PROCEEDINGS A

royalsocietypublishing.org/journal/rspa

Research

Cite this article: Yavari A. 2021 Universal deformations in inhomogeneous isotropic nonlinear elastic solids. *Proc. R. Soc. A* **477**: 20210547.
https://doi.org/10.1098/rspa.2021.0547

Received: 6 July 2021
Accepted: 2 September 2021

Subject Areas:
mechanical engineering, mechanics, mathematical physics

Keywords:
universal deformations, nonlinear elasticity, isotropic solids, inhomogeneous solids

Author for correspondence:
Arash Yavari
e-mail: arash.yavari@ce.gatech.edu

Universal deformations in inhomogeneous isotropic nonlinear elastic solids

Arash Yavari

School of Civil and Environmental Engineering, and The George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA

AY, 0000-0002-7088-7984

Universal (controllable) deformations of an elastic solid are those deformations that can be maintained for all possible strain-energy density functions and suitable boundary tractions. Universal deformations have played a central role in nonlinear elasticity and anelasticity. However, their classification has been mostly established for homogeneous isotropic solids following the seminal works of Ericksen. In this article, we extend Ericksen's analysis of universal deformations to inhomogeneous compressible and incompressible isotropic solids. We show that a necessary condition for the known universal deformations of homogeneous isotropic solids to be universal for inhomogeneous solids is that inhomogeneities respect the symmetries of the deformations. Symmetries of a deformation are encoded in the symmetries of its pulled-back metric (the right Cauchy–Green strain). We show that this necessary condition is sufficient as well for all the known families of universal deformations except for Family 5.

1. Introduction

For a given class of solids, it turns out that one cannot deform an elastic body to an arbitrary shape by only applying boundary tractions; most likely body forces are needed to maintain the desired deformation. Those deformations that can be maintained by only applying boundary tractions are called universal or controllable [1,2]. The set of universal deformations explicitly depends on the class of materials. In the case of (unconstrained) compressible isotropic elastic solids, Ericksen [3] proved that the only universal deformations are homogeneous deformations. In the

THE ROYAL SOCIETY PUBLISHING

case of incompressible isotropic solids, motivated by the earlier works of Rivlin [4–6], Ericksen [7] found four families of universal deformations. In his analysis, he conjectured that a deformation whose principal invariants are constant must be homogeneous. This conjecture turned out to be incorrect [8]. A fifth family of inhomogeneous universal deformations with constant principal invariants was discovered independently by Singh & Pipkin [9] and Klingbeil & Shield [10]. The six known families of universal deformations are as follows:

— Family 0: Homogeneous deformations.
— Family 1: Bending, stretching and shearing of a rectangular block.
— Family 2: Straightening, stretching and shearing of a sector of a cylindrical shell.
— Family 3: Inflation, bending, torsion, extension and shearing of a sector of an annular wedge.
— Family 4: Inflation/inversion of a sector of a spherical shell.
— Family 5: Inflation, bending, extension and azimuthal shearing of an annular wedge.

For incompressible isotropic solids, Ericksen's problem has not been completely solved to this date; the case of deformations with constant principal invariants is still open. However, the conjecture is that there is no other family of inhomogeneous isochoric universal deformations with constant principal invariants other than Family 5.

Ericksen's work has recently been generalized to anelasticity. In the case of compressible anelastic solids, universal deformations are covariantly homogeneous [11]. In the case of incompressible isotropic solids with finite eigenstrains, Goodbrake *et al.* [12] suggested that universal eigenstrain distributions (that are modelled by a material Riemannian metric) should follow the same symmetry as the deformations. In particular, they showed that all the six known families of universal deformations are invariant under the action of certain Lie subgroups of the special Euclidean group.

Yavari & Goriely [13] extended Ericksen's analysis to compressible and incompressible transversely isotropic, orthotropic and monoclinic solids. They showed that the universality constraints of incompressible anisotropic solids include those of incompressible isotropic solids. For each known family of universal deformations for isotropic solids they obtained the corresponding universal material preferred directions. The analogue of universal deformations in linear elasticity are universal displacements [14–16]. It turns out that universal displacements explicitly depend on the symmetry class of the material. More specifically, the smaller the material symmetry group, the smaller the corresponding space of universal displacements [16].

Golgoon & Yavari [17] observed that radial deformations of spherical shells are universal even for radially inhomogeneous transversely isotropic spherical shells with radial material preferred direction. This means that, in particular, Family 4 is universal for radially inhomogeneous incompressible isotropic solids. To this date, the study of universal deformations has been restricted to homogeneous solids. One may ask if Family 4 can admit other forms of material inhomogeneity. The more general question is: What are the universal deformations for inhomogeneous compressible and incompressible isotropic solids? And what forms of inhomogeneity can accommodate universal deformations? These questions will be answered in this article.

We will consider both inhomogeneous compressible and incompressible isotropic solids. We find the universality constraints that are imposed by the equilibrium equations in the absence of body forces and the arbitrariness of the inhomogeneous energy function. It will be seen that the set of universality constraints for each material class includes those of the corresponding homogenous solids. For compressible solids, the universality constraints force the universal deformations to be homogeneous. The extra universality constraints force the energy function to be homogeneous. This implies that inhomogeneous compressible isotropic solids do not admit universal deformations. In the case of incompressible solids for each of the six known families of universal deformations, we find the corresponding universal material inhomogeneity.

This article is organized as follows. In §2, we briefly review nonlinear elasticity. In §3, we consider inhomogeneous compressible isotropic solids. In §4, the universal deformations and universal inhomogeneities of incompressible isotropic solids are analysed for each of the known six families. Conclusions are given in §5.

2. Nonlinear elasticity

(a) Kinematics

In nonlinear elasticity, a body $\mathcal{B}$ is identified with a flat Riemannian manifold $(\mathcal{B},\mathbf{G})$, which is a submanifold of the Euclidean 3-space $(\mathcal{S},\mathbf{g})$ [18]. $\mathbf{G}$ is the material metric, which is induced from the ambient space metric $\mathbf{g}$. A deformation is a mapping $\varphi:\mathcal{B}\to\mathcal{S}$. The deformation gradient is the tangent map (or derivative) of φ and is denoted by $\mathbf{F}=T\varphi$. The deformation gradient at each material point $\mathbf{X}\in\mathcal{B}$ is a linear map $\mathbf{F}(\mathbf{X}):T_{\mathbf{X}}\mathcal{B}\to T_{\varphi(\mathbf{X})}\mathcal{S}$. With respect to local (curvilinear) coordinates $\{x^a\}$ and $\{X^A\}$ on $\mathcal{S}$ and $\mathcal{B}$, respectively, the deformation gradient has the following components:

$$F^a{}_A(\mathbf{X})=\frac{\partial\varphi^a}{\partial X^A}(\mathbf{X}). \tag{2.1}$$

The transpose of deformation gradient is defined as follows:

$$\mathbf{F}^{\mathsf{T}}:T_{\mathbf{x}}\mathcal{S}\to T_{\mathbf{X}}\mathcal{B},\quad \langle\!\langle\mathbf{FV},\mathbf{v}\rangle\!\rangle_{\mathbf{g}}=\langle\!\langle\mathbf{V},\mathbf{F}^{\mathsf{T}}\mathbf{v}\rangle\!\rangle_{\mathbf{G}},\quad \forall\mathbf{V}\in T_{\mathbf{X}}\mathcal{B},\mathbf{v}\in T_{\mathbf{x}}\mathcal{S}, \tag{2.2}$$

which in components reads

$$(F^{\mathsf{T}}(\mathbf{X}))^A{}_a=g_{ab}(\mathbf{x})F^b{}_B(\mathbf{X})G^{AB}(\mathbf{X}). \tag{2.3}$$

Another measure of strain is the right Cauchy–Green deformation tensor (or strain), which is defined as $\mathbf{C}(X)=\mathbf{F}(\mathbf{X})^{\mathsf{T}}\mathbf{F}(\mathbf{X}):T_{\mathbf{X}}\mathcal{B}\to T_{\mathbf{X}}\mathcal{B}$ and has components $C^A{}_B=(F^{\mathsf{T}})^A{}_aF^a{}_B$. Note that $C_{AB}=(g_{ab}\circ\varphi)F^a{}_AF^b{}_B$, which implies that the right Cauchy–Green strain is the pulled-back metric, i.e. $\mathbf{C}^\flat=\varphi^*(\mathbf{g})$, where $\flat$ is the flat operator induced by the metric $\mathbf{g}$ and is used for lowering indices. The left Cauchy–Green strain is defined as $\mathbf{B}^\sharp=\varphi^*(\mathbf{g}^\sharp)$ and has components $B^{AB}=(F^{-1})^A{}_a(F^{-1})^B{}_bg^{ab}$. The spatial analogues of $\mathbf{C}^\flat$ and $\mathbf{B}^\sharp$ are denoted by $\mathbf{c}^\flat$ and $\mathbf{b}^\sharp$, respectively, and are defined as follows:

$$\left.\begin{aligned}&\mathbf{c}^\flat=\varphi_*(\mathbf{G}),\quad c_{ab}=(F^{-1})^A{}_a(F^{-1})^B{}_bG_{AB}\\ \text{and}\qquad &\mathbf{b}^\sharp=\varphi_*(\mathbf{G}^\sharp),\quad b^{ab}=F^a{}_AF^b{}_BG^{AB}.\end{aligned}\right\} \tag{2.4}$$

