4\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + 2W_5\left[\mathbf{N}_1\otimes(\mathbf{C}\cdot\mathbf{N}_1) + (\mathbf{C}\cdot\mathbf{N}_1)\otimes\mathbf{N}_1\right] \\ + 2W_6\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + 2W_7\left[\mathbf{N}_2\otimes(\mathbf{C}\cdot\mathbf{N}_2) + (\mathbf{C}\cdot\mathbf{N}_2)\otimes\mathbf{N}_2\right] .\end{aligned} \tag{2.31}$$

Similarly, the Cauchy stress tensor is given as

$$\sigma^{ab} = -pg^{ab} + 2W_1 b^{ab} - 2W_2\, c^{ab} + 2W_4\, n_1^a\, n_1^b + 2W_5\, \ell_1^{ab} + 2W_6\, n_2^a\, n_2^b + 2W_7\, \ell_2^{ab}\,, \tag{2.32}$$

where $\ell_1^{ab} = n_1^a\, b^{bc}\, n_1^d\, g_{cd} + n_1^b\, b^{ac}\, n_1^d\, g_{cd}$, and $\ell_2^{ab} = n_2^a\, b^{bc}\, n_2^d\, g_{cd} + n_2^b\, b^{ac}\, n_2^d\, g_{cd}$.

Monoclinic solids. An example of a transversely isotropic solid is a composite that is made of an isotropic matrix reinforced by a single family of aligned fibers (Spencer, 1986). At the macroscopic scale fibers are the integral curves of the vector field **N**. Similarly, an orthotropic solid can be visualized as an isotropic matrix reinforced by two orthogonal families of fibers. For a monoclinic solid $\mathbf{N}_1\cdot\mathbf{N}_2 \neq 0$ but $\mathbf{N}_3$ is still normal to the plane of $\mathbf{N}_1$ and $\mathbf{N}_2$ (Merodio and Ogden, 2020). For such solids, the energy function depends on nine invariants (Spencer, 1986). Seven of them are identical to the orthotropic invariants (2.27). The two extra invariants are

$$I_8 = g\,\mathbf{N}_1\cdot\mathbf{C}\cdot\mathbf{N}_2, \quad I_9 = g^2\,, \tag{2.33}$$

where $g = \mathbf{N}_1\cdot\mathbf{N}_2$. The term g is included in the expression of I_8 to ensure that I_8 is invariant under both transformations $\mathbf{N}_1 \to -\mathbf{N}_1$, and $\mathbf{N}_2 \to -\mathbf{N}_2$. Note that

$$\frac{\partial I_8}{\partial \mathbf{C}^\flat} = \frac{g}{2}\left(\mathbf{N}_1\otimes\mathbf{N}_2 + \mathbf{N}_2\otimes\mathbf{N}_1\right)\,, \quad \frac{\partial I_9}{\partial \mathbf{C}^\flat} = \mathbf{0}\,. \tag{2.34}$$

From $W = W(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9)$, one obtains

$$\begin{aligned}\mathbf{S} =& 2W_1\mathbf{G}^\sharp + 2W_2\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + 2W_3 I_3\mathbf{C}^{-1} + 2W_4\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + 2W_5\left[\mathbf{N}_1\otimes(\mathbf{C}\cdot\mathbf{N}_1) + (\mathbf{C}\cdot\mathbf{N}_1)\otimes\mathbf{N}_1\right] \\ &+ 2W_6\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + 2W_7\left[\mathbf{N}_2\otimes(\mathbf{C}\cdot\mathbf{N}_2) + (\mathbf{C}\cdot\mathbf{N}_2)\otimes\mathbf{N}_2\right] + gW_8\left(\mathbf{N}_1\otimes\mathbf{N}_2 + \mathbf{N}_2\otimes\mathbf{N}_1\right) .\end{aligned} \tag{2.35}$$

The Cauchy stress has the component representation

$$\begin{aligned}\sigma^{ab} =& \frac{2}{\sqrt{I_3}}\Big[\, W_1 b^{ab} + (I_2W_2 + I_3W_3)g^{ab} - I_3W_2\, c^{ab} + W_4\, n_1^a n_1^b + W_5\left(n_1^a\, b^{bc}\, n_1^d\, g_{cd} + n_1^b\, b^{ac}\, n_1^d\, g_{cd}\right) \\ &+ W_6\, n_2^a n_2^b + W_7\left(n_2^a\, b^{bc}\, n_2^d\, g_{cd} + n_2^b\, b^{ac}\, n_2^d\, g_{cd}\right) + gW_8\left(n_1^a\, n_2^b + n_1^b\, n_2^a\right)\Big] .\end{aligned} \tag{2.36}$$

For incompressible monoclinic solids ($I_3 = 1$), $W = W\left(I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9\right)$. Therefore, using (2.29), one obtains the following representation

$$\begin{aligned}\mathbf{S} = -p\mathbf{C}^{-1} + 2W_1\mathbf{G}^\sharp + 2W_2\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + 2W_4\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + 2W_5\left[\mathbf{N}_1\otimes(\mathbf{C}\cdot\mathbf{N}_1) + (\mathbf{C}\cdot\mathbf{N}_1)\otimes\mathbf{N}_1\right] \\ + 2W_6\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + 2W_7\left[\mathbf{N}_2\otimes(\mathbf{C}\cdot\mathbf{N}_2) + (\mathbf{C}\cdot\mathbf{N}_2)\otimes\mathbf{N}_2\right] + gW_8\left(\mathbf{N}_1\otimes\mathbf{N}_2 + \mathbf{N}_2\otimes\mathbf{N}_1\right) .\end{aligned} \tag{2.37}$$

Similarly, the Cauchy stress tensor is given as

$$\sigma^{ab} = -pg^{ab} + 2W_1 b^{ab} - 2I_3W_2\, c^{ab} + 2W_4\, n_1^a\, n_1^b + 2W_5\, \ell_1^{ab} + 2W_6\, n_2^a\, n_2^b + 2W_7\, \ell_2^{ab} + W_8\, \ell_3^{ab}\,, \tag{2.38}$$

where $\ell_3^{ab} = g(n_1^a\, n_2^b + n_1^b\, n_2^a)$.

Remark 2.1. In many references (Merodio and Ogden, 2006; Vergori et al., 2013) the dependence of the energy function on I_9 is ignored since from $(2.34)_2$ it does not enter the expression of stress. However, in finding the universality constraints one cannot ignore this dependence as we will see in Section 6.

3. Compressible anisotropic solids

Transversely isotropic solids. Let us consider a body made of a compressible transversely isotropic solid. At this point we do not specify the material preferred direction **N**. In the absence of body forces, the equilibrium equations are $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$, and in components $\sigma^{ab}{}_{|b} = \sigma^{ab}{}_{,b} + \gamma^a{}_{bc}\sigma^{cb} + \gamma^b{}_{bc}\sigma^{ac} = 0$, where $\gamma^a{}_{bc} = \frac{1}{2}g^{ak}\left(g_{kb,c} + g_{kc,b} - g_{bc,k}\right)$ are the Christoffel symbols of the Levi-Civita connection associated with the metric $\mathbf{g}$. It is convenient to use Cartesian coordinates in the ambient space, and hence, $\sigma^{ab}{}_{,b} = 0$. The Cauchy stress has the representation (2.24). Substituting (2.24) into the equilibrium equations one obtains

$$\begin{aligned} & - I_3^{-\frac{3}{2}} I_{3,b} \left[W_1 b^{ab} + (I_2 W_2 + I_3 W_3)\delta^{ab} - I_3 W_2\, c^{ab} + W_4\, n^a n^b + W_5 \ell^{ab} \right] \\ & + 2 I_3^{-\frac{1}{2}} \Big[(I_{2,b} W_2 + I_2 W_{2,b} + I_{3,b} W_3 + I_3 W_{3,b})\delta^{ab} + W_1 b^{ab}{}_{,b} + W_{1,b} b^{ab} - I_{3,b} W_2\, c^{ab} - I_3 W_{2,b}\, c^{ab} - I_3 W_2\, c^{ab}{}_{,b} \\ & \qquad + W_{4,b}\, n^a\, n^b + W_4\, n^a{}_{,b}\, n^b + W_4\, n^a\, n^b{}_{,b} + W_{5,b}\, \ell^{ab} + W_5\, \ell^{ab}{}_{,b} \Big] = 0\,. \end{aligned} \tag{3.1}$$

This should hold for an arbitrary energy function. As W is an arbitrary function of its arguments, the coefficient of W_1, W_2, W_3, W_4, and W_5 must vanish separately. Therefore

$$\begin{aligned} W_1 :&\quad b^{ab}{}_{,b} = 0\,, \\ W_2 :&\quad I_{2,b}\delta^{ab} - I_3 c^{ab}{}_{,b} = 0\,, \\ W_3 :&\quad I_{3,b} = 0\,, \\ W_4 :&\quad (n^a n^b)_{,b} = 0\,, \\ W_5 :&\quad \ell^{ab}{}_{,b} = 0\,. \end{aligned} \tag{3.2}$$

Hence, (3.1) is simplified to read

$$b^{ab} W_{1,b} + (I_2 \delta^{ab} - I_3\, c^{ab}) W_{2,b} + I_3 \delta^{ab} W_{3,b} + n^a\, n^b\, W_{4,b} + \ell^{ab}\, W_{5,b} = 0\,. \tag{3.3}$$

Note that ($I_{3,b} = 0$)

$$\begin{aligned} W_{1,b} &= W_{11} I_{1,b} + W_{12} I_{2,b} + W_{14} I_{4,b} + W_{15} I_{5,b}\,, \\ W_{2,b} &= W_{12} I_{1,b} + W_{22} I_{2,b} + W_{24} I_{4,b} + W_{25} I_{5,b}\,, \\ W_{3,b} &= W_{13} I_{1,b} + W_{23} I_{2,b} + W_{34} I_{4,b} + W_{35} I_{5,b}\,, \\ W_{4,b} &= W_{14} I_{1,b} + W_{24} I_{2,b} + W_{44} I_{4,b} + W_{45} I_{5,b}\,, \\ W_{5,b} &= W_{15} I_{1,b} + W_{25} I_{2,b} + W_{45} I_{4,b} + W_{55} I_{5,b}\,, \end{aligned} \tag{3.4}$$

where $W_{ij} = \frac{\partial^2 W}{\partial I_i \partial I_j}$. Substituting the above relations into (3.3) the coefficients of W_{13} and W_{23} read

$$\begin{aligned} W_{13} :&\quad I_3 I_{1,b} \delta^{ab} = 0\,, \\ W_{23} :&\quad I_3 I_{2,b} \delta^{ab} = 0\,. \end{aligned} \tag{3.5}$$

Thus, $I_{1,b} = I_{2,b} = 0$. Substituting these into (3.4) and using (3.3) the coefficients of W_{34} and W_{35} read

$$\begin{aligned} W_{34} :&\quad I_3 I_{4,b} \delta^{ab} = 0\,, \\ W_{35} :&\quad I_3 I_{5,b} \delta^{ab} = 0\,. \end{aligned} \tag{3.6}$$

Therefore, $I_{4,b} = I_{5,b} = 0$. In summary, we have the following universality constraints

$$I_1,\ I_2,\ \text{and } I_3 \text{ are constant}, \tag{3.7}$$

$$b^{ab}{}_{,b} = c^{ab}{}_{,b} = 0\,, \tag{3.8}$$

$$I_4,\ \text{and } I_5 \text{ are constant}, \tag{3.9}$$

$$(n^a n^b)_{,b} = \ell^{ab}{}_{,b} = 0\,. \tag{3.10}$$

Note that (3.7) and (3.8) are the universality constraints for isotropic solids (Ericksen, 1955; Yavari and Goriely, 2016) and imply that $F^a{}_{A|B} = 0$, i.e., homogeneous deformations. Note that $I_{4,b} = I_{4,A} (F^{-1})^A{}_b = 0$, and hence $I_{4,A} = 0$. Similarly, $I_{5,A} = 0$.

Suppose $\mathbf{C}^\flat$ has eigenvalues $\lambda_1^2 \geq \lambda_2^2 \geq \lambda_3^2$. Let us consider a homogeneous deformation for which the eigenvalues are distinct, i.e., $\lambda_1^2 > \lambda_2^2 > \lambda_3^2$, and choose a Cartesian coordinate system $\{X^A\}$ for the reference configuration whose axes are the principal directions of $\mathbf{C}^\flat$. With respect to this coordinate system $\mathbf{N}$ has components N^A. Knowing that $\mathbf{N}$ is a unit vector we have

$$\left(N^1\right)^2 + \left(N^2\right)^2 + \left(N^3\right)^2 = 1\,. \tag{3.11}$$

The constraint $I_4 = \alpha^2$ reads

$$\lambda_1^2 \left(N^1\right)^2 + \lambda_2^2 \left(N^2\right)^2 + \lambda_3^2 \left(N^3\right)^2 = \alpha^2\,. \tag{3.12}$$

Similarly, the constraint $I_5 = \beta^2$ reads

$$\lambda_1^4 \left(N^1\right)^2 + \lambda_2^4 \left(N^2\right)^2 + \lambda_3^4 \left(N^3\right)^2 = \beta^2\,, \tag{3.13}$$

where α and β are constants. These three constraints can be written as a system of linear equations for dN^A:

$$\begin{cases} N^1dN^1 + N^2dN^2 + N^3dN^3 & = 0\,, \\ \lambda_1^2N^1dN^1 + \lambda_2^2N^2dN^2 + \lambda_3^2N^3dN^3 & = 0\,, \\ \lambda_1^4N^1dN^1 + \lambda_2^4N^2dN^2 + \lambda_3^4N^3dN^3 & = 0\,. \end{cases} \tag{3.14}$$

The determinant of this linear system is $N^1N^2N^3(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)$. If $N^1N^2N^3 \neq 0$, then $d\mathbf{N}=\mathbf{0}$, and hence $\mathbf{N}$ is a constant unit vector. Suppose $N^3=0$ ($N^1=0$ or $N^2=0$ would be similar). Thus

$$\begin{cases} N^1dN^1 + N^2dN^2 & = 0\,, \\ \lambda_1^2N^1dN^1 + \lambda_2^2N^2dN^2 & = 0\,, \\ \lambda_1^4N^1dN^1 + \lambda_2^4N^2dN^2 & = 0\,. \end{cases} \tag{3.15}$$

Using the first equation, the second and third equations are simplified to read $(\lambda_1^2-\lambda_2^2)N^1dN^1=0$, and $(\lambda_1^4-\lambda_2^4)N^1dN^1=0$, respectively. Thus, $N^1dN^1=0$. If $N^1=0$, then $\left(N^2\right)^2=1$, and hence $\mathbf{N}$ is a constant unit vector. If $dN^1=0$, then $N^2dN^2=0$. If $N^2=0$, then $(N^1)^2=1$, and hence $\mathbf{N}$ is a constant unit vector. If $dN^2=0$, then $\mathbf{N}$ is a constant unit vector. Therefore, we conclude that $\mathbf{N}$ is a constant unit vector.

