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On applications of generalized functions to the analysis of Euler–Bernoulli beam–columns with jump discontinuities

Arash Yavari[a], Shahram Sarkani[b,*]

[a]Department of Mechanical and Aerospace Engineering, The George Washington University, Washington, DC 20052, USA

[b]Department of Civil, Mechanical & Environmental Engineering, School of Engineering and Applied Science, The George Washington University, Washington, DC 20052, USA

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Abstract

In this article some applications of the distribution theory of Schwarz to the analysis of beam–columns with various jump discontinuities are offered. The governing differential equation of an Euler–Bernoulli beam–column with jump discontinuities in flexural stiffness, displacement, and rotation, and under an axial force at the point of discontinuities, is obtained in the space of generalized functions. The auxiliary beam–column method is introduced. Using this method, instead of solving the differential equation of the beam–column in the space of generalized functions, another differential equation can be solved in the space of classical functions. Some examples of beam–columns and columns with various jump discontinuities are solved. Deflections of beam–columns and buckling loads for columns with jump discontinuities are calculated using the Laplace transform method in the space of generalized functions. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Beam-columns; Generalized functions; Distribution theory

1. Introduction

In practical applications, sometimes one has to analyze beam–columns with jump discontinuities in slope, deflection, or flexural stiffness and in some instances the beam–columns are under discontinuous loading conditions. The classical method for solving these problems is to partition the beam–column into beam–column segments between any two successive discontinuity points.

* Corresponding author. Tel.: 001-202-994-9825; fax: 001-202-994-0092.
E-mail address: sarkani@seas.gwu.edu (S. Sarkani).

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Nomenclature

A, B	constants of integration
EI	flexural stiffness of an Euler–Bernoulli beam–column
EI_1, EI_2	flexural stiffnesses of beam–column segments
K_r	stiffness of a rotational spring
K_t	stiffness of a translational spring
L	length of a beam–column
M_1, M_2	bending moments
M_0	bending moment at a discontinuity point
P, P_0, P'	concentrated forces
V_0	shear force at a discontinuity point
V_1, V_2	shear forces
X_1, X_2	functions of x
k^2	P/EI
q	a distributed force
s	a variable for the Laplace transform
x	longitudinal axis of a beam–column
x_0	position of a discontinuity point
x_0^-	$x_0 - \varepsilon$ for a very small ε
x_0^+	$x_0 + \varepsilon$ for a very small ε
w	deflection of an Euler–Bernoulli beam–column
$\bar{w}$	deflection of the auxiliary beam–column of an Euler–Bernoulli beam–column
Δ	strength of a jump discontinuity in deflection of an Euler–Bernoulli beam–column
Θ	strength of a jump discontinuity in rotation of an Euler–Bernoulli beam–column
$\bar{\Theta}$	rotation at x_0^+
α	ratio of flexural stiffnesses of two beam–column segments
β	ratio of axial forces applied at the end points of a beam–column
λ	a parameter

Mathematical symbols

D	space of test functions
D'	space of distributions
D'_R	space of right-sided distributions
$H(x - x_0)$	Heaviside's unit step function
P	space of all polynomials
W(s)	Laplace transform of w
f, g	distributions
$\delta(x - x_0)$	Dirac delta function
$\delta^{(n)}(x - x_0)$	*nth* distributional derivative of delta function
φ	a test function
*	convolution symbol
$< >$	distribution symbol
$\mathscr{L}$	Laplace transform operator

Solving the differential equation of each beam–column segment and enforcing boundary and continuity conditions then yields the beam–column deflection equation. Clebsch [1] was the first to simplify these problems in beam bending by writing a single expression for the bending moment. Later, Macaulay [2] introduced the so-called Macaulay's bracket, also referred to in the literature as the singularity function method. The advantage of this method is that it reduces an uncoupled system of ordinary second-order differential equations to a single ordinary second-order equation.

The singularity function method was later generalized to two-dimensional problems by Wittrick [3], Mahing [4], Conway [5], and Selek and Conway [6]. Wittrick [3] analyzed beams with lateral loads and circular plates with axisymmetric lateral loads. Mahing [4] used the method for rectangular plates whose opposite sides are simply supported under a point load and for circular plates with axisymmetric loading. Conway [5] and Selek and Conway [6] generalized the singularity function method to two-dimensional problems governed by partial differential equations.

The singularity function method was generalized by Arbabi [7] for a beam with an internal hinge and for a beam with jump discontinuities in flexural stiffness. He did not solve these beam deflection problems as boundary-value problems; instead, he started the analysis from the bending moment expression. Lowe [8] introduced the "discontinuity method" for analyzing beams and columns with jump discontinuities. Lowe's idea is to find particular solutions that satisfy the discontinuities of the problem. The discontinuity method is efficient for practical problems and usually needs to satisfy fewer continuity conditions than the singularity function method does. Lowe uses the classical beam and column governing differential equations. Here, we find the boundary-value problem describing an Euler–Bernoulli beam–column with jump discontinuities in the space of generalized functions.

Schwarz's distribution theory [9] provides a rigorous justification for a number of very common formal mathematical manipulations published in the engineering literature. Certain types of distributions, in particular the Dirac delta function and its derivatives, were used in engineering problems years before the development of distribution theory. Reference to the delta function dates back to the 19^{th} century and the works of Hermite, Cauchy, Poisson, Kirchhoff, Helmholtz, Lord Kelvin, and Heaviside [10, pp. 62–66]. In 1930 Dirac [11] introduced this function in quantum mechanics and since then the function has been known as the Dirac delta function.

Yavari et al. [12], studied the applications of the distribution theory of Schwarz in beam bending problems. They found the equivalent distributed force for a general class of singular loading conditions. Finding the equivalent distributed force for a distributed moment, they offered a mathematical explanation for the corner condition in classical plate theory. They also obtained the governing differential equation of an Euler–Bernoulli beam with jump discontinuities in slope, deflection, and flexural stiffness. They introduced the auxiliary beam method and used it to solve, not the governing differential equation of the beam in the space of generalized functions, but another differential equation in the space of classical functions. They investigated the same problem for Timoshenko beams and showed that the governing differential equations of the beam can always be written in terms of a transverse deflection and a rotation function.

In this paper we investigate Euler–Bernoulli beam–columns with jump discontinuities in slope, deflection, and flexural stiffness under an axial force at the point of discontinuity. Schwarz distribution theory is used to find the governing differential equation of a beam–column with jump discontinuities in the space of generalized functions. It is shown that the governing differential equation of an Euler–Bernoulli beam–column with jump discontinuities can be written in terms of a single transverse deflection function in two cases. In the first case there is no jump discontinuity in deflection and in the second case there is a special relation between the relative flexural stiffnesses and the ratio of the internal and external axial forces. Then the auxiliary beam–column method is introduced. Using this method, instead of solving the fourth-order governing differential equation of the beam–column in the space of generalized functions another fourth-order differential equation can be solved in the space of classical functions. However, it is pointed out that for beam–columns this method is not efficient, and it is more convenient to solve the governing equations directly without reference to the auxiliary beam–column. Some examples of finding the deflection of beam–columns and buckling loads for columns with jump discontinuities are solved using the Laplace transform method in the space of generalized functions.

