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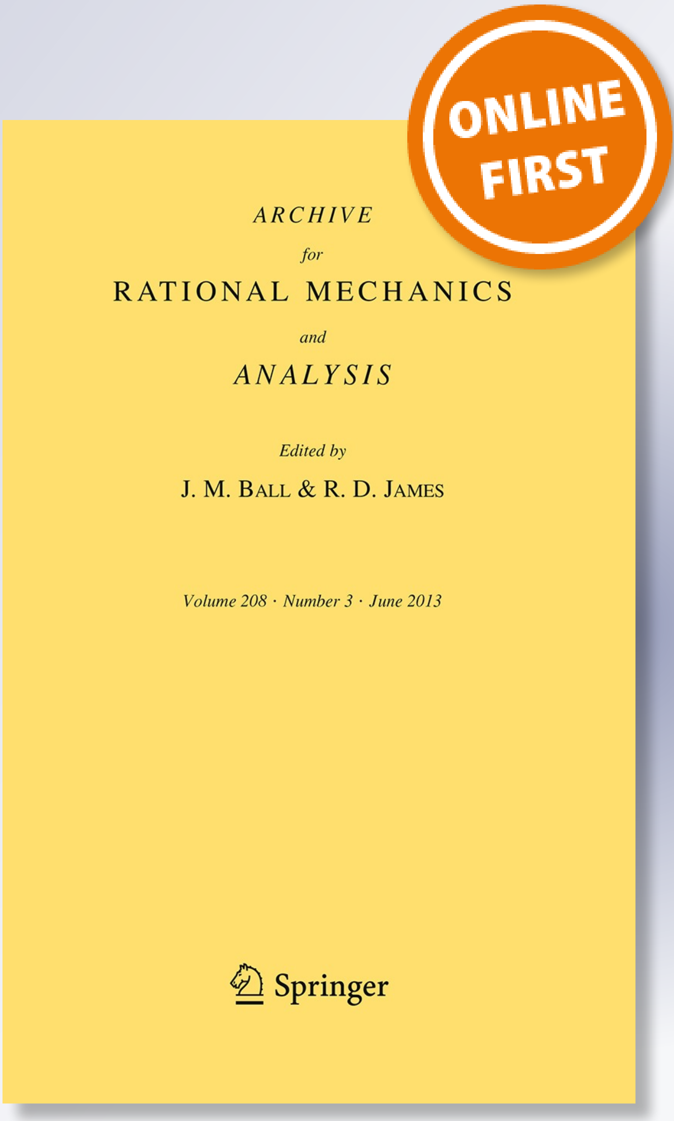

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Compatibility Equations of Nonlinear Elasticity for Non-Simply-Connected Bodies

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Abstract

Compatibility equations of elasticity are almost 150 years old. Interestingly, they do not seem to have been rigorously studied, to date, for non-simply-connected bodies. In this paper we derive necessary and sufficient compatibility equations of nonlinear elasticity for arbitrary non-simply-connected bodies when the ambient space is Euclidean. For a non-simply-connected body, a measure of strain may not be compatible, even if the standard compatibility equations ("bulk" compatibility equations) are satisfied. It turns out that there may be topological obstructions to compatibility; this paper aims to understand them for both deformation gradient $\mathbf{F}$ and the right Cauchy-Green strain $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$. We show that the necessary and sufficient conditions for compatibility of deformation gradient $\mathbf{F}$ are the vanishing of its exterior derivative and all its periods, that is, its integral over generators of the first homology group of the material manifold. We will show that not every non-null-homotopic path requires supplementary compatibility equations for $\mathbf{F}$ and linearized strain $\mathbf{e}$. We then find both necessary and sufficient compatibility conditions for the right Cauchy-Green strain tensor $\mathbf{C}$ for arbitrary non-simply-connected bodies when the material and ambient space manifolds have the same dimensions. We discuss the well-known necessary compatibility equations in the linearized setting and the Cesàro-Volterra path integral. We then obtain the sufficient conditions of compatibility for the linearized strain when the body is not simply-connected. To summarize, the question of compatibility reduces to two issues: i) an integrability condition, which is $\mathrm{d}(\mathbf{F}\ \mathrm{d}\mathbf{X}) = \mathbf{0}$ for the deformation gradient and a curvature vanishing condition for $\mathbf{C}$, and ii) a topological condition. For $\mathbf{F}\,\mathrm{d}\mathbf{X}$ this is a homological condition because the equation one is trying to solve takes the form $\mathrm{d}\varphi = \mathbf{F}\,\mathrm{d}\mathbf{X}$. For $\mathbf{C}$, however, parallel transport is involved, which means that one needs to solve an equation of the form $\mathrm{d}\mathbf{R}/\,\mathrm{d}s = \mathbf{R}\mathbf{K}$, where $\mathbf{R}$ takes values in the orthogonal group. This is, therefore, a question about an orthogonal representation of the fundamental group, which, as the orthogonal group is not commutative, cannot, in general, be reduced to a homological question.