There are two more universality constraints (3.10) that need to be checked. Note that

$$(n^an^b)_{,b} = \left(F^a{}_AF^b{}_BN^AN^B\right)_{,C}(F^{-1})^C{}_b\,, \tag{3.16}$$

which trivially vanishes for homogeneous deformations and constant $\mathbf{N}$. Similarly

$$\ell^{ab}{}_{,b} = \ell^{ab}{}_{,M}(F^{-1})^M{}_b\,, \tag{3.17}$$

and

$$\ell^{ab} = N^AN^DC_{CD}\delta^{BC}(F^a{}_AF^b{}_B + F^b{}_AF^a{}_B)\,. \tag{3.18}$$

For homogeneous deformations and constant $\mathbf{N}$, $\ell^{ab}{}_{,M}=0$, and hence $\ell^{ab}{}_{,b}=0$ is trivially satisfied. In summary, we have proved the following proposition.

Proposition 3.1. *For compressible nonlinear transversely isotropic solids the only universal deformations are homogeneous deformations, and the anisotropy must be homogeneous, i.e., the material preferred direction is everywhere the same constant unit vector* $\mathbf{N}$.

Orthotropic solids. Using a similar argument, the universality constraints coming from the equilibrium equations for arbitrary compressible orthotropic solids are

$$I_1,\ I_2,\ \text{and}\ I_3\ \text{are constant}\,, \tag{3.19}$$

$$b^{ab}{}_{,b} = c^{ab}{}_{,b} = 0\,, \tag{3.20}$$

$$I_4,\ \text{and}\ I_5\ \text{are constant}\,, \tag{3.21}$$

$$(n_1^a\,n_1^b)_{,b} = \ell_1^{ab}{}_{,b} = 0\,, \tag{3.22}$$

$$I_6,\ \text{and}\ I_7\ \text{are constant}\,, \tag{3.23}$$

$$(n_2^a\,n_2^b)_{,b} = \ell_2^{ab}{}_{,b} = 0\,. \tag{3.24}$$

The first two universality constraints imply that universal deformations must be homogeneous and the remaining universality constraints force the material preferred directions to be uniform.

Proposition 3.2. *For compressible nonlinear orthotropic solids the only universal deformations are homogeneous deformations, and the anisotropy must be homogeneous, i.e., the material preferred directions are everywhere the same three mutually orthogonal constant unit vectors* $\mathbf{N}_1, \mathbf{N}_2$*, and* $\mathbf{N}_3$.

Monoclinic solids. In deriving the constraints (3.19)–(3.24) orthogonality of the material preferred directions was not used. This means that the same universality constraints must hold for monoclinic solids as well. In addition to (3.19)–(3.24), one has the following extra universality constraints:

$$I_8,\ \text{and}\ I_9\ \text{are constant}\,, \tag{3.25}$$

$$(n_3^a\,n_3^b)_{,b} = \ell_3^{ab}{}_{,b} = 0\,. \tag{3.26}$$

Therefore, universal deformations are homogeneous and $\mathbf{N}_1, \mathbf{N}_2$, and $\mathbf{N}_3$ are constant unit vectors. This in turn implies that I_8 and I_9 are constant, and (3.25), (3.26) are trivially satisfied. Hence, equilibrium equations hold for arbitrary monoclinic energy functions. Therefore, in Proposition 3.2 "orthotropic" can be replaced by "monoclinic".

4. Incompressible transversely isotropic elastic solids

In the absence of body forces, and using (2.26), the equilibrium equations read

$$\frac{1}{2}p_{,b}\, g^{ab} = \left[W_1 b^{ab} - W_2\, c^{ab} + W_4\, n^a n^b + W_5\, \ell^{ab}\right]_{|b} \,. \tag{4.1}$$

Or

$$\frac{1}{2}p_{,a} = g_{am}\left[W_1 b^{mn} - W_2\, c^{mn} + W_4\, n^m n^n + W_5\, \ell^{mn}\right]_{|n} \,. \tag{4.2}$$

Thus

$$\frac{1}{2}\boldsymbol{d}p = \frac{1}{2}p_{,a}dx^a = g_{am}\left[W_1 b^{mn} - W_2\, c^{mn} + W_4\, n^m n^n + W_5\, \ell^{mn}\right]_{|n} dx^a \,, \tag{4.3}$$

where $\boldsymbol{d}$ is the exterior derivative. In other words, $\boldsymbol{\xi} = g_{am}\left[W_1 b^{mn} - W_2\, c^{mn} + W_4\, n^m n^n + W_5\, \ell^{mn}\right]_{|n} dx^a$ is an exact 1-form. A necessary condition for $\boldsymbol{\xi}$ to be an exact form is that $\boldsymbol{d\xi} = \boldsymbol{0}$ (Yavari, 2013). This is equivalent to $\xi_{a,b} = \xi_{b,a}$. But note that $\xi_{a|b} = \xi_{a,b} - \gamma^c{}_{ab}\,\xi_c$. Therefore, $\xi_{a,b} = \xi_{b,a}$ is equivalent to $\xi_{a|b} = \xi_{b|a}$, which is more convenient to use in curvilinear coordinates as the metric of the ambient space is covariantly constant, i.e., $g_{ab|c} = 0$. Thus, the universality constraints read

$$g_{am}\left[W_1 b^{mn} - W_2\, c^{mn} + W_4\, n^m n^n + W_5\, \ell^{mn}\right]_{|nb} = g_{bm}\left[W_1 b^{mn} - W_2\, c^{mn} + W_4\, n^m n^n + W_5\, \ell^{mn}\right]_{|na} \,. \tag{4.4}$$

One can write

$$\begin{aligned}\xi_{a|b} = g_{am}\Big(& W_1 b^{mn}{}_{|nb} - W_2\, c^{mn}{}_{|nb} + W_4\,(n^m n^n)_{|nb} + W_5\,\ell^{mn}{}_{|nb} + W_{1,n} b^{mn}{}_{|b} - W_{2,n}\, c^{mn}{}_{|b} + W_{4,n}\,(n^m n^n)_{|b} + W_{5,n}\,\ell^{mn}{}_{|b} \\ & + W_{1,b} b^{mn}{}_{|n} - W_{2,b}\, c^{mn}{}_{|n} + W_{4,b}\,(n^m n^n)_{|n} + W_{5,b}\,\ell^{mn}{}_{|n} + W_{1|nb} b^{mn} - W_{2|nb}\, c^{mn} + W_{4|nb}\, n^m n^n + W_{5|nb}\,\ell^{mn} \Big) \,.\end{aligned} \tag{4.5}$$

Note that $W_i = W_i(I_1, I_2, I_4, I_5)$, $i = 1, 2, 4, 5$, and hence

$$\begin{aligned} W_{1,b} &= W_{11} I_{1,b} + W_{12} I_{2,b} + W_{14} I_{4,b} + W_{15} I_{5,b} \,, \\ W_{2,b} &= W_{12} I_{1,b} + W_{22} I_{2,b} + W_{24} I_{4,b} + W_{25} I_{5,b} \,, \\ W_{4,b} &= W_{14} I_{1,b} + W_{24} I_{2,b} + W_{44} I_{4,b} + W_{45} I_{5,b} \,, \\ W_{5,b} &= W_{15} I_{1,b} + W_{25} I_{2,b} + W_{45} I_{4,b} + W_{55} I_{5,b} \,. \end{aligned} \tag{4.6}$$

Note also that

$$W_{1|bn} = W_{11} I_{1|bn} + W_{12} I_{2|bn} + W_{14} I_{4|bn} + W_{15} I_{5|bn} + W_{11,n} I_{1,b} + W_{12,n} I_{2,b} + W_{14,n} I_{4,b} + W_{15,n} I_{5,b} \,. \tag{4.7}$$

Denoting the independent third order derivatives of the energy function by $W_{ijk} = \frac{\partial^3 W}{\partial I_i \partial I_j \partial I_k}$, $(i \leq j \leq k)$, we have

$$\begin{aligned} W_{11,n} &= W_{111} I_{1,n} + W_{112} I_{2,n} + W_{114} I_{4,n} + W_{115} I_{5,n} \,, \\ W_{12,n} &= W_{112} I_{1,n} + W_{122} I_{2,n} + W_{124} I_{4,n} + W_{125} I_{5,n} \,, \\ W_{14,n} &= W_{114} I_{1,n} + W_{124} I_{2,n} + W_{144} I_{4,n} + W_{145} I_{5,n} \,, \\ W_{15,n} &= W_{115} I_{1,n} + W_{125} I_{2,n} + W_{145} I_{4,n} + W_{155} I_{5,n} \,. \end{aligned} \tag{4.8}$$

Therefore,

$$\begin{aligned} W_{1|bn} = {} & W_{11} I_{1|bn} + W_{12} I_{2|bn} + W_{14} I_{4|bn} + W_{15} I_{5|bn} \\ & + W_{111} I_{1,n} I_{1,b} + W_{112}(I_{2,n} I_{1,b} + I_{1,n} I_{2,b}) + W_{114}(I_{4,n} I_{1,b} + I_{1,n} I_{4,b}) \\ & + W_{115}(I_{5,n} I_{1,b} + I_{1,n} I_{5,b}) + W_{122} I_{2,n} I_{2,b} + W_{124}(I_{4,n} I_{2,b} + I_{4,b} I_{2,n}) \\ & + W_{125}(I_{5,n} I_{2,b} + I_{2,n} I_{5,b}) + W_{144} I_{4,n} I_{4,b} + W_{145}(I_{5,n} I_{4,b} + I_{5,b} I_{4,n}) \\ & + W_{155} I_{5,n} I_{5,b} \,. \end{aligned} \tag{4.9}$$

Similarly,

$$\begin{aligned} W_{2|bn} = {} & W_{12} I_{1|bn} + W_{22} I_{2|bn} + W_{24} I_{4|bn} + W_{25} I_{5|bn} \\ & + W_{112} I_{1,n} I_{1,b} + W_{122}(I_{2,n} I_{1,b} + I_{1,n} I_{2,b}) + W_{222} I_{2,n} I_{2,b} \\ & + W_{244} I_{4,n} I_{4,b} + W_{255} I_{5,n} I_{5,b} + W_{124}(I_{4,n} I_{1,b} + I_{4,b} I_{1,n}) \\ & + W_{125}(I_{5,n} I_{1,b} + I_{1,n} I_{5,b}) + W_{224}(I_{4,n} I_{2,b} + I_{4,b} I_{2,n}) \\ & + W_{225}(I_{5,n} I_{2,b} + I_{5,b} I_{2,n}) + W_{245}(I_{5,n} I_{4,b} + I_{5,b} I_{4,n}) \,, \end{aligned} \tag{4.10}$$

$$\begin{aligned} W_{4|bn} = {} & W_{14} I_{1|bn} + W_{24} I_{2|bn} + W_{44} I_{4|bn} + W_{45} I_{5|bn} \\ & + W_{114} I_{1,n} I_{1,b} + W_{224} I_{2,n} I_{2,b} + W_{444} I_{4,n} I_{4,b} + W_{455} I_{5,n} I_{5,b} \\ & + W_{124}(I_{2,n} I_{1,b} + I_{1,n} I_{2,b}) + W_{144}(I_{4,n} I_{1,b} + I_{4,b} I_{1,n}) \\ & + W_{244}(I_{4,n} I_{2,b} + I_{2,n} I_{4,b}) + W_{145}(I_{5,n} I_{1,b} + I_{5,b} I_{1,n}) \\ & + W_{245}(I_{5,n} I_{2,b} + I_{2,n} I_{5,b}) + W_{445}(I_{5,n} I_{4,b} + I_{5,b} I_{4,n}) \,, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} W_{5|bn} &= W_{15} I_{1|bn} + W_{25} I_{2|bn} + W_{45} I_{4|bn} + W_{55} I_{5|bn} \\ &+ W_{115} I_{1,n} I_{1,b} + W_{225} I_{2,n} I_{2,b} + W_{445} I_{4,n} I_{4,b} + W_{555} I_{5,n} I_{5,b} \\ &+ W_{125}(I_{2,n} I_{1,b} + I_{1,n} I_{2,b}) + W_{145}(I_{4,n} I_{1,b} + I_{4,b} I_{2,1}) \\ &+ W_{155}(I_{5,n} I_{1,b} + I_{1,n} I_{5,b}) + W_{245}(I_{4,n} I_{2,b} + I_{4,b} I_{2,n}) \\ &+ W_{255}(I_{5,n} I_{2,b} + I_{2,n} I_{5,b}) + W_{455}(I_{5,n} I_{4,b} + I_{5,b} I_{4,n}) . \end{aligned} \tag{4.12}$$

For $\xi_{a|b} = \xi_{b|a}$ to hold the coefficient of each partial derivative must be symmetric. We define $\mathcal{A}^{\kappa}_{ab}$ as the matrix of coefficient of W_{κ}, where κ is a multi-index. For isotropic solids there are 9 terms: $\kappa \in \mathcal{K}_{\text{iso}} = \{1, 2, 11, 22, 12, 111, 222, 112, 122\}$. In the case of transversely isotropic solids there are 25 extra terms:

$$\mathcal{K} = \{4, 5, 44, 55, 14, 15, 24, 25, 45, 444, 555, 114, 115, 124, 125, 144, 145, 155, 224, 225, 244, 245, 255, 445, 455\} . \tag{4.13}$$

Each matrix provides 3 conditions so that there are, in total, 102 equations for the 8 unknowns given by the 6 components of the Finger tensor $\mathbf{b}^{\sharp}$ and the 2 independent components of the unit vector $\mathbf{N}$. A deformation φ is universal with universal material preferred direction $\mathbf{N}$ if and only if $\mathcal{A}^{\kappa}_{ab}$ is symmetric for all $\kappa \in \mathcal{K} \cup \mathcal{K}_{\text{iso}}$.