This article is organized as follows. In Section 2, the governing differential equation of an Euler–Bernoulli beam–column with various jump discontinuities is obtained in the space of generalized functions. The auxiliary beam–column method is presented in Section 3. In Section 4, three examples are solved to show the capabilities of generalized functions to simplify the analysis of beam–columns with jump discontinuities. Conclusions are given in Section 5. In the Appendix basic definitions and some theorems of the distribution theory of Schwarz are presented.

2. Euler–Bernoulli beam–columns with jump discontinuities in slope, deflection, and flexural stiffness

In this section the governing differential equation of an Euler–Bernoulli beam–column with various jump discontinuities is obtained in the space of generalized functions. Basic definitions and some theorems of the distribution theory of Schwarz are given in the Appendix.

The classical method for analyzing beam–columns with jump discontinuities in slope, deflection, and flexural stiffness with point axial forces applied at internal points is to partition the beam–column into several sub-beam–columns in such a way that each sub-beam–column is free of any jump discontinuity. Then by analyzing each sub-beam–column and enforcing the continuity conditions the whole system is analyzed [13]. For beam bending problems Macaulay's bracket has been used for years to simplify the analysis. As we will see in the sequel, for some instances using generalized functions makes the analysis of beam–columns with jump discontinuities easier, especially when calculating buckling loads.

Consider the beam–column shown in Fig. 1. This beam–column has jump discontinuities at point B. For the sake of simplicity only one point of jump discontinuity is considered. Generalization to the case of several points of discontinuity is straightforward. An arbitrary distributed transverse force $q(x)$ is applied and a point force P_0 is applied at B. In the most general case, a translational spring and a rotational spring are considered at the point of discontinuity. At point B displacement and rotation have jump discontinuities:

$$w(x_0^+) - w(x_0^-) = \Delta, \tag{1a}$$

$$\frac{\mathrm{d}w(x_0^+)}{\mathrm{d}x} - \frac{\mathrm{d}w(x_0^-)}{\mathrm{d}x} = \Theta. \tag{1b}$$

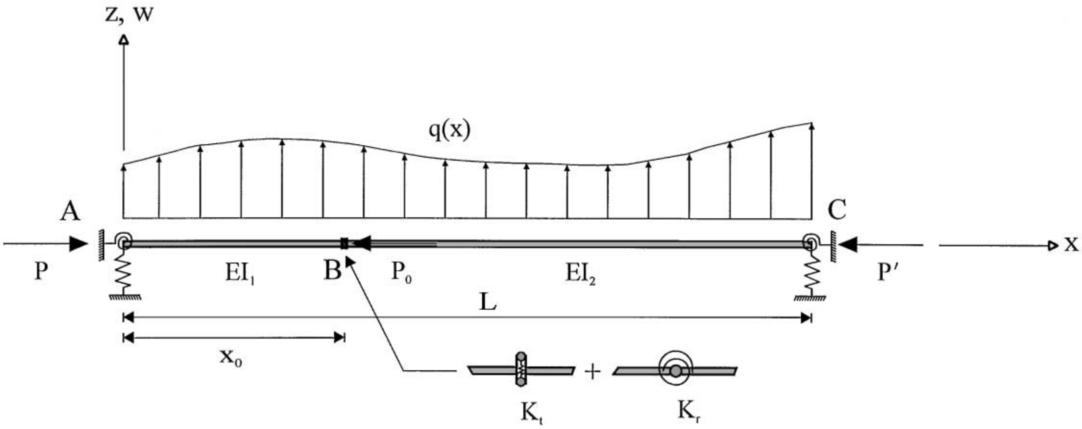

Fig. 1. A beam–column with jump discontinuities in slope, deflection, and flexural stiffness with arbitrary boundary conditions under a distributed transverse force and a concentrated axial force at the point of jump discontinuities.

The governing differential equation of an Euler–Bernoulli beam column without any jump discontinuity may be written as [13–15]

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) + P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = q(x). \tag{2}$$

For the special case of a beam–column with a uniform cross-section and constant Young's modulus, the governing differential equation has the following form:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \frac{P}{EI}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{q(x)}{EI}. \tag{3}$$

The beam AC of Fig. 1 is composed of two beam–column segments, AB and BC. Hence, using Heaviside's function

$$w(x) = w_1(x) + [w_2(x) - w_1(x)]H(x - x_0), \tag{4}$$

where w is the deflection of the beam–column and w_1 and w_2 are the deflections of the beam–column segments, AB and BC, respectively. The governing differential equations of beam–column segments may be written as

$$\frac{\mathrm{d}^4 w_1}{\mathrm{d}x^4} + \frac{P}{EI_1}\frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} = \frac{q(x)}{EI_1}, \quad 0 \leqslant x < x_0, \tag{5a}$$

$$\frac{\mathrm{d}^4 w_2}{\mathrm{d}x^4} + \frac{P'}{EI_2}\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2} = \frac{q(x)}{EI_2}, \quad x_0 < x \leqslant L. \tag{5b}$$

Differentiating both sides of Eq. (4), we obtain

$$\frac{\bar{d}w}{dx} = \frac{dw_1}{dx} + \left(\frac{dw_2}{dx} - \frac{dw_1}{dx}\right)H(x - x_0) + (w_2 - w_1)\delta(x - x_0)$$
$$= \frac{dw_1}{dx} + \left(\frac{dw_2}{dx} - \frac{dw_1}{dx}\right)H(x - x_0) + (w_2 - w_1)_{x=x_0}\,\delta(x - x_0)$$
$$= \frac{dw_1}{dx} + \left(\frac{dw_2}{dx} - \frac{dw_1}{dx}\right)H(x - x_0) + \Delta\delta(x - x_0) \tag{6}$$

and

$$\frac{\bar{d}^2w}{dx^2} = \frac{d^2w_1}{dx^2} + \left[\frac{d^2w_2}{dx^2} - \frac{d^2w_1}{dx^2}\right]H(x - x_0) + \left[\frac{dw_2}{dx} - \frac{dw_1}{dx}\right]_{x=x_0}\delta(x - x_0) + \Delta\delta^{(1)}(x - x_0)$$
$$= \frac{d^2w_1}{dx^2} + \left[\frac{d^2w_2}{dx^2} - \frac{d^2w_1}{dx^2}\right]H(x - x_0) + \Theta\delta(x - x_0) + \Delta\delta^{(1)}(x - x_0), \tag{7}$$

where a bar over the differentiation symbol means distributional differentiation. We know that

$$M_1(x_0) = \left[EI_1\frac{d^2w_1}{dx^2}\right]_{x=x_0}, \quad M_2(x_0) = \left[EI_2\frac{d^2w_2}{dx^2}\right]_{x=x_0}. \tag{8}$$

