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BENDING OF UNBONDED MULTILAYERED BEAMS AND PLATES

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Introduction

In this article, the bending of unbonded multilayered beams and plates are studied. It is shown that under specific loading and various boundary conditions adjacent layers remain fully in contact. Here, an unbonded multilayered beam (plate) is composed of Euler-Bernoulli beams (Kirchhoff-type plates) loosely placed on each other. The analysis is performed using a semi-inverse method.

Unbonded Multilayered Beams

Consider a system composed of n Euler-Bernoulli beams loosely placed on each other with the following hypotheses:

H_1. The length of all beams are equal.

H_2. The beams may have different flexural stiffnesses.

H_3. At each of the two boundary points, the boundary conditions are the same for all beam layers.

Such a system is called an Unbonded Multilayered Beam with n layers (UMB^n). For a UMB^n there are six possible boundary conditions (FIG. 1). For the sake of simplicity, at first a UMB^2 with C-F boundary condition is considered. A downward force is applied at point $x = x_0$ (FIG. 2). At the outset the contact force distribution, F(x), is unknown. The total potential energy of the system must be minimized with the following inequality constraint

$$w_1(x) \geq w_2(x) \quad \forall x \in [0, \ell] \tag{1}$$

Here, instead of solving a variational inequality problem, a semi-inverse method [1-2] is used. We assume that the contact force distribution is nonzero only for $x = x_0$, i.e.,

$$F(x) = F_0 \delta(x - x_0) \tag{2}$$

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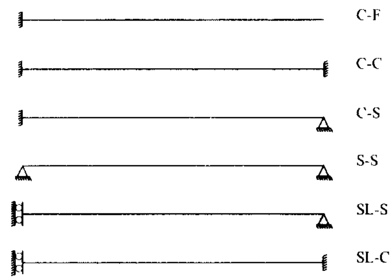

Figure 1. Six possible boundary conditions for a UMB^n

It should be noted that, implicitly we have assumed that $w_1(x_0) = w_2(x_0)$; because

$$F(x^*) \neq 0 \Rightarrow w_1(x^*) = w_2(x^*) \tag{3}$$

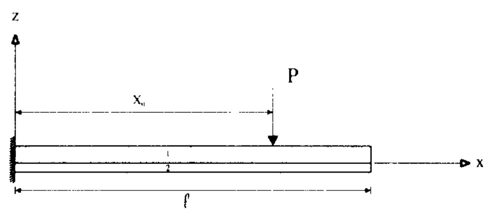

Figure 2. UMB^2 with C-F boundary condition under a concentrated force

where x^* is an arbitrary point in the interval $(0,\ell]$. However, there is no restriction for w_1 and w_2 at other points as long as they satisfy the constraint (1). If $F(x^*) = 0$, we cannot conclude that $w_1(x^*) \neq w_2(x^*)$; this means that it is possible for the two beams to be in contact in a point without any contact force at that point. By this assumption on contact force distribution, equilibrium equations can be written as

$$\begin{aligned} EI_1 w_1^{(IV)} &= (F_0 - P)\delta(x - x_0) \\ EI_2 w_2^{(IV)} &= -F_0\delta(x - x_0) \end{aligned} \tag{4}$$

and with the following boundary conditions

$$w_1(0) = w_1'(0) = w_2(0) = w_2'(0) = 0 \tag{5.a}$$

$$w_1(x_0) = w_2(x_0), \; w_1''(\ell) = w_2''(\ell) = 0 \tag{5.b}$$

$$EI_1 w_1'''(\ell) + EI_2 w_2'''(\ell) = 0, \; w_1(\ell) = w_2(\ell) \; or \; w_1'''(\ell) = w_2'''(\ell) = 0 \tag{5.c}$$

The end condition (5.c) means that if $w_1(\ell) = w_2(\ell)$ we have $V_1(\ell)+V_2(\ell) = 0$, otherwise $V_1(\ell)= V_2(\ell) = 0$. Solving the uncoupled system of differential equations (4) and applying boundary conditions (5), yields

$$w_1(x) = w_2(x) \qquad \forall x \in [0, \ell] \tag{6}$$

which satisfies the constraint (1). Therefore, the assumption (2) is correct. This means that the two beams deform simultaneously, i.e., the contact between the two beams is full. Also from (4) and (6) we have

$$F_0 = \frac{EI_2}{EI_1 + EI_2} \tag{7}$$

Therefore, the distribution of contact stresses can be written as

$$\sigma^c(x,y) = \frac{EI_2}{EI_1 + EI_1}\frac{P}{b}\delta(x - x_0) \tag{8}$$

where b is the width of the beam layers. This is also true for all the other boundary conditions.