1. Introduction

Compatibility equations in elasticity have an old history. Given a measure of strain, compatibility equations are the necessary and sufficient conditions that guarantee existence of a deformation mapping or a single-valued displacement field. In the language of differential geometry this is closely related to the existence of an isometry between two Riemannian manifolds. When one manifold is Euclidean, Riemann showed that an isometry exists locally if and only if the Riemann curvature tensor vanishes.

Given a field of "strain" one observes that the system of PDEs governing the deformation field is overdetermined. Existence of a deformation (or displacement) field corresponding to a strain field requires some integrability equations, which have traditionally been called compatibility equations in continuum mechanics. LOVE [21] credits Saint Venant (1864) for the derivation of the "bulk" compatibility equations. MICHELL [24] studied the compatibility equations of linearized elasticity in two dimensions for non-simply-connected bodies. He showed that compatibility requires vanishing of certain integrals on each "independent irreducible circuit".[1] CESÀRO [8] and VOLTERRA [35] studied compatibility equations for non-simply-connected bodies and the possibility of multi-valuedness of displacements when the body is not simply-connected.[2] LOVE [21] (Article 17) and later on GREEN and ZERNA [17] and SEUGLING [28][3] realized that the classical compatibility equations of elasticity can be written as vanishing of the curvature tensor of the Levi-Civita connection of strain (understood as a metric). Note that it is known that in a simply-connected open subset of $\mathbb{R}^3$, vanishing of the curvature tensor of $\mathbf{C}$ is also sufficient for compatibility [10]. SHIELD [29] derived a system of PDEs for the rotation field in the polar decomposition of the deformation gradient. PIETRASZKIEWICZ [25] and PIETRASZKIEWICZ AND BADUR [26] studied the problem of calculating the deformation mapping when the right Cauchy-Green strain is given (see also [9] for the case of two-dimensional elasticity). In particular, they obtained a nonlinear analogue of the Cesàro integral. BLUME [5] discussed the compatibility equations in terms of the left Cauchy-Green strain $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathsf{T}}$ in two

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[1] An independent irreducible circuit is a generator of the fundamental group in the language of algebraic topology. Michell's statement is correct only for plane problems of elasticity. Any embedded 2-submanifold of $\mathbb{R}^2$ is a (topological) disk with a finite number of holes. The fundamental group of a planar region obtained by removing k disjoint disks from it is the *free* group on k generators, while the first homology group is the *free abelian* group on k generators. These are isomorphic only if $k = 1$. As we will see in the sequel, for deformation gradient $\mathbf{F}$ and linearized strain $\mathbf{e}$ each generator of the first homology group requires supplementary compatibility equations. However, in general, not every generator of the fundamental group requires complementary compatibility equations, as we will see in an example.

[2] It is interesting that this is almost the same time period at which algebraic topology was being created by POINCARÉ [27,13].