Therefore

$$\frac{d^2w_1(x_0)}{dx^2} = \frac{M_1(x_0)}{EI_1}, \quad \frac{d^2w_2(x_0)}{dx^2} = \frac{M_2(x_0)}{EI_2}. \tag{9}$$

Assuming $I_1 = I$, $I_2 = \alpha I$, and also considering $M_1(x_0) = M_2(x_0) = K_r\Theta$, we have

$$\frac{d^2w_1}{dx^2} = \frac{K_r\Theta}{EI}, \quad \frac{d^2w_2}{dx^2} = \frac{K_r\Theta}{\alpha EI}. \tag{10}$$

Therefore, differentiating both sides of Eq. (7) yields

$$\frac{\bar{d}^3w}{dx^3} = \frac{d^3w_1}{dx^3} + \left[\frac{d^3w_2}{dx^3} - \frac{d^3w_1}{dx^3}\right]H(x - x_0) + \left[\frac{d^2w_2}{dx^2} - \frac{d^2w_1}{dx^2}\right]_{x=x_0}\delta(x - x_0)$$
$$+ \Theta\delta^{(1)}(x - x_0) + \Delta\delta^{(2)}(x - x_0)$$
$$= \frac{d^3w_1}{dx^3} + \left[\frac{d^3w_2}{dx^3} - \frac{d^3w_1}{dx^3}\right]H(x - x_0) + \frac{K_r\Theta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta(x - x_0)$$
$$+ \Theta\delta^{(1)}(x - x_0) + \Delta\delta^{(2)}(x - x_0). \tag{11}$$

We also know that

$$\left(EI_1\frac{d^3w_1}{dx^3}+P\frac{dw_1}{dx}\right)_{x=x_0}=\left(EI_2\frac{d^3w_2}{dx^3}+P'\frac{dw_2}{dx}\right)_{x=x_0}=V(x_0)=K_t\Delta. \tag{12}$$

Now let $P'=P+P_0=\beta P$. From Eqs. (1b) and (12), we obtain

$$\left(\frac{d^3w_2}{dx^3}-\frac{d^3w_1}{dx^3}\right)_{x=x_0}=\frac{(1-\alpha)K_t\Delta-\alpha P\Theta}{\alpha EI}+\frac{P}{EI}\left(1-\frac{\beta}{\alpha}\right)\frac{dw_2(x_0)}{dx}. \tag{13}$$

Now differentiating both sides of Eq. (11), we obtain

$$\begin{aligned}\frac{\bar{d}^4w}{dx^4}&=\frac{d^4w_1}{dx^4}+\left[\frac{d^4w_2}{dx^4}-\frac{d^4w_1}{dx^4}\right]H(x-x_0)+\left[\frac{d^3w_2}{dx^3}-\frac{d^3w_1}{dx^3}\right]_{x=x_0}\delta(x-x_0)\\&\quad+\frac{K_r\Theta}{EI}\left(\frac{1}{\alpha}-1\right)\delta^{(1)}(x-x_0)+\Theta\delta^{(2)}(x-x_0)+\Delta\delta^{(3)}(x-x_0).\end{aligned} \tag{14}$$

Hence

$$\begin{aligned}\frac{\bar{d}^4w}{dx^4}&=\frac{d^4w_1}{dx^4}+\left[\frac{d^4w_2}{dx^4}-\frac{d^4w_1}{dx^4}\right]H(x-x_0)\\&\quad+\left[\frac{(1-\alpha)K_t\Delta-\alpha P\Theta}{\alpha EI}+\frac{P}{EI}\left(1-\frac{\beta}{\alpha}\right)\frac{dw_2(x_0)}{dx}\right]\delta(x-x_0)\\&\quad+\frac{K_r\Theta}{EI}\left(\frac{1}{\alpha}-1\right)\delta^{(1)}(x-x_0)+\Theta\delta^{(2)}(x-x_0)+\Delta\delta^{(3)}(x-x_0).\end{aligned} \tag{15}$$

Now let

$$\frac{dw_2(x_0)}{dx}=\frac{\bar{d}w(x_0^+)}{dx}=\bar{\Theta}. \tag{16}$$

From Eqs. (5a) and (5b) we have

$$\frac{d^4w_1}{dx^4}+\frac{P}{EI}\frac{d^2w_1}{dx^2}=\frac{q(x)}{EI},\quad 0\leqslant x<x_0, \tag{17a}$$

$$\frac{d^4w_2}{dx^4}+\frac{\beta}{\alpha}\frac{P}{EI}\frac{d^2w_2}{dx^2}=\frac{1}{\alpha}\frac{q(x)}{EI},\quad x_0<x\leqslant L. \tag{17b}$$

From Eqs. (17a) and (17b) we can write

$$\begin{aligned}&\frac{d^4w_1}{dx^4}+\left(\frac{d^4w_2}{dx^4}-\frac{d^4w_1}{dx^4}\right)H(x-x_0)+\frac{P}{EI}\left[\frac{d^2w_1}{dx^2}+\left(\frac{d^2w_2}{dx^2}-\frac{d^2w_1}{dx^2}\right)H(x-x_0)\right]\\&\quad+\left(\frac{\beta}{\alpha}-1\right)\frac{P}{EI}\frac{d^2w_2}{dx^2}H(x-x_0)=\frac{q(x)}{EI}\left[1+\left(\frac{1}{\alpha}-1\right)H(x-x_0)\right].\end{aligned} \tag{18}$$

Substituting Eqs. (7) and (15) into Eq. (18) we obtain

$$\frac{\bar{d}^4 w}{dx^4} + \frac{P}{EI}\frac{\bar{d}^2 w}{dx^2} + \left(\frac{\beta}{\alpha} - 1\right)\frac{P}{EI}\frac{d^2 w_2}{dx^2} H(x - x_0) = \frac{q(x)}{EI}\left[1 + \left(\frac{1}{\alpha} - 1\right) H(x - x_0)\right]$$
$$+ \left[\frac{(1-\alpha)K_t\Delta}{\alpha\, EI} + \frac{P}{EI}\left(1 - \frac{\beta}{\alpha}\right)\bar{\Theta}\right]\delta(x - x_0)$$
$$+ \frac{K_r\Theta(1/\alpha - 1) + P\Delta}{EI}\delta^{(1)}(x - x_0) + \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0). \tag{19}$$

Multiplying both sides of Eq. (7) by $H(x - x_0)$ yields

$$\frac{\bar{d}^2 w}{dx^2} H(x - x_0) = \frac{d^2 w_2}{dx^2} H(x - x_0) + \Theta\delta(x - x_0)H(x - x_0) + \Delta\delta(x - x_0)H(x - x_0)$$
$$= \frac{d^2 w_2}{dx^2} H(x - x_0) + \frac{\Theta}{2}\delta(x - x_0) + \Delta\delta(x - x_0)H(x - x_0). \tag{20}$$