$\mathbf{b}^\sharp$ is called the Finger deformation tensor. The tensors $\mathbf{C}$ and $\mathbf{b}$ have the same principal invariants I_1, I_2 and I_3, which are defined as follows [18,19]:

$$\left.\begin{aligned}&I_1=\operatorname{tr}\mathbf{b}=b^a{}_a=b^{ab}g_{ab},\\ &I_2=\frac{1}{2}(I_1^2-\operatorname{tr}\mathbf{b}^2)=\frac{1}{2}(I_1^2-b^a{}_bb^b{}_a)=\frac{1}{2}(I_1^2-b^{ab}b^{cd}g_{ac}g_{bd})\\ \text{and}\qquad &I_3=\det\mathbf{b}.\end{aligned}\right\} \tag{2.5}$$

(b) Balance of linear and angular momenta

The balance of linear and angular momenta in the absence of inertial effects in material form read

$$\operatorname{Div}\mathbf{P}+\rho_0\mathbf{B}=\mathbf{0},\quad \mathbf{PF}^{\mathsf{T}}=\mathbf{FP}^{\mathsf{T}}, \tag{2.6}$$

where $\mathbf{B}$ is body force per unit undeformed volume, ρ_0 is the material mass density and $\mathbf{P}$ is the first Piola–Kirchhoff stress. $\mathbf{P}$ is related to the Cauchy stress $\boldsymbol{\sigma}$ as $J\sigma^{ab}=P^{aA}F^b{}_A$, where J is the Jacobian of deformation that relates the material (dV) and spatial (dv) Riemannian volume forms as $dv = JdV$ and is defined as follows

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}. \tag{2.7}$$

In terms of the Cauchy stress $\boldsymbol{\sigma}$, the balance of linear and angular momenta read

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \mathbf{0}, \quad \boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma}, \tag{2.8}$$

where $\mathbf{b} = \mathbf{B} \circ \varphi_t^{-1}$, and $\rho = J^{-1}\rho_0$ is the spatial mass density. In components, balance of linear momentum reads $\sigma^{ab}{}_{|b} + \rho b^a = 0$, where

$$\sigma^{ab}{}_{|b} = \sigma^{ab}{}_{,b} + \gamma^a{}_{bc}\sigma^{cb} + \gamma^b{}_{bc}\sigma^{ac}. \tag{2.9}$$

$\gamma^c{}_{ab}$ are the Christoffel symbols of the ambient space metric $\mathbf{g}$ and are defined as follows:

$$\gamma^a{}_{bc} = \frac{1}{2} g^{ak}(g_{kb,c} + g_{kc,b} - g_{bc,k}). \tag{2.10}$$

(c) Constitutive equations

In the case of an inhomogeneous isotropic hyperelastic solid, the energy function (per unit undeformed volume) is written as $W = \hat{W}(\mathbf{X}, \mathbf{C}^\flat, \mathbf{G})$. For an isotropic solid, the energy function can be rewritten as $W = W(\mathbf{X}, I_1, I_2, I_3)$, where I_1, I_2 and I_3 are the principal invariants of the right Cauchy–Green deformation tensor that are given in (2.5). The Cauchy stress has the following representation [20]:

$$\sigma^{ab} = \frac{2}{\sqrt{I_3}}[W_1 b^{ab} + (I_2 W_2 + I_3 W_3) g^{ab} - I_3 W_2 c^{ab}], \tag{2.11}$$

where

$$W_i = W_i(\mathbf{X}, I_1, I_2, I_3) = \frac{\partial W(\mathbf{X}, I_1, I_2, I_3)}{\partial I_i}, \quad i = 1, 2, 3, \tag{2.12}$$

and $c^{ab} = (F^{-1})^M{}_m (F^{-1})^N{}_n G_{MN} g^{am} g^{bn}$. For incompressible isotropic solids ($I_3 = J^2 = 1$), the Cauchy stress has the following representation [20]:

$$\sigma^{ab} = -p g^{ab} + 2W_1 b^{ab} - 2W_2 c^{ab}, \tag{2.13}$$

where p is a Lagrange multiplier associated with the incompressibility constraint $J = \sqrt{I_3} = 1$.

3. Inhomogeneous compressible isotropic solids

For an inhomogeneous compressible isotropic solid, the Cauchy stress representation is given in (2.11). The ambient space is Euclidean, and hence one can use a single Cartesian coordinate chart $\{x^a\}$, so that $g_{ab} = \delta_{ab}$. Thus,

$$\sigma^{ab} = \frac{2}{\sqrt{I_3}}[W_1 b^{ab} + (I_2 W_2 + I_3 W_3)\delta^{ab} - I_3 W_2 c^{ab}]. \tag{3.1}$$

In the absence of body forces, the equilibrium equations in Cartesian coordinates read $\sigma^{ab}{}_{,b} = 0$. Note that

$$\left.\begin{aligned} W_{1,b} &= (F^{-1})^A{}_b \frac{\partial^2 W}{\partial X^A \partial I_1} + \frac{\partial^2 W}{\partial I_1^2} I_{1,b} + \frac{\partial^2 W}{\partial I_1 \partial I_2} I_{2,b} + \frac{\partial^2 W}{\partial I_1 \partial I_3} I_{3,b}, \\ W_{2,b} &= (F^{-1})^A{}_b \frac{\partial^2 W}{\partial X^A \partial I_2} + \frac{\partial^2 W}{\partial I_1 \partial I_2} I_{1,b} + \frac{\partial^2 W}{\partial I_2^2} I_{2,b} + \frac{\partial^2 W}{\partial I_2 \partial I_3} I_{3,b} \\ \text{and} \quad W_{3,b} &= (F^{-1})^A{}_b \frac{\partial^2 W}{\partial X^A \partial I_3} + \frac{\partial^2 W}{\partial I_1 \partial I_3} I_{1,b} + \frac{\partial^2 W}{\partial I_2 \partial I_3} I_{2,b} + \frac{\partial^2 W}{\partial I_3^2} I_{3,b}. \end{aligned}\right\} \tag{3.2}$$

These can be written more compactly as follows

$$\left.\begin{aligned} W_{1,b} &= (F^{-1})^A{}_b W_{1,A} + W_{11} I_{1,b} + W_{12} I_{2,b} + W_{13} I_{3,b}, \\ W_{2,b} &= (F^{-1})^A{}_b W_{2,A} + W_{12} I_{1,b} + W_{22} I_{2,b} + W_{23} I_{3,b} \\ \text{and} \quad W_{3,b} &= (F^{-1})^A{}_b W_{3,A} + W_{13} I_{1,b} + W_{23} I_{2,b} + W_{33} I_{3,b}, \end{aligned}\right\} \tag{3.3}$$

where

$$W_{i,A} = \frac{\partial W_i}{\partial X^A} \quad \text{and} \quad W_{ij} = \frac{\partial^2 W}{\partial I_i \partial I_j}, \quad i \le j. \tag{3.4}$$

The first term on the right-hand side of each equation in (3.3) vanishes for homogeneous solids [3,11]. Substituting (3.3) into the equilibrium equations, one obtains

$$\begin{aligned} &\left[-\frac{I_{3,b}}{2I_3} b^{ab} + b^{ab}{}_{,b}\right] W_1 + \left[-\frac{I_{3,b}}{2I_3}(I_2 \delta^{ab} - I_3 c^{ab}) + I_{2,b}\delta^{ab} - I_{3,b} c^{ab} - I_3 c^{ab}{}_{,b}\right] W_2 \\ &+ \frac{1}{2} I_{3,b}\delta^{ab} W_3 + b^{ab} I_{1,b} W_{11} + I_{2,b}(I_2\delta^{ab} - I_3 c^{ab}) W_{22} + I_3 I_{3,b}\delta^{ab} W_{33} \\ &+ [I_{1,b}(I_2\delta^{ab} - I_3 c^{ab}) + I_{2,b} b^{ab}] W_{12} + (b^{ab} I_{3,b} + \delta^{ab} I_{1,b} I_3) W_{13} \\ &+ [I_{3,b}(I_2\delta^{ab} - I_3 c^{ab}) + I_3 I_{2,b}\delta^{ab}] W_{23} \\ &+ (F^{-1})^A{}_b b^{ab} W_{1,A} + (F^{-1})^A{}_b (I_2\delta^{ab} - I_3 c^{ab}) W_{2,A} + (F^{-1})^A{}_b I_3 \delta^{ab} W_{3,A} = 0. \end{aligned} \tag{3.5}$$

The aforementioned identity must hold for any choice of $W = W(\mathbf{X}, I_1, I_2, I_3)$. This means that the partial derivatives of W can vary independently. Thus, in particular, the coefficients of the partial derivatives W_1, W_2, W_3, W_{11}, W_{22}, W_{33}, W_{12}, W_{23} and W_{31} must vanish independently, and hence, one obtains Ericksen's universality constraints for homogeneous compressible isotropic solids [3,11]:

$$-\frac{I_{3,b}}{2I_3} b^{ab} + b^{ab}{}_{,b} = 0, \tag{3.6}$$

$$-\frac{I_{3,b}}{2I_3}(I_2\delta^{ab} - I_3 c^{ab}) + I_{2,b}\delta^{ab} - I_{3,b} c^{ab} - I_3 c^{ab}{}_{,b} = 0, \tag{3.7}$$

$$I_{3,b}\delta^{ab} = 0, \tag{3.8}$$

$$b^{ab} I_{1,b} = 0, \tag{3.9}$$

$$I_{2,b}\,(I_2\delta^{ab} - I_3 c^{ab}) = 0, \tag{3.10}$$

$$I_3 I_{3,b}\delta^{ab} = 0, \tag{3.11}$$

$$I_{1,b}\,(I_2\delta^{ab} - I_3 c^{ab}) + I_{2,b} b^{ab} = 0, \tag{3.12}$$

$$b^{ab} I_{3,b} + \delta^{ab} I_{1,b} I_3 = 0 \tag{3.13}$$

and

$$I_{3,b}\,(I_2\delta^{ab} - I_3 c^{ab}) + I_3 I_{2,b}\delta^{ab} = 0. \tag{3.14}$$