[3] Truesdell in a review of another paper in *Mathematical Reviews* (MR0040940 (12,770b)) mentions that this paper is the fourteenth since 1902 to derive the compatibility equations. Since then there have been at least a dozen more similar papers, all restricted to simply-connected bodies.

dimensions. This was studied later on in three dimensions by ACHARYA [1] who provided necessary and sufficient conditions for compatibility of **B**.[4] SKALAK et al. [30] realized the importance of compatibility equations for non-simply-connected bodies in growth mechanics applications. They pointed out that the compatibility equations for **C** in "multiply-connected" bodies[5] are not known. In this paper we find these compatibility equations. For the deformation gradient they correctly write the necessary and sufficient compatibility equations as

$$\int_c (\mathbf{F} - \mathbf{I})\ d\mathbf{X} = \mathbf{0}, \tag{1.1}$$

for all closed paths c in the body. This is obviously equivalent to $\int_c \mathbf{F}\ d\mathbf{X} = \mathbf{0}$ for any closed path c in the body. They mention that it is not clear what closed paths should be chosen to guarantee compatibility of **F**. In this paper, we will show that one only needs to consider generators of the first homology group of the body manifold $\mathcal{B}$.[6] They also discuss sufficient compatibility conditions for linearized elasticity. Their argument is flawed, as we will explain in Section 2.3; they provide only half of the complementary compatibility equations in three dimensions.

DELPHENICH [12] discussed some topological ideas relevant to compatibility equations, although he did not give anything other than the well-known compatibility equations for simply-connected bodies. For a non-simply-connected body, a measure of strain may not be compatible even if the standard compatibility equations ("bulk" compatibility equations) are satisfied; there may be topological obstructions to compatibility; this paper aims to understand them for both deformation gradient **F** and the right Cauchy-Green strain **C**. It is strange that such a fundamental problem has not been rigorously studied to date. It is also surprising and unfortunate that topological methods have not been systematically used in elasticity to date.[7] Compatibility equations for simply-connected bodies are well-known and

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[4] One striking result regarding the difference between **C** and **B** is that if two deformations have the same **C**, then the deformations differ by at most a rigid body motion. This is not the case for the corresponding **B** case [15].

[5] A comment is in order here. In complex analysis (of one complex variable) a multiply-connected domain in the complex plane is one whose complement is not connected. Here, we are interested in elasticity of arbitrary embedded 2- and 3-submanifolds of Euclidean space, and hence refrain from using the term "multiply-connected", which is meaningless for general manifolds, and instead use "non-simply-connected".

[6] We will see that not every generator of the fundamental group needs complementary compatibility equations for deformation gradient **F** and linearized strain **e**. The number of complementary compatibility equations is proportional to the number of the generators of the first homology group of the material manifold.

[7] Topological ideas already existed implicitly in the work of Maxwell on electromagnetism [23]. Maxwell calls the number of independent cycles in a graph its "cyclomatic number". He clearly had the idea of deformation retract and invariance of the "cyclomatic number" under a deformation retract. When an embedded 3-submanifold in $\mathbb{R}^3$ has a boundary with more than one connected component, Maxwell calls it a "periphractic region". He calls the period of a differential form over a loop its "cyclic constant". Homotopic paths are called "reconcilable curves". He calls a compatible strain a "non-rotational strain". More recently, it has been observed that algebraic topology is crucial in a deeper understanding of electromagnetism and more efficient numerical implementations [18].

have been studied by many. Interestingly, they do not seem to have been rigorously studied for non-simply-connected bodies. This is certainly an important problem as there are many examples of non-simply-connected bodies in Nature, for example arteries, which are thick hollow cylinders. In structural mechanics applications, for example, multi-compartment thin-walled sections can be in both torsion and bending. In such structures, in torsion for example, the number of extra compatibility equations is proportional to the number of holes in the cross section [19]. Such bodies are deformation retracts of a bouquet of a finite number of circles.[8] We follow STERNBERG [32] and call the extra compatibility equations, complementary "compatibility equations". Let us emphasize that for solids of arbitrary shapes with an arbitrary number of holes (with arbitrary shapes), compatibility equations are not known in the literature.

