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Affine development of closed curves in Weitzenböck manifolds and the Burgers vector of dislocation mechanics

Arkadas Ozakin
Georgia Tech Research Institute, Atlanta, GA, USA

Arash Yavari
School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA, USA

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Abstract

In the theory of dislocations, the Burgers vector is usually defined by referring to a crystal structure. Using the notion of affine development of curves on a differential manifold with a connection, we give a differential geometric definition of the Burgers vector directly in the continuum setting, without making use of an underlying crystal structure. As opposed to some other approaches to the continuum definition of the Burgers vector, our definition is completely geometric, in the sense that it involves no ambiguous operations such as the integration of a vector field: when we integrate a vector field, it is a vector field living in the tangent space at a given point in the manifold. For a body with distributed dislocations, the material manifold, which describes the geometry of the stress-free state of the body, is commonly taken to be a Weitzenböck manifold, i.e. a manifold with a metric-compatible, flat connection with torsion. We show that for such a manifold, the density of the Burgers vector calculated according to our definition reproduces the commonly stated relation between the density of dislocations and the torsion tensor.

Keywords

Burgers vector, defects, dislocation mechanics, geometric elasticity

1. Introduction

In much of the literature, the Burgers vector, a central object in the theory of dislocations, is defined by referring to an underlying crystal structure. One traverses a closed path in a material with a dislocation by taking steps on the crystal structure. The ‘same’ steps, when taken in a ‘corresponding’ perfect crystal, fail to form a closed path, and the amount of failure is then defined as the Burgers vector of the dislocation. While this classical definition is certainly useful, it seems to use the crystal structure in an essential way. Many of the fundamental concepts of the continuum theory of solids can be formulated directly, without reference to an underlying crystal structure.[1] One hopes to be able to define a fundamental concept such as the Burgers vector directly in the continuum case, as well.

Volterra [1] gave a classification of line defects, where dislocations are presented in terms of a cut-and-paste operation, analogous to the common crystal description (see Figure 1). Starting from Volterra’s picture

Corresponding author:
Arash Yavari, School of Civil and Environmental Engineering, Georgia Institute of Technology, 790 Atlantic Drive, Atlanta, GA 30332, USA.
Email: arash.yavari@ce.gatech.edu

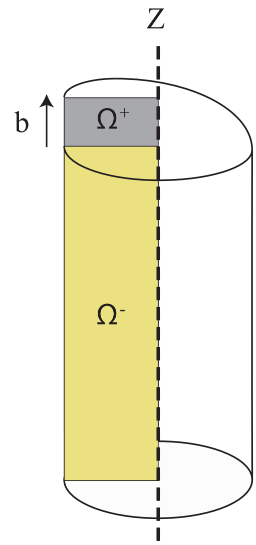

Figure 1. Volterra's cut-and-weld construction of a screw dislocation.

as a continuum version of screw dislocations in crystals, one may attempt to define the Burgers vector in a continuous medium with analogy to the definition in the crystal case: take a closed path in the continuous material, and look at the 'same' path in a 'corresponding, perfect medium'. The amount of failure of the second path to close will be the continuous version of the Burgers vector. However, the notions of a *corresponding* perfect medium and the *same* path in this corresponding medium are both ambiguous, especially in the case of a material with a continuous distribution of defects. How can we make precise, geometric sense of these notions? In this paper, we provide an answer to this question in terms of the operation of *curve development*, as defined in the differential geometry of affine connections. There have been other approaches to the definition of the Burgers vector in continuum mechanics, however, we believe our approach is the first one that is rigorous in terms of the underlying differential (Riemann–Cartan) geometry.

In the classical, non-linear theory of residual stresses (due to plasticity, distributed defects, thermal expansion, etc.), the starting point is the decomposition

$$\mathbf{F} = \mathbf{F}_e \mathbf{F}_p, \tag{1}$$

of the deformation gradient $\mathbf{F}^2$ into the factors $\mathbf{F}_p$, which represents a local change in the relaxed state of the material (due to, e.g., plastic deformation), and an 'elastic piece', $\mathbf{F}_e$, which represents an elastic stretch in the material from its new, locally relaxed state. It is this latter piece, $\mathbf{F}_e$, which is the source of the stress. The map $\mathbf{F}_p$ is thought of as taking the initial, 'perfect' material, and sending it *locally* to its new, 'imperfect' state, whose stress-free *local* configuration is called the intermediate configuration. It is worth emphasizing that this description is only local, and no global map really exists that sends the initial configuration of the material to some global, intermediate configuration that is stress-free. The small pieces, each of which has a new stress-free state after the plastic deformation, do not in general mesh together to form a new, global stress-free state. The differential-geometric meaning of the decomposition (Equation (1)) has not always been clear. The need for an initial, 'perfect' state, from which one obtains the state with defects by a specific plastic deformation, can likewise be a source of confusion. What if the material was created in its 'imperfect' state with residual stresses? What is the meaning of an 'initial', perfect state in that case? (See [2] for a clarification of the geometric meaning of $\mathbf{F}_p$ for the case of thermal stresses.)