In addition to the aforementioned constraints, for inhomogeneous solids from (3.5), one has the following extra universality constraints

$$\left.\begin{aligned} &b^{ab}(F^{-1})^A{}_b W_{1,A} = 0, \\ &(I_2\delta^{ab} - I_3 c^{ab})(F^{-1})^A{}_b W_{2,A} = 0 \\ \text{and} \quad &I_3\delta^{ab}(F^{-1})^A{}_b W_{3,A} = 0. \end{aligned}\right\} \tag{3.15}$$

From the constraints (3.6)–(3.14), one obtains Erisksen's conditions:

$$I_1, I_2, I_3 \text{ are constants, and } b^{ab}{}_{,b} = c^{ab}{}_{,b} = 0. \tag{3.16}$$

By using Ericksen's conditions and the compatibility equations, one can show that the universal deformations must be homogeneous [3]. Knowing that the tensors b^{ab}, $I_2\delta^{ab} - I_3c^{ab}$ and $I_3\delta^{ab}$ are invertible,[1] from (3.15), one concludes that

$$(F^{-1})^A{}_b W_{1,A} = (F^{-1})^A{}_b W_{2,A} = (F^{-1})^A{}_b W_{3,A} = 0, \tag{3.18}$$

which in turn implies that

$$W_{1,A} = W_{2,A} = W_{3,A} = 0. \tag{3.19}$$

Note that

$$W_{1,A} = \frac{\partial}{\partial X^A}\frac{\partial W}{\partial I_1} = \frac{\partial}{\partial I_1}\frac{\partial W}{\partial X^A} = 0. \tag{3.20}$$

Similarly, using $W_{2,A} = W_{3,A} = 0$, one obtains

$$\frac{\partial}{\partial I_2}\frac{\partial W}{\partial X^A} = \frac{\partial}{\partial I_3}\frac{\partial W}{\partial X^A} = 0. \tag{3.21}$$

This means that

$$\frac{\partial W}{\partial X^1} = f_1(\mathbf{X}), \quad \frac{\partial W}{\partial X^2} = f_2(\mathbf{X}), \quad \frac{\partial W}{\partial X^3} = f_3(\mathbf{X}), \tag{3.22}$$

for some scalar functions f_A. In particular, note that $\partial f_1/\partial X^2 = \partial f_2/\partial X^1$, and $\partial f_1/\partial X^3 = \partial f_3/\partial X^1$. From $(3.22)_1$, one concludes that

$$W(\mathbf{X}, I_1, I_2, I_3) = \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3)\,\mathrm{d}X^1 + h(X^2, X^3, I_1, I_2, I_3), \tag{3.23}$$

where h is some scalar function and X_0^1 is some fixed value of X^1. Taking the partial derivative with respect to X^2 of both sides, one obtains

$$\begin{aligned}
\frac{\partial W}{\partial X^2} &= \frac{\partial}{\partial X^2}\int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3)\,\mathrm{d}X^1 + \frac{\partial h(X^2, X^3, I_1, I_2, I_3)}{\partial X^2},\\
&= \int_{X_0^1}^{X^1} \frac{\partial f_1(X^1, X^2, X^3)}{\partial X^2}\,\mathrm{d}X^1 + \frac{\partial h(X^2, X^3, I_1, I_2, I_3)}{\partial X^2},\\
&= \int_{X_0^1}^{X^1} \frac{\partial f_2(X^1, X^2, X^3)}{\partial X^1}\,\mathrm{d}X^1 + \frac{\partial h(X^2, X^3, I_1, I_2, I_3)}{\partial X^2},\\
&= f_2(X^1, X^2, X^3) - f_2(X_0^1, X^2, X^3) + \frac{\partial h(X^2, X^3, I_1, I_2, I_3)}{\partial X^2}.
\end{aligned} \tag{3.24}$$

From (3.24) and $(3.22)_2$, one concludes that

$$\frac{\partial h(X^2, X^3, I_1, I_2, I_3)}{\partial X^2} = f_2(X_0^1, X^2, X^3). \tag{3.25}$$

Thus,

$$\int_{X_0^2}^{X^2} \frac{\partial h(X^2, X^3, I_1, I_2, I_3)}{\partial X^2}\,\mathrm{d}X^2 = \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3)\,\mathrm{d}X^2, \tag{3.26}$$

[1]When expressed in the principal directions of c^{ab}, one has

$$\begin{aligned}
[I_2\delta^{ab} - I_3c^{ab}] &= (\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \lambda_1^2\lambda_2^2\lambda_3^2\begin{bmatrix}\frac{1}{\lambda_1^2} & 0 & 0\\ 0 & \frac{1}{\lambda_2^2} & 0\\ 0 & 0 & \frac{1}{\lambda_3^2}\end{bmatrix}\\
&= \begin{bmatrix}\lambda_1^2(\lambda_2^2+\lambda_3^2) & 0 & 0\\ 0 & \lambda_2^2(\lambda_1^2+\lambda_3^2) & 0\\ 0 & 0 & \lambda_3^2(\lambda_1^2+\lambda_2^2)\end{bmatrix},
\end{aligned} \tag{3.17}$$

where λ_1^2, λ_2^2 and λ_3^2 are the eigenvalues of $\mathbf{b}^\sharp$. Clearly, $[I_2\delta^{ab} - I_3c^{ab}]$ is invertible.

where X_0^2 is some fixed value of X^2. Hence,

$$h(X^2, X^3, I_1, I_2, I_3) = h(X_0^2, X^3, I_1, I_2, I_3) + \int_{X_0^2}^{X^2} f_2(X_0^1, X^2, X^3)\, dX^2. \tag{3.27}$$

This implies that

$$h(X^2, X^3, I_1, I_2, I_3) = H(X^3, I_1, I_2, I_3) + K(X^2, X^3). \tag{3.28}$$

By using the aforementioned relation in (3.23), one has

$$W(\mathbf{X}, I_1, I_2, I_3) = \int_{X_0^1}^{X^1} f_1(X^1, X^2, X^3)\, dX^1 + H(X^3, I_1, I_2, I_3) + K(X^2, X^3). \tag{3.29}$$

Taking the partial derivative with respect to X^3 of the aforementioned relation, one can show that

$$H(X^3, I_1, I_2, I_3) = \bar{W}(I_1, I_2, I_3) + M(X^3). \tag{3.30}$$

The aforementioned relation and (3.29) imply that $W(\mathbf{X}, I_1, I_2, I_3) = \hat{W}(\mathbf{X}) + \bar{W}(I_1, I_2, I_3)$. Note that the inhomogeneous term $\hat{W}(\mathbf{X})$ is mechanically inconsequential. In summary, we have proved the following result.

Proposition 3.1. *Inhomogeneous compressible nonlinear isotropic solids do not admit universal deformations.*

4. Inhomogeneous incompressible isotropic solids

For an incompressible isotropic solid, the equilibrium equations in the absence of body forces read

$$\frac{1}{2} p_{,b} g^{ab} = [W_1 b^{ab} - W_2 c^{ab}]_{|b} \quad \text{or} \quad \frac{1}{2} p_{,a} = g_{am} [W_1 b^{mn} - W_2 c^{mn}]_{|n}. \tag{4.1}$$

Hence,

$$\frac{1}{2} \mathbf{d}p = \frac{1}{2} p_{,a}\, dx^a = g_{am} [W_1 b^{mn} - W_2 c^{mn}]_{|n}\, dx^a, \tag{4.2}$$

where $\mathbf{d}$ is the exterior derivative. This means that[2]

$$\boldsymbol{\xi} = g_{am} [W_1 b^{mn} - W_2 c^{mn}]_{|n}\, dx^a = [W_1 b_a^n - W_2 c_a^n]_{|n}\, dx^a, \tag{4.3}$$

is an exact 1-form. Note that $\mathbf{d}\boldsymbol{\xi} = \mathbf{0}$, or equivalently $\xi_{a,b} = \xi_{b,a}$, is a necessary condition for $\boldsymbol{\xi}$ to be an exact form [21]. However, from $\xi_{a|b} = \xi_{a,b} - \gamma^c{}_{ab} \xi_c$, one concludes that $\xi_{a,b} = \xi_{b,a}$ is equivalent to $\xi_{a|b} = \xi_{b|a}$. The latter constraints are more convenient in curvilinear coordinates as the metric of the ambient space is covariantly constant, i.e. $g_{ab|c} = 0$. One can simplify ξ_a as follows:

$$\xi_a = [W_1 b_a^n - W_2 c_a^n]_{|n} = W_{1|n} b_a^n - W_{2|n} c_a^n + W_1 b_a^n{}_{|n} - W_2 c_a^n{}_{|n}. \tag{4.4}$$

Note that $W_i = W_i(\mathbf{X}, I_1, I_2)$, $i = 1, 2$, and hence,

$$\left.\begin{aligned} W_{1|n} &= (F^{-1})^A{}_n W_{1,A} + W_{11} I_{1,n} + W_{12} I_{2,n} \\ \text{and} \qquad W_{2|n} &= (F^{-1})^A{}_n W_{2,A} + W_{12} I_{1,n} + W_{22} I_{2,n}. \end{aligned}\right\} \tag{4.5}$$