Contributions of this paper. In this paper, we derive the necessary and sufficient compatibility equations for $\mathbf{F}$ and $\mathbf{C}$ using homology and homotopy group techniques. When the ambient space is Euclidean, it turns out that a simple generalization of a celebrated theorem by de Rham can be used to find all the compatibility equations of $\mathbf{F}$. The number of complementary compatibility equations will be shown to be equal to $N\beta_1(\mathcal{B})$, where $\mathcal{B}$ is the material manifold, $N = \dim \mathcal{S}$ (dimension of the ambient space), and $\beta_1(\mathcal{B})$ is the first Betti number of $\mathcal{B}$, that is, the dimension of its first homology group with real coefficients $H_1(\mathcal{B}; \mathbb{R})$ or, equivalently, the rank of its first homology group with integer coefficients $H_1(\mathcal{B}, \mathbb{Z})$.

The more familiar method for deriving compatibility equations is to use the fact that integral of some function of "strain" must vanish over any closed path in $\mathcal{B}$. A closed path may be continuously deformed to a class of paths. Thus, this would then force one to work with the first homotopy group (fundamental group) $\pi_1(\mathcal{B})$. In simple words, this group tells us about the equivalence classes of those closed paths that can be continuously deformed to each other. In the case of compact manifolds (for us this means bounded bodies) it is known that $\pi_1(\mathcal{B})$ has a finite presentation in the sense of combinatorial group theory.

When $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$ is given, the deformation gradient is not known a priori. For simply-connected bodies it is known how to construct $\mathbf{F}$ [26,29]. Basically, using the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$, one easily calculates the stretch tensor $\mathbf{U} = \sqrt{\mathbf{C}}$. One can then find $\mathbf{R}$ by solving a system of linear first-order PDEs. Solution of $\mathbf{R}$ is written in terms of a path integral (the nonlinear analogue of the classical Cesàro-Volterra path integral) that should be path independent. For simply-connected bodies, vanishing of the curvature tensor of $\mathbf{C}$ is the necessary and sufficient condition (when $\dim \mathcal{B} = \dim \mathcal{S}$) [26]. In the case of non-simply-connected bodies, in addition to this one has some complementary compatibility equations (topological conditions), which we will obtain using a finite presentation of $\pi_1(\mathcal{B})$. We end the paper by finding the necessary and sufficient compatibility

[8] Consider a solid cylinder with h tubular holes. Note that this body is homeomorphic to a genus h handlebody and has a deformation retract to a bouquet of h circles; hence its fundamental group is the free group on h generators. If this is a solid body, for example a hollow bar under torsion and bending, we will show that because all group generators are free, each would require its own complementary compatibility equations.

conditions for linearized strain. We will see that the number of these complementary compatibility equations is consistent with Weingarten's theorem.

2. Compatibility Equations in Nonlinear Elasticity

We assume that a compact material manifold $(\mathcal{B}, \mathbf{G})$ is given. We also assume that both $H_1(\mathcal{B})$ and $\pi_1(\mathcal{B})$ are known.[9] We first find the compatibility equations for $\mathbf{F}$ using a very simple generalization of de Rham's theorem.[10] Next we revisit the calculations of Shield [29] and Pietraszkiewicz and Badur [26] on deriving a system of linear PDEs governing $\mathbf{R}$ when $\mathbf{C}$ is given. We then find the compatibility equations when $\dim \mathcal{B} = \dim \mathcal{S}$ and the body is not simply-connected.

2.1. Compatibility Equations for Deformation Gradient F

An old question in vector calculus is the following. Given a vector field on some bounded domain in the Euclidean 3-space, how can one tell if the vector field is the gradient of some function? It turns out that the topology of the domain of definition of the vector field plays a crucial role here. The question that we will answer in this section is compatibility of $\mathbf{F}$ for non-simply-connected bodies: Given a body $\mathcal{B} \subset \mathbb{R}^3$, find the conditions that guarantee existence of a map $\varphi : \mathcal{B} \to \mathbb{R}^3$ such that $\mathbf{F} = \mathrm{d}\varphi$.