The analysis of this problem is greatly simplified by first considering the coefficients of the 9 terms that appear in the isotropic case as well, which are (Ericksen, 1954):

$$\begin{aligned} \mathcal{A}^{1}_{ab} &= b^{n}_{a|bn} , \\ \mathcal{A}^{2}_{ab} &= -c^{n}_{a|bn} , \\ \mathcal{A}^{11}_{ab} &= b^{n}_{a|n} I_{1,b} + \left(b^{n}_{a} I_{1,n}\right)_{|b} , \\ \mathcal{A}^{22}_{ab} &= -c^{n}_{a|n} I_{2,b} - \left(c^{n}_{a} I_{2,n}\right)_{|b} , \\ \mathcal{A}^{12}_{ab} &= \left(b^{n}_{a} I_{2,n}\right)_{|b} + b^{n}_{a|n} I_{2,b} - \left[\left(c^{n}_{a} I_{1,n}\right)_{|b} + c^{n}_{a|n} I_{1,b}\right] , \\ \mathcal{A}^{111}_{ab} &= b^{n}_{a} I_{1,n} I_{1,b} , \\ \mathcal{A}^{222}_{ab} &= -c^{n}_{a} I_{2,n} I_{2,b} , \\ \mathcal{A}^{112}_{ab} &= b^{n}_{a} \left(I_{1,b} I_{2,n} + I_{1,n} I_{2,b}\right) - c^{n}_{a} I_{1,n} I_{1,b} , \\ \mathcal{A}^{122}_{ab} &= b^{n}_{a} I_{2,b} I_{2,n} - c^{n}_{a} \left(I_{1,b} I_{2,n} + I_{1,n} I_{2,b}\right) , \end{aligned} \tag{4.14}$$

where $b^{n}_{a} = b^{mn} g_{ma}$, and $c^{n}_{a} = c^{mn} g_{ma}$.[1] Symmetry of the nine terms in Eqs. (4.14), in addition to homogeneous deformations, admit five classes of deformations (Ericksen, 1954; Singh and Pipkin, 1965; Klingbeil and Shield, 1966). In the sequel, we will find the universal preferred material directions for these six families of deformations. The case of constant I_1 and I_2 is still an open problem, for which we will not be able to say anything about the universal preferred material directions other than those of the Family 5 deformations.

For transversely isotropic solids, in addition to symmetry of these 9 terms, the following 25 terms must be symmetric as well. The coefficients of the first-order and second-order derivatives of the energy function are:

$$\begin{aligned} \mathcal{A}^{4}_{ab} &= (n_a n^n)_{|nb} , \\ \mathcal{A}^{5}_{ab} &= \ell^{n}_{a|nb} , \\ \mathcal{A}^{44}_{ab} &= (n_a n^n)_{|n} I_{4,b} + (n_a n^n I_{4,n})_{|b} , \\ \mathcal{A}^{55}_{ab} &= \ell^{n}_{a|n} I_{5,b} + (\ell^{n}_{a} I_{5,n})_{|b} , \\ \mathcal{A}^{14}_{ab} &= b^{n}_{a|n} I_{4,b} + (b^{n}_{a} I_{4,n})_{|b} + (n_a n^n)_{|n} I_{1,b} + (n_a n^n I_{1,n})_{|b} , \\ \mathcal{A}^{15}_{ab} &= b^{n}_{a|n} I_{5,b} + (b^{n}_{a} I_{5,n})_{|b} + \ell^{n}_{a|n} I_{1,b} + (\ell^{n}_{a} I_{1,n})_{|b} , \\ \mathcal{A}^{24}_{ab} &= (n_a n^n)_{|n} I_{2,b} + (n_a n^n I_{2,n})_{|b} - \left[c^{n}_{a|n} I_{4,b} + (c^{n}_{a} I_{4,n})_{|b}\right] , \\ \mathcal{A}^{25}_{ab} &= \ell^{n}_{a|n} I_{2,b} + (\ell^{n}_{a} I_{2,n})_{|b} - \left[c^{n}_{a|n} I_{5,b} + (c^{n}_{a} I_{5,n})_{|b}\right] , \\ \mathcal{A}^{45}_{ab} &= (n_a n^n)_{|n} I_{5,b} + (n_a n^n I_{5,n})_{|b} + \ell^{n}_{a|n} I_{4,b} + (\ell^{n}_{a} I_{4,n})_{|b} . \end{aligned} \tag{4.15}$$

[1] Note that $b^{n}{}_{a} = b^{nm} g_{ma}$, and $b_{a}{}^{n} = g_{am} b^{mn}$, which are equal. Thus, we use the notation $b^{n}_{a} = b^{n}{}_{a} = b_{a}{}^{n}$. Similarly, the same notation is used for $\mathbf{c}$.

The coefficients of the third-order derivatives of the energy function are:

$$\begin{aligned}
\mathcal{A}_{ab}^{444} &= n_a n^n I_{4,n} I_{4,b}\,,\\
\mathcal{A}_{ab}^{555} &= \ell^n_{\ a} I_{5,n} I_{5,b}\,,\\
\mathcal{A}_{ab}^{114} &= b^n_{\ a}\left(I_{4,n} I_{1,b} + I_{4,b} I_{1,n}\right) + n_a n^n I_{1,n} I_{1,b}\,,\\
\mathcal{A}_{ab}^{115} &= b^n_{\ a}\left(I_{5,n} I_{1,b} + I_{5,b} I_{1,n}\right) + \ell^n_{\ a} I_{1,n} I_{1,b}\,,\\
\mathcal{A}_{ab}^{124} &= b^n_{\ a}\left(I_{4,n} I_{2,b} + I_{4,b} I_{2,n}\right) - c^n_{\ a}\left(I_{4,n} I_{1,b} + I_{4,b} I_{1,n}\right) + n_a n^n\left(I_{2,n} I_{1,b} + I_{2,b} I_{1,n}\right)\,,\\
\mathcal{A}_{ab}^{125} &= b^n_{\ a}\left(I_{5,n} I_{2,b} + I_{5,b} I_{2,n}\right) - c^n_{\ a}\left(I_{5,n} I_{1,b} + I_{5,b} I_{1,n}\right) + \ell^n_{\ a}\left(I_{2,n} I_{1,b} + I_{2,b} I_{1,n}\right)\,,\\
\mathcal{A}_{ab}^{144} &= b^n_{\ a} I_{4,n} I_{4,b} + n_a n^n\left(I_{4,n} I_{1,b} + I_{4,b} I_{1,n}\right)\,,\\
\mathcal{A}_{ab}^{145} &= b^n_{\ a}\left(I_{5,n} I_{4,b} + I_{5,b} I_{4,n}\right) + n_a n^n\left(I_{5,n} I_{1,b} + I_{5,b} I_{1,n}\right) + \ell^n_{\ a}\left(I_{4,n} I_{1,b} + I_{4,b} I_{1,n}\right)\,,\\
\mathcal{A}_{ab}^{155} &= b^n_{\ a} I_{5,n} I_{5,b} + \ell^n_{\ a}\left(I_{5,n} I_{1,b} + I_{5,b} I_{1,n}\right)\,,\\
\mathcal{A}_{ab}^{224} &= n_a n^n I_{2,n} I_{2,b} - c^n_{\ a}\left(I_{4,n} I_{2,b} + I_{4,b} I_{2,n}\right)\,,\\
\mathcal{A}_{ab}^{225} &= \ell^n_{\ a} I_{2,n} I_{2,b} - c^n_{\ a}\left(I_{5,n} I_{2,b} + I_{5,b} I_{2,n}\right)\,,\\
\mathcal{A}_{ab}^{244} &= -c^n_{\ a} I_{4,n} I_{4,b} + n_a n^n\left(I_{4,n} I_{2,b} + I_{4,b} I_{2,n}\right)\,,\\
\mathcal{A}_{ab}^{245} &= n_a n^n\left(I_{5,n} I_{2,b} + I_{5,b} I_{2,n}\right) + \ell^n_{\ a}\left(I_{4,n} I_{2,b} + I_{4,b} I_{2,n}\right) - c^n_{\ a}\left(I_{5,n} I_{4,b} + I_{5,b} I_{4,n}\right)\,,\\
\mathcal{A}_{ab}^{255} &= \ell^n_{\ a}\left(I_{5,n} I_{2,b} + I_{5,b} I_{2,n}\right) - c^n_{\ a} I_{5,n} I_{5,b}\,,\\
\mathcal{A}_{ab}^{445} &= n_a n^n\left(I_{5,n} I_{4,b} + I_{5,b} I_{4,n}\right) + \ell^n_{\ a} I_{4,n} I_{4,b}\,,\\
\mathcal{A}_{ab}^{455} &= n_a n^n I_{5,n} I_{5,b} + \ell^n_{\ a}\left(I_{5,n} I_{4,b} + I_{5,b} I_{4,n}\right)\,.
\end{aligned} \tag{4.16}$$

Goodbrake et al. (2020) showed that all the known universal deformations are symmetric with respect to Lie subgroups of the special Euclidean group. In order to find universal eigenstrains corresponding to each family, they assumed that the material metric has the same symmetry as the classical universal deformations do. Note that the symmetry of a universal deformation $\varphi : \mathcal{B} \to \varphi(\mathcal{B}) \subset \mathcal{S}$ is encoded in the symmetry of $\mathbf{C}^\flat = \varphi^*\mathbf{g}$. Here, we use the same strategy and assume that the material preferred direction vector $\mathbf{N}$ has the same symmetries. This symmetry reduction will make the above systems of nonlinear PDEs tractable.

4.1. Family 0: Homogeneous deformations

Homogeneous deformations have the form $x^a(\mathbf{X}) = F^a{}_A X^A + c^a$, where $[F^a{}_A]$ is a constant matrix and c^a are constants. The incompressibility constraint in Cartesian coordinates is written as $\det[F^a{}_A] = 1$. In Cartesian coordinates the right Cauchy–Green tensor has components $C_{AB} = F^a{}_A F^a{}_A\, \delta_{ab}$, which are constants. This means that $\mathbf{C}^\flat$ is invariant under the action of $T(3) \subset SE(3)$ — the group of translations. We assume that $N^A(\mathbf{X})$ are invariant under $T(3)$ as well, or in other words $\mathbf{N}$ is a constant vector. In this case all the universality constraints are satisfied. Therefore, for isochoric homogeneous deformations uniform material preferred directions are universal.

4.2. Family 1: Bending, stretching, and shearing of a rectangular block

With respect to the Cartesian (X, Y, Z) and cylindrical (r, θ, z) coordinates in the reference and current configurations, respectively, this family of deformations have the following representation

$$r(X,Y,Z) = \sqrt{C_1(2X + C_4)}\,, \quad \theta(X,Y,Z) = C_2(Y + C_5)\,, \quad z(X,Y,Z) = \frac{Z}{C_1 C_2} - C_2 C_3 Y + C_6\,. \tag{4.17}$$

Thus

$$[C_{AB}] = \begin{bmatrix} \frac{C_1}{2X+C_4} & 0 & 0 \\ 0 & C_2^2\left[C_1(2X+C_4) + C_3^2\right] & -\frac{C_3}{C_1} \\ 0 & -\frac{C_3}{C_1} & \frac{1}{C_1^2 C_2^2} \end{bmatrix}, \tag{4.18}$$

which is independent of Y and Z, i.e., $\mathbf{C}^\flat$ is invariant under the action of $T(2) \subset SE(3)$. We assume that $\mathbf{N}$ has the same symmetry, i.e.,

$$\mathbf{N}(X,Y,Z) = \begin{bmatrix} N^1(X) \\ N^2(X) \\ N^3(X) \end{bmatrix}, \tag{4.19}$$

such that $\left(N^1(X)\right)^2 + \left(N^2(X)\right)^2 + \left(N^3(X)\right)^2 = 1$.

Symmetry of the coefficients of W_{224} for $(a,b)=(1,2)$ and $(a,b)=(1,3)$ gives[2]

$$\frac{C_1\left[1+C_1^2C_2^4C_3^2-C_2^2(C_4+2X)^2\right]^2}{C_2^3[C_1(C_4+2X)]^{5/2}}N_1(X)N_2(X)=0, \tag{4.20}$$

$$\frac{\sqrt{C_1(C_4+2X)}\left[1+C_1^2C_2^4C_3^2-C_2^2(C_4+2X)^2\right]^2}{C_1^4C_2^5(C_4+2X)^4}N_1(X)\left[C_1C_2^2C_3N_2(X)-\sqrt{1-N_1(X)^2-N_2(X)^2}\right]=0. \tag{4.21}$$

From (4.20) either $N_1(X)=0$, or $N_2(X)=0$. If $N_2(X)=0$, from (4.21), either $N_1(X)=0$ ($N_3(X)=\pm1$), or $N_1(X)=\pm1$ ($N_3(X)=0$). If $N_1(X)=0$, both equations are satisfied. Therefore, we have the following two possibilities:

$$\mathbf{N}=\begin{bmatrix}\pm1\\0\\0\end{bmatrix},\quad \mathbf{N}=\begin{bmatrix}0\\f(X)\\\pm\sqrt{1-f^2(X)}\end{bmatrix},\quad \text{for any } f(X) \text{ such that } f^2(X)\le 1. \tag{4.22}$$

Or equivalently

$$\mathbf{N}=\begin{bmatrix}\pm1\\0\\0\end{bmatrix},\quad \mathbf{N}=\begin{bmatrix}0\\\cos\psi(X)\\\pm\sin\psi(X)\end{bmatrix}, \tag{4.23}$$

for some function $\psi(X)$. These two vector fields satisfy all the other universally constraints. If either I_4 or I_5 (or both) are constant, still symmetry of the coefficients of W_{224} for $(a,b)=(1,2)$ and $(a,b)=(1,3)$ gives (4.20) and (4.21). This means that still (4.23) are solutions. However, for neither solution I_4 or I_5 is constant. Therefore, the only solutions for $\mathbf{N}$ that respect the symmetry of the Family 1 deformations are (4.23).