It is known that the product of $\delta^{(1)}$ and H is not defined. Suppose that $\Delta = 0$. Hence

$$\frac{\bar{d}^2 w_2}{dx^2} H(x - x_0) = \frac{d^2 w_2}{dx^2} H(x - x_0) - \frac{\Theta}{2}\delta(x - x_0). \tag{21}$$

Inserting Eq. (21) into Eq. (19) and considering $\Delta = 0$, we obtain the governing differential equation of the beam–column in terms of a single displacement function w as

$$\frac{\bar{d}^4 w}{dx^4} + \frac{P}{EI}\left[1 + \left(\frac{\beta}{\alpha} - 1\right) H(x - x_0)\right]\frac{\bar{d}^2 w}{dx^2} = \frac{q(x)}{EI}\left[1 + \left(\frac{1}{\alpha} - 1\right) H(x - x_0)\right]$$
$$+ \left[\frac{(1-\alpha)V_0}{\alpha\, EI} + \frac{P}{EI}\left(1 - \frac{\beta}{\alpha}\right)\bar{\Theta}\right]\delta\,(x - x_0) + \frac{K_r\Theta(1/\alpha - 1)}{EI}\delta^{(1)}(x - x_0) + \Theta\delta^{(2)}(x - x_0), \tag{22}$$

where $V_0 = EI\mathrm{d}^3 w(x_0^-)/\mathrm{d}x^3 + P\mathrm{d}w(x_0^-)/\mathrm{d}x$. As can be seen in this case, the operator of the differential equation is different from that of the classical one (3). If $\Delta \neq 0$, the governing differential equation can be written in terms of a single displacement function w if and only if $\alpha = \beta$. In this case, from Eq. (19), the governing differential equation may be expressed as

$$\frac{\bar{d}^4 w}{dx^4} + \frac{P}{EI}\frac{\bar{d}^2 w}{dx^2} = \frac{q(x)}{EI}\left[1 + \left(\frac{1}{\alpha} - 1\right) H(x - x_0)\right] + \frac{(1-\alpha)K_t\Delta}{\alpha\, EI}\delta(x - x_0)$$
$$+ \frac{K_r\Theta\,(1/\alpha - 1) + P\Delta}{EI}\delta^{(1)}(x - x_0) + \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0). \tag{23}$$

It may be seen that in this case, the operator of the governing differential equation is the same as that of the classical one (3) — only the force term is changed. Equation (22) or (23) is the governing differential equation of an Euler–Bernoulli beam–column with one point of jump discontinuity in the space of generalized functions. The continuity conditions can be expressed as

$$EI\,\frac{\mathrm{d}^2 w(x_0^-)}{\mathrm{d}x^2} = K_r\Theta, \quad EI\,\frac{\mathrm{d}^3 w(x_0^-)}{\mathrm{d}x^3} + P\,\frac{\mathrm{d}w(x_0^-)}{\mathrm{d}x} = K_t\Delta \; = V_0, \quad \frac{\bar{\mathrm{d}}w(x_0^+)}{\mathrm{d}x} = \bar{\Theta}. \tag{24}$$

Applying the four boundary conditions at $x = 0$ and L, and the continuity conditions (24), gives us the beam–column deflection w. Therefore, instead of solving two differential equations and applying eight boundary and continuity conditions, only one differential equation with five or six boundary and continuity conditions need to be solved (when $\Delta = 0$ there are only two continuity conditions). In practical problems usually only one of the jump discontinuities exists at a point. In this case using generalized functions makes the analysis even more efficient because there will be fewer continuity conditions.

In summary, in two cases the governing equilibrium equation of an Euler–Bernoulli beam–column can be written in terms of a single displacement function w: (i) when $\Delta = 0$, and (ii) when $\alpha = \beta$. For the special case of a column the governing differential equations may be expressed in case (i) by

$$\frac{\bar{\mathrm{d}}^4 w}{\mathrm{d}x^4} + \frac{P}{EI}\left[1 + \left(\frac{\beta}{\alpha} - 1\right)H(x - x_0)\right]\frac{\bar{\mathrm{d}}^2 w}{\mathrm{d}x^2} = \left[\frac{(1-\alpha)V_0}{\alpha\,EI} + \frac{P}{EI}\left(1 - \frac{\beta}{\alpha}\right)\bar{\theta}\right]\delta(x - x_0)$$
$$+ \frac{K_r\Theta(1/\alpha - 1)}{EI}\delta^{(1)}(x - x_0) + \Theta\delta^{(2)}(x - x_0) \tag{25}$$

and in case (ii) by

$$\frac{\bar{\mathrm{d}}^4 w}{\mathrm{d}x^4} + \frac{P}{EI}\frac{\bar{\mathrm{d}}^2 w}{\mathrm{d}x^2} = \frac{(1-\alpha)K_t\Delta}{\alpha\,EI}\delta(x - x_0) + \frac{K_r\Theta\,(1/\alpha - 1) + P\Delta}{EI}\delta^{(1)}(x - x_0)$$
$$+ \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0). \tag{26}$$

In subsequent sections we show that the above equations may be used to calculate the buckling load of columns with jump discontinuities.

It should be noted that in Eqs. (22) and (25) the Heaviside function appears as a coefficient of the differential operator $\mathrm{d}^2/\mathrm{d}x^2$. For solving the differential equation in this case, we should first assume that $w(x) = X_1(x) + X_2(x)H(x - x_0)$ and then substitute it in the differential equation and then find the functions X_1 and X_2. Clearly, using the classical method for this case is more efficient. Therefore, although theoretically we can express the governing differential equation in terms of a single displacement function w, practically classical method is preferable for this case. Hence here we consider only case (ii) in subsequent sections.

A common method of solving differential equations in the space of generalized functions is the Laplace transform method [16]. But before solving Eq. (26), we try to find an equivalent boundary-value problem in the space of classical functions by defining an auxiliary beam–column.

3. The auxiliary beam–column method

The auxiliary beam method was introduced in Ref. [12] for beams with jump discontinuities. In this section, we generalize the idea for beam–columns with jump discontinuities.

Here only case (ii), Eq. (23), is considered; for case (i) it is not possible to define an equivalent boundary-value problem in the space of classical functions. As was mentioned, for case (i) it is easier to use the classical method to analyze the beam–column. Suppose that w is the deflection of an Euler–Bernoulli beam–column with jump discontinuities in slope, deflection, and flexural stiffness at a point $x = x_0$ and under a point axial force at this point. The deflection of the auxiliary beam–column is defined as follows:

$$\bar{w}(x) = w(x) - \Delta H(x - x_0) - \Theta(x - x_0)H(x - x_0) - \frac{K_r\Theta}{2EI}\left(\frac{1}{\alpha} - 1\right)(x - x_0)^2 H(x - x_0) - \left[\frac{(1-\alpha)K_t\Delta - \alpha P\Theta}{6\alpha EI}\right](x - x_0)^3 H(x - x_0). \tag{27}$$