From (4.4), one can write

$$\begin{aligned} \xi_{a|b} = {} & (W_{1|n})_{|b} b_a^n - (W_{2|n})_{|b} c_a^n + W_{1|n} b_a^n{}_{|b} - W_{2|n} c_a^n{}_{|b} \\ & + W_{1|b} b_a^n{}_{|n} - W_{2|b} c_a^n{}_{|n} + W_1 b_a^n{}_{|nb} - W_2 c_a^n{}_{|nb}. \end{aligned} \tag{4.6}$$

[2]Note that $b^n{}_a = b^{nm} g_{ma}$, and $b_a{}^n = g_{am} b^{mn}$, which are equal. Thus, we use the notation $b_a^n = b^n{}_a = b_a{}^n$. Similarly, the same notation is used for $\mathbf{c}$.

royalsocietypublishing.org/journal/rspa Proc. R. Soc. A 477: 20210547

By using (4.5), we have

$$
\begin{aligned}
(W_{1|n})_{|b} = {} & W_{11} I_{1|nb} + W_{12} I_{2|nb} + W_{111} I_{1,b} I_{1,n} \\
& + W_{112}(I_{2,b} I_{1,n} + I_{1,b} I_{2,n}) + W_{122} I_{2,b} I_{2,n} \\
& + [(F^{-1})^B{}_b (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb} (F^{-1})^A{}_m] W_{1,A} \\
& + [(F^{-1})^A{}_n I_{1,b} + (F^{-1})^A{}_b I_{1,n}] W_{11,A} \\
& + [(F^{-1})^A{}_n I_{2,b} + (F^{-1})^A{}_b I_{2,n}] W_{12,A} \\
& + [(F^{-1})^A{}_n (F^{-1})^B{}_b + (F^{-1})^B{}_n (F^{-1})^A{}_b] W_{1,AB},
\end{aligned} \tag{4.7}
$$

and

$$
\begin{aligned}
(W_{2|n})_{|b} = {} & W_{12} I_{1|nb} + W_{22} I_{2|nb} + W_{112} I_{1,b} I_{1,n} \\
& + W_{122}(I_{2,b} I_{1,n} + I_{1,b} I_{2,n}) + W_{222} I_{2,b} I_{2,n} \\
& + [(F^{-1})^B{}_b (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb} (F^{-1})^A{}_m] W_{2,A} \\
& + [(F^{-1})^A{}_n I_{2,b} + (F^{-1})^A{}_b I_{2,n}] W_{22,A} \\
& + [(F^{-1})^A{}_n I_{1,b} + (F^{-1})^A{}_b I_{1,n}] W_{12,A} \\
& + [(F^{-1})^A{}_n (F^{-1})^B{}_b + (F^{-1})^B{}_n (F^{-1})^A{}_b] W_{2,AB}.
\end{aligned} \tag{4.8}
$$

Therefore,

$$
\begin{aligned}
\xi_{a|b} = {} & (b^n_{a\,|nb}) W_1 - (c^n_{a\,|nb}) W_2 \\
& + [b^n_{a\,|n} I_{1,b} + (b^n_a I_{1,n})_{|b}] W_{11} - [c^n_{a\,|n} I_{2,b} + (c^n_a I_{2,n})_{|b}] W_{22} \\
& + \{b^n_{a\,|n} I_{2,b} + (b^n_a I_{2,n})_{|b} - [c^n_{a\,|n} I_{1,b} + (c^n_a I_{1,n})_{|b}]\} W_{12} \\
& + (b^n_a I_{1,n} I_{1,b}) W_{111} - (c^n_a I_{2,n} I_{2,b}) W_{222} \\
& + [b^n_a (I_{1,b} I_{2,n} + I_{1,n} I_{2,b}) - c^n_a I_{1,n} I_{1,b}] W_{112} \\
& + [b^n_a I_{2,b} I_{2,n} - c^n_a (I_{1,b} I_{2,n} + I_{1,n} I_{2,b})] W_{122} \\
& + \{(F^{-1})^A{}_n b^n_{a\,|b} + (F^{-1})^A{}_b b^n_{a\,|n} + b^n_a [(F^{-1})^B{}_b (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb} (F^{-1})^A{}_m]\} W_{1,A} \\
& - \{(F^{-1})^A{}_n c^n_{a\,|b} + (F^{-1})^A{}_b c^n_{a\,|n} + c^n_a [(F^{-1})^B{}_b (F^{-1})^A{}_{n,B} - \gamma^m{}_{nb} (F^{-1})^A{}_m]\} W_{2,A} \\
& + b^n_a [(F^{-1})^A{}_n I_{1,b} + (F^{-1})^A{}_b I_{1,n}] W_{11,A} - c^n_a [(F^{-1})^A{}_n I_{2,b} + (F^{-1})^A{}_b I_{2,n}] W_{22,A} \\
& + \{b^n_a [(F^{-1})^A{}_n I_{2,b} + (F^{-1})^A{}_b I_{2,n}] - c^n_a [(F^{-1})^A{}_n I_{1,b} + (F^{-1})^A{}_b I_{1,n}]\} W_{12,A} \\
& + b^n_a [(F^{-1})^A{}_n (F^{-1})^B{}_b + (F^{-1})^B{}_n (F^{-1})^A{}_b] W_{1,AB} \\
& - c^n_a [(F^{-1})^A{}_n (F^{-1})^B{}_b + (F^{-1})^B{}_n (F^{-1})^A{}_b] W_{2,AB}.
\end{aligned} \tag{4.9}
$$

The first nine terms appear for homogeneous solids as well. As W is an arbitrary function of its arguments, for $\xi_{a|b} = \xi_{b|a}$ to hold, it is necessary that the coefficients of W_κ, where κ is a multi-index, $\kappa \in \{1, 2, 11, 22, 12, 111, 222, 112, 122\}$, be symmetric. Therefore, the following nine terms must be symmetric [7]:

$$
\left.\begin{aligned}
\mathcal{A}^{1}_{ab} &= b^{n}_{a\,|bn},\\
\mathcal{A}^{2}_{ab} &= c^{n}_{a\,|bn},\\
\mathcal{A}^{11}_{ab} &= b^{n}_{a\,|n} I_{1,b} + (b^{n}_{a} I_{1,n})_{|b},\\
\mathcal{A}^{22}_{ab} &= c^{n}_{a\,|n} I_{2,b} + (c^{n}_{a} I_{2,n})_{|b},\\
\mathcal{A}^{12}_{ab} &= (b^{n}_{a} I_{2,n})_{|b} + b^{n}_{a\,|n} I_{2,b} - [(c^{n}_{a} I_{1,n})_{|b} + c^{n}_{a\,|n} I_{1,b}],\\
\mathcal{A}^{111}_{ab} &= b^{n}_{a} I_{1,n} I_{1,b},\\
\mathcal{A}^{222}_{ab} &= c^{n}_{a} I_{2,n} I_{2,b},\\
\mathcal{A}^{112}_{ab} &= b^{n}_{a}(I_{1,b} I_{2,n} + I_{1,n} I_{2,b}) - c^{n}_{a} I_{1,n} I_{1,b}\\
\text{and}\quad \mathcal{A}^{122}_{ab} &= b^{n}_{a} I_{2,b} I_{2,n} - c^{n}_{a}(I_{1,b} I_{2,n} + I_{1,n} I_{2,b}).
\end{aligned}\right\}\tag{4.10}
$$

It is known that symmetry of the aforementioned nine terms, in addition to homogenous deformations, admit five classes of deformations [7,9,10].

For inhomogeneous solids, in addition to Ericksen's symmetry conditions (4.10), from (4.9), the following seven groups of terms (for $A = 1, 2, 3$, and $B \geq A$) must be symmetric as well:

$$
\left.\begin{aligned}
\mathcal{C}^{1A}_{ab} &= (F^{-1})^{A}{}_{n} b^{n}_{a\,|b} + (F^{-1})^{A}{}_{b} b^{n}_{a\,|n}\\
&\quad + b^{n}_{a}[(F^{-1})^{B}{}_{b}(F^{-1})^{A}{}_{n,B} - \gamma^{m}{}_{nb}(F^{-1})^{A}{}_{m}],\\
\mathcal{C}^{2A}_{ab} &= (F^{-1})^{A}{}_{n} c^{n}_{a\,|b} + (F^{-1})^{A}{}_{b} c^{n}_{a\,|n}\\
&\quad + c^{n}_{a}[(F^{-1})^{B}{}_{b}(F^{-1})^{A}{}_{n,B} - \gamma^{m}{}_{nb}(F^{-1})^{A}{}_{m}],\\
\mathcal{C}^{11A}_{ab} &= b^{n}_{a}[(F^{-1})^{A}{}_{n} I_{1,b} + (F^{-1})^{A}{}_{b} I_{1,n}],\\
\mathcal{C}^{22A}_{ab} &= c^{n}_{a}[(F^{-1})^{A}{}_{n} I_{2,b} + (F^{-1})^{A}{}_{b} I_{2,n}],\\
\mathcal{C}^{12A}_{ab} &= b^{n}_{a}[(F^{-1})^{A}{}_{n} I_{2,b} + (F^{-1})^{A}{}_{b} I_{2,n}] - c^{n}_{a}[(F^{-1})^{A}{}_{n} I_{1,b} + (F^{-1})^{A}{}_{b} I_{1,n}],\\
\mathcal{C}^{1AB}_{ab} &= b^{n}_{a}[(F^{-1})^{A}{}_{n}(F^{-1})^{B}{}_{b} + (F^{-1})^{B}{}_{n}(F^{-1})^{A}{}_{b}]\\
\text{and}\quad \mathcal{C}^{2AB}_{ab} &= c^{n}_{a}[(F^{-1})^{A}{}_{n}(F^{-1})^{B}{}_{b} + (F^{-1})^{B}{}_{n}(F^{-1})^{A}{}_{b}].
\end{aligned}\right\}\tag{4.11}
$$

The aforementioned 27 symmetry constraints restrict the form of the inhomogeneity of the elastic body. For a family of deformations consistent with (4.10), we call the corresponding inhomogeneities that respect (4.11) the *universal inhomogeneities*. In the sequel, for each of the six known families of universal deformations, we will find the corresponding universal inhomogeneities. More specifically, for a given family, if a term in (4.11) cannot be symmetric, then the corresponding derivative of W must vanish. This will then restrict the form of the inhomogeneity, i.e. the explicit dependence of W on X^A.