A general downward load is defined as a general distributed force per unit length, q(x) for which

$$q(x) \leq 0 \qquad \forall x \in [0, \ell] \tag{9}$$

Now consider a UMB^2 with C-F boundary condition under a general downward load q(x) (FIG. 3). In the interval [x, x+Δx], this loading condition is equivalent to a concentrated force $\Delta P = q(x)\Delta x$. Under this concentrated force, contact is full; therefore by using the principle of superposition it can be concluded that under a general downward load q(x) contact is full, too. The distribution of contact stresses can be written as

$$\sigma^c(x,y) = \frac{EI_2}{EI_1 + EI_2}\frac{q(x)}{b} \tag{10}$$

It should be noted that a UMB^2 is a nonlinear system but under a general downward load it acts like a linear system and the principle of superposition can be used. This result is also true for all the other boundary conditions and it can be easily generalized for a UMB^n with general boundary conditions.

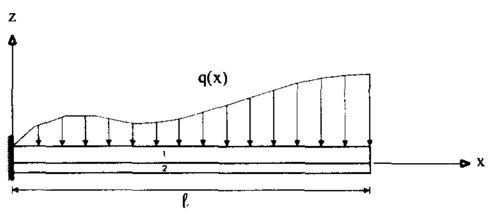

Figure 3. UMB^2 with C-F boundary condition under a general downward load

For a UMB^n under a general downward load q(x), the contact stress distribution can be written as

$$\sigma_i{}^c(x,y) = R_i q(x)\,,\ i = 1,2,3,\ldots,n-1$$

$$R_i = \frac{\sum_{j=i+1}^{n} EI_j}{b\sum_{j=1}^{n} EI_j} \tag{11}$$

where $\sigma_i^c(x,y)$ is the contact stress between *ith* and (i+1)*th* beams. Beams are numbered from top to bottom (FIG. 4).

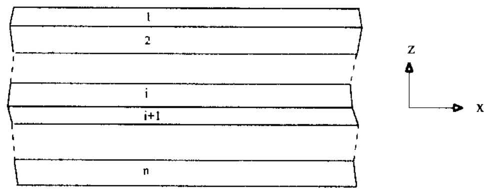

Figure 4. Numbering of the layers of a UMB^n or UMP^n

Unbonded Multilayered Plates

Consider a system of n Kirchhoff-type plates loosely placed on each other with the following hypotheses:
H_1. All the plate layers are identical in shape.
H_2. The plates may have different flexural stiffnesses.
H_3. Friction between any two successive plates is neglected.
H_4. At any point on the boundary, the boundary conditions are the same for all the plate layers.
Such a system is called an Unbonded Multilayered Plate with n layers (UMP^n). For the sake of simplicity, at first a UMP^2 is considered (FIG. 5).

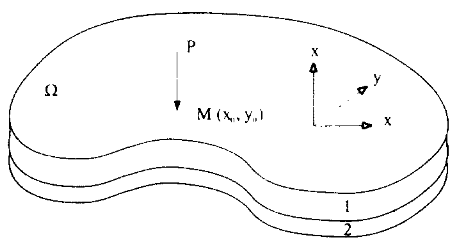

Figure 5. UMP^2 with arbitrary shape and general boundary conditions under a downward concentrated force

Suppose that this system is under the action of a concentrated downward force P at point $M(x_0,y_0)$. Again, at the outset, the contact stress distribution is unknown. We have also the following inequality constraint

$$w_1(x,y) \geq w_2(x,y) \qquad \forall (x,y) \in \Omega \tag{12}$$

Using a semi-inverse method, the following assumption is made on the contact force distribution:

$$F(x,y) = F_0 \delta(x - x_0, y - y_0) \tag{13}$$

where $\delta(x-x_0,y-y_0)$ is the Dirac delta function. We have also, implicitly, assumed that $w_1(x_0,y_0) =$