Therefore

$$\frac{d^2\bar{w}}{dx^2} = \frac{\bar{d}^2 w}{dx^2} - \Delta\delta^{(1)}(x - x_0) - \Theta\delta(x - x_0) - \frac{K_r\Theta}{EI}\left(\frac{1}{\alpha} - 1\right)H(x - x_0) - \left[\frac{(1-\alpha)K_t\Delta - \alpha P\Theta}{\alpha EI}\right](x - x_0)H(x - x_0), \tag{28a}$$

$$\frac{d^4\bar{w}}{dx^4} = \frac{\bar{d}^4 w}{dx^4} - \Delta\delta^{(3)}(x - x_0) - \Theta\delta^{(2)}(x - x_0) - \frac{K_r\Theta}{EI}\left(\frac{1}{\alpha} - 1\right)\delta^{(1)}(x - x_0) - \left[\frac{(1-\alpha)K_t\Delta - \alpha P\Theta}{\alpha EI}\right]\delta(x - x_0). \tag{28b}$$

Clearly, $w(x)$ is a classical function. Substituting Eqs. (28a) and (28b) into Eq. (23) yields

$$\frac{d^4\bar{w}(x)}{dx^4} + \frac{P}{EI}\frac{d^2\bar{w}(x)}{dx^2} = \frac{q(x)}{EI}\left[1 + \left(\frac{1}{\alpha} - 1\right)H(x - x_0)\right] + \frac{P}{EI}\left\{\frac{K_r\Theta}{EI}\left(\frac{1}{\alpha} - 1\right) + \left[\frac{(1-\alpha)K_t\Delta - \alpha P\Theta}{\alpha EI}\right](x - x_0)\right\}H(x - x_0). \tag{29}$$

Also, from Eq. (27) we have

$$\bar{w}(0) = w(0), \quad \bar{w}(L) = w(L) - \Delta - \Theta(L - x_0) - \frac{K_r\Theta}{2EI}\left(\frac{1}{\alpha} - 1\right)(L - x_0)^2 - \left[\frac{(1-\alpha)K_t\Delta - \alpha P\Theta}{6\alpha EI}\right](L - x_0)^3, \tag{30a}$$

$$\frac{d\bar{w}(0)}{dx} = \frac{dw(0)}{dx}, \quad \frac{d\bar{w}(L)}{dx} = \frac{dw(L)}{dx} - \Theta - \frac{K_r\Theta}{EI}\left(\frac{1}{\alpha} - 1\right)(L - x_0)$$
$$- \left[\frac{(1-\alpha)K_t\Delta - \alpha\, P\Theta}{2\alpha\, EI}\right](L - x_0)^2, \tag{30b}$$

$$\frac{d^2\bar{w}(0)}{dx^2} = \frac{d^2w(0)}{dx^2}, \quad \frac{d^2\bar{w}(L)}{dx^2} = \frac{d^2w(L)}{dx^2} - \frac{K_r\Theta}{EI}\left(\frac{1}{\alpha} - 1\right)$$
$$- \left[\frac{(1-\alpha)K_t\Delta - \alpha\, P\Theta}{\alpha\, EI}\right](L - x_0). \tag{30c}$$

The continuity conditions for the auxiliary beam are

$$\frac{d^2\bar{w}(x_0)}{dx^2} = \frac{d^2w(x_0^-)}{dx^2} = \frac{K_r\Theta}{EI}, \quad EI\frac{d^3\bar{w}(x_0)}{dx^3} + P\frac{d\bar{w}(x_0)}{dx} = EI\frac{d^3w(x_o^-)}{dx^3} + P\frac{dw(x_0^-)}{dx} = K_t\Delta. \tag{31}$$

Therefore, instead of solving two differential equations for the two beam segments and applying eight boundary and continuity conditions, we solve only one differential equation with six boundary and continuity conditions. As can be seen from Eq. (29), Heaviside's function appears in the differential equation, and because we have both second- and fourth-order derivatives of the dependent variable in the differential equation, direct integration is not possible. It follows that for beam–columns, in contrast to beam bending problems, using the auxiliary beam–column does not make the analysis easier. Therefore, the governing differential equation (23) is directly solved in the space of generalized functions using the Laplace transform method. To clarify the method, some examples are solved in the next section.

4. Examples

In this section three examples are solved. In the first two examples buckling loads of columns with jump discontinuities are calculated and in the last example the deflection of a beam–column with jump discontinuities is calculated. For all three examples the governing differential equation is solved in the space of generalized functions using the Laplace transform method.

Example 1. Consider the simply supported column shown in Fig. 2a. The column has an internal hinge with a rotational spring at $x = L/2$. The cross section of the column is uniform; i.e., $\alpha = 1$. There is no internal axial force; i.e., $P_0 = 0$, hence $\beta = 1$. The governing differential equation of the column is found by substituting $\alpha = \beta = 1$ and $\Delta = 0$ in Eq. (26):

$$\frac{\bar{d}^4w}{dx^4} + \frac{P}{EI}\frac{\bar{d}^2w}{dx^2} = \Theta\delta^{(2)}\left(x - \frac{L}{2}\right). \tag{32}$$