The continuum definitions of the Burgers vector encountered in the literature utilize the decomposition (Equation (1)). Given a material with a distribution of dislocations and a corresponding $\mathbf{F}_p$, the total Burgers vector $\mathbf{b}$ enclosed by a curve C in the reference configuration is commonly defined in terms of an integral involving $\mathbf{F}_p$ [3, 4]:

$$\mathbf{b} = \int_C \mathbf{F}_p \, d\mathbf{X}. \tag{2}$$

While it is certainly possible to pretend that Equation (2) is an ordinary integral and evaluate the result for a given matrix field $\mathbf{F}_p$, there are conceptual difficulties that make it hard to make geometric, invariant sense of what this integral means. First, the object that is being integrated, $\mathbf{F}_p\, d\mathbf{X}$, lives in the intermediate configuration, which is only locally defined. One does not have a global region over which to integrate this object. Second, if one treats the integrand as a vector field living on a curve in some space with possibly non-Euclidean geometry, one has to face the fact that the integration of a vector along a curve is not in general defined for such geometries: unless one uses a connection to perform parallel transport. There is no apparent use of parallel transport in Equation (2).

In the geometric framework, a body with a distribution of dislocations is represented by a Riemann–Cartan manifold with a metric-compatible connection that has non-zero torsion and vanishing curvature, i.e. a Weitzenböck manifold [5]. The torsion tensor is the geometric counterpart of the dislocation density tensor, as argued in [3, 6–8]. Given a connection, one can define parallel transport of vectors, and the notion of parallel transport allows one to define the notion of curve development [9]. In the next section, we define the Burgers vector corresponding to a given closed curve in a material manifold in terms of curve development, and show that the standard relation between the torsion tensor and the dislocation density is reproduced by this definition. To make the paper relatively self-contained, we briefly review the geometry of the material manifold for a solid with distributed dislocations in the appendix.

2. Burgers vector in geometric dislocation mechanics

Given a closed curve C in a material body, we would like to define the notion of a corresponding curve in an ideal (defect-free) version of the material, and define the Burgers vector corresponding to C as the failure of this corresponding curve to close. The relevant notion we will use is that of *curve development*, or *affine development of curves*. This can be intuitively thought of as the process of finding a curve in Euclidean space that has the same pattern of velocities as the original curve C. The comparison of velocity vectors at different points in a manifold can be made through the notion of a connection, and this is what we need to use. The *material manifold* is a differential manifold with some additional geometric structure such as a metric tensor and a connection, representing the intrinsic, relaxed state of a material body. Given a configuration of the material in an ambient space, stresses occur (in general) when the distances between material points as measured by the metric of the material manifold differ from the distances as measured by the metric of the ambient space. We represent the material manifold as a Riemann–Cartan manifold, $(\mathcal{B}, \nabla, \mathbf{G})$, where $\mathcal{B}$ is a differential manifold, $\mathbf{G}$ is a Riemannian metric, and ∇ is a connection compatible with the metric, i.e. $\nabla\mathbf{G} = 0$. For a quick review of the relevant definitions, see the appendix.

Given a connection ∇, the parallel transport of a vector $\mathbf{V}_0 \in T_p\mathcal{B}$ along a curve $C : [0, \ell] \to \mathcal{B}$ with $C(0) = p$ is defined to be a vector field $\mathbf{V}$ on $C([0, \ell])$ that has vanishing covariant derivative along the curve: $\nabla_{dC(s)/ds} V = 0$. In components