Proposition 2.1. *The following conditions are both necessary and sufficient for compatibility of* $\mathbf{F}$

$$\mathrm{d}\mathbf{F} = \mathbf{0}, \quad \int_{c_i} \mathbf{F}\, \mathrm{d}\mathbf{X} = \mathbf{0}, \quad i = 1, \ldots, \beta_1(\mathcal{B}), \tag{2.2}$$

where $c_i, \ i = 1, \ldots, \beta_1(\mathcal{B})$ *are generators of* $H_1(\mathcal{B}; \mathbb{R})$.

Proof. $\mathbf{F}$ is a vector-valued 1-form with the coordinate representation

$$\mathbf{F} = \partial_a \otimes F^a{}_A \, \mathrm{d}X^A. \tag{2.3}$$

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[9] Heinrich F. F. Tietze (1908) showed that fundamental group of any compact, finite-dimensional, path-connected manifold is finitely presented. One forms the abelization of a group by taking the quotient over the subgroup generated by all commutators $g^{-1}h^{-1}gh$. Poincaré isomorphism theorem tells us that (Poincaré, 1895)

$$\pi_1(M)/[\pi_1(M), \pi_1(M)] \cong H_1(M, \mathbb{Z}). \tag{2.1}$$

If $\gamma_1^{n_1}\gamma_2^{n_2}\ldots\gamma_k^{n_k} = 1$, Poincaré observed that $n_1\gamma_1 + n_2\gamma_2 + \ldots + n_k\gamma_k$ is null-homologous [13]. Given a group G with the presentation $G = \langle a_1, \ldots, a_m;\ r_1, \ldots, r_n \rangle$, its Abelianization is obtained by adding the relations $a_i a_j = a_j a_i$ and is independent of the presentation of G.

[10] This was conjectured by Cartan in 1928 and was proved later on by de Rham [13]. This theorem can be summarized as follows. If for a closed form ω, $(c, \omega) = 0$ for all p-cycles, then ω is exact. If for a p-cycle c, $(c, \omega) = 0$ for all closed p-forms, then c is a boundary.

$\mathbf{F}$ is compatible if and only if $F^a{}_A\,\mathrm{d}X^A$'s are exact. We assume that the ambient space is Euclidean, that is, $\mathbf{F}$ is an $\mathbb{R}^N$-valued 1-form. In this case the compatibility of $\mathbf{F}$ is reduced to that of its 1-form components, that is, we need to see when $F^a{}_A\,\mathrm{d}X^A$ $(a = 1, \ldots, N)$ are exact 1-forms. We know that a necessary condition is closedness of $\mathbf{F}$, that is, $\mathrm{d}(F^a{}_A\,\mathrm{d}X^A) = 0$ $(a = 1, \ldots, N)$. Thus, a necessary condition for $\mathbf{F}$ to be compatible is $\mathrm{d}\mathbf{F} = \mathbf{0}$.[11] Now if $\{c_i\}_{i=1,\ldots,k}$ are generators of $H_1(\mathcal{B};\mathbb{R})$ from de Rham's theorem [11] we know that the closed forms $F^a{}_A\,\mathrm{d}X^A$ $(a = 1, \ldots, N)$ are exact if and only if

$$\int_{c_i} F^a_A\,\mathrm{d}X^A = 0, \quad i = 1, \ldots, \beta_1(\mathcal{B}). \tag{2.4}$$

□

Remark 2.2. We now show that not every generator of the fundamental group requires complementary compatibility equations.[12] Assuming that the position of a point $X_0 \in \mathcal{B}$ in the deformed configuration $x_0 \in \mathcal{S}$ is known, the position of an arbitrary point $X \in \mathcal{B}$ in the deformed configuration is obtained as

$$x = x_0 + \int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{X}, \tag{2.5}$$

where the ambient space is assumed to be Euclidean, hence the integration makes sense for an arbitrary curve γ joining X_0 to X. For $\mathbf{F}$ to be compatible, the above integral must be path-independent. This is equivalent to