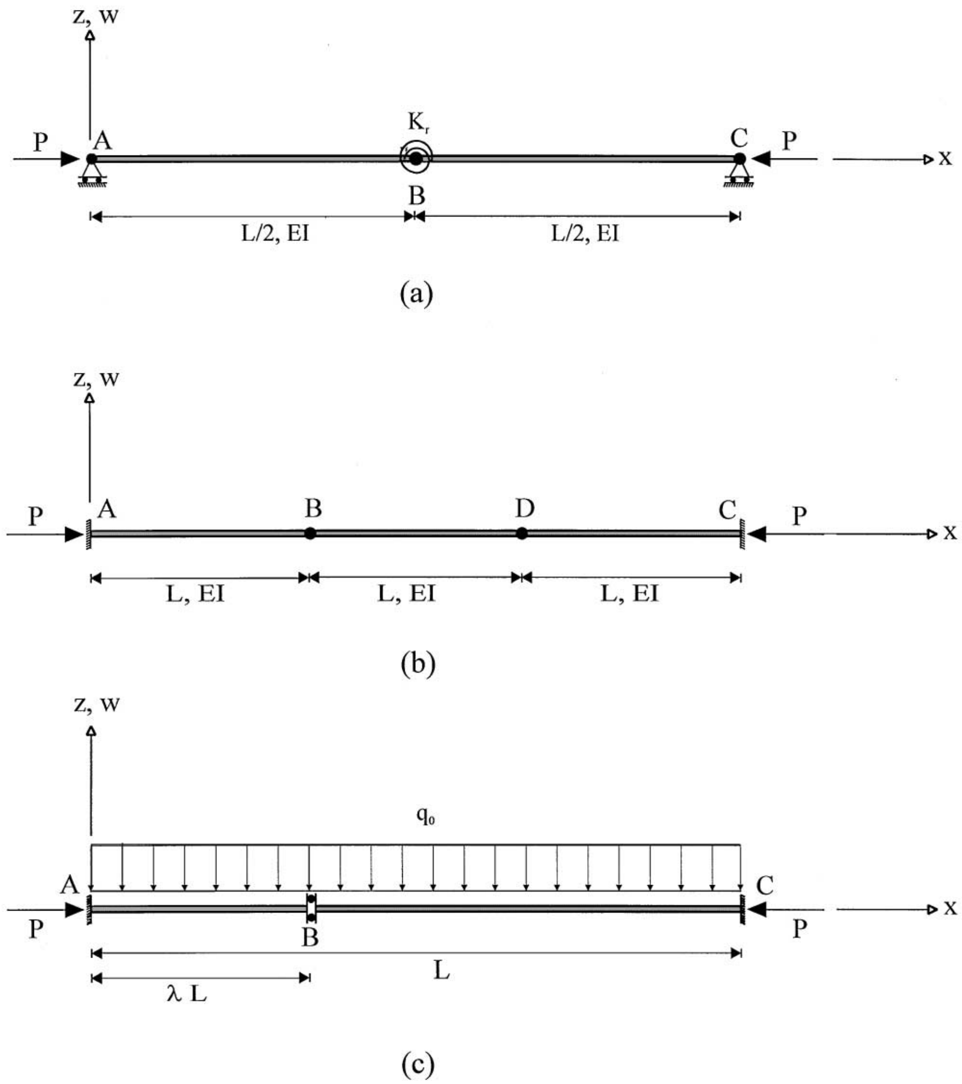

Fig. 2. (a) A simply supported column with an internal hinge and rotational spring. (b) A clamped–clamped column with two internal hinges. (c) A clamped–clamped beam–column with an internal shear–free connection under a uniform distributed force.

Now let $k^2 = P/EI$ and $\mathscr{L}\{w\} = W(s)$, where $\mathscr{L}$ is the Laplace transform operator. Taking the Laplace transform from both sides of Eq. (32) and considering the boundary conditions $w(0) = w''(0) = 0$, we obtain

$$W(s) = \frac{A}{s^2 + k^2} + \frac{B + Ak^2}{s^2(s^2 + k^2)} + \Theta\frac{e - (L/2)s}{s^2 + k^2}, \tag{33}$$

where $A = w'(0)$ and $B = w'''(0)$. After some manipulations and taking inverse Laplace transforms from both sides of Eq. (33) we find

$$w(x) = \left(\frac{B}{k^2} + A\right)x - \frac{B}{k^3}\sin kx + \frac{\Theta}{k}\sin k\left(x - \frac{L}{2}\right)H\left(x - \frac{L}{2}\right) \tag{34}$$

Applying the boundary conditions $w(L) = w''(L) = 0$ and the continuity condition $w''(L^-/2) = K_r/EI$ yields

$$LA + \frac{L - \sin kL/k}{k^2}B + \frac{\sin kL/2}{k}\Theta = 0, \tag{35a}$$

$$\frac{\sin kL}{k}B + k\sin\frac{kL}{2}\Theta = 0, \tag{35b}$$

$$\frac{\sin kL/2}{k}B - \frac{K_r}{EI}\Theta = 0. \tag{35c}$$

For the system of linear equations (35) to have nontrivial solutions, the determinant of the coefficient matrix must be zero; hence

$$\begin{vmatrix} L & \frac{1}{k^2}\left(L - \frac{\sin kL}{k}\right) & \frac{\sin kL/2}{k} \\ 0 & \frac{\sin kL}{k} & k\sin\frac{kL}{2} \\ 0 & \frac{\sin kL/2}{k} & -\frac{K_r}{EI} \end{vmatrix} = 0. \tag{36}$$

Therefore

$$\sin kL + \frac{kEI}{K_r}\left(\sin\frac{kL}{2}\right)^2 = 0. \tag{37}$$

From Eq. (37) the buckling load P_{cr} can be calculated by trial and error. For the special case when K_r is very large ($K_r \to \infty$), from Eq. (37) we obtain $\sin kL = 0$, which gives the buckling load of a simply supported column with length L:

$$P_{cr} = \frac{n^2\pi^2 EI}{L^2}, \quad n = 1, 2, 3, \ldots. \tag{38}$$

as we expected.

Example 2. A clamped–clamped column with two internal hinges at $x = L$ and $2L$ is shown in Fig. 2b. The column has a uniform cross-section. The governing differential equation of the column may be written as

$$\frac{\bar{d}^4 w}{dx^4} + \frac{P}{EI}\frac{\bar{d}^2 w}{dx^2} = +\Theta_1\delta^{(2)}(x - L) + \Theta_2\delta^{(2)}(x - 2L), \tag{39}$$

where

$$\Theta_1 = \frac{\mathrm{d}w(L^+)}{\mathrm{d}x} - \frac{\mathrm{d}w(L^-)}{\mathrm{d}x}, \quad \Theta_2 = \frac{\mathrm{d}w(2L^+)}{\mathrm{d}x} - \frac{\mathrm{d}w(2L^-)}{\mathrm{d}x}. \tag{40}$$

Again, let $k^2 = P/EI$ and $\mathscr{L}\{w\} = W(s)$. Taking the Laplace transform from both sides of Eq. (39) and considering the boundary conditions at $x = 0$ yields

$$W(s) = \frac{A}{s(s^2+k^2)} + \frac{B}{s^2(s^2+k^2)} + \Theta_1 \frac{\mathrm{e}^{-sL}}{s^2+k^2} + \Theta_2 \frac{\mathrm{e}^{-2sL}}{s^2+k^2}, \tag{41}$$

where$A = w''(0)$ and $B = w'''(0)$. After some manipulations and taking the inverse Laplace transform from both sides of Eq. (41) we obtain

$$w(x) = \frac{A}{k^2}(1 - \cos kx) + \frac{B}{k^2}\left(x - \frac{1}{k}\sin kx\right) + \Theta_1 \frac{1}{k}\sin k(x-L)\,H(x-L)$$
$$+ \Theta_2 \frac{1}{k}\sin k(x-2L)\,H(x-2L). \tag{42}$$

Enforcing boundary conditions $w(3L) = w'(3L) = 0$ and continuity conditions $w''(L^-) = w''(2L^-) = 0$ yields

$$\frac{1-\cos 3kL}{k^2}A + \frac{3L - \sin 3kL/k}{k^2}B + \frac{\sin 2kL}{k}\Theta_1 + \frac{\sin kL}{k}\Theta_2 = 0, \tag{43a}$$

$$\frac{\sin 3kL}{k}A + \frac{1-\cos 3kL}{k^2}B + (\cos 2kL)\Theta_1 + (\cos kL)\Theta_2 = 0, \tag{43b}$$

$$\cos kL\, A + \frac{\sin kL}{k}B = 0, \tag{43c}$$

$$\cos 2kL\, A + \frac{\sin 2kL}{k}B - k\sin kL\Theta_1 = 0. \tag{43d}$$

The buckling load can be calculated from the following equation:

$$\begin{vmatrix} \frac{1-\cos 3kL}{k^2} & \frac{3L-\sin 3kL/k}{k^2} & \frac{\sin 2kL}{k} & \frac{\sin kL}{k} \\ \frac{\sin 3kL}{k} & \frac{1-\cos 3kL}{k^2} & \cos 2kL & \cos kL \\ \cos kL & \frac{\sin kL}{k} & 0 & 0 \\ \cos 2kL & \frac{\sin 2kL}{k} & -k\sin kL & 0 \end{vmatrix} = 0 \tag{44a}$$