$$\frac{dV^A}{dt} + \Gamma^A{}_{BC}\frac{dC^B}{dt}V^C = 0, \tag{3}$$

where $t \in [0, \ell]$ is the curve parameter, $\Gamma^A{}_{BC}$ are the connection coefficients for ∇ defined in the appendix, and we have used the shorthand notation dC^B/dt for $dX^B(C(t))/dt$. Generalizing this definition to an arbitrary starting point τ instead of $t = 0$, we can define an operator $\mathsf{p}(C)^t_\tau : T_{C(\tau)}\mathcal{B} \to T_{C(t)}\mathcal{B}$ that parallel transports vectors tangent to the manifold at $C(\tau)$ to those at $C(t)$. The operators $\mathsf{p}(C)^t_\tau$ satisfy the identities $\mathsf{p}(C)^t_t = Id$ and $\mathsf{p}(C)^\tau_u \circ \mathsf{p}(C)^u_t = \mathsf{p}(C)^\tau_t$, for $t, \tau, u \in [0, \ell]$, where Id denotes the identity operator. Dropping the reference to the curve C, we will denote the components of the operator $\mathsf{p}(C)^t_0$ by $\mathsf{p}^A{}_B(t)$, or simply, $\mathsf{p}^A{}_B$. Thus, given a vector $\mathbf{V}_0 = \mathbf{V}(0) \in T_p\mathcal{B}$ with components $V^A(0)$ in the chart $\{X^A\}$, the parallel transported vector $\mathbf{V}(t) \in T_{C(t)}\mathcal{B}$ has components $V^A(t) = \mathsf{p}^A{}_B(s)V^B(0)$. Plugging this into the parallel transport equation (3), we see that the operator components $\mathsf{p}^A{}_B(t)$ satisfy the equation

$$\frac{d\mathsf{p}^A{}_B}{dt} + \Gamma^A{}_{CD}\frac{dC^C}{dt}\mathsf{p}^D{}_B = 0. \tag{4}$$

Let us now assume that the connection ∇ is flat, but possibly with torsion, as discussed at the end of our introduction. Then, parallel transport between two points is the same for two curves that connect the two points, if these curves can be smoothly deformed to one another. For a simply-connected neighborhood U of p, this

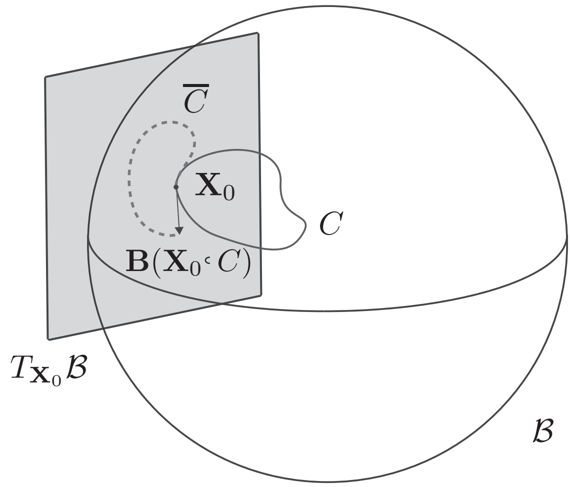

Figure 2. In a manifold with torsion, the affine development of a closed curve in the manifold is not a closed curve in the tangent space in general. The solid closed curve lies in the manifold and the dotted curve is its affine development in the tangent space and is not necessarily closed. The lack of closure is related to the Burgers vector.

allows us to unambiguously define a point-dependent parallel transport operator that transports vectors at p to all the points in U, by connecting each point $q \in U$ to p by an arbitrary curve lying in U. In this way, the components $\mathsf{p}^A{}_B$ of the parallel transport operator become functions of the point q, or, for a given coordinate chart, of the coordinates.

We now define the *affine development* of C to be the unique curve $\overline{C} : [0,\ell] \to T_p\mathcal{B}$ satisfying the following equations [9]

$$\begin{cases} \overline{C}(0) = \mathbf{0}, \\ \dot{\overline{C}}(t) = p(C)^0_s \cdot \dot{C}(t), \end{cases} \tag{5}$$

where we have denoted the vector tangent to the curve C at $C(t)$ by $\dot{C}(t)$. Note that the affine development $\overline{C}$ lies in the vector space $T_p\mathcal{B}$. Now suppose that the curve C is closed: $C(0) = C(\ell) = p$. It turns out that affine development $\overline{C}$ is not necessarily closed. We represent a material with distributed dislocations by a material manifold with a flat connection ∇ that possibly has non-zero torsion, and define the Burgers vector $\mathbf{B}(p; C)$ corresponding to the curve C based at p to be a measure of the failure of $\overline{C}$ to close (see Figure 2):[3]

$$\mathbf{B}(p; C) = \overline{C}(\ell) - \overline{C}(0) = \overline{C}(\ell). \tag{6}$$