Remark 4.1. Ericksen and Rivlin (1954) analyzed a special subset of this family ($C_3=0$) and assumed the following two cases

$$\mathbf{N}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{N}=\begin{bmatrix}0\\\cos\zeta\\\sin\zeta\end{bmatrix}, \tag{4.24}$$

where ζ is a constant. Clearly, these are special cases of (4.23).

Remark 4.2. An example of a transversely isotropic solid is a unidirectional fiber composite. One can think of the material preferred direction unit vector $\mathbf{N}(X^1,X^2,X^3)$ as the tangent vector to the fiber at the point (X^1,X^2,X^3) in an isotropic matrix. The solution $(4.23)_1$ corresponds to a uniform distribution of fibers parallel to the X-axis. In the solution $(4.23)_2$ for fixed X fibers are distributed uniformly in the YZ-plane and make an angle $\psi(X)$ with the Y-axis.

4.3. Family 2: Straightening, stretching, and shearing of a sector of a cylindrical shell

With respect to the cylindrical (R,Θ,Z) and Cartesian (x,y,z) coordinates in the reference and current configurations, respectively, this family of deformations have the following representation

$$x(R,\Theta,Z)=\frac{1}{2}C_1C_2^2R^2+C_4,\quad y(R,\Theta,Z)=\frac{\Theta}{C_1C_2}+C_5,\quad z(R,\Theta,Z)=\frac{C_3}{C_1C_2}\Theta+\frac{1}{C_2}Z+C_6. \tag{4.25}$$

Thus

$$[C_{AB}]=\begin{bmatrix}C_1^2C_2^4R^2 & 0 & 0\\ 0 & \frac{C_3^2+1}{C_1^2C_2^2} & \frac{C_3}{C_1C_2^2}\\ 0 & \frac{C_3}{C_1C_2^2} & \frac{1}{C_2^2}\end{bmatrix}, \tag{4.26}$$

which is independent of Θ, and Z. We assume that $\mathbf{N}$ has the same symmetry, i.e.,

$$\mathbf{N}(R,\Theta,Z)=\begin{bmatrix}N^1(R)\\N^2(R)\\N^3(R)\end{bmatrix}, \tag{4.27}$$

such that $\left(N^1(R)\right)^2+R^2\left(N^2(R)\right)^2+\left(N^3(R)\right)^2=1$. Symmetry of the coefficients of W_{224} for $(a,b)=(1,2)$ and $(a,b)=(1,3)$ gives

$$\frac{\left(C_1^4C_2^6R^4-1\right)^2}{C_1^6C_2^{11}R^7}N_1(R)N_2(R)=0, \tag{4.28}$$

[2] Symbolic computations were done with Mathematica Version 12.3.0.0, Wolfram Research, Champaign, IL.

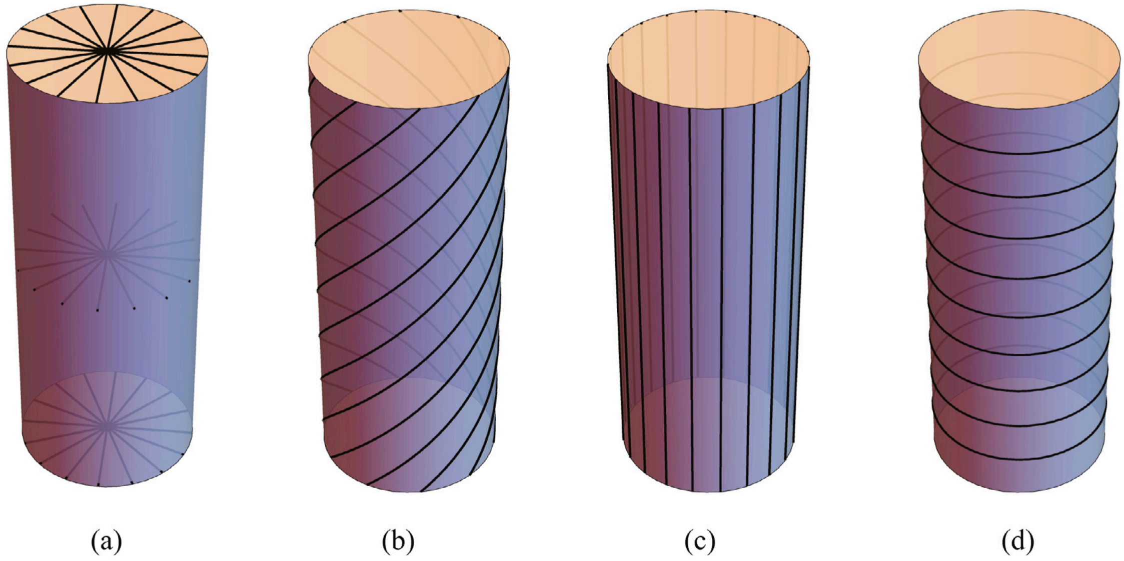

Fig. 1. (a) Universal material preferred directions $(4.32)_1$ for Family 2. (b–d) Universal material preferred directions $(4.32)_2$ for Family 2: (b) $\cos\psi(R) \neq 0, \pm 1$, (c) $\cos\psi(R) = 0$, and (d) $\cos\psi(R) = \pm 1$.

$$\frac{\left(C_1^4 C_2^6 R^4 - 1\right)^2}{C_1^6 C_2^{11} R^7} N_1(R) \left[C_1 \sqrt{1 - N_1(R)^2 - R^2 N_2(R)^2} + C_3 N_2(R)\right] = 0\,. \tag{4.29}$$

From (4.28) either $N_1(R) = 0$, or $N_2(R) = 0$. If $N_2(R) = 0$, from (4.29) either $N_1(R) = 0$ ($N_3(R) = \pm 1$), or $N_1(R) = \pm 1$ ($N_3(R) = 0$). If $N_1(R) = 0$, both equations are satisfied. Therefore, we have the following two possibilities:

$$\mathbf{N} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 \\ f(R)/R \\ \pm\sqrt{1 - f^2(R)} \end{bmatrix}, \quad \text{for any } f(R) \text{ such that } f^2(R) \le 1\,, \tag{4.30}$$

or equivalently

$$\mathbf{N} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 \\ \frac{1}{R}\cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix}, \tag{4.31}$$

for some function $\chi(R)$. Replacing the components of $\mathbf{N}$ with the corresponding physical components and denoting the resulting array by $\hat{\mathbf{N}}$ the two solutions are

$$\hat{\mathbf{N}} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} 0 \\ \cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix}. \tag{4.32}$$

These two vector fields satisfy all the other universality constraints. If either I_4 or I_5 (or both) are constant, then symmetry of the coefficients of W_{224} for $(a,b) = (1,2)$ and $(a,b) = (1,3)$ gives (4.28) and (4.29), and (4.32) are still solutions. However, for neither solution I_4 or I_5 is constant. Therefore, the only solutions for $\mathbf{N}$ that respect the symmetry of the Family 2 deformations are (4.32). In the solution $(4.32)_1$ fibers are distributed radially. The material preferred direction in the solution (4.32) are sketched in Fig. 1.

Remark 4.3. Assuming that the cylindrical shell is made of a unidirectional fiber composite, the solution $(4.32)_1$ corresponds to a uniform radial distribution of fibers. In the solution $(4.32)_2$ for fixed R fibers are arranged helically when $\psi(R) \neq \frac{n\pi}{2}$, $n \in \mathbb{Z}$ (Fig. 1(b)). When $\cos\psi(R) = 0$, fibers are distributed uniformly parallel to the axis of the cylindrical shell (Fig. 1(c)). When $\sin\psi(R) = 0$, fibers are concentric circles parallel to the (R,Θ) plane (Fig. 1(d)). Examples of fiber-reinforced composite with one or two families of helical fibers can be found in biological systems (Goriely and Tabor, 2011), in gels (Demirkoparan and Pence, 2007, 2008, 2015), and in the McKibben actuators (Daerden and Lefeber, 2002; Liu and Rahn, 2003) as described in (Goriely, 2017). Note that, in this universal solution, helical fibers can change orientation as a function of R. However, in the limit of $R \to 0$, one must have $\chi(R) \to \pi/2$ or the vector direction becomes ill-defined.

4.4. Family 3: Inflation, bending, torsion, extension, and shearing of a sector of an annular wedge

With respect to the cylindrical coordinates (R,Θ,Z) and (r,θ,z) in the reference and current configurations, respectively, this family of deformations have the following representation

$$r(R,\Theta,Z)=\sqrt{\frac{R^2}{C_1C_4-C_2C_3}+C_5}\,,\quad \theta(R,\Theta,Z)=C_1\Theta+C_2Z+C_6\,,\quad z(R,\Theta,Z)=C_3\Theta+C_4Z+C_7\,. \tag{4.33}$$

Thus

$$[C_{AB}]=\begin{bmatrix}\frac{R^2}{K(KC_5+R^2)} & 0 & 0\\ 0 & C_3^2+C_1^2\left[\frac{R^2}{K}+C_5\right] & C_1C_2\left[\frac{R^2}{K}+C_5\right]+C_3C_4\\ 0 & C_1C_2\left[\frac{R^2}{K}+C_5\right]+C_3C_4 & C_4^2+C_2^2\left[\frac{R^2}{K}+C_5\right]\end{bmatrix}, \tag{4.34}$$

where $K=C_1C_4-C_2C_3$. Note that $\mathbf{C}^\flat$ only depends on R. We assume that $\mathbf{N}$ has the same symmetry, i.e.,

$$\mathbf{N}(R,\Theta,Z)=\begin{bmatrix}N^1(R)\\ N^2(R)\\ N^3(R)\end{bmatrix}, \tag{4.35}$$

such that $\left(N^1(R)\right)^2+R^2\left(N^2(R)\right)^2+\left(N^3(R)\right)^2=1$. Symmetry of the coefficients of W_{224} for $(a,b)=(1,2)$ gives

$$N_1(R)\left(C_1N_2(R)+C_2\sqrt{1-N_1(R)^2-R^2N_2(R)^2}\right)=0\,. \tag{4.36}$$

This implies that either $N_1(R)=0$, or $C_1N_2(R)+C_2\sqrt{1-N_1(R)^2-R^2N_2(R)^2}=0$. If $N_1(R)=0$, then

$$\mathbf{N}(R,\Theta,Z)=\begin{bmatrix}0\\ f(R)/R\\ \pm\sqrt{1-f^2(R)}\end{bmatrix},\quad \text{for any } f(R) \text{ such that } f^2(R)\le 1, \tag{4.37}$$

or equivalently

$$\mathbf{N}(R,\Theta,Z)=\begin{bmatrix}0\\ \frac{1}{R}\cos\psi(R)\\ \pm\sin\psi(R)\end{bmatrix}, \tag{4.38}$$

for some function $\psi(R)$. One can check that (4.38) satisfies all the other universality constraints. Suppose $C_1N_2(R)+C_2\sqrt{1-N_1(R)^2-R^2N_2(R)^2}=0$, or $C_1N_2(R)+C_2N_3(R)=0$. Symmetry of the coefficients of W_{114} for $(a,b)=(1,3)$ gives $C_3N_2(R)+C_4N_3(R)=0$. Therefore, we have the following system of equations for $N_2(R)$ and $N_3(R)$:

$$\begin{cases}C_1N_2(R)+C_2N_3(R)=0\,,\\ C_3N_2(R)+C_4N_3(R)=0\,.\end{cases} \tag{4.39}$$

The determinant of the coefficient matrix is $C_1C_4-C_2C_3\neq 0$ (see (4.33)). Thus, $N_2(R)=N_3(R)=0$, and hence

$$\mathbf{N}(R,\Theta,Z)=\begin{bmatrix}\pm 1\\ 0\\ 0\end{bmatrix}, \tag{4.40}$$

which satisfies all the other universality constraints. In summary, (4.38) and (4.40) are those universal material preferred directions that respect the symmetry of the universal deformations (4.33). The universal material preferred directions of Families 2 and 3 are identical, see Remark 4.3.

Remark 4.4. Ericksen and Rivlin (1954) analyzed this family assuming the solution (4.37) for the special choice of $f(R)=0$.

4.5. Family 4: Inflation/inversion of a sector of a spherical shell

With respect to the spherical coordinates (R,Θ,Φ) and (r,θ,ϕ) in the reference and current configurations, respectively, this family of deformations have the following representation

$$r(R,\Theta,\Phi)=(\pm R^3+C_1^3)^{\frac{1}{3}}\,,\quad \theta(R,\Theta,\Phi)=\pm\Theta\,,\quad \phi(R,\Theta,\Phi)=\Phi\,. \tag{4.41}$$

Thus

$$[C_{AB}]=\begin{bmatrix}\frac{R^4}{\left(C_1^3\pm R^3\right)^{4/3}} & 0 & 0\\ 0 & \left(C_1^3\pm R^3\right)^{2/3} & 0\\ 0 & 0 & \left(C_1^3\pm R^3\right)^{2/3}\sin^2\Theta\end{bmatrix}. \tag{4.42}$$

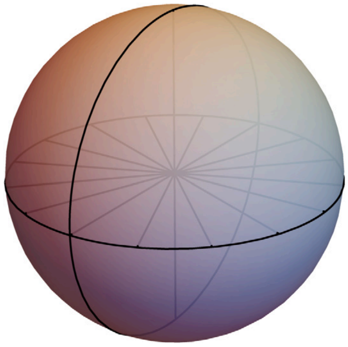

Fig. 2. Radial universal material preferred directions for Family 4.