(a) Family 0: homogeneous deformations

For homogeneous deformations, the deformation mapping has the component form $x^a(\mathbf{X}) = F^a{}_A X^A + c^a$, where $[F^a{}_A]$ is a constant matrix and c^a are components of a constant vector. The incompressibility constraint in Cartesian coordinates reads $\det[F^a{}_A] = 1$. The right Cauchy–Green strain in Cartesian coordinates has components $C_{AB} = F^a{}_A F^b{}_B \delta_{ab}$, which are constants. $[b^{ab}]$ and $[c^{ab}]$ are constant matrices and I_1 and I_2 are constant as well. For isochoric homogeneous deformations, the universality constraints (4.10) are trivially satisfied. The first five sets of constraints in (4.11) are trivially satisfied as well, and only the last two need to be checked, i.e.

$\mathcal{C}^{1AB}_{[ab]} = \mathcal{C}^{2AB}_{[ab]} = 0$, where we use the standard notation $(.)_{[ab]} = \frac{1}{2}[(.)_{ab} - (.)_{ba}]$. Thus,

$$\left.\begin{aligned} & b^n_{[a}[(F^{-1})^A{}_n(F^{-1})^B{}_{b]} + (F^{-1})^B{}_n(F^{-1})^A{}_{b]}] = 0 \\ \text{and}\quad & c^n_{[a}[(F^{-1})^A{}_n(F^{-1})^B{}_{b]} + (F^{-1})^B{}_n(F^{-1})^A{}_{b]}] = 0. \end{aligned}\right\} \tag{4.12}$$

$[b^{ab}]$ and $[c^{ab}]$ have the same principal directions. With respect to the principal directions of $\mathbf{b}^\sharp$, $\mathbf{F}^{-1}$ has the representation

$$\mathbf{F}^{-1} = \begin{bmatrix} \mathsf{f}_{11} & \mathsf{f}_{12} & \mathsf{f}_{13} \\ \mathsf{f}_{21} & \mathsf{f}_{22} & \mathsf{f}_{23} \\ \mathsf{f}_{31} & \mathsf{f}_{32} & \mathsf{f}_{33} \end{bmatrix}, \quad [b^{ab}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \quad [c^{ab}] = \begin{bmatrix} \frac{1}{\lambda_1^2} & 0 & 0 \\ 0 & \frac{1}{\lambda_2^2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3^2} \end{bmatrix}, \tag{4.13}$$

where $\lambda_1^2\lambda_2^2\lambda_3^2 = 1$ and $\det \mathbf{F}^{-1} = 1$. The constraints $(4.12)_1$ for $(A, B) = (1, 1)$ read

$$\mathsf{f}_{11}\mathsf{f}_{12}(\lambda_1^2 - \lambda_2^2) = 0, \quad \mathsf{f}_{12}\mathsf{f}_{13}(\lambda_2^2 - \lambda_3^2) = 0, \quad \mathsf{f}_{11}\mathsf{f}_{13}(\lambda_3^2 - \lambda_1^2) = 0. \tag{4.14}$$

Clearly, one solution is $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = 1$, which corresponds to the identity deformation, which satisfies all the other constraints. We show that this is the only solution. If not, as all isochoric homogeneous deformations satisfy the symmetry of all the terms in (4.10), we can assume that the eigenvalues of $\mathbf{b}^\sharp$ are distinct. Thus,

$$\mathsf{f}_{11}\mathsf{f}_{12} = \mathsf{f}_{12}\mathsf{f}_{13} = \mathsf{f}_{11}\mathsf{f}_{13} = 0. \tag{4.15}$$

Similarly, the constraints $(4.12)_1$ for $(A, B) = (2, 2)$ and $(A, B) = (3, 3)$ give

$$\left.\begin{aligned} & \mathsf{f}_{21}\mathsf{f}_{22} = \mathsf{f}_{22}\mathsf{f}_{23} = \mathsf{f}_{21}\mathsf{f}_{23} = 0 \\ \text{and}\quad & \mathsf{f}_{31}\mathsf{f}_{32} = \mathsf{f}_{32}\mathsf{f}_{33} = \mathsf{f}_{31}\mathsf{f}_{33} = 0. \end{aligned}\right\} \tag{4.16}$$

It is straightforward to show that (4.15) and (4.16), and the constraint $\det \mathbf{F}^{-1} = 1$ imply that $\mathsf{f}_{12} = \mathsf{f}_{21} = \mathsf{f}_{13} = \mathsf{f}_{31} = \mathsf{f}_{23} = \mathsf{f}_{32} = 0$. The remaining constraints cannot be satisfied unless $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = 1$. Thus, we have proved the following result.

Proposition 4.1. *Homogeneous deformations are not universal for inhomogeneous incompressible nonlinear isotropic solids.*

Remark 4.2. Note that symmetry of a family of deformations $\varphi : \mathcal{B} \to \varphi(\mathcal{B}) \subset \mathcal{S}$ is encoded in the symmetry of its pulled-back metric $\mathbf{C}^\flat = \varphi^*\mathbf{g}$. Goodbrake *et al.* [12] observed that for homogeneous deformations $\mathbf{C}^\flat$ is invariant under the action of $T(3) \subset \mathrm{SE}(3)$—the group of translations. Proposition 4.1 tells us that the inhomogeneous energy function must respect the same symmetry, i.e. $W(\mathbf{X} + \mathbf{a}, I_1, I_2) = W(\mathbf{X}, I_1, I_2)$, $\forall\ \mathbf{a} \in \mathbb{R}^3$. In other words, the energy function must be homogeneous.

(b) Family 1: bending, stretching and shearing of a rectangular block

This family of deformations, with respect to the Cartesian (X, Y, Z) and cylindrical (r, θ, z) coordinates in the reference and current configurations, respectively, has the following representation:

$$\begin{aligned} & r(X, Y, Z) = \sqrt{C_1(2X + C_4)}, \quad \theta(X, Y, Z) = C_2(Y + C_5), \\ & z(X, Y, Z) = \frac{Z}{C_1C_2} - C_2C_3Y + C_6, \end{aligned} \tag{4.17}$$

where the Cartesian coordinate planes are parallel to the faces of the undeformed rectangular block.[3]

[3]Note that any connected subset of a rectangular block can undergo these deformations as long as appropriate surface tractions are applied. This is also the case for subsets of cylindrical shells, spherical shells and annular wedges for Families 2–5.

$\mathbf{C}^\flat$, $\mathbf{b}^\sharp$ and $\mathbf{c}^\sharp$ have the following representations.

$$[C_{AB}] = \begin{bmatrix} \frac{C_1}{2X+C_4} & 0 & 0 \\ 0 & C_2^2[C_1(2X+C_4)+C_3^2] & -\frac{C_3}{C_1} \\ 0 & -\frac{C_3}{C_1} & \frac{1}{C_1^2C_2^2} \end{bmatrix}, \tag{4.18}$$

$$[b^{ab}] = \begin{bmatrix} \frac{C_1}{C_4+2X} & 0 & 0 \\ 0 & C_2^2 & -C_2^2C_3 \\ 0 & -C_2^2C_3 & \frac{1}{C_1^2C_2^2}+C_2^2C_3^2 \end{bmatrix} \tag{4.19}$$

and

$$[c^{ab}] = \begin{bmatrix} \frac{C_4+2X}{C_1} & 0 & 0 \\ 0 & C_1^2C_2^2C_3^2+\frac{1}{C_2^2} & C_1^2C_2^2C_3 \\ 0 & C_1^2C_2^2C_3 & C_1^2C_2^2 \end{bmatrix}. \tag{4.20}$$

Note that $\mathbf{C}^\flat$ is independent of Y and Z, i.e. it is invariant under the action of $T(2) \subset \mathrm{SE}(3)$.