Note that $\mathbf{B}(p; C) \in T_p\mathcal{B}$. In components, the Burgers vector is given as

$$B^A(p; C) = \overline{C}(\ell) - \overline{C}(0) = \int_0^\ell \frac{d\overline{C}^A(t)}{dt}\, dt = \int_0^\ell \mathsf{p}_B{}^A(t) \frac{dC^B(t)}{dt}\, dt, \tag{7}$$

where we have denoted the components of the operator $\mathsf{p}(C)^0_t$ by $\mathsf{p}_B{}^A(t)$. In other words, $\mathsf{p}_B{}^A(t)$ is the inverse of the matrix $\mathsf{p}^C{}_D(t)$, so that $\mathsf{p}_B{}^A \mathsf{p}^B{}_C = \delta^A_C$ and $\mathsf{p}_B{}^A \mathsf{p}^C{}_A = \delta^C_B$. We will next connect our definition of the Burgers vector with a classical result on the relation between the torsion tensor and the density of dislocations. We will proceed by looking at a family (homotopy) of curves that start and end at the same point p, and perform a limiting procedure to investigate the behavior of the Burgers vector as the curves get smaller and smaller.

Consider a smooth, one-parameter family of curves $C_s(t) = C(s,t)$, $0 \leq s \leq 1$, $0 \leq t \leq 1$ lying on an embedded, two-dimensional submanifold of $\mathcal{B}$, such that $C_0(t) = p$ is a constant curve. It is known that if γ is a simple, null-homotopic (contractible) loop on a surface, then it is the boundary of a topological disk (a genus-zero surface with one boundary curve) [10, 11]. Thus, we know that for $0 \leq s \leq 1$, $C([s,1),[0,1])$ defines a smooth surface $\Omega_s \in \mathcal{B}$ with a boundary given by the curve C_s. As mentioned above, since the connection on $\mathcal{B}$ is flat, we can define a path-independent parallel transport from a given point p to all other points q in a simply-connected region. Given a small open neighborhood U of the surface C_s, this allows us to define a parallel transport operator $\mathsf{p}(q)$ that transports vectors tangent at p to the tangent space at $q \in U$. Let the matrix representation of this operator in a given coordinate system $\{X^A\}$ be $\mathsf{p}^A{}_B(q)$. Since the functions $\mathsf{p}^A{}_B(q)$ are defined on an open neighborhood of Ω_s, we can calculate the partial derivatives $\partial \mathsf{p}^A{}_B/\partial X^C$ and $\partial \mathsf{p}_A{}^B/\partial X^C$. This allows us to turn Equation (7) into a surface integral over Ω_s by using Stokes' theorem[4]

$$B^A(p; C_s) = \int_{C_s} \mathsf{p}_B{}^A \, dX^B = \int_{\Omega_s} d\left(\mathsf{p}_B{}^A \, dX^B\right) = \int_{\Omega_s} \frac{\partial \mathsf{p}_B{}^A}{\partial X^D} \, dX^D \wedge dX^B. \tag{8}$$

Now, differentiating $\mathsf{p}_B{}^A \mathsf{p}^C{}_A = \delta^C_B$, we see that

$$\frac{\partial \mathsf{p}_B{}^A}{\partial X^D} = -\mathsf{p}_C{}^A \mathsf{p}_B{}^E \frac{\partial \mathsf{p}^C{}_E}{\partial X^D}. \tag{9}$$

This allows us to rewrite Equation (8) as

$$B^A(p; C_s) = -\int_{\Omega_s} \mathsf{p}_C{}^A \mathsf{p}_B{}^E \frac{\partial \mathsf{p}^C{}_E}{\partial X^D} \, dX^D \wedge dX^B. \tag{10}$$

Now, for a given point $q \in \Omega_s$, let us consider a curve γ in Ω_s joining p at $\xi = 0$ to q at $\xi = 1$, such that the tangent vector to γ at[5] q is $\partial/\partial X^D$. Then, using $\dot{\gamma}^M = d\gamma^M/d\xi = \delta^M_D$ and Equation (4), we get

$$\left.\frac{d\mathsf{p}^C{}_E(\gamma(\xi))}{d\xi}\right|_{\xi=1} = \left.\frac{\partial \mathsf{p}^C{}_E}{\partial X^D}\right|_q = -\Gamma^C{}_{MN} \dot{\gamma}^M \mathsf{p}^N{}_E = -\Gamma^C{}_{MN} \delta^M_D \mathsf{p}^N{}_E = -\Gamma^C{}_{DN} \mathsf{p}^N{}_E. \tag{11}$$