$w_2(x_0,y_0)$. With this assumption, equilibrium equations can be written as

$$\nabla^2\nabla^2 w_1(x,y) = \frac{-P+F_0}{D_1}\delta(x-x_0,y-y_0)$$
$$\nabla^2\nabla^2 w_2(x,y) = \frac{-F_0}{D_2}\delta(x-x_0,y-y_0) \tag{14}$$

where D_1 and D_2 are flexural stiffnesses. w_1 and w_2 are expressed by

$$w_1(x,y) = \iint_\Omega G(x,y;\xi,\eta)\frac{-P+F_0}{D_1}\delta(\xi-x_0,\eta-y_0)d\xi d\eta$$
$$w_2(x,y) = \iint_\Omega G(x,y;\xi,\eta)\frac{-F_0}{D_2}\delta(\xi-x_0,\eta-y_0)d\xi d\eta \tag{15}$$

The function $G(x,y;\xi,\eta)$ is Green's function which satisfies

$$L^*G(x,y;\xi,\eta) = \delta(x-\xi,y-\eta) \tag{16}$$

where $L = \nabla^2\nabla^2$, and L^* is the adjoint operator of L but in this case $L = L^*$ (self-adjoint) [3]. The Green's function, $G(x,y;\xi,\eta)$ must satisfy the homogeneous boundary conditions. It should be noted that, since boundary conditions are the same for the two plates, only one Green's function, $G(x,y;\xi,\eta)$ has been used. From (15), we obtain

$$w_1(x,y) = -\frac{P-F_0}{D_1}G(x,y;x_0,y_0)$$
$$w_2(x,y) = -\frac{F_0}{D_2}G(x,y;x_0,y_0) \tag{17}$$

Applying the condition $w_1(x_0,y_0) = w_2(x_0,y_0)$, yields

$$\frac{P-F_0}{D_1} = \frac{F_0}{D_2} \tag{18}$$

Therefore,

$$w_1(x,y) = w_2(x,y) \qquad \forall(x,y)\in\Omega \tag{19}$$

which satisfies the constraint (12); therefore the assumption (13) is correct. Thus the two plate layers deform simultaneously; i.e., contact remains full. Also

$$F_0 = \frac{D_2}{D_1+D_2}P \tag{20}$$

This is also true for a UMP^n.

Now consider a UMP^n under a general downward load $q(x,y)$, i.e.,

$$q(x,y) \le 0 \qquad \forall(x,y)\in\Omega \tag{21}$$

Since the load $q(x,y)$ can be considered as a combination of differential concentrated forces $q(x,y)dxdy$, all plate layers deform simultaneously, i.e., contact between any two successive plates is full. Also the contact stress distribution can be written as

$$\sigma_i^c(x,y) = R_i q(x,y) \qquad ,i = 1,2,\dots,n-1$$

$$R_i = \frac{\sum_{j=i+1}^{n} D_j}{\sum_{j=1}^{n} D_j} \tag{22}$$

The plates are numbered from top to bottom (FIG. 4).

For deflection analysis of a UMB^n or UMP^n under a general downward load q, one can consider only the upper beam or plate under the general downward load q_1 which is defined as

$$q_1 = \frac{S_1}{\sum_{i=1}^{n} S_i} \tag{23}$$

where S_1 and S_i are the flexural stiffnesses of the upper and the ith beams or plates, respectively. The deflection of this system is the solution.

Conclusions

It was shown that, under a general downward load, the n layers of an unbonded multilayered beam or plate deform simultaneously. Also the contact stress distribution was found. From the contact mechanics point of view, this is a frictionless stationary contact problem [4-6].

In loose terms, what has been shown is that under the given conditions the individual elements of the beam (or plate) remain in contact, each carrying a load proportional to its own stiffness. While this may be intuitively accepted as true on physical grounds, the necessary precise statement and proof have here been provided.

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