The universality constraint $\mathcal{C}^{1A}_{[ab]}=0$, for $A=2$, and $(a,b)=(1,2)$ requires[4] $C_1^2[1+3C_2^2(C_4+2X)]=0$, which is not possible. This implies that

$$\frac{\partial W_1}{\partial Y} = \frac{\partial}{\partial I_1}\frac{\partial W}{\partial Y} = 0. \tag{4.21}$$

$\mathcal{C}^{1A}_{[ab]}=0$, for $A=3$, and $(a,b)=(1,2)$ requires $C_1^3C_2C_3[1+C_2^2(C_4+2X)]=0$, which cannot be satisfied. This implies that

$$\frac{\partial W_1}{\partial Z} = \frac{\partial}{\partial I_1}\frac{\partial W}{\partial Z} = 0. \tag{4.22}$$

$\mathcal{C}^{2A}_{[ab]}=0$, for $A=2$, and $(a,b)=(1,2)$ requires $(C_1^4C_2^4C_3^2+C_1^2-C_2^2)\sqrt{C_1(C_4+2X)}=0$, which is not possible, and hence,

$$\frac{\partial W_2}{\partial Y} = \frac{\partial}{\partial I_2}\frac{\partial W}{\partial Y} = 0. \tag{4.23}$$

$\mathcal{C}^{2A}_{[ab]}=0$, for $A=3$, and $(a,b)=(1,3)$ requires $(C_1^4C_2^4C_3^2+C_1^2-3C_2^2)\sqrt{C_1(C_4+2X)}=0$, which does not hold, and thus,

$$\frac{\partial W_2}{\partial Z} = \frac{\partial}{\partial I_2}\frac{\partial W}{\partial Z} = 0. \tag{4.24}$$

Equations (4.21)–(4.24) imply that up to a mechanically inconsequential function of (X,Y,Z), the energy function must have the form $W=W(X,I_1,I_2)$. For this form of the energy function in (4.11), only the symmetry of the terms with $A=1$ and $A=B=1$ needs to be checked. It turns out that those terms are all symmetric.

Proposition 4.3. *For inhomogeneous incompressible nonlinear isotropic solids, Family 1 deformations are universal for any energy function of the form* $W=W(X,I_1,I_2)$.

Remark 4.4. Note that for Family 1 deformations, $\mathbf{C}^\flat$ is independent of Y and Z, i.e. it is invariant under the action of the transformations $Y \to Y+a_2$, and $Z \to Z+a_3$, $\forall\, a_2,a_3 \in \mathbb{R}$. We have shown that for Family 1 deformations to be universal for inhomogeneous solids, the energy function must be invariant under the same group of transformations.

(c) Family 2: straightening, stretching and shearing of a sector of a cylindrical shell

This family of deformations, with respect to the cylindrical (R,Θ,Z) and Cartesian (x,y,z) coordinates in the reference and current configurations, respectively, have the following

[4]All the symbolic computations in this article were performed using Mathematica Version 12.3.0.0, Wolfram Research, Champaign, IL.

royalsocietypublishing.org/journal/rspa Proc. R. Soc. A 477: 20210547

representation

$$\begin{aligned} x(R,\Theta,Z) &= \frac{1}{2}C_1C_2^2R^2 + C_4, \quad y(R,\Theta,Z) = \frac{\Theta}{C_1C_2} + C_5, \\ z(R,\Theta,Z) &= \frac{C_3}{C_1C_2}\Theta + \frac{1}{C_2}Z + C_6, \end{aligned} \tag{4.25}$$

where the Z-coordinate line is the axis of the cylindrical shell sector. Thus,

$$[C_{AB}] = \begin{bmatrix} C_1^2C_2^4R^2 & 0 & 0 \\ 0 & \frac{C_3^2+1}{C_1^2C_2^2} & \frac{C_3}{C_1C_2^2} \\ 0 & \frac{C_3}{C_1C_2^2} & \frac{1}{C_2^2} \end{bmatrix}, \tag{4.26}$$

$$[b^{ab}] = \begin{bmatrix} C_1^2C_2^4R^2 & 0 & 0 \\ 0 & \frac{1}{C_1^2C_2^2R^2} & \frac{C_3}{C_1^2C_2^2R^2} \\ 0 & \frac{C_3}{C_1^2C_2^2R^2} & \frac{1}{C_2^2} + \frac{C_3^2}{C_1^2C_2^2R^2} \end{bmatrix} \tag{4.27}$$

and

$$[c^{ab}] = \begin{bmatrix} \frac{1}{C_1^2C_2^4R^2} & 0 & 0 \\ 0 & C_1^2C_2^2R^2 + C_2^2C_3^2 & -C_2^2C_3 \\ 0 & -C_2^2C_3 & C_2^2 \end{bmatrix}. \tag{4.28}$$

Note that $\mathbf{C}^\flat$ is independent of Θ and Z, i.e. it is invariant under the action of $SO(2) \times T(1) \subset SE(3)$.

The universality constraint $C^{1A}_{[ab]} = 0$, for $A = 2$, and $(a,b) = (1,3)$ requires $C_1^4C_2^6 + (1/R^4) = 0$, which cannot be satisfied. This implies that

$$\frac{\partial W_1}{\partial \Theta} = \frac{\partial}{\partial I_1}\frac{\partial W}{\partial \Theta} = 0. \tag{4.29}$$

Similarly

$$\left.\begin{aligned} C^{1A}_{[ab]} &= 0, \quad \text{for } (A,a,b) = (3,1,2) \Rightarrow C_1C_2^3 = 0, \\ C^{2A}_{[ab]} &= 0, \quad \text{for } (A,a,b) = (2,1,2) \Rightarrow C_1^2C_2 + \frac{1}{C_1^2C_2^5R^4} = 0 \\ \text{and} \quad C^{2A}_{[ab]} &= 0, \quad \text{for } (A,a,b) = (3,1,3) \Rightarrow \frac{1}{C_1^3C_2^5R^4} = 0. \end{aligned}\right\} \tag{4.30}$$

None of the aforementioned constraints can be satisfied, and hence,

$$\frac{\partial W_1}{\partial Z} = \frac{\partial}{\partial I_1}\frac{\partial W}{\partial Z} = 0, \quad \frac{\partial W_2}{\partial \Theta} = \frac{\partial}{\partial I_2}\frac{\partial W}{\partial \Theta} = 0, \quad \frac{\partial W_2}{\partial Z} = \frac{\partial}{\partial I_2}\frac{\partial W}{\partial Z} = 0. \tag{4.31}$$

Equations (4.29) and (4.31) imply that up to a mechanically inconsequential function of (R,Θ,Z), the energy function must have the form $W = W(R, I_1, I_2)$. For this form of the energy in (4.11), only the symmetry of the terms with $A = 1$, and $A = B = 1$ needs to be checked. One can check that all those terms are symmetric.

Proposition 4.5. *For inhomogeneous incompressible nonlinear isotropic solids, Family 2 deformations are universal for any energy function of the form $W = W(R, I_1, I_2)$.*

Remark 4.6. Note that for Family 2 deformations, $\mathbf{C}^\flat$ is independent of Θ and Z, i.e. it is invariant under the action of the transformations $\Theta \to \Theta + \Theta_0$ and $Z \to Z + Z_0$, $\forall\, \Theta_0, Z_0 \in \mathbb{R}$. We have shown that for Family 2 deformations to be universal for inhomogeneous solids, the energy function must be invariant under the same group of transformations.

(d) Family 3: inflation, bending, torsion, extension and shearing of a sector of an annular wedge

With respect to the cylindrical coordinates (R,Θ,Z) and (r,θ,z) in the reference and current configurations, respectively, this family of deformations has the following representation

$$\begin{aligned} r(R,\Theta,Z) &= \sqrt{\frac{R^2}{C_1C_4-C_2C_3}+C_5}, \quad \theta(R,\Theta,Z)=C_1\Theta+C_2Z+C_6, \\ z(R,\Theta,Z) &= C_3\Theta+C_4Z+C_7, \end{aligned} \tag{4.32}$$

where the Z-coordinate line is the axis of the annular wedge. Thus,

$$[C_{AB}] = \begin{bmatrix} \frac{R^2}{K(KC_5+R^2)} & 0 & 0 \\ 0 & C_3^2+C_1^2\left[\frac{R^2}{K}+C_5\right] & C_1C_2\left[\frac{R^2}{K}+C_5\right]+C_3C_4 \\ 0 & C_1C_2\left[\frac{R^2}{K}+C_5\right]+C_3C_4 & C_4^2+C_2^2\left[\frac{R^2}{K}+C_5\right] \end{bmatrix}, \tag{4.33}$$

$$[b^{ab}] = \begin{bmatrix} \frac{R^2}{K(C_5K+R^2)} & 0 & 0 \\ 0 & \frac{C_1^2}{R^2}+C_2^2 & \frac{C_1C_3}{R^2}+C_2C_4 \\ 0 & \frac{C_1C_3}{R^2}+C_2C_4 & \frac{C_3^2}{R^2}+C_4^2 \end{bmatrix} \tag{4.34}$$

and

$$[c^{ab}] = \begin{bmatrix} \frac{C_5K^2}{R^2}+K & 0 & 0 \\ 0 & \frac{C_3^2+C_4^2R^2}{K^2} & -\frac{C_1C_3+C_2C_4R^2}{K^2} \\ 0 & -\frac{C_1C_3+C_2C_4R^2}{K^2} & \frac{C_1^2+C_2^2R^2}{K^2} \end{bmatrix}, \tag{4.35}$$

where $K=C_1C_4-C_2C_3$. Note that $\mathbf{C}^\flat$ only depends on R.