$$\int_\gamma \mathbf{F}\,\mathrm{d}\mathbf{X} = \mathbf{0}, \tag{2.6}$$

for any closed path γ based at X_0.[13] Suppose $\{\gamma_i\}_{i=1,\ldots,m}$ are generators of the fundamental group $\pi_1(\mathcal{B})$. When $\mathcal{B}$ is simply-connected and compact, we know that fundamental group has a finite presentation [33]

$$\pi_1(\mathcal{B}) = \langle \gamma_1, \ldots, \gamma_m; \quad r_1, \ldots, r_n \rangle, \tag{2.7}$$

where

$$r_i = \gamma_{i_1}^{\varepsilon_{i_1}} \ldots \gamma_{j_i}^{\varepsilon_{j_i}} = 1, \quad i = 1, \ldots, n, \quad \varepsilon_k = \pm 1, \tag{2.8}$$

are relators of the fundamental group. Note that if (2.6) holds on each generator of the fundamental group, then $\mathbf{F}$ is compatible. However, these are not necessary. The fact that r_i is a relation in the fundamental group means that it represents a loop that is null-homotopic, and therefore null-homologous. Thus, the conditions $\int_{r_i} \mathbf{F}\,\mathrm{d}\mathbf{X} = \mathbf{0}$ follow from the fact that $\mathrm{d}(\mathbf{F}\,\mathrm{d}\mathbf{X}) = \mathbf{0}$. Some of the relations

[11] Note that $\mathrm{d}\mathbf{F} = \mathbf{0}$ is what is usually written as $\mathsf{Curl}\,\mathbf{F} = \mathbf{0}$ in the elasticity literature.

[12] This fact is ignored in some previous works as we discussed earlier in Section 1.

[13] As we mentioned in the introduction, the condition (2.6) has been known in the literature, but what is not known is how to rewrite it in terms of the generators of the fundamental group.

$\int_{\gamma_i} \mathbf{F}\, \mathrm{d}\mathbf{X} = \mathbf{0}$ may be redundant, but this has nothing to do with relations in the fundamental group itself. It follows from the fact that the real conditions are the homological conditions in (2.4). Thus, we need to consider only a collection of loops whose images modulo the commutator subgroup are independent and do not represent torsion.

Remark 2.3. To Abelianize the fundamental group, one adds relations of the form $\gamma_i\gamma_j = \gamma_j\gamma_i$ in the presentation of the fundamental group. These obviously do not introduce any new compatibility equations. Note, also, that the generators of the torsion subgroup do not contribute to compatibility equations because, for γ, an element of the torsion subgroup $\gamma^n = 1$ for some $n \in \mathbb{N}$ and hence, trivially, $\int_{\gamma} \mathbf{F}\, \mathrm{d}\mathbf{X} = \mathbf{0}$. Thus, we need to have $\int_{\gamma} \mathbf{F}\, \mathrm{d}X = \mathbf{0}$ only on each generator of the first homology group with real coefficients. This means that the number of complementary compatibility equations is equal to $N\beta_1(\mathcal{B})$, where $N = \dim \mathcal{S}$.

Example 2.4. Let us look at two-dimensional elasticity on a torus and a punctured torus and derive their **F**-compatibility equations. The first homology group of a torus is generated by the loops γ_1 and γ_2 in Fig. 1a. Thus, the compatibility equations read

$$\mathrm{d}\mathbf{F} = \mathbf{0}, \quad \int_{\gamma_1} \mathbf{F}\, \mathrm{d}\mathbf{X} = \int_{\gamma_2} \mathbf{F}\, \mathrm{d}\mathbf{X} = \mathbf{0}. \tag{2.9}$$

The fundamental group of a torus (see Fig. 1) has the following presentation

$$\pi_1(T^2) = \langle \gamma_1, \gamma_2; \gamma_1\gamma_2 = \gamma_2\gamma_1 \rangle. \tag{2.10}$$