Using this in Equation (10), we obtain

$$\begin{aligned} B^A(p; C_s) &= \int_{\Omega_s} \mathsf{p}_C{}^A \Gamma^C{}_{DB} \, dX^D \wedge dX^B \\ &= \int_{\Omega_s} \mathsf{p}_C{}^A \left(\Gamma^C{}_{DB} - \Gamma^C{}_{BD}\right) (dX^D \wedge dX^B) \\ &= \int_{\Omega_s} \mathsf{p}_C{}^A \mathcal{T}^C, \end{aligned} \tag{12}$$

where $\{(dX^D \wedge dX^B)\} = \{dX^D \wedge dX^B\}_{D<B}$ is a basis for two-forms and $\mathcal{T}^C$ is the torsion tensor, which is a vector-valued two-form. Informally speaking, the vector index of $\mathcal{T}^C$ is transported to the tangent space at p by the parallel transport operator, so that the integrand is a two-form with values in $T_p\mathcal{B}$. Note that this relation holds for any closed curve passing through p in U.

Next, we prove a lemma that will allow us to relate the torsion two-form to the area density of the Burgers vector via Equation (12). This seems to be a standard argument in Riemannian geometry; see [12] for a related discussion in the context of Riemannian curvature and holonomy.

Lemma 2.1. *On a Riemannian manifold $\mathcal{B}$ with metric tensor $\mathbf{G}$, let $C_s(t) = C(s,t)$ with $0 \leq s \leq 1$, $0 \leq t \leq 1$ denote a one-parameter family of curves such that $C_0(t) = p$ for $t \in [0,1]$, $C_s(0) = C_s(1) = p$ for $s \in [0,1]$, and let $\Omega_s = C([0,s],[0,1])$, $\Omega = \Omega_1$. Let ω be a two-form on Ω. Then*

$$\lim_{s \to 0} \frac{\int_{\Omega_s} \omega}{|\Omega_s|} = \omega(\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p),$$

where $\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p \in T_pM$ are orthonormal vectors tangent to Ω at p, and $|\Omega_s|$ is the area of Ω_s.

Proof. Let $\mathbf{S} = \partial_s C$ and $\mathbf{T} = \partial_t C$ be vector fields tangent to Ω and define $\mathbf{J} = \mathbf{T} - \mathbf{G}(\mathbf{S},\mathbf{T})/\mathbf{G}(\mathbf{S},\mathbf{S})\mathbf{S}$. Using the notation $\mathbf{U} \cdot \mathbf{V}$ for $\mathbf{G}(\mathbf{U},\mathbf{V})$ for tangent vectors $\mathbf{U}$ and $\mathbf{V}$, we have

$$|\Omega_s| = \int_0^s ds \int_0^1 dt \sqrt{(\mathbf{S}\cdot\mathbf{S})(\mathbf{T}\cdot\mathbf{T}) - (\mathbf{S}\cdot\mathbf{T})^2} = \int_0^s ds \int_0^1 dt \sqrt{(\mathbf{S}\cdot\mathbf{S})(\mathbf{J}\cdot\mathbf{J})} = \int_0^s ds \int_0^1 dt \, |\mathbf{S}||\mathbf{J}|,$$

where $|\mathbf{S}|$ and $|\mathbf{T}|$ denote the Riemannian lengths of the tangent vectors. Now, we can write

$$\int_{\Omega_s} \omega = \int_0^s ds \int_0^1 dt\, \omega(\mathbf{S}, \mathbf{T}) = \int_0^s ds \int_0^1 dt\, \omega(\mathbf{S}, \mathbf{J}) = \int_0^s ds \int_0^1 dt\, \omega(\mathbf{S}/|\mathbf{S}|, \mathbf{J}/|\mathbf{J}|)|\mathbf{S}||\mathbf{J}|,$$

where the second identity follows from the fact that $\omega(\mathbf{S}, \mathbf{S}) = 0$. Defining the orthonormal vectors $\boldsymbol{\sigma} = \mathbf{S}/|\mathbf{S}|$ and $\boldsymbol{\tau} = \mathbf{T}/|\mathbf{T}|$, we have

$$\int_{\Omega_s} \omega = \int_0^s ds \int_0^1 dt\, \omega(\boldsymbol{\sigma}, \boldsymbol{\tau})|\mathbf{S}||\mathbf{J}|.$$

As $s \to 0$, we have

$$\int_{\Omega_s} \omega \approx \omega(\boldsymbol{\sigma}, \boldsymbol{\tau})|_p \int_0^s ds \int_0^1 dt\, |\mathbf{S}||\mathbf{J}| = \omega(\boldsymbol{\sigma}, \boldsymbol{\tau})|_p |\Omega_s|,$$

which proves the lemma. □

We can immediately apply this lemma to Equation (12), since A is a vector index in the space $T_p\mathcal{B}$, and so for each A, the integrand is a usual (real-valued) two-form on Ω_s. We obtain

$$\lim_{s\to 0} \frac{B^A(p; C_s)}{|\Omega_s|} = \mathsf{p}_C{}^A \mathcal{T}^C(\boldsymbol{\sigma}, \boldsymbol{\tau})\Big|_p = \mathcal{T}^A(\boldsymbol{\sigma}, \boldsymbol{\tau})\Big|_p,$$

where in the second equality we used the fact that parallel transport from p to p is the identity operator. Thus, we have shown that the area density of the Burgers vector as calculated by our definition is given by the torsion tensor.