$\mathbf{C}^\flat$ can be written as (Goodbrake et al., 2020)

$$\mathbf{C}^\flat(\mathbf{X}) = \frac{R^4}{\left(C_1^3 \pm R^3\right)^{4/3}} \hat{\mathbf{R}} \otimes \hat{\mathbf{R}} + \frac{\left(C_1^3 \pm R^3\right)^{2/3}}{R^2} (\mathbf{1} - \hat{\mathbf{R}} \otimes \hat{\mathbf{R}}), \tag{4.43}$$

where $\mathbf{1}$ is the identity tensor, and $\hat{\mathbf{R}} = \frac{\mathbf{X}}{|\mathbf{X}|}$. This means that at a point $\mathbf{X}$, $\mathbf{C}^\flat$ is invariant under all those rotations that fix $\mathbf{X}$. We assume that $\mathbf{N}(\mathbf{X})$ has the same symmetry, i.e., it is invariant under all those rotations that fix $\mathbf{X}$. This implies that $\mathbf{N}(\mathbf{X})$ must be parallel to $\mathbf{X}$, and because it is a unit vector we conclude that

$$\mathbf{N}(\mathbf{X}) = \pm \frac{\mathbf{X}}{|\mathbf{X}|} = \pm \hat{\mathbf{R}}. \tag{4.44}$$

Thus, in spherical coordinates

$$\mathbf{N}(\mathbf{X}) = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}. \tag{4.45}$$

These two vector fields satisfy all the other universality constraints, see Fig. 2.

Remark 4.5. Golgoon and Yavari (2021) had observed that radial deformations are universal for transversely isotropic spherical shells with radial material preferred direction.

4.6. Family 5: Inflation, bending, extension, and azimuthal shearing of an annular wedge

With respect to the cylindrical coordinates (R, Θ, Z) and (r, θ, z) in the reference and current configurations, respectively, this family of deformations have the following representation

$$r(R, \Theta, Z) = C_1 R, \quad \theta(R, \Theta, Z) = C_2 \log R + C_3 \Theta + C_4, \quad z(R, \Theta, Z) = \frac{1}{C_1^2 C_3} Z + C_5. \tag{4.46}$$

Thus

$$[C_{AB}] = \begin{bmatrix} C_1^2\left(C_2^2+1\right) & C_1^2 C_2 C_3 R & 0 \\ C_1^2 C_2 C_3 R & C_1^2 C_3^2 R^2 & 0 \\ 0 & 0 & \frac{1}{C_1^4 C_3^2} \end{bmatrix}, \tag{4.47}$$

which only depends on R. We assume that $\mathbf{N}$ has the same symmetry, i.e.,

$$\mathbf{N}(R, \Theta, Z) = \begin{bmatrix} N^1(R) \\ N^2(R) \\ N^3(R) \end{bmatrix}, \tag{4.48}$$

such that $\left(N^1(R)\right)^2 + \left(N^2(R)\right)^2 + \left(N^3(R)\right)^2 = 1$. Symmetry of the coefficients of W_{444} for $(a, b) = (1, 2)$ gives

$$\begin{aligned} & N^1(R)[C_2 N^1(R) + C_3 R N^2(R)] \\ & \times \frac{d}{dR}\left\{-\left[N^1(R)^2 + (RN^2(R))^2\right] + C_1^6 C_3^2\left[N^1(R)^2 + (C_2 N^1(R) + C_3 R N^2(R))^2\right]\right\} = 0. \end{aligned} \tag{4.49}$$

If $N^1(R) = 0$, symmetry of the coefficients of W_{555} for $(a, b) = (1, 2)$ gives

$$N^2(R)[N^2(R) + \frac{d}{dR}(RN^2(R))] = 0. \tag{4.50}$$

This implies that

$$N^2(R) = \frac{k}{R}. \tag{4.51}$$

If $R_1 \le R \le R_2$, we must have $k^2 \le R_1^2$. All the other symmetry constraints are satisfied and thus one solution is[3]

$$\mathbf{N}(R,\Theta,Z) = \begin{bmatrix} 0 \\ \frac{k}{R} \\ \pm\sqrt{1-k^2} \end{bmatrix}, \tag{4.52}$$

or equivalently

$$\mathbf{N}(R,\Theta,Z) = \begin{bmatrix} 0 \\ \frac{1}{R}\cos\eta \\ \pm\sin\eta \end{bmatrix}, \tag{4.53}$$

for some constant η. In (4.49) if $C_2 N^1(R) + C_3 R N^2(R) = 0$, arbitrariness of C_2, and C_3 implies that $N^1(R) = N^2(R) = 0$, which is already included in the solution (4.53). If

$$-\frac{d}{dR}\left[N^1(R)^2 + (RN^2(R))^2\right] + C_1^6 C_3^2 \frac{d}{dR}\left[N^1(R)^2 + (C_2 N^1(R) + C_3 R N^2(R))^2\right] = 0, \tag{4.54}$$

because C_1 and C_3 are arbitrary one concludes that

$$\begin{aligned} &\left[N^1(R)^2 + (RN^2(R))^2\right]' = 0, \\ &\left[N^1(R)^2 + (C_2 N^1(R) + C_3 R N^2(R))^2\right]' = 0. \end{aligned} \tag{4.55}$$

The first equation implies that $N^3(R)$ is constant. For $N^3(R) = constant$, symmetry of the coefficients of W_4 for $(a,b) = (1,2)$ gives

$$N^3\left[R^2\frac{d^2}{dR^2}N^1(R) + R\frac{d}{dR}N^1(R) - N^1(R)\right] = 0. \tag{4.56}$$

Thus, either $N^3 = 0$, or $N^1(R) = k_1 R + \frac{k_2}{R}$. If $N^3 = 0$, symmetry of the coefficients of W_{255} for $(a,b) = (1,2)$ implies that $N^1(R)$ is constant. The unit vectors

$$\mathbf{N}(R,\Theta,Z) = \begin{bmatrix} \alpha \\ \pm\frac{1}{R}\sqrt{1-\alpha^2} \\ 0 \end{bmatrix}, \tag{4.57}$$

or equivalently

$$\mathbf{N}(R,\Theta,Z) = \begin{bmatrix} \cos\xi \\ \pm\frac{1}{R}\sin\xi \\ 0 \end{bmatrix}, \tag{4.58}$$

for some constant ξ, satisfy all the other universality constraints and, hence, are universal material preferred directions. If $N^1(R) = k_1 R + \frac{k_2}{R}$, symmetry of the coefficients of W_5 for $(a,b) = (1,3)$ implies that

$$N^3\left[(N^3)^2 - 1\right]\left[(N^3)^2 - 1 + 4k_1k_2\right] = 0. \tag{4.59}$$

Therefore, either $\left(N^3\right)^2 = 1$, which is already included in the solution (4.53) when $k = 1$, or $\left(N^3\right)^2 = 1 - 4k_1k_2$. If $\left(N^3\right)^2 = 1 - 4k_1k_2$, one has

$$(N^1)^2 + (N^3)^2 = 1 + \left(k_1 R - \frac{k_2}{R}\right)^2 \le 1. \tag{4.60}$$

Therefore, $k_1 R - \frac{k_2}{R} = 0$, which implies that $k_1 = k_2 = 0$, or $N^1(R) = 0$.

If either I_4 or I_5 (or both) is constant, symmetry of the coefficient of W_2 for $(a,b) = (1,2)$ results in the following second-order ODE:

$$\begin{aligned} &R\left\{N^1(R)'\left[2C_2N^1(R)' + 2C_3RN^2(R)' + 5C_3N^2(R)\right] + C_3RN^2(R)N^1(R)''\right\} \\ &+ N^1(R)\left\{R\left[2C_2N^1(R)'' + 5C_3N^2(R)' + C_3RN^2(R)''\right] + 6C_2N^1(R)' + 3C_3N^2(R)\right\} = 0. \end{aligned} \tag{4.61}$$

If $N^1(R) \neq 0$, this gives $N_2(R)$ in terms of $N_1(R)$ as:

$$N_2(R) = \frac{k_1}{R^3 N_1(R)} + \frac{k_2}{R N_1(R)} - \frac{C_2 N_1(R)}{C_3 R}, \tag{4.62}$$

[3] If we restrict ourselves to the $C_2 = 0$ subset of Family 5, the larger class of material preferred directions (4.38) is a solution.

Table 1
Universal deformations and universal material preferred directions for incompressible transversely isotropic solids for the six known families of universal deformations. Note that for Family 3, $K = C_1C_4 - C_2C_3$.

Family	Universal deformations	$\mathbf{C}^\flat$	Universal material preferred directions
0	$x^a(X) = F^a{}_A X^A + c^a$	$C_{AB} = F^a{}_A F^a{}_A \delta_{ab}$	Any constant unit vector $\mathbf{N}$
1	$\begin{cases} r(X,Y,Z) = \sqrt{C_1(2X+C_4)} \\ \theta(X,Y,Z) = C_2(Y+C_5) \\ z(X,Y,Z) = \frac{Z}{C_1C_2} - C_2C_3Y + C_6 \end{cases}$	$[C_{AB}] = \begin{bmatrix} \frac{C_1}{2X+C_4} & 0 & 0 \\ 0 & C_2^2\left[C_1(2X+C_4)+C_3^2\right] & -\frac{C_3}{C_1} \\ 0 & -\frac{C_3}{C_1} & \frac{1}{C_1^2C_2^2} \end{bmatrix}$	$\hat{\mathbf{N}} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} 0 \\ \cos\psi(X) \\ \pm\sin\psi(X) \end{bmatrix}$
2	$\begin{cases} x(R,\Theta,Z) = \frac{1}{2}C_1C_2^2R^2 + C_4 \\ y(R,\Theta,Z) = \frac{\Theta}{C_1C_2} + C_5 \\ z(R,\Theta,Z) = \frac{C_3}{C_1C_2}\Theta + \frac{1}{C_2}Z + C_6 \end{cases}$	$[C_{AB}] = \begin{bmatrix} C_1^2C_2^4R^2 & 0 & 0 \\ 0 & \frac{C_3^2+1}{C_1^2C_2^2} & \frac{C_3}{C_1C_2^2} \\ 0 & \frac{C_3}{C_1C_2^2} & \frac{1}{C_2^2} \end{bmatrix}$	$\hat{\mathbf{N}} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} 0 \\ \cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix}$
3	$\begin{cases} r(R,\Theta,Z) = \sqrt{\frac{R^2}{C_1C_4-C_2C_3} + C_5} \\ \theta(R,\Theta,Z) = C_1\Theta + C_2Z + C_6 \\ z(R,\Theta,Z) = C_3\Theta + C_4Z + C_7 \end{cases}$	$[C_{AB}] = \begin{bmatrix} \frac{R^2}{K(KC_5+R^2)} & 0 & 0 \\ 0 & C_1^2\left(\frac{R^2}{K}+C_5\right)+C_3^2 & C_1C_2\left(\frac{R^2}{K}+C_5\right)+C_3C_4 \\ 0 & C_1C_2\left(\frac{R^2}{K}+C_5\right)+C_3C_4 & C_2^2\left(\frac{R^2}{K}+C_5\right)+C_4^2 \end{bmatrix}$	$\hat{\mathbf{N}} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} 0 \\ \cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix}$
4	$\begin{cases} r(R,\Theta,\Phi) = (\pm R^3 + C_1^3) \\ \theta(R,\Theta,\Phi) = \pm\Theta \\ \phi(R,\Theta,\Phi) = \Phi \end{cases}$	$[C_{AB}] = \begin{bmatrix} \frac{R^4}{(C_1^3\pm R^3)^{4/3}} & 0 & 0 \\ 0 & \left(C_1^3\pm R^3\right)^{2/3} & 0 \\ 0 & 0 & \left(C_1^3\pm R^3\right)^{2/3}\sin^2\Theta \end{bmatrix}$	$\hat{\mathbf{N}} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}$
5	$\begin{cases} r(R,\Theta,Z) = C_1R \\ \theta(R,\Theta,Z) = C_2\log R + C_3\Theta + C_4 \\ z(R,\Theta,Z) = \frac{1}{C_1^2C_3}Z + C_5 \end{cases}$	$[C_{AB}] = \begin{bmatrix} C_1^2\left(C_2^2+1\right) & C_1^2C_2C_3R & 0 \\ C_1^2C_2C_3R & C_1^2C_3^2R^2 & 0 \\ 0 & 0 & \frac{1}{C_1^4C_3^2} \end{bmatrix}$	$\hat{\mathbf{N}} = \begin{bmatrix} 0 \\ \cos\eta \\ \pm\sin\eta \end{bmatrix}, \quad \hat{\mathbf{N}} = \begin{bmatrix} \cos\xi \\ \pm\sin\xi \\ 0 \end{bmatrix}$

which is clearly not a universal solution as $\mathbf{N}$ should not depend on the parameters of the universal deformations. Therefore, $N^1(R) = 0$. Symmetries of the coefficients of W_2 for $(a,b) = (1,2)$, and W_5 for $(a,b) = (1,3)$, imply that

$$\mathbf{N}(R,\Theta,Z) = \begin{bmatrix} 0 \\ \pm\frac{1}{R} \\ 0 \end{bmatrix}, \tag{4.63}$$

which is already included in (4.53). Therefore, (4.53) and (4.58) are the only solutions.

Remark 4.6. Assuming that the annular wedge is made of a unidirectional fiber composite, in the solution (4.53) fibers are arranged helically when $\cos\eta \neq 0$ (Fig. 1(b)). When $\cos\eta = 0$, fibers are distributed uniformly parallel to the axis of the wedge (Fig. 1(c)). When $\sin\eta = 0$, fibers are concentric circles parallel to the (R,Θ) plane (Fig. 1(d)). For the solution (4.58) fibers are parallel to the (R,Θ) plane and are distributed uniformly in some fixed direction. Table 1 summarizes our results for incompressible transversely isotropic solids.