The universality constraint $\mathcal{C}^{1A}_{[ab]}=0$, for $A=2$, and $(a,b)=(1,2)$ requires that

$$\begin{aligned} &C_4R^6(C_1^2+2C_2^2C_5K-1)+C_5KR^4[C_4(C_2^2C_5K+2)-2C_1C_2C_3] \\ &\quad +C_1C_5^2K^2R^2(4C_2C_3-3C_1C_4)-2C_1C_5^3K^4+C_2^2C_4R^8=0, \end{aligned} \tag{4.36}$$

which is not possible, and hence,

$$\frac{\partial W_1}{\partial \Theta}=\frac{\partial}{\partial I_1}\frac{\partial W}{\partial \Theta}=0. \tag{4.37}$$

$\mathcal{C}^{1A}_{[ab]}=0$, for $A=3$, and $(a,b)=(1,3)$ requires

$$C_1\{C_1^2(C_5K+R^2)^2+C_5KR^2[C_2^2(C_5K+2R^2)-2]+C_2^2R^6-R^4\}=0, \tag{4.38}$$

which cannot be satisfied. This implies that

$$\frac{\partial W_1}{\partial Z}=\frac{\partial}{\partial I_1}\frac{\partial W}{\partial Z}=0. \tag{4.39}$$

Similarly, $\mathcal{C}^{2A}_{[ab]}=0$, for $A=2$, and $(a,b)=(1,2)$, and $\mathcal{C}^{2A}_{[ab]}=0$, for $A=3$, and $(a,b)=(1,3)$ cannot be satisfied, and hence,

$$\frac{\partial W_2}{\partial \Theta}=\frac{\partial}{\partial I_2}\frac{\partial W}{\partial \Theta}=0 \quad \text{and} \quad \frac{\partial W_2}{\partial Z}=\frac{\partial}{\partial I_2}\frac{\partial W}{\partial Z}=0. \tag{4.40}$$

Therefore, up to a mechanically inconsequential function of (R,Θ,Z), the energy function must have the form $W=W(R,I_1,I_2)$. For this form of the energy in (4.11), only the symmetry of the terms with $A=1$ and $A=B=1$ needs to be checked. All those terms are symmetric. In summary, in proposition 4.5, 'Family 2' can be replaced by 'Family 3'. Similar to Family 2, for Family 3 deformations to be universal, the energy function must respect the symmetry of $\mathbf{C}^\flat$.

(e) Family 4: inflation/inversion of a sector of a spherical shell

With respect to the spherical coordinates (R, Θ, Φ) and (r, θ, ϕ) in the reference and current configurations, respectively, this family of deformations has the following representation:

$$r(R, \Theta, \Phi) = (\pm R^3 + C_1^3)^{1/3}, \quad \theta(R, \Theta, \Phi) = \pm\Theta, \quad \phi(R, \Theta, \Phi) = \Phi, \tag{4.41}$$

where the spherical coordinates are centred at the centre of the spherical shell sector. Thus,

$$[C_{AB}] = \begin{bmatrix} \frac{R^4}{(C_1^3 \pm R^3)^{4/3}} & 0 & 0 \\ 0 & (C_1^3 \pm R^3)^{2/3} & 0 \\ 0 & 0 & (C_1^3 \pm R^3)^{2/3}\sin^2\Theta \end{bmatrix}, \tag{4.42}$$

$$[b^{ab}] = \begin{bmatrix} \frac{R^4}{(C_1^3 \pm R^3)^{4/3}} & 0 & 0 \\ 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & \frac{1}{R^2\sin^2\Theta} \end{bmatrix} \quad \text{and} \quad [c^{ab}] = \begin{bmatrix} \frac{(C_1^3 \pm R^3)^{4/3}}{R^4} & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2\sin^2\Theta \end{bmatrix}. \tag{4.43}$$

$\mathbf{C}^\flat$ can be written as follows [12]:

$$\mathbf{C}^\flat(\mathbf{X}) = \frac{R^4}{(C_1^3 \pm R^3)^{4/3}}\hat{\mathbf{R}} \otimes \hat{\mathbf{R}} + \frac{(C_1^3 \pm R^3)^{2/3}}{R^2}(\mathbf{1} - \hat{\mathbf{R}} \otimes \hat{\mathbf{R}}), \tag{4.44}$$

where $\mathbf{1}$ is the identity tensor and $\hat{\mathbf{R}} = \mathbf{X}/|\mathbf{X}|$. This means that at a point $\mathbf{X}$, $\mathbf{C}^\flat$ is invariant under all those rotations that fix $\mathbf{X}$.

The universality constraint $\mathcal{C}^{1A}_{[ab]} = 0$, for $A = 2$, and $(a, b) = (1, 3)$ requires that $4C_1^3R^6 \pm 3C_1^6R^3 + 2C_1^9 = 0$, which cannot be satisfied. This implies that

$$\frac{\partial W_1}{\partial \Theta} = \frac{\partial}{\partial I_1}\frac{\partial W}{\partial \Theta} = 0. \tag{4.45}$$

$\mathcal{C}^{1A}_{[ab]} = 0$, for $A = 3$, and $(a, b) = (1, 2)$ requires $(C_1^3 \pm R^3)^{2/3}\cot\Theta = 0$, which is not possible, and thus,

$$\frac{\partial W_1}{\partial \Phi} = \frac{\partial}{\partial I_1}\frac{\partial W}{\partial \Phi} = 0. \tag{4.46}$$

$\mathcal{C}^{2A}_{[ab]} = 0$, for $A = 2$, and $(a, b) = (1, 2)$ requires

$$(5R^9 \pm 2C_1^3R^6)(C_1^3 \pm R^3)^{1/3} \mp R^6 + 3C_1^3R^3 \pm 4C_1^6 = 0, \tag{4.47}$$

which is not possible. Thus,

$$\frac{\partial W_2}{\partial \Theta} = \frac{\partial}{\partial I_2}\frac{\partial W}{\partial \Theta} = 0. \tag{4.48}$$

$\mathcal{C}^{2A}_{[ab]} = 0$, for $A = 3$, and $(a, b) = (2, 3)$ requires $R^2(C_1^3 \pm R^3)^{2/3}\cos\Theta\sin^3\Theta = 0$, which is not possible. This implies that

$$\frac{\partial W_2}{\partial \Phi} = \frac{\partial}{\partial I_2}\frac{\partial W}{\partial \Phi} = 0. \tag{4.49}$$

Equations (4.45), (4.46), (4.48) and (4.49) imply that up to a mechanically inconsequential function of (R, Θ, Φ), the energy function must have the form $W = W(R, I_1, I_2)$. For this form of the energy in (4.11), only the symmetry of the terms with $A = 1$ and $A = B = 1$ needs to be checked. One can check that all those terms are symmetric. In summary, in proposition 4.5, 'Family 2' can be replaced by 'Family 4'. Similar to Families 2 and 3, for Family 4 deformations to be universal, the energy function must respect the symmetry of $\mathbf{C}^\flat$.

royalsocietypublishing.org/journal/rspa *Proc. R. Soc. A* **477**: 20210547

(f) Family 5: inflation, bending, extension and azimuthal shearing of an annular wedge

With respect to the cylindrical coordinates (R, Θ, Z) and (r, θ, z) in the reference and current configurations, respectively, this family of deformations has the following representation:

$$r(R, \Theta, Z) = C_1 R, \quad \theta(R, \Theta, Z) = C_2 \log R + C_3 \Theta + C_4, \quad z(R, \Theta, Z) = \frac{1}{C_1^2 C_3} Z + C_5, \tag{4.50}$$

where the Z-coordinate line is the axis of the annular wedge. Thus,

$$[C_{AB}] = \begin{bmatrix} C_1^2(C_2^2+1) & C_1^2 C_2 C_3 R & 0 \\ C_1^2 C_2 C_3 R & C_1^2 C_3^2 R^2 & 0 \\ 0 & 0 & \frac{1}{C_1^4 C_3^2} \end{bmatrix} \tag{4.51}$$

and

$$[b^{ab}] = \begin{bmatrix} C_1^2 & \frac{C_1 C_2}{R} & 0 \\ \frac{C_1 C_2}{R} & \frac{C_2^2 + C_3^2}{R^2} & 0 \\ 0 & 0 & \frac{1}{C_1^4 C_3^2} \end{bmatrix}, \quad [c^{ab}] = \begin{bmatrix} \frac{C_2^2}{C_1^2 C_3^2} + \frac{1}{C_1^2} & -\frac{C_2 R}{C_1 C_3^2} & 0 \\ -\frac{C_2 R}{C_1 C_3^2} & \frac{R^2}{C_3^2} & 0 \\ 0 & 0 & C_1^4 C_3^2 \end{bmatrix}. \tag{4.52}$$

Note that $\mathbf{C}^\flat$ only depends on R. For homogeneous incompressible isotropic solids, this is the only known family of inhomogeneous universal deformations for which I_1 and I_2 are constant. Let us consider the following universality constraints:

$$\left.\begin{aligned} &\mathcal{C}^{1A}_{[ab]} = 0, \quad \text{for } (A, a, b) = (1,1,2) \Rightarrow C_1 C_2 = 0, \\ &\mathcal{C}^{1A}_{[ab]} = 0, \quad \text{for } (A, a, b) = (2,1,2) \Rightarrow C_1(1 + C_2^2 - C_3^2) = 0, \\ &\mathcal{C}^{1A}_{[ab]} = 0, \quad \text{for } (A, a, b) = (3,2,3) \Rightarrow C_1^4 C_2 C_3 = 0, \\ &\mathcal{C}^{2A}_{[ab]} = 0, \quad \text{for } (A, a, b) = (1,1,2) \Rightarrow R^2 C_2 = 0, \\ &\mathcal{C}^{2A}_{[ab]} = 0, \quad \text{for } (A, a, b) = (2,1,2) \Rightarrow C_2^2(1 - 6C_1^2 R^2) - 5C_1^4 R^4 + C_3^2 = 0 \\ \text{and} \quad &\mathcal{C}^{2A}_{[ab]} = 0, \quad \text{for } (A, a, b) = (3,2,3) \Rightarrow R^2 C_1^2 C_2 = 0. \end{aligned}\right\} \tag{4.53}$$

None of the above six universality constraints can be satisfied. This means that the energy function must be homogeneous.