Remark 2.2. Note that instead of a curve in the material manifold $\mathcal{B}$, one can start with a spatial closed curve $C(t) \in \mathcal{S}$ and evaluate the affine development of the corresponding curve in $T_p\mathcal{B}$ after pulling the curve C to the manifold $\mathcal{B}$ by using the current configuration. Note that this would also result in a Burgers vector defined in the material manifold. Of course, if needed, one can push forward the Burgers vector to the spatial manifold $\mathcal{S}$ by using the current configuration.

Remark 2.3. Torsion two-form is similar to stress two-form [13] in the following sense. Stress two-form when acting on a small piece of a two-manifold (more precisely, a two-plane section of the tangent space at the point) gives the force one-form (in the current configuration) that acts on the deformed two-manifold [13, 14]. Similarly, when torsion two-form acts on the same small two-manifold in the reference configuration it gives the total Burgers vector (in the reference configuration) on that surface.

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Conflict of interest

None declared.

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Notes

1. The properties of an underlying crystal structure, if it exists, can show up in the continuum description as symmetry properties of the elasticity tensors, etc.
2. The deformation gradient is defined as the derivative of the deformation map.
3. In [6], $\mathbf{B}(p; C)$ was called the 'cumulative Burgers vector'.
4. In the integral (Equation (8)) over C_s, the integrand is a vector-valued one-form taking values in the linear space $T_p\mathcal{B}$. Thus, we are using a simple generalization of the classical Stokes' theorem for vector-valued forms, or, equivalently, we can treat each component of $B^A(p; C_s)$ as a separate function and apply the usual Stokes' theorem.
5. We assume that γ passes through q only once.

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Appendix: Geometric theory of solids with distributed defects

To make the paper self-contained, in this appendix we briefly review the geometry of bodies with distributed dislocations. For more details on non-symmetric connections see [15–19]. For more details on the geometric foundations of non-linear dislocation mechanics the reader is referred to [5]. A linear (affine) connection on a manifold $\mathcal{B}$ is an operation $\nabla : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ is the set of vector fields on $\mathcal{B}$, such that $\forall f, f_1, f_2 \in C^\infty(\mathcal{B})$, $\forall a_1, a_2 \in \mathbb{R}$:

$$\text{(i)}\ \nabla_{f_1\mathbf{X}_1+f_2\mathbf{X}_2}\mathbf{Y} = f_1\nabla_{\mathbf{X}_1}\mathbf{Y} + f_2\nabla_{\mathbf{X}_2}\mathbf{Y}, \tag{13}$$

$$\text{(ii)}\ \nabla_{\mathbf{X}}(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2) = a_1\nabla_{\mathbf{X}}(\mathbf{Y}_1) + a_2\nabla_{\mathbf{X}}(\mathbf{Y}_2), \tag{14}$$

$$\text{(iii)}\ \nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y}. \tag{15}$$

In a local chart $\{X^A\}$, $\nabla_{\partial_A}\partial_B = \Gamma^C{}_{AB}\partial_C$, where $\Gamma^C{}_{AB}$ are Christoffel symbols of the connection and $\partial_A = \partial/\partial x^A$ are the natural bases for the tangent space corresponding to a coordinate chart $\{x^A\}$. A linear connection is said to be compatible with a metric $\mathbf{G}$ of the manifold if $\nabla_{\mathbf{X}} \langle\!\langle \mathbf{Y}, \mathbf{Z} \rangle\!\rangle_{\mathbf{G}} = \langle\!\langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle\!\rangle_{\mathbf{G}} + \langle\!\langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z} \rangle\!\rangle_{\mathbf{G}}$, where $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathbf{G}}$ is the inner product induced by the metric $\mathbf{G}$. It can be shown that ∇ is compatible with $\mathbf{G}$ if and only if $\nabla\mathbf{G} = \mathbf{0}$, or in components

$$G_{AB|C} = \frac{\partial G_{AB}}{\partial X^C} - \Gamma^S{}_{CA}G_{SB} - \Gamma^S{}_{CB}G_{AS} = 0. \tag{16}$$