5. Incompressible orthotropic elastic solids

For orthotropic solids there are seven invariants. Note that I_4 and I_6 have identical forms, and similarly, I_5 and I_7 have identical forms, see (2.27). This means that the forms of the universality constraints associated with the pair (I_6, I_7) are identical to those associated with (I_4, I_5). Since most of the analysis relies on the previous case, we only explain briefly the underlying computations and give the main results. In the case of orthotropic solids

$$\xi_a = g_{am}\left[W_1 b^{mn} - W_2\, c^{mn} + W_4\, n_1^m n_1^n + W_5\, \ell_1^{mn} + W_6\, n_2^m n_2^n + W_7\, \ell_2^{mn}\right]_{\|n}. \tag{5.1}$$

For $\xi_{a|b} = \xi_{b|a}$ to hold for arbitrary energy functions the coefficient of each partial derivative of W must be symmetric. There are four groups of terms:

(i) Nine terms that must be symmetric for isotropic solids as well:

$$\mathcal{K}_{\text{iso}} = \{1,2,11,22,12,111,222,112,122\}. \tag{5.2}$$

(ii) 25 terms corresponding to $\mathbf{N}_1$:

$$\mathcal{K}_{i} = \{4,5,44,55,14,15,24,25,45,444,555,114,115,124,125,144,145,155,224,225,244,245,255,445,455\}. \tag{5.3}$$

(iii) 25 terms corresponding to $\mathbf{N}_2$:

$$\mathcal{K}_{ii} = \{6,7,66,77,16,17,26,27,67,666,777,116,117,126,127,166,167,177,226,227,266,267,277,667,677\}. \tag{5.4}$$

(iv) 24 terms corresponding to coupling of $\mathbf{N}_1$ and $\mathbf{N}_2$:

$$\mathcal{K}_{iii} = \{46, 47, 56, 57, 146, 147, 156, 157, 246, 247, 256, 257, 446, 447, 456, 457, 556, 557, 466, 467, 566, 567, 477, 577\}. \tag{5.5}$$

The universality constraints corresponding to the sets $\mathcal{K}_i$ and $\mathcal{K}_{ii}$ are identical in form to those corresponding to the extra symmetry constraints of transversely isotopic solids (4.13). This means that if there are three mutually orthogonal universal material preferred directions $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$ for transversely isotropic solids, they are universal for orthotropic solids as well if the three pairs $(\mathbf{N}_1, \mathbf{N}_2)$, $(\mathbf{N}_2, \mathbf{N}_3)$, and $(\mathbf{N}_3, \mathbf{N}_1)$ satisfy the universality conditions corresponding to the set $\mathcal{K}_{iii}$.

In order to write the constraint equations more compactly, let us denote the pair of vectors $(\boldsymbol{n}, \boldsymbol{m}) = (\mathbf{n}_1, \mathbf{n}_2)$. Also, $\ell^{ab} = \ell_1^{ab}$, and $\hbar^{ab} = \ell_2^{ab}$. The coefficients of the four second-order derivatives of the energy function corresponding to the set $\mathcal{K}_{iii}$ are:

$$\begin{aligned}
\mathcal{A}_{ab}^{46} &= [n_a\, I_{6,n}\, n^n]_{|b} + I_{6,b}[n_a\, n^n]_{|n} + [m_a\, I_{4,n}\, m^n]_{|b} + I_{4,b}[m_a\, m^n]_{|n}\,, \\
\mathcal{A}_{ab}^{47} &= [n_a\, I_{7,n}\, n^n]_{|b} + I_{7,b}[n_a\, n^n]_{|n} + (\hbar_a^n I_{4,n})_{|b} + \hbar^n_{a|n} I_{4,b}\,, \\
\mathcal{A}_{ab}^{56} &= (\ell_a^n I_{6,n})_{|b} + \ell^n_{a|n} I_{6,b} + (m_a I_{5,n} m^n)_{|b} + (m_a m^n)_{|n} I_{5,b}\,, \\
\mathcal{A}_{ab}^{57} &= (\ell_a^n I_{7,n})_{|b} + \ell^n_{a|n} I_{7,b} + (\hbar_a^n I_{5,n})_{|b} + \hbar^n_{a|n} I_{5,b}\,.
\end{aligned} \tag{5.6}$$

The coefficients of the twenty third-order derivatives of the energy function corresponding to the set $\mathcal{K}_{iii}$ read:

$$\begin{aligned}
\mathcal{A}_{ab}^{146} &= b_a^n \left(I_{4,b} I_{6,n} + I_{4,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{147} &= b_a^n \left(I_{4,b} I_{7,n} + I_{4,n} I_{7,b}\right), \\
\mathcal{A}_{ab}^{156} &= b_a^n \left(I_{5,b} I_{6,n} + I_{5,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{157} &= b_a^n \left(I_{5,b} I_{7,n} + I_{5,n} I_{7,b}\right), \\
\mathcal{A}_{ab}^{246} &= c_a^n \left(I_{4,b} I_{6,n} + I_{4,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{247} &= c_a^n \left(I_{4,b} I_{7,n} + I_{4,n} I_{7,b}\right), \\
\mathcal{A}_{ab}^{256} &= c_a^n \left(I_{5,b} I_{6,n} + I_{5,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{257} &= c_a^n \left(I_{5,b} I_{7,n} + I_{5,n} I_{7,b}\right), \\
\mathcal{A}_{ab}^{446} &= n_a n^n \left(I_{4,b} I_{6,n} + I_{4,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{447} &= n_a n^n \left(I_{4,b} I_{7,n} + I_{4,n} I_{7,b}\right), \\
\mathcal{A}_{ab}^{456} &= n_a n^n \left(I_{5,b} I_{6,n} + I_{5,n} I_{6,b}\right) + \ell_a^n \left(I_{4,b} I_{6,n} + I_{4,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{457} &= n_a n^n \left(I_{5,b} I_{7,n} + I_{5,n} I_{7,b}\right) + \ell_a^n \left(I_{4,b} I_{7,n} + I_{4,n} I_{7,b}\right), \\
\mathcal{A}_{ab}^{466} &= m_a m^n \left(I_{4,b} I_{6,n} + I_{4,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{467} &= m_a m^n \left(I_{4,b} I_{7,n} + I_{4,n} I_{7,b}\right) + \hbar_a^n \left(I_{4,b} I_{6,n} + I_{4,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{477} &= \hbar_a^n \left(I_{4,b} I_{7,n} + I_{4,n} I_{7,b}\right), \\
\mathcal{A}_{ab}^{556} &= \ell_a^n \left(I_{5,b} I_{6,n} + I_{5,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{557} &= \ell_a^n \left(I_{5,b} I_{7,n} + I_{5,n} I_{7,b}\right), \\
\mathcal{A}_{ab}^{566} &= m_a m^n \left(I_{5,b} I_{6,n} + I_{5,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{567} &= m_a m^n \left(I_{5,b} I_{7,n} + I_{5,n} I_{7,b}\right) + \hbar_a^n \left(I_{5,b} I_{6,n} + I_{5,n} I_{6,b}\right), \\
\mathcal{A}_{ab}^{577} &= \hbar_a^n \left(I_{5,b} I_{7,n} + I_{5,n} I_{7,b}\right).
\end{aligned} \tag{5.7}$$

Family 0. We saw that for transversely isotropic solids any constant unit vector is a universal material preferred direction. For any pair of constant unit vectors, all the terms in (5.6) and (5.7) are trivially symmetric. This means that any three constant unit vectors $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$ that are mutually orthogonal are universal material preferred directions for isochoric homogeneous deformations.

Family 1. Let us consider two solutions in the family of solutions (4.23), namely

$$\mathbf{N}_2 = \begin{bmatrix} 0 \\ f(X) \\ \pm\sqrt{1 - f^2(X)} \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0 \\ g(X) \\ \pm\sqrt{1 - g^2(X)} \end{bmatrix}. \tag{5.8}$$

$\mathbf{N}_2$ and $\mathbf{N}_3$ are orthogonal if and only if $f^2(X) + g^2(X) = 1$. Thus, $f(X) = \cos\psi(X)$, and $g(X) = \sin\psi(X)$, for an arbitrary function $\psi(X)$. Therefore, we have the following set of mutually orthogonal universal material preferred directions for transversely isotropic solids.

$$\mathbf{N}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} 0 \\ \cos\psi(X) \\ \pm\sin\psi(X) \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0 \\ \sin\psi(X) \\ \mp\cos\psi(X) \end{bmatrix}. \tag{5.9}$$

One can check that for any pair of mutually orthogonal vectors in the above set all the terms in (5.6) and (5.7) are symmetric. Therefore, (5.9) is a family of universal material preferred directions for orthotropic solids.

Families 2 and 3. The solutions for material preferred directions for transversely isotropic solids for Families 2 and 3 are very similar to those of Family 1. Therefore, we have the following family of universal material preferred directions

$$\mathbf{N}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} 0 \\ \frac{\cos\chi(R)}{R} \\ \pm\sin\chi(R) \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0 \\ \frac{\sin\chi(R)}{R} \\ \mp\cos\chi(R) \end{bmatrix}, \tag{5.10}$$

for an arbitrary function $\chi(R)$. One can check that for any pair of mutually orthogonal vectors in the above set all the terms in (5.6) and (5.7) are symmetric. Therefore, these are universal material preferred directions.

Family 4. In the case of transversely isotropic solids, there are only two solutions for the material preferred directions (4.45) that are parallel. This means that in the case of orthotropic solids Family 4 is not universal.

Family 5. Let us consider arbitrary members of the two families of solutions (4.53) and (4.58)

$$\mathbf{N}_1 = \begin{bmatrix} 0 \\ \frac{k}{R} \\ \pm\sqrt{1-k^2} \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} \alpha \\ \pm\frac{1}{R}\sqrt{1-\alpha^2} \\ 0 \end{bmatrix}. \tag{5.11}$$

Note that $\mathbf{N}_1 \cdot \mathbf{N}_2 = \pm k\sqrt{1-\alpha^2} = 0$ implies that $k = 0$ or $\alpha = \pm 1$. Therefore, we have the following two classes of universal material preferred directions:

$$\begin{cases} \mathbf{N}_1 = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} \cos\xi \\ \pm\frac{1}{R}\sin\xi \\ 0 \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} \sin\xi \\ \mp\frac{1}{R}\cos\xi \\ 0 \end{bmatrix}, \\ \mathbf{N}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{N}_2 = \begin{bmatrix} 0 \\ \frac{1}{R}\cos\eta \\ \pm\sin\eta \end{bmatrix}, \quad \mathbf{N}_3 = \begin{bmatrix} 0 \\ \frac{1}{R}\sin\eta \\ \mp\cos\eta \end{bmatrix}. \end{cases} \tag{5.12}$$

One can check that for any pair of mutually orthogonal vectors in each row in the above set all the terms in (5.6) and (5.7) are symmetric. Table 2 summarizes our results for incompressible orthotropic solids.

6. Incompressible monoclinic elastic solids

For monoclinic solids

$$\xi_a = g_{am}\left[W_1 b^{mn} - W_2 c^{mn} + W_4 n_1^m n_1^n + W_5 \ell_1^{mn} + W_6 n_2^m n_2^n + W_7 \ell_2^{mn} + \frac{1}{2} W_8 \ell_3^{mn}\right]_{|n}. \tag{6.1}$$

For $\xi_{a|b} = \xi_{b|a}$ to hold for arbitrary monoclinic energy functions the coefficient of each partial derivative of W must be symmetric. In addition to the terms corresponding to the sets $\mathcal{K}_{\text{iso}}$, $\mathcal{K}_i$, $\mathcal{K}_{ii}$, $\mathcal{K}_{iii}$, there are an extra 78 terms corresponding to the following set:

$$\begin{aligned} \mathcal{K}_{iv} = \{&8, 18, 19, 28, 29, 48, 49, 58, 59, 68, 69, 78, 79, 88, 89, \\ &118, 119, 128, 129, 148, 149, 158, 159, 168, 169, 178, 179, 188, 189, 199, 228, 229, \\ &248, 249, 258, 259, 268, 269, 278, 279, 288, 289, 299, 448, 449, 458, 459, 468, 469, \\ &478, 479, 488, 489, 499, 558, 559, 568, 569, 578, 579, 588, 589, 599, 668, 669, \\ &678, 679, 688, 689, 699, 778, 779, 788, 789, 799, 888, 889, 999\}. \end{aligned} \tag{6.2}$$

Similar to the analysis for orthotropic solids, in order to write the constraint equations more compactly, we denote the pair of vectors $(\boldsymbol{n}, \boldsymbol{m}) = (\mathbf{n}_1, \mathbf{n}_2)$. Also, $\ell^{ab} = \ell_1^{ab}$, $\hbar^{ab} = \ell_2^{ab}$, and $q^{ab} = \ell_3^{ab}$. The coefficients of the first and second-order derivatives of the energy function corresponding to the set $\mathcal{K}_{iv}$ are:

$$\begin{aligned}
\mathcal{A}_{ab}^{8} &= q^n_{a|nb}\,,\\
\mathcal{A}_{ab}^{18} &= q^n_{a|n}\,I_{1,b} + (q^n_a\,I_{1,n})_{|b} + (b^n_a I_{8,n})_{|b} + b^n_{a|n} I_{8,b}\,,\\
\mathcal{A}_{ab}^{19} &= (b^n_a I_{9,n})_{|b} + b^n_{a|n} I_{9,b}\,,\\
\mathcal{A}_{ab}^{28} &= q^n_{a|n}\,I_{2,b} + (q^n_a\,I_{2,n})_{|b} - (c^n_a I_{8,n})_{|b} - c^n_{a|n} I_{8,b}\,,\\
\mathcal{A}_{ab}^{29} &= -(c^n_a I_{9,n})_{|b} - c^n_{a|n} I_{9,b}\,,\\
\mathcal{A}_{ab}^{48} &= q^n_{a|n}\,I_{4,b} + (q^n_a\,I_{4,n})_{|b} + (n_a n^n I_{8,n})_{|b} + (n_a n^n)_{|n} I_{8,b}\,,\\
\mathcal{A}_{ab}^{49} &= (n_a n^n I_{9,n})_{|b} + (n_a n^n)_{|n} I_{9,b}\,,\\
\mathcal{A}_{ab}^{58} &= q^n_{a|n}\,I_{5,b} + (q^n_a\,I_{5,n})_{|b} + (\ell^n_a I_{8,n})_{|b} + \ell^n_{a|n} I_{8,b}\,,\\
\mathcal{A}_{ab}^{59} &= (\ell^n_a I_{9,n})_{|b} + \ell^n_{a|n} I_{9,b}\,,\\
\mathcal{A}_{ab}^{68} &= q^n_{a|n}\,I_{6,b} + (q^n_a\,I_{6,n})_{|b} + (m_a m^n I_{8,n})_{|b} + (m_a m^n)_{|n} I_{8,b}\,,\\
\mathcal{A}_{ab}^{69} &= (m_a m^n I_{9,n})_{|b} + (m_a m^n)_{|n} I_{9,b}\,,\\
\mathcal{A}_{ab}^{78} &= q^n_{a|n}\,I_{7,b} + (q^n_a\,I_{7,n})_{|b} + (k^n_a I_{8,n})_{|b} + k^n_{a|n} I_{8,b}\,,\\
\mathcal{A}_{ab}^{79} &= (k^n_a I_{9,n})_{|b} + k^n_{a|n} I_{9,b}\,,\\
\mathcal{A}_{ab}^{88} &= q^n_{a|n}\,I_{8,b} + (q^n_a\,I_{8,n})_{|b}\,,\\
\mathcal{A}_{ab}^{89} &= q^n_{a|n}\,I_{9,b} + (q^n_a\,I_{9,n})_{|b}\,.
\end{aligned} \tag{6.3}$$