Proposition 4.7. *For inhomogeneous incompressible nonlinear isotropic solids, Family 5 deformations are not universal.*

Remark 4.8. This family of deformations is peculiar in the sense that it is inhomogeneous yet it does not accommodate universal inhomogeneity. This result is consistent with what Yavari & Goriely [13] observed for transversely isotropic solids. For Family 1, one of the universal solutions is a uniform distribution of fibres for fixed X. For Families 2 and 3 and Family 5, they showed that the integral curves of the material preferred directions are circular helices. For Families 2 and 3, the helices can be R dependent but not for Family 5. Goodbrake *et al.* [12] observed another peculiarity of this family: $\mathbf{C}^\flat$ corresponding to other inhomogeneous families (i.e. Families 1–4) has an eigenvector parallel to the inhomogeneity direction, but not for this family.

5. Concluding remarks

In this article, we extended Ericksen's analysis of universal deformations in homogeneous isotropic solids to inhomogeneous isotropic solids. The set of universality constraints of inhomogeneous solids include those of the corresponding homogeneous solids. We showed that inhomogeneous compressible isotropic solids do not admit universal deformations. For

Table 1. Universal deformations and universal material inhomogeneities for incompressible isotropic solids for the six known families of universal deformations. For Family 3, $K = C_1C_4 - C_2C_3$. Note that Families 0 and 5 are not universal for inhomogeneous solids.

family	universal deformations	$\mathbf{C}^\flat$	universal inhomogeneity
1	$\begin{cases} r(X,Y,Z) = \sqrt{C_1(2X+C_4)} \\ \theta(X,Y,Z) = C_2(Y+C_5) \\ z(X,Y,Z) = \frac{Z}{C_1C_2} - C_2C_3Y + C_6 \end{cases}$	$[C_{AB}] = \begin{bmatrix} \frac{C_1}{2X+C_4} & 0 & 0 \\ 0 & C_2^2[C_1(2X+C_4)+C_3^2] & -\frac{C_3}{C_1} \\ 0 & -\frac{C_3}{C_1} & \frac{1}{C_1^2C_2^2} \end{bmatrix}$	$W = W(X, I_1, I_2)$
2	$\begin{cases} x(R,\Theta,Z) = \frac{1}{2}C_1C_2^2R^2 + C_4 \\ y(R,\Theta,Z) = \frac{\Theta}{C_1C_2} + C_5 \\ z(R,\Theta,Z) = \frac{C_3}{C_1C_2}\Theta + \frac{1}{C_2}Z + C_6 \end{cases}$	$[C_{AB}] = \begin{bmatrix} C_1^2C_2^4R^2 & 0 & 0 \\ 0 & \frac{C_3^2+1}{C_1^2C_2^2} & \frac{C_3}{C_1C_2^2} \\ 0 & \frac{C_3}{C_1C_2^2} & \frac{1}{C_2^2} \end{bmatrix}$	$W = W(R, I_1, I_2)$
3	$\begin{cases} r(R,\Theta,Z) = \sqrt{\frac{R^2}{C_1C_4-C_2C_3} + C_5} \\ \theta(R,\Theta,Z) = C_1\Theta + C_2Z + C_6 \\ z(R,\Theta,Z) = C_3\Theta + C_4Z + C_7 \end{cases}$	$[C_{AB}] = \begin{bmatrix} \frac{R^2}{K(KC_5+R^2)} & 0 & 0 \\ 0 & C_1^2(\frac{R^2}{K}+C_5)+C_3^2 & C_1C_2(\frac{R^2}{K}+C_5)+C_3C_4 \\ 0 & C_1C_2(\frac{R^2}{K}+C_5)+C_3C_4 & C_2^2(\frac{R^2}{K}+C_5)+C_4^2 \end{bmatrix}$	$W = W(R, I_1, I_2)$
4	$\begin{cases} r(R,\Theta,\Phi) = (\pm R^3 + C_1^3)^{1/3} \\ \theta(R,\Theta,\Phi) = \pm\Theta \\ \phi(R,\Theta,\Phi) = \Phi \end{cases}$	$[C_{AB}] = \begin{bmatrix} \frac{R^4}{(C_1^3 \pm R^3)^{4/3}} & 0 & 0 \\ 0 & (C_1^3 \pm R^3)^{2/3} & 0 \\ 0 & 0 & (C_1^3 \pm R^3)^{2/3}\sin^2\Theta \end{bmatrix}$	$W = W(R, I_1, I_2)$

incompressible solids, we considered each of the six known families of universal deformations for homogeneous isotropic solids and showed that:

— Family 0 is not universal for inhomogeneous solids.
— Family 1 is universal for inhomogeneous solids as long as the energy function respects the symmetry of these deformations, i.e. when $W=W(X,I_1,I_2)$ with respect to the natural Cartesian coordinates (X,Y,Z) in the reference configuration of a rectangular block.
— Families 2, 3 and 4 are universal for those inhomogeneous solids for which $W=W(R,I_1,I_2)$ with respect to the natural referential cylindrical coordinates (R,Θ,Z) for Families 2 and 3, and with respect to the natural referential spherical coordinates (R,Θ,Φ) for Family 4.
— Family 5 is not universal for inhomogeneous solids.

Table 1 summarizes our results for inhomogeneous incompressible isotropic solids.

Data accessibility. This article has no additional data.
Competing interests. I declare I have no competing interests.
Funding. This research was supported by ARO W911NF-18-1-0003 and NSF—grant no. CMMI 1939901.

References

1. Beatty MF. Introduction to nonlinear elasticity. In *Nonlinear effects in fluids and solids* (eds MM Carroll, M Hayes). New York, NY: Plenum Press.
2. Saccomandi G. 2001 Universal solutions and relations in finite elasticity. In *Topics in finite elasticity* (eds M Hayes, G Saccomandi), pp. 95–130. Berlin, Germany: Springer.
3. Ericksen JL. 1955 Deformations possible in every compressible, isotropic, perfectly elastic material. *J. Math. Phys.* **34**, 126–128. (doi:10.1002/sapm1955341126)
4. Rivlin RS. 1948 Large elastic deformations of isotropic materials IV. Further developments of the general theory. *Philos. Trans. R. Soc. Lond. A* **241**, 379–397. (doi:10.1098/rsta.1948.0024)
5. Rivlin RS. 1949 Large elastic deformations of isotropic materials. V. The problem of flexure. *Proc. R. Soc. Lond. A* **195**, 463–473. (doi:10.1098/rspa.1949.0004)
6. Rivlin RS. 1949 A note on the torsion of an incompressible highly elastic cylinder. *Math. Proc. Camb. Philos. Soc.* **45**, 485–487.
7. Ericksen JL. 1954 Deformations possible in every isotropic, incompressible, perfectly elastic body. *Z. Angew. Math. Phys. (ZAMP)* **5**, 466–489. (doi:10.1007/bf01601214)
8. Fosdick RL. 1966 Remarks on compatibility. *Modern developments in the mechanics of continua*, pp. 109–127. New York, NY: Academic Press.
9. Singh M, Pipkin AC. 1965 Note on Ericksen's problem. *Z. Angew. Math. Phys. (ZAMP)* **16**, 706–709. (doi:10.1007/bf01590971)
10. Klingbeil WW, Shield RT. 1966 On a class of solutions in plane finite elasticity. *Z. Angew. Math. Phys. (ZAMP)* **17**, 489–511. (doi:10.1007/bf01595984)
11. Yavari A, Goriely A. 2016 The anelastic Ericksen problem: universal eigenstrains and deformations in compressible isotropic elastic solids. *Proc. R. Soc. A* **472**, 20160690. (doi:10.1098/rspa.2016.0690)
12. Goodbrake C, Yavari A, Goriely A. 2020 The anelastic Ericksen problem: universal deformations and universal eigenstrains in incompressible nonlinear anelasticity. *J. Elast.* **142**, 291–381. (doi:10.1007/s10659-020-09797-2)
13. Yavari A, Goriely A. 2021 Universal deformations in anisotropic nonlinear elastic solids. *J. Mech. Phys. Solids* **156**, 104598. (doi:10.1016/j.jmps.2021.104598)
14. Truesdell C. 1966 *The elements of continuum mechanics*. Berlin, Germany: Springer-Verlag.
15. Gurtin ME. 1972 The linear theory of elasticity. In *Handbuch der physik, band VIa/2* (eds S Flügge, C Truesdell), pp. 1–296. Berlin, Germany: Springer-Verlag.
16. Yavari A, Goodbrake C, Goriely A. 2020 Universal displacements in linear elasticity. *J. Mech. Phys. Solids* **135**, 103782. (doi:10.1016/j.jmps.2019.103782)

17. Golgoon A, Yavari A. 2021 On Hashin's hollow cylinder and sphere assemblages in anisotropic nonlinear elasticity. *J. Elast.* (doi:10.1007/s10659-021-09856-2)
18. Marsden JE, Hughes TJR. 1994 *Mathematical foundations of elasticity*. New York, NY: Dover Publications.
19. Ogden RW. 1984 *Non-linear elastic deformations*. Mineola, NY: Dover.
20. Doyle TC, Ericksen JL. 1956 Nonlinear elasticity. *Adv. Appl. Mech.* **4**, 53–115.
21. Yavari A. 2013 Compatibility equations of nonlinear elasticity for non-simply-connected bodies. *Arch. Ration. Mech. Anal.* **209**, 237–253. (doi:10.1007/s00205-013-0621-0)