Torsion of a connection is a map $T : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$ defined by $T(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]$. In components in a local chart $\{X^A\}$, $T^A{}_{BC} = \Gamma^A{}_{BC} - \Gamma^A{}_{CB}$. The connection is said to be symmetric if it is torsion-free, i.e. $\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]$. It can be shown that on any Riemannian manifold $(\mathcal{B}, \mathbf{G})$ there is a unique linear connection ∇, which is compatible with $\mathbf{G}$ and is torsion-free. This is the Levi–Civita connection. In a manifold with a connection, the Riemann curvature is a map $\boldsymbol{\mathcal{R}} : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$ defined by $\boldsymbol{\mathcal{R}}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}$, or, in components

$$\mathcal{R}^A{}_{BCD} = \frac{\partial \Gamma^A{}_{CD}}{\partial X^B} - \frac{\partial \Gamma^A{}_{BD}}{\partial X^C} + \Gamma^A{}_{BM}\Gamma^M{}_{CD} - \Gamma^A{}_{CM}\Gamma^M{}_{BD}. \tag{17}$$

A metric-affine manifold is a manifold equipped with both a connection and a metric: $(\mathcal{B}, \nabla, \mathbf{G})$. If the connection is metric-compatible the manifold is called a Riemann–Cartan manifold [20, 21]. If the connection is torsion-free but has non-vanishing curvature $\mathcal{B}$ is called a Riemannian manifold. If the curvature of the connection vanishes but it has torsion $\mathcal{B}$ is called a Weitzenböck manifold. If both torsion and curvature vanish $\mathcal{B}$ is a flat (Euclidean) manifold.

Cartan's moving frames. A frame field $\{\mathbf{e}_\alpha\}_{\alpha=1}^n$ at every point of an n-dimensional manifold $\mathcal{B}$ forms a basis for the tangent space. We assume that this frame is orthonormal, i.e. $\langle\!\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle\!\rangle_{\mathbf{G}} = \delta_{\alpha\beta}$. This frame is, in general, a non-coordinate basis for the tangent space. Given a coordinate basis $\{\partial_A\}$ an arbitrary frame field $\{\mathbf{e}_\alpha\}$ is obtained by an orientation-preserving $GL(N, \mathbb{R})$-rotation of $\{\partial_A\}$ as $\mathbf{e}_\alpha = \mathsf{F}_\alpha{}^A \partial_A$ and $\det \mathsf{F}_\alpha{}^A > 0$. We know that for the coordinate frame $[\partial_A, \partial_B] = 0$ but for the non-coordinate frame field we have $[\mathbf{e}_\alpha, \mathbf{e}_\alpha] = -c^\gamma{}_{\alpha\beta}\mathbf{e}_\gamma$, where $c^\gamma{}_{\alpha\beta}$ are components of the object of anhonolomy. Note that for scalar fields f, g and vector fields $\mathbf{X}, \mathbf{Y}$ on $\mathcal{B}$ we have $[f\mathbf{X}, g\mathbf{Y}] = fg[\mathbf{X}, \mathbf{Y}] + f(\mathbf{X}[g])\mathbf{Y} - g(\mathbf{Y}[f])\mathbf{X}$ and, hence

$$c^\gamma{}_{\alpha\beta} = \mathsf{F}_\alpha{}^A \mathsf{F}_\beta{}^B \left(\partial_A \mathsf{F}^\gamma{}_B - \partial_B \mathsf{F}^\gamma{}_A\right), \tag{18}$$

where $\mathsf{F}^\gamma{}_B$ is the inverse of $\mathsf{F}_\gamma{}^B$. The frame field $\{\mathbf{e}_\alpha\}$ defines the coframe field $\{\vartheta^\alpha\}_{\alpha=1}^n$ such that $\vartheta^\alpha(\mathbf{e}_\beta) = \delta^\alpha_\beta$. The object of anholonomy is defined as $c^\gamma = d\vartheta^\gamma$.