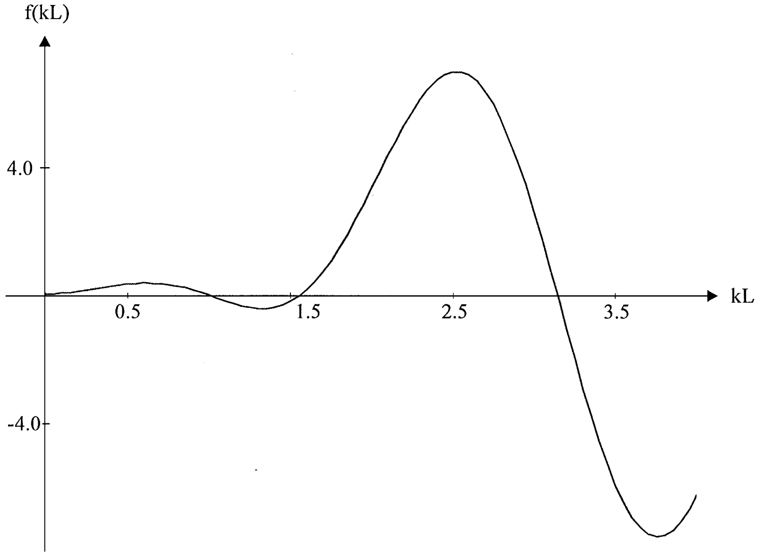

Fig. 3. The graph of $f(kL)$ for Example 2. It is seen that $f(kL)$ has roots in [0.5, 1.5] and [1.5, 2.5].

or

$$f(kL) = 2\sin^2 kL(2\cos^2 kL - \cos kL) + \sin 2kL(3kL\cos kL - \sin kL - \sin 2kL) = 0. \quad (44b)$$

A plot of $f(kL)$ is shown in Fig. 3 for $0 \leqslant kL \leqslant 4$. By trial and error the first two roots of Eq. (44b) are $kL = 0.967408, 1.57079$ or $P_{cr} = 0.3793\,\pi^2 EI/4L^2, \pi^2 EI/4L^2$.

It is observed that in this example, instead of 12 boundary and continuity conditions of the classical method, only six boundary and continuity conditions needed to be enforced.

Example 3. In this example a clamped–clamped beam–column with an internal shear-free connection under a uniformly distributed force is considered (Fig. 2c). The beam–column has a uniform cross-section ($\alpha = \beta = 1$). From Eq. (23), the governing differential equation of the beam–column can be written as

$$\frac{\bar{d}^4 w}{dx^4} + \frac{P}{EI}\frac{\bar{d}^2 w}{dx^2} = -\frac{q_0}{EI} + \frac{P}{EI}\Delta\delta^{(1)}(x - \lambda L) + \Delta\delta^{(3)}(x - \lambda L). \quad (45)$$

As was done in the previous examples, the Laplace transform of w, $\mathscr{L}\{w\} = W(s)$, is found; here, it is

$$W(s) = \frac{A}{s(s^2 + k^2)} + \frac{B}{s^2(s^2 + k^2)} - \frac{q_0}{EI}\frac{1}{s^3(s^2 + k^2)} + \frac{k^2\Delta}{s(s^2 + k^2)}\,e^{-\lambda Ls} + \Delta\frac{s}{s^2 + k^2}\,e^{-\lambda Ls} \quad (46)$$

where $A = w''(0)$ and $B = w'''(0)$. Taking the inverse Laplace transform from both sides of Eq. (46) yields

$$w(x) = \frac{A}{k^2}(1 - \cos kx) + \frac{B}{k^2}\left(x - \frac{1}{k}\sin kx\right) - \frac{q_0}{EIk^4}(k^2 x - 1 + \cos kx) + \Delta H(x - \lambda\,L). \quad (47)$$

Applying the boundary conditions $w(L) = w'(L) = 0$ and the continuity condition $EIw'''(\lambda L^-) - Pw'(\lambda L^-) = 0$ yields

$$\frac{1-\cos kL}{k^2}A + \frac{L-(1/k)\sin kL}{k^2}B + \Delta = \frac{q_0}{EIk^4}(k^2L - 1 + \cos kL), \tag{48a}$$

$$\frac{\sin kL}{k}A + \frac{1-\cos kL}{k^2}B = \frac{q_0}{EIk^3}(k - \sin kL), \tag{48b}$$

$$2\,k \sin k\lambda L\; A + (1 - 2\cos k\lambda\, L)B = \frac{q_0}{EI}(1 - 2\sin k\lambda L). \tag{48c}$$

When $q_0 = 0$, the buckling load can be calculated from the following equation:

$$2\sin k(\lambda - 1)L + \sin kL - 2\sin k\lambda\, L = 0. \tag{49}$$

When $\lambda = 1$, the buckling load is

$$\sin kL = 0 \;\Rightarrow\; P_{cr} = \frac{\pi^2 EI}{L^2} \tag{50}$$

which is the correct answer [13].

5. Conclusions

In this article the distribution theory of Schwarz is used to obtain the governing differential equation of Euler–Bernoulli beam–columns with various jump discontinuities. It is demonstrated that the governing equilibrium equation of beam–columns with jump discontinuities in slope, deflection, and flexural stiffness, and with a point load applied at the point of discontinuity, can be written in terms of a single displacement function in two cases: (i) when there is no jump discontinuity in deflection ($\Delta = 0$), and (ii) when $\alpha = \beta$. In the first case the operator of the differential equation is different from that of the classical equation, but in the second case the operator remains unchanged.

It is observed that distributional derivatives of the Dirac delta function appear in the governing differential equations. The auxiliary beam–column method is defined for case (ii). The displacement function of the auxiliary beam–column is always a classical function. It is found that, in contrast to the case of beam bending problems, the auxiliary beam–column method has no superiority; indeed, it is more convenient to solve the governing differential equation of the beam–column in the space of generalized functions. A good way to solve these differential equations is the Laplace transform method. This method is applied in two examples to calculate buckling loads for columns with jump discontinuities, and in a third example to calculate the deflection of a beam–column. This method is applicable only to case (ii); for case (i) the classical method should be used to analyze the beam–column.

This investigation shows that distribution theory improves one's understanding of discontinuous problems, and in some cases offers efficient methods for analyzing these problems.

Acknowledgements

We are grateful to the reviewer who called our attention to the work of Lowe [8] on buckling analysis of columns with jump discontinuities using the discontinuity method.