The coefficients of the third-order derivatives of the energy function corresponding to the set $\mathcal{K}_{iv}$ read:

$$\begin{aligned}
\mathcal{A}_{ab}^{118} &= b^n_a\left(I_{1,b}I_{8,n} + I_{1,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{119} &= b^n_a\left(I_{1,b}I_{9,n} + I_{1,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{128} &= b^n_a\left(I_{2,b}I_{8,n} + I_{2,n}I_{8,b}\right) - c^n_a\left(I_{1,b}I_{8,n} + I_{1,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{129} &= b^n_a\left(I_{2,b}I_{9,n} + I_{2,n}I_{9,b}\right) - c^n_a\left(I_{1,b}I_{9,n} + I_{1,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{148} &= b^n_a\left(I_{4,b}I_{8,n} + I_{4,n}I_{8,b}\right) + n_a n^n\left(I_{1,b}I_{8,n} + I_{1,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{149} &= b^n_a\left(I_{4,b}I_{9,n} + I_{4,n}I_{9,b}\right) + n_a n^n\left(I_{1,b}I_{9,n} + I_{1,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{158} &= b^n_a\left(I_{5,b}I_{8,n} + I_{5,n}I_{8,b}\right) + \ell^n_a\left(I_{1,b}I_{8,n} + I_{1,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{159} &= b^n_a\left(I_{5,b}I_{9,n} + I_{5,n}I_{9,b}\right) + \ell^n_a\left(I_{1,b}I_{9,n} + I_{1,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{168} &= b^n_a\left(I_{6,b}I_{8,n} + I_{6,n}I_{8,b}\right) + m_a m^n\left(I_{1,b}I_{8,n} + I_{1,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{169} &= b^n_a\left(I_{6,b}I_{9,n} + I_{6,n}I_{9,b}\right) + m_a m^n\left(I_{1,b}I_{9,n} + I_{1,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{178} &= b^n_a\left(I_{7,b}I_{8,n} + I_{7,n}I_{8,b}\right) + k^n_a\left(I_{1,b}I_{8,n} + I_{1,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{179} &= b^n_a\left(I_{7,b}I_{9,n} + I_{7,n}I_{9,b}\right) + k^n_a\left(I_{1,b}I_{9,n} + I_{1,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{188} &= b^n_a\,I_{8,b}I_{8,n} + q^n_a\left(I_{1,b}I_{8,n} + I_{1,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{189} &= b^n_a\left(I_{8,b}I_{9,n} + I_{8,n}I_{9,b}\right) + q^n_a\left(I_{1,b}I_{9,n} + I_{1,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{199} &= b^n_a\,I_{9,b}I_{9,n}\,,
\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
\mathcal{A}_{ab}^{228} &= -c^n_a\left(I_{2,b}I_{8,n} + I_{2,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{229} &= -c^n_a\left(I_{2,b}I_{9,n} + I_{2,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{248} &= -c^n_a\left(I_{4,b}I_{8,n} + I_{4,n}I_{8,b}\right) + n_a n^n\left(I_{2,b}I_{8,n} + I_{2,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{249} &= -c^n_a\left(I_{4,b}I_{9,n} + I_{4,n}I_{9,b}\right) + n_a n^n\left(I_{2,b}I_{9,n} + I_{2,n}I_{9b}\right)\,,\\
\mathcal{A}_{ab}^{258} &= -c^n_a\left(I_{5,b}I_{8,n} + I_{5,n}I_{8,b}\right) + \ell^n_a\left(I_{2,b}I_{8,n} + I_{2,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{259} &= -c^n_a\left(I_{5,b}I_{9,n} + I_{5,n}I_{9,b}\right) + \ell^n_a\left(I_{2,b}I_{9,n} + I_{2,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{268} &= -c^n_a\left(I_{6,b}I_{8,n} + I_{6,n}I_{8,b}\right) + m_a m^n\left(I_{2,b}I_{8,n} + I_{2,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{269} &= -c^n_a\left(I_{6,b}I_{9,n} + I_{6,n}I_{9,b}\right) + m_a m^n\left(I_{2,b}I_{9,n} + I_{2,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{278} &= -c^n_a\left(I_{7,b}I_{8,n} + I_{7,n}I_{8,b}\right) + k^n_a\left(I_{2,b}I_{8,n} + I_{2,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{279} &= -c^n_a\left(I_{7,b}I_{9,n} + I_{7,n}I_{9,b}\right) + k^n_a\left(I_{2,b}I_{9,n} + I_{2,n}I_{9,b}\right)\,,\\
\mathcal{A}_{ab}^{288} &= -c^n_a\,I_{8,b}I_{8,n} + q^n_a\left(I_{2,b}I_{8,n} + I_{2,n}I_{8,b}\right)\,,\\
\mathcal{A}_{ab}^{289} &= -c^n_a\left(I_{8,b}I_{9,n} + I_{8,n}I_{9,b}\right) + q^n_a\left(I_{2,b}I_{9,n} + I_{2,n}I_{9,b}\right)\,,
\end{aligned}$$

Table 2
Universal deformations and universal material preferred directions for incompressible orthotropic solids for the six known families of universal deformations. Note that for Family 3, $K = C_1C_4 - C_2C_3$. For orthotropic solids Family 4 is not universal.

Family	Universal deformations	$\mathbf{C}^\flat$	Universal material preferred directions
0	$x^a(X) = F^a{}_A X^A + c^a$	$C_{AB} = F^a{}_A F^a{}_A \delta_{ab}$	Any three mutually orthogonal constant unit vectors $(\hat{\mathbf{N}}_1, \hat{\mathbf{N}}_2, \hat{\mathbf{N}}_3)$
1	$\begin{cases} r(X,Y,Z) = \sqrt{C_1(2X+C_4)} \\ \theta(X,Y,Z) = C_2(Y+C_5) \\ z(X,Y,Z) = \frac{Z}{C_1C_2} - C_2C_3Y + C_6 \end{cases}$	$[C_{AB}] = \begin{bmatrix} \frac{C_1}{2X+C_4} & 0 & 0 \\ 0 & C_2^2\left[C_1(2X+C_4)+C_3^2\right] & -\frac{C_3}{C_1} \\ 0 & -\frac{C_3}{C_1} & \frac{1}{C_1^2C_2^2} \end{bmatrix}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\psi(X) \\ \pm\sin\psi(X) \end{bmatrix},$ $\hat{\mathbf{N}}_3 = \begin{bmatrix} 0 \\ \sin\psi(X) \\ \mp\cos\psi(X) \end{bmatrix}$
2	$\begin{cases} x(R,\Theta,Z) = \frac{1}{2}C_1C_2^2R^2 + C_4 \\ y(R,\Theta,Z) = \frac{\Theta}{C_1C_2} + C_5 \\ z(R,\Theta,Z) = \frac{C_3}{C_1C_2}\Theta + \frac{1}{C_2}Z + C_6 \end{cases}$	$[C_{AB}] = \begin{bmatrix} C_1^2C_2^4R^2 & 0 & 0 \\ 0 & \frac{C_3^2+1}{C_1^2C_2^2} & \frac{C_3}{C_1C_2^2} \\ 0 & \frac{C_3}{C_1C_2^2} & \frac{1}{C_2^2} \end{bmatrix}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix},$ $\hat{\mathbf{N}}_3 = \begin{bmatrix} 0 \\ \sin\chi(R) \\ \mp\cos\chi(R) \end{bmatrix}$
3	$\begin{cases} r(R,\Theta,Z) = \sqrt{\frac{R^2}{C_1C_4-C_2C_3} + C_5} \\ \theta(R,\Theta,Z) = C_1\Theta + C_2Z + C_6 \\ z(R,\Theta,Z) = C_3\Theta + C_4Z + C_7 \end{cases}$	$[C_{AB}] = \begin{bmatrix} \frac{R^2}{K(KC_5+R^2)} & 0 & 0 \\ 0 & C_1^2\left(\frac{R^2}{K}+C_5\right)+C_3^2 & C_1C_2\left(\frac{R^2}{K}+C_5\right)+C_3C_4 \\ 0 & C_1C_2\left(\frac{R^2}{K}+C_5\right)+C_3C_4 & C_2^2\left(\frac{R^2}{K}+C_5\right)+C_4^2 \end{bmatrix}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\chi(R) \\ \pm\sin\chi(R) \end{bmatrix},$ $\hat{\mathbf{N}}_3 = \begin{bmatrix} 0 \\ \sin\chi(R) \\ \mp\cos\chi(R) \end{bmatrix}$
5	$\begin{cases} r(R,\Theta,Z) = C_1R \\ \theta(R,\Theta,Z) = C_2\log R + C_3\Theta + C_4 \\ z(R,\Theta,Z) = \frac{1}{C_1^2C_3}Z + C_5 \end{cases}$	$[C_{AB}] = \begin{bmatrix} C_1^2\left(C_2^2+1\right) & C_1^2C_2C_3R & 0 \\ C_1^2C_2C_3R & C_1^2C_3^2R^2 & 0 \\ 0 & 0 & \frac{1}{C_1^4C_3^2} \end{bmatrix}$	$\begin{cases} \hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} \cos\xi \\ \pm\sin\xi \\ 0 \end{bmatrix}, \\ \hat{\mathbf{N}}_3 = \begin{bmatrix} \sin\xi \\ \mp\cos\xi \\ 0 \end{bmatrix} \\ \hat{\mathbf{N}}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\eta \\ \pm\sin\eta \end{bmatrix}, \\ \hat{\mathbf{N}}_3 = \begin{bmatrix} 0 \\ \sin\eta \\ \mp\cos\eta \end{bmatrix} \end{cases}$

$$
\begin{aligned}
\mathcal{A}^{299}_{ab} &= -c^n_a\, I_{9,b}I_{9,n}, \\
\mathcal{A}^{448}_{ab} &= n_a n^n \left(I_{4,b}I_{8,n} + I_{4,n}I_{8,b} \right), \\
\mathcal{A}^{449}_{ab} &= n_a n^n \left(I_{4,b}I_{9,n} + I_{4,n}I_{9,b} \right),
\end{aligned} \tag{6.5}
$$

and

$$
\begin{aligned}
\mathcal{A}^{458}_{ab} &= n_a n^n \left(I_{5,b}I_{8,n} + I_{5,n}I_{8,b} \right) + \ell^n_a \left(I_{4,b}I_{8,n} + I_{4,n}I_{8,b} \right), \\
\mathcal{A}^{459}_{ab} &= n_a n^n \left(I_{5,b}I_{9,n} + I_{5,n}I_{9,b} \right) + \ell^n_a \left(I_{4,b}I_{9,n} + I_{4,n}I_{9,b} \right), \\
\mathcal{A}^{468}_{ab} &= n_a n^n \left(I_{6,b}I_{8,n} + I_{6,n}I_{8,b} \right) + m_a m^n \left(I_{4,b}I_{8,n} + I_{4,n}I_{8,b} \right), \\
\mathcal{A}^{469}_{ab} &= n_a n^n \left(I_{6,b}I_{9,n} + I_{6,n}I_{9,b} \right) + m_a m^n \left(I_{4,b}I_{9,n} + I_{4,n}I_{9,b} \right), \\
\mathcal{A}^{478}_{ab} &= n_a n^n \left(I_{7,b}I_{8,n} + I_{7,n}I_{8,b} \right) + k^n_a \left(I_{4,b}I_{8,n} + I_{4,n}I_{8,b} \right), \\
\mathcal{A}^{479}_{ab} &= n_a n^n \left(I_{7,b}I_{9,n} + I_{7,n}I_{9,b} \right) + k^n_a \left(I_{4,b}I_{9,n} + I_{4,n}I_{9,b} \right), \\
\mathcal{A}^{488}_{ab} &= n_a n^n\, I_{8,b}I_{8,n} + q^n_a \left(I_{4,b}I_{8,n} + I_{4,n}I_{8,b} \right), \\
\mathcal{A}^{489}_{ab} &= n_a n^n \left(I_{8,b}I_{9,n} + I_{8,n}I_{9,b} \right) + q^n_a \left(I_{4,b}I_{9,n} + I_{4,n}I_{9,b} \right), \\
\mathcal{A}^{499}_{ab} &= n_a n^n\, I_{9,b}I_{9,n}, \\
\mathcal{A}^{558}_{ab} &= \ell^n_a \left(I_{5,b}I_{8,n} + I_{5,n}I_{8,b} \right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{559}_{ab} &= \ell^n_a \left(I_{5,b}I_{9,n} + I_{5,n}I_{9,b}\right) ,\\
\mathcal{A}^{568}_{ab} &= \ell^n_a \left(I_{6,b}I_{8,n} + I_{6,n}I_{8,b}\right) + m_a m^n \left(I_{5,b}I_{8,n} + I_{5,n}I_{8,b}\right) ,\\
\mathcal{A}^{569}_{ab} &= \ell^n_a \left(I_{6,b}I_{9,n} + I_{6,n}I_{9,b}\right) + m_a m^n \left(I_{5,b}I_{9,n} + I_{5,n}I_{9,b}\right) ,\\
\mathcal{A}^{578}_{ab} &= \ell^n_a \left(I_{7,b}I_{8,n} + I_{7,n}I_{8,b}\right) + \mathscr{k}^n_a \left(I_{5,b}I_{8,n} + I_{5,n}I_{8,b}\right) ,\\
\mathcal{A}^{579}_{ab} &= \ell^n_a \left(I_{7,b}I_{9,n} + I_{7,n}I_{9,b}\right) + \mathscr{k}^n_a \left(I_{5,b}I_{9,n} + I_{5,n}I_{9,b}\right) ,
\end{aligned}
\tag{6.6}
$$