Therefore, $r_1 = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} = 1$. Note that

$$\begin{aligned} \int_{r_1} \mathbf{F}\, \mathrm{d}\mathbf{X} &= \int_{\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}} \mathbf{F}\, \mathrm{d}\mathbf{X} \\ &= \int_{\gamma_1} \mathbf{F}\, \mathrm{d}\mathbf{X} + \int_{\gamma_2} \mathbf{F}\, \mathrm{d}\mathbf{X} - \int_{\gamma_1} \mathbf{F}\, \mathrm{d}\mathbf{X} - \int_{\gamma_2} \mathbf{F}\, \mathrm{d}\mathbf{X} = \mathbf{0}, \end{aligned} \tag{2.11}$$

which is trivially satisfied, that is, (2.9) are the necessary and sufficient compatibility equations as expected; the relator of the fundamental group does not affect the complementary compatibility equations.

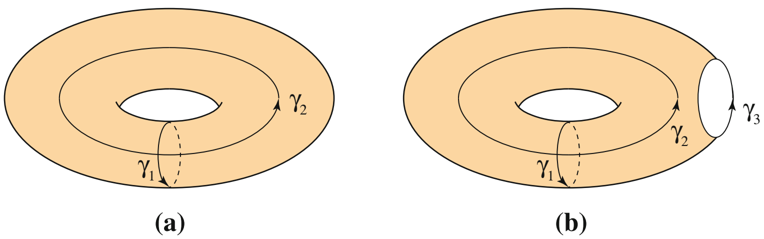

Fig. 1. **a** γ_1 and γ_2 are generators of both the first homology and first homotopy groups of a torus. **b** A punctured torus. γ_1, γ_2, and γ_3 are generators of the fundamental group

For a punctured torus (see Fig. 1b) the fundamental group has three generators and the following presentation [33]

$$\pi_1(\mathcal{H}) = \langle \gamma_1, \gamma_2, \gamma_3; \gamma_3 = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} \rangle. \tag{2.12}$$

Therefore, $r_1 = \gamma_3\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1} = 1$. Note that

$$\begin{aligned} \int_{r_1} \mathbf{F}\,\mathrm{d}\mathbf{X} &= \int_{\gamma_3\gamma_2\gamma_1\gamma_2^{-1}\gamma_1^{-1}} \mathbf{F}\,\mathrm{d}\mathbf{X} \\ &= \int_{\gamma_3} \mathbf{F}\,\mathrm{d}\mathbf{X} + \int_{\gamma_2} \mathbf{F}\,\mathrm{d}\mathbf{X} + \int_{\gamma_1} \mathbf{F}\,\mathrm{d}\mathbf{X} - \int_{\gamma_2} \mathbf{F}\,\mathrm{d}\mathbf{X} - \int_{\gamma_1} \mathbf{F}\,\mathrm{d}\mathbf{X} \\ &= \int_{\gamma_3} \mathbf{F}\,\mathrm{d}\mathbf{X} = \mathbf{0}. \end{aligned} \tag{2.13}$$

Therefore, the following are the necessary and sufficient compatibility equations

$$\mathrm{d}\mathbf{F} = \mathbf{0}, \quad \int_{\gamma_1} \mathbf{F}\,\mathrm{d}\mathbf{X} = \int_{\gamma_2} \mathbf{F}\,\mathrm{d}\mathbf{X} = \mathbf{0}. \tag{2.14}$$

It is seen that γ_3 is a generator of the fundamental group but does not correspond to any complementary compatibility equations. The boundary of the hole in the torus (boundary of a handle) is an example of a null-homologous path that is not null-homotopic.