Appendix: Schwarz's distribution theory

This appendix gives some definitions and operations in the Schwarz theory of distributions that are used in this paper. We restrict our discussion to distributions with a one-dimensional independent real variable. For more details, the reader may refer to Zemanian [16], Kanwal [17], Stakgold [18] and Lighthill [19].

Definition 1. *The Heaviside function* $H(x - x_0)$ is defined as

$$H(x - x_0) = \begin{cases} 0, & x < x_0, \\ 1, & x > x_0. \end{cases} \tag{A.1}$$

It has a jump discontinuity at $x = x_0$. Its value at $x = x_0$ is usually taken to be 1/2 Clearly,

$$H(x_0 - x) = 1 - H(x - x_0). \tag{A.2}$$

The Heaviside function is very useful in the study of functions with jump discontinuities. For example, let $F(x)$ be a function that is continuous everywhere except at the point $x = x_0$, where it has a jump discontinuity

$$F(x) = \begin{cases} F_1(x), & x < x_0, \\ F_2(x), & x > x_0. \end{cases} \tag{A.3}$$

Then the function can be written as

$$\begin{aligned} F(x) &= F_1(x)H(x_0 - x) + F_2(x)H(x - x_0) \\ &= F_1(x) + [F_2(x) - F_1(x)]H(x - x_0). \end{aligned} \tag{A.4}$$

Definition 2. *Test functions* are real-valued functions $\varphi(x)$ with the following two properties: (1) φ is infinitely smooth; (2) φ is zero outside a finite interval; i.e., φ has compact support. The space of the test functions is denoted by D.

Definition 3. A *distribution* is a continuous linear functional on the space D of test functions. The space of all distributions is denoted by D'; D' is itself a linear space and is called the dual space of D, but is a larger space than D. The space D' forms a generalization of the class of locally integrable functions because it contains functions that are not locally integrable. Here the terms "distribution" and "generalized function" are used interchangeably.

A locally integrable function is integrable in the Lebesgue sense over every finite interval. Every locally integrable function $f(x)$ generates a distribution by means of the formula

$$\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(x)\, \varphi(x)\, \mathrm{d}x. \tag{A.5}$$

This is called a *regular distribution*. All other distributions are called *singular distributions*.

Two distributions in D', f and g are said to be equal if

$$\langle f, \varphi \rangle = \langle g, \varphi \rangle \tag{A.6}$$

for every test function $\varphi(x)$ in D.

Definition 4. *The Dirac delta function* is a singular generalized function defined as

$$\langle \delta(x - x_0), \varphi(x) \rangle = \varphi(x_0). \tag{A.7}$$

Definition 5. The nth derivative $f^{(n)}(x)$ of any generalized function $f(x)$ is given by

$$\langle f^{(n)}(x), \varphi(x) \rangle = \langle f(x), (-1)^n \varphi^{(n)}(x) \rangle, \quad \varphi \in D. \tag{A.8}$$

The nth distributional derivative of the delta function is therefore defined as

$$\langle \delta^{(n)}(x - x_0), \varphi(x) \rangle = \langle \delta(x - x_0), (-1)^n \varphi^{(n)}(x) \rangle = (-1)^n \varphi(x_0). \tag{A.9}$$

Corollary.

$$\begin{aligned} \langle H^{(1)}(x - x_0), \varphi(x) \rangle = \langle H(x - x_0), -\varphi^{(1)}(x) \rangle &= -\int_{x_0}^{+\infty} \varphi^{(1)}(x)\, \mathrm{d}x \\ &= \varphi(x_0) = \langle \delta(x - x_0), \varphi(x) \rangle. \end{aligned} \tag{A.10}$$

Hence

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}x} H(x - x_0) = \delta(x - x_0). \tag{A.11}$$

Theorem 1. *If f(x) is a classical function and*

$$f(x) + a_0 \delta(x - x_0) + \cdots + a_n \delta^{(n)}(x - x_0) = 0 \tag{A.12}$$

on the whole axis, $-\infty < x < +-\infty$, *then* $f(x) = 0$ *and* $a_0 = \cdots = a_{\mathrm{n}} = 0$.

Theorem 2. *Let a function $f(x)$ be n times continuously differentiable; then*

$$\begin{aligned} f(x)\, \delta^{(n)}(x - x_0) = (-1)^n f^{(n)}(x_0)\, \delta(x - x_0) + (-1)^{n-1}\, n f^{(n-1)}(x_0)\, \delta^{(1)}(x - x_0) \\ + (-1)^{n-2} \frac{n(n-1)}{2!} f^{(n-2)}(x_0)\, \delta^{(2)}(x - x_0) + \cdots + f(x_0)\, \delta^{(n)}(x - x_0). \end{aligned} \tag{A.13}$$

Corollary.

$$[f(x)H(x-x_0)]^{(n)} = f^{(n)}(x)H(x-x_0) + f^{(n-1)}(x_0)\delta(x-x_0) + f^{(n-2)}(x_0)\,\delta^{(1)}(x-x_0) + \cdots + f(x_0)\,\delta^{(n-1)}(x-x_0). \quad (A.14)$$

Definition 6. The space of distributions D'_R having their supports bounded on the left is called the space of *right-sided distributions*, $D'_R \subset D'$ (proper subspace).

Definition 7. *The convolution* of two right-sided distributions f and g, $f*g$ is defined as

$$\langle h, \varphi\rangle = \langle f*g, \varphi\rangle = \langle f(x), \langle g(\tau), \varphi(x+\tau)\rangle\rangle = \langle g*f, \varphi\rangle. \quad (A.15)$$

Theorem 3. *The convolution of the nth derivative of the delta function with any distribution yields the nth derivative of that distribution, i.e.,*

$$\delta^{(n)}*f = f^{(n)}. \quad (A.16)$$

Proof.

$$\langle \delta^{(n)}*f, \varphi\rangle = \langle f*\delta^{(n)}, \varphi\rangle = \langle f(x), \langle \delta^{(n)}(\tau), \varphi(x+\tau)\rangle\rangle = \langle f(x), (-1)^n\varphi^{(n)}(x)\rangle = \langle f^{(n)}(x), \varphi(x)\rangle.$$

Definition 8. The Laplace transform of a distribution f whose support is bounded on the left is defined as

$$F(s) = Lf(x) = \langle f(x), e^{-sx}\rangle. \quad (A.17)$$

The Laplace transform of the Heaviside function, the delta function and its distributional derivatives, can be computed directly from the definition.

$$L\{H(x-x_0)\} = \frac{1}{s}\,e^{-sx_0}, \quad (A.18a)$$

$$L\{\delta(x-x_0)\} = e^{-sx_0}, \quad (A.18b)$$

$$L\{\delta^{(k)}(x-x_0)\} = s^k e^{-sx_0}. \quad (A.18c)$$

Theorem 4. *If $f^{(k)}(x)$ exists and is continuous for all x, then*

$$L\{f^{(k)}(x)\} = s^kF(s) - f(0^+)s^{k-1} - f^{(1)}(0^+)s^{k-2} - \cdots - f^{(k-1)}(0^+). \quad (A.19)$$

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