and

$$
\begin{aligned}
\mathcal{A}^{588}_{ab} &= \ell^n_a\, I_{8,b}I_{8,n} + q^n_a \left(I_{5,b}I_{8,n} + I_{5,n}I_{8,b}\right) ,\\
\mathcal{A}^{589}_{ab} &= \ell^n_a \left(I_{8,b}I_{9,n} + I_{8,n}I_{9,b}\right) + q^n_a \left(I_{5,b}I_{9,n} + I_{5,n}I_{9,b}\right) ,\\
\mathcal{A}^{599}_{ab} &= \ell^n_a\, I_{9,b}I_{9,n} ,\\
\mathcal{A}^{668}_{ab} &= m_a m^n \left(I_{6,b}I_{8,n} + I_{6,n}I_{8,b}\right) ,\\
\mathcal{A}^{669}_{ab} &= m_a m^n \left(I_{6,b}I_{9,n} + I_{6,n}I_{9,b}\right) ,\\
\mathcal{A}^{678}_{ab} &= m_a m^n \left(I_{7,b}I_{8,n} + I_{7,n}I_{8,b}\right) + \mathscr{k}^n_a \left(I_{6,b}I_{8,n} + I_{6,n}I_{8,b}\right) ,\\
\mathcal{A}^{679}_{ab} &= m_a m^n \left(I_{7,b}I_{9,n} + I_{7,n}I_{9,b}\right) + \mathscr{k}^n_a \left(I_{6,b}I_{9,n} + I_{6,n}I_{9,b}\right) ,\\
\mathcal{A}^{688}_{ab} &= m_a m^n\, I_{8,b}I_{8,n} + q^n_a \left(I_{6,b}I_{8,n} + I_{6,n}I_{8,b}\right) ,\\
\mathcal{A}^{689}_{ab} &= m_a m^n \left(I_{8,b}I_{9,n} + I_{8,n}I_{9,b}\right) + q^n_a \left(I_{6,b}I_{9,n} + I_{6,n}I_{9,b}\right) ,\\
\mathcal{A}^{699}_{ab} &= m_a m^n\, I_{9,b}I_{9,n} ,\\
\mathcal{A}^{778}_{ab} &= \mathscr{k}^n_a \left(I_{7,b}I_{8,n} + I_{7,n}I_{8,b}\right) ,\\
\mathcal{A}^{779}_{ab} &= \mathscr{k}^n_a \left(I_{7,b}I_{9,n} + I_{7,n}I_{9,b}\right) ,\\
\mathcal{A}^{788}_{ab} &= \mathscr{k}^n_a\, I_{8,b}I_{8,n} + q^n_a \left(I_{7,b}I_{8,n} + I_{7,n}I_{8,b}\right) ,\\
\mathcal{A}^{789}_{ab} &= \mathscr{k}^n_a \left(I_{8,b}I_{9,n} + I_{8,n}I_{9,b}\right) + q^n_a \left(I_{7,b}I_{9,n} + I_{7,n}I_{9,b}\right) ,\\
\mathcal{A}^{799}_{ab} &= \mathscr{k}^n_a\, I_{9,b}I_{9,n} ,\\
\mathcal{A}^{888}_{ab} &= q^n_a\, I_{8,b}I_{8,n} ,\\
\mathcal{A}^{889}_{ab} &= q^n_a \left(I_{8,b}I_{9,n} + I_{8,n}I_{9,b}\right) ,\\
\mathcal{A}^{999}_{ab} &= q^n_a\, I_{9,b}I_{9,n} .
\end{aligned}
\tag{6.7}
$$

Family 1. Let us consider two arbitrary but distinct members of the set $(4.23)_2$, namely

$$
\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\psi_1(X) \\ \pm\sin\psi_1(X) \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\psi_2(X) \\ \pm\sin\psi_2(X) \end{bmatrix}. \tag{6.8}
$$

These two vectors satisfy all the universality symmetry conditions for arbitrary $\psi_1(X)$ and $\psi_2(X)$, $\psi_1(X) \neq \psi_2(X)$. This means that (6.8) are universal material preferred directions for Family 1.

Families 2 and 3. For Families 2 and 3, let us consider two arbitrary but distinct members of the set $(4.32)_2$, namely

$$
\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\chi_1(R) \\ \pm\sin\chi_1(R) \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\chi_2(R) \\ \pm\sin\chi_2(R) \end{bmatrix}. \tag{6.9}
$$

The above two vectors satisfy all the universality conditions for arbitrary $\chi_1(R)$ and $\chi_2(R)$, $\chi_1(R) \neq \chi_2(R)$, i.e., (6.9) are universal material preferred directions for Families 2 & 3.

Family 5. In the case of orthotropic solids, Family 5 has two classes of universal material preferred directions (5.12). Let us consider two arbitrary but distinct members in Class 1 of universal solutions, namely

$$
\hat{\mathbf{N}}_1 = \begin{bmatrix} \cos\xi_1 \\ \pm\sin\xi_1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} \cos\xi_2 \\ \pm\sin\xi_2 \\ 0 \end{bmatrix}. \tag{6.10}
$$

These vectors satisfy all the universality symmetry conditions for arbitrary ξ_1 and ξ_2, $\xi_1 \neq \xi_2$, i.e., (6.10) are universal material preferred directions for Family 5.

Next we consider two arbitrary but distinct members in Class 2 of the transversely isotropic universal solutions, namely

$$
\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\eta_1 \\ \pm\sin\eta_1 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\eta_2 \\ \pm\sin\eta_2 \end{bmatrix}. \tag{6.11}
$$

It turns out that the above two vectors satisfy all the universality constraints other than $\mathscr{A}^8_{ab} = \mathscr{A}^8_{ba}$, which gives the following universality condition:

$$C_2 \sin\eta_1 \cos\eta_2 = 0\,. \tag{6.12}$$

If $C_2 = 0$, (6.11) are universal material preferred directions for arbitrary η_1 and η_2 as long as $\eta_1 \neq \eta_2$. This is similar to what was observed in footnote 3 for transversely isotropic solids. Considering the full set of universal deformations (4.46), the universality conditions are: $\sin\eta_1 \cos\eta_2 = 0$. The cases $\sin\eta_1 = 0$, and $\cos\eta_2 = 0$ were discussed in Remark 4.6. Therefore, we have the following two classes of universal material preferred directions

$$\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\eta \\ \pm\sin\eta \end{bmatrix}, \quad \sin\eta \neq 0\,, \tag{6.13}$$

and

$$\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\eta \\ \pm\sin\eta \end{bmatrix}, \quad \cos\eta \neq 0\,. \tag{6.14}$$

In summary, we have the following three classes of universal material preferred directions for Family 5:

$$\text{Class (i)}: \quad \hat{\mathbf{N}}_1 = \begin{bmatrix} \cos\xi_1 \\ \pm\sin\xi_1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} \cos\xi_2 \\ \pm\sin\xi_2 \\ 0 \end{bmatrix}, \quad \xi_1 \neq \xi_2\,, \tag{6.15}$$

$$\text{Class (ii)}: \quad \hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\eta \\ \pm\sin\eta \end{bmatrix}, \quad \sin\eta \neq 0\,, \tag{6.16}$$

$$\text{Class (iii)}: \quad \hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\eta \\ \pm\sin\eta \end{bmatrix}, \quad \cos\eta \neq 0\,. \tag{6.17}$$

Class (i) corresponds to two families of fibers that are parallel to the (R, Θ) plane and are distributed uniformly in two distinct fixed directions. In Class (ii) one family of fibers are concentric circles parallel to the (R, Θ) plane, and the second family of fibers are arranged helically, i.e., a combination of fibers in Figs. 1(b) and (d). Note that the two families of fibers are not mechanically equivalent, in general. In Class (iii) one family of fibers are distributed uniformly parallel to the axis of the wedge, and the second family of fibers are arranged helically, i.e., a combination of fibers in Figs. 1(b) and (c). Table 3 summarizes our results for incompressible monoclinic solids.

Remark 6.1. For Families 1, 2, and 3, the monoclinic universal material preferred directions are reduced to those of orthotropic solids when $\mathbf{N}_1 \cdot \mathbf{N}_2 = 0$. For Family 5, the same thing happens for Class (i) solutions. However, for Class (ii) solutions the monoclinic universality constraints force one family of fibers to be either parallel lines or concentric circles. When $\mathbf{N}_1 \cdot \mathbf{N}_2 = 0$, this recovers only a subset of the corresponding orthotropic solutions.

7. Concluding remarks

We have shown that the universal deformations for compressible transversely isotropic, orthotropic, and monoclinic solids are homogeneous and the universal material preferred directions are uniform. In the case of incompressible transversely isotropic, orthotropic, and monoclinic solids, in addition to the nine universality constraints for isotropic solids that were derived by Ericksen (1954), there are extra 25, 74, and 152, respectively, extra universality constraints that need to be satisfied. For each of the six known families of universal deformations for isotropic solids we obtained the corresponding universal material preferred directions assuming that the material preferred directions share the symmetries of the right Cauchy–Green strain. Tables 1, 2, and 3 summarize our results for incompressible transversely isotropic, orthotropic, and monoclinic solids. This classification of universal solutions provides a collection of solutions that can be used for applications and restrict the possible choice of new solutions to material preferred directions that do not preserve the underlying symmetry of the deformations. We believe that these solutions are unlikely to exist and we conjecture that this classification, like the cases of isotropic incompressible solids, and isotropic anelastic solids is complete.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Table 3
Universal deformations and universal material preferred directions for incompressible monoclinic solids for the six known families of universal deformations. Note that for Family 3, $K = C_1 C_4 - C_2 C_3$. For monoclinic solids Family 4 is not universal. Also, note that $\hat{\mathbf{N}}_3$ is normal to the plane of $\hat{\mathbf{N}}_1$ and $\hat{\mathbf{N}}_2$.

Family	Universal deformations	$\mathbf{C}^\flat$	Universal material preferred directions
0	$x^a(X) = F^a{}_A X^A + c^a$	$C_{AB} = F^a{}_A F^a{}_A \delta_{ab}$	Any two non-parallel constant unit vectors $\hat{\mathbf{N}}_1$, and $\hat{\mathbf{N}}_2$
1	$\begin{cases} r(X,Y,Z) = \sqrt{C_1(2X+C_4)} \\ \theta(X,Y,Z) = C_2(Y+C_5) \\ z(X,Y,Z) = \frac{Z}{C_1C_2} - C_2C_3Y + C_6 \end{cases}$	$[C_{AB}] = \begin{bmatrix} \frac{C_1}{2X+C_4} & 0 & 0 \\ 0 & C_2^2\left[C_1(2X+C_4)+C_3^2\right] & -\frac{C_3}{C_1} \\ 0 & -\frac{C_3}{C_1} & \frac{1}{C_1^2C_2^2} \end{bmatrix}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\psi_1(X) \\ \pm\sin\psi_1(X) \end{bmatrix}$, $\hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\psi_2(X) \\ \pm\sin\psi_2(X) \end{bmatrix}$
2	$\begin{cases} x(R,\Theta,Z) = \frac{1}{2}C_1C_2^2R^2 + C_4 \\ y(R,\Theta,Z) = \frac{\Theta}{C_1C_2} + C_5 \\ z(R,\Theta,Z) = \frac{C_3}{C_1C_2}\Theta + \frac{1}{C_2}Z + C_6 \end{cases}$	$[C_{AB}] = \begin{bmatrix} C_1^2C_2^4R^2 & 0 & 0 \\ 0 & \frac{C_3^2+1}{C_1^2C_2^2} & \frac{C_3}{C_1C_2^2} \\ 0 & \frac{C_3}{C_1C_2^2} & \frac{1}{C_2^2} \end{bmatrix}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\chi_1(R) \\ \pm\sin\chi_1(R) \end{bmatrix}$, $\hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\chi_2(R) \\ \pm\sin\chi_2(R) \end{bmatrix}$, $\chi_1(R) \neq \chi_2(R)$
3	$\begin{cases} r(R,\Theta,Z) = \sqrt{\frac{R^2}{C_1C_4-C_2C_3} + C_5} \\ \theta(R,\Theta,Z) = C_1\Theta + C_2Z + C_6 \\ z(R,\Theta,Z) = C_3\Theta + C_4Z + C_7 \end{cases}$	$[C_{AB}] = \begin{bmatrix} \frac{R^2}{K(KC_5+R^2)} & 0 & 0 \\ 0 & C_1^2\left(\frac{R^2}{K}+C_5\right)+C_3^2 & C_1C_2\left(\frac{R^2}{K}+C_5\right)+C_3C_4 \\ 0 & C_1C_2\left(\frac{R^2}{K}+C_5\right)+C_3C_4 & C_2^2\left(\frac{R^2}{K}+C_5\right)+C_4^2 \end{bmatrix}$	$\hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \cos\chi_1(R) \\ \pm\sin\chi_1(R) \end{bmatrix}$, $\hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\chi_2(R) \\ \pm\sin\chi_2(R) \end{bmatrix}$, $\chi_1(R) \neq \chi_2(R)$
5	$\begin{cases} r(R,\Theta,Z) = C_1R \\ \theta(R,\Theta,Z) = C_2\log R + C_3\Theta + C_4 \\ z(R,\Theta,Z) = \frac{1}{C_1^2C_3}Z + C_5 \end{cases}$	$[C_{AB}] = \begin{bmatrix} C_1^2\left(C_2^2+1\right) & C_1^2C_2C_3R & 0 \\ C_1^2C_2C_3R & C_1^2C_3^2R^2 & 0 \\ 0 & 0 & \frac{1}{C_1^4C_3^2} \end{bmatrix}$	$\begin{cases} \hat{\mathbf{N}}_1 = \begin{bmatrix} \cos\xi_1 \\ \pm\sin\xi_1 \\ 0 \end{bmatrix}, \\ \hat{\mathbf{N}}_2 = \begin{bmatrix} \cos\xi_2 \\ \pm\sin\xi_2 \\ 0 \end{bmatrix}, \quad \xi_1 \neq \xi_2, \\ \hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\eta \\ \pm\sin\eta \end{bmatrix}, \\ \sin\eta \neq 0, \\ \hat{\mathbf{N}}_1 = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \quad \hat{\mathbf{N}}_2 = \begin{bmatrix} 0 \\ \cos\eta \\ \pm\sin\eta \end{bmatrix}, \\ \cos\eta \neq 0. \end{cases}$

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