Connection one-forms are defined as $\nabla \mathbf{e}_\alpha = \mathbf{e}_\gamma \otimes \omega^\gamma{}_\alpha$. The corresponding connection coefficients are defined as $\nabla_{\mathbf{e}_\beta}\mathbf{e}_\alpha = \langle \omega^\gamma{}_\alpha, \mathbf{e}_\beta \rangle \mathbf{e}_\gamma = \omega^\gamma{}_{\beta\alpha}\mathbf{e}_\gamma$. In other words, $\omega^\gamma{}_\alpha = \omega^\gamma{}_{\beta\alpha}\vartheta^\beta$. Similarly, $\nabla \vartheta^\alpha = -\omega^\alpha{}_\gamma \vartheta^\gamma$, and $\nabla_{\mathbf{e}_\beta}\vartheta^\alpha = -\omega^\alpha{}_{\beta\gamma}\vartheta^\gamma$. The relation between the connection coefficients in the two coordinate systems is $\omega^\gamma{}_{\alpha\beta} = \mathsf{F}_\alpha{}^A \mathsf{F}_\beta{}^B \mathsf{F}^\gamma{}_C \Gamma^C{}_{AB} - \mathsf{F}_\alpha{}^A \mathsf{F}_\beta{}^B \partial_A \mathsf{F}^\gamma{}_B$. Equivalently, $\Gamma^A{}_{BC} = \mathsf{F}^\beta{}_B \mathsf{F}^\gamma{}_C \mathsf{F}_\alpha{}^A \omega^\alpha{}_{\beta\gamma} + \mathsf{F}_\alpha{}^A \partial_B \mathsf{F}^\alpha{}_C$. In the non-coordinate basis torsion has the following components

$$T^\alpha{}_{\beta\gamma} = \omega^\alpha{}_{\beta\gamma} - \omega^\alpha{}_{\gamma\beta} + c^\alpha{}_{\beta\gamma}.$$

Similarly, the curvature tensor has the following components with respect to the frame field

$$\mathcal{R}^\alpha{}_{\beta\lambda\mu} = \partial_\beta \omega^\alpha{}_{\lambda\mu} - \partial_\lambda \omega^\alpha{}_{\beta\mu} + \omega^\alpha{}_{\beta\xi}\omega^\xi{}_{\lambda\mu} - \omega^\alpha{}_{\lambda\xi}\omega^\xi{}_{\beta\mu} + \omega^\alpha{}_{\xi\mu}c^\xi{}_{\beta\lambda}.$$

In the orthonormal frame $\{\mathbf{e}_\alpha\}$, the metric tensor has the simple representation $\mathbf{G} = \delta_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$. Assuming that the connection ∇ is metric compatible, i.e. $\nabla\mathbf{G} = \mathbf{0}$, metric compatibility constraints on the connection one-forms read:

$$\delta_{\alpha\gamma}\omega^\gamma{}_\beta + \delta_{\beta\gamma}\omega^\gamma{}_\alpha = 0.$$

Torsion and curvature two-forms are defined using *Cartan's structural equations* as

$$\mathcal{T}^\alpha = d\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta,$$
$$\mathcal{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta,$$

where d is the exterior derivative. Torsion two-form is written as $\boldsymbol{\mathcal{T}} = \mathbf{e}_\alpha \otimes \mathcal{T}^\alpha = \partial_A \otimes T^A$, where $\mathcal{T}^\alpha = \mathsf{F}^\alpha{}_A T^A$. Bianchi identities read:

$$D\mathcal{T}^\alpha := d\mathcal{T}^\alpha + \omega^\alpha{}_\beta \wedge \mathcal{T}^\beta = \mathcal{R}^\alpha{}_\beta \wedge \vartheta^\beta,$$
$$D\mathcal{R}^\alpha{}_\beta := d\mathcal{R}^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \mathcal{R}^\gamma{}_\beta - \omega^\gamma{}_\beta \wedge \mathcal{R}^\alpha{}_\gamma = 0,$$

where D is the covariant exterior derivative. Note that for a flat manifold the first Bianchi identity tells us that $D\mathcal{T}^\alpha = 0$.

For a given frame field $\{\mathbf{e}_\alpha\}$ one may be interested in a connection ∇ such that in $(\mathcal{B}, \nabla)$ the frame field is parallel everywhere. This means that $\nabla \mathbf{e}_\alpha = \omega^\beta{}_\alpha \mathbf{e}_\beta = \mathbf{0}$, i.e. the connection one-forms vanish with respect to the frame field or $\omega^\beta{}_{\gamma\alpha} = 0$. Using this we have the following connection coefficients in the coordinate frame

$$\Gamma^C{}_{AB} = \mathsf{F}_\alpha{}^C \partial_A \mathsf{F}^\alpha{}_B.$$

This is called the Weitzenböck connection [22, 23], which has the following torsion components

$$T^C{}_{AB} = \mathsf{F}_\alpha{}^C \left(\partial_A \mathsf{F}^\gamma{}_B - \partial_B \mathsf{F}^\gamma{}_A \right).$$

For a body with distributed dislocations material manifold, where the body is stress-free, is a Weitzenböck manifold [5].