2.2. *Compatibility Equations for the Right Cauchy-Green Strain* C

Let us consider motion of a body $\varphi_t : \mathcal{B} \to \mathcal{S}$ and assume that $\dim \mathcal{B} = \dim \mathcal{S}$. If the ambient space $(\mathcal{S}, \mathbf{g})$ is Euclidean, its Riemann curvature tensor vanishes, that is, $\boldsymbol{\mathcal{R}}(\mathbf{g}) = \mathbf{0}$. Thus, pull-back of curvature to $\mathcal{B}$ vanishes as well, that is, $\varphi_t^*\boldsymbol{\mathcal{R}}(\mathbf{g}) = \mathbf{0}$. But note that [22]

$$\varphi_t^*\boldsymbol{\mathcal{R}}(\mathbf{g}) = \boldsymbol{\mathcal{R}}(\varphi_t^*\mathbf{g}) = \boldsymbol{\mathcal{R}}(\mathbf{C}). \tag{2.15}$$

Therefore, a necessary condition for compatibility of $\mathbf{C}$ is the vanishing of its Riemann curvature (thinking of $\mathbf{C}$ as a metric in $\mathcal{B}$). MARSDEN and HUGHES [22] showed that this is locally sufficient, as well. Note that when the body is simply-connected, vanishing curvature guarantees a global isometry [10]. In dimension three, Ricci curvature algebraically determines the entire curvature tensor. FOSDICK [16] showed that compatibility equations can be rewritten in terms of Ricci curvature or Einstein tensor. In dimension two, a weaker requirement is sufficient [4]: A metric is flat if and only if its scalar curvature (the Ricci scalar) is zero. This is the geometric reason behind the fact that in two dimensions there is only one compatibility equation, while in three dimensions there are six.[14]

[14] In this paper, we restrict ourselves to the case $\dim \mathcal{B} = \dim \mathcal{S}$, for which the metric (the first fundamental form) is specified. When $\dim \mathcal{B} < \dim \mathcal{S}$, in addition to the metric, the second fundamental form should be considered and it must satisfy its own compatibility equations.

Remark 2.5. Suppose the ambient space is $\mathbb{R}^3$. The left Cauchy-Green strain $\mathbf{B}$ is a push-forward of $\mathbf{G}$, the metric in the reference configuration. Assuming that $\mathbf{G}$ is flat and that a deformed configuration exists, curvature of $\mathbf{B}$ vanishes. This is a necessary condition for compatibility of $\mathbf{B}$ (even if the body is simply-connected). The sufficiency question requires one to construct a deformed configuration whose $\mathbf{B}$ tensor matches the prescribed field. See [1] for more details.[15]

In coordinate charts $\{X^A\}$ and $\{x^a\}$ for $\mathcal{B}$ and $\mathcal{S}$, respectively, let us denote the Levi-Civita connection coefficients of $\mathbf{g}$ and $\mathbf{C} = \varphi^*\mathbf{g}$ by $\gamma^a{}_{bc}$ and $\Gamma^A{}_{BC}$, respectively. These connection coefficients are related as

$$\Gamma^A{}_{BC} = \frac{\partial X^A}{\partial x^a}\frac{\partial x^b}{\partial X^B}\frac{\partial x^c}{\partial X^C}\gamma^a{}_{bc} + \frac{\partial^2 x^a}{\partial X^B \partial X^C}\frac{\partial X^A}{\partial x^a}. \tag{2.16}$$

If $\{x^a\}$ is a Cartesian coordinate chart for the Euclidean ambient space, then $\gamma^a{}_{bc} = 0$, and hence

$$\Gamma^A{}_{BC} = \frac{\partial^2 x^a}{\partial X^B \partial X^C}\frac{\partial X^A}{\partial x^a}. \tag{2.17}$$

Therefore

$$\frac{\partial^2 x^a}{\partial X^B \partial X^C} = \frac{\partial}{\partial X^C} F^a{}_B = F^a{}_A \Gamma^A{}_{BC}. \tag{2.18}$$

Substituting the polar decomposition in the above identity we obtain

$$R^a{}_{A,B} = R^a{}_C \Omega^C{}_{AB}, \tag{2.19}$$

where

$$\Omega^C{}_{AB} = \left(\Gamma^M{}_{BN} U^C{}_M - U^C{}_{N,B}\right) U_A{}^N, \tag{2.20}$$

$$\Gamma^C{}_{AB} = \frac{1}{2} C^{CD}(C_{BD,A} + C_{AD,B} - C_{AB,D}), \tag{2.21}$$

and U_A^N are components of $\mathbf{U}^{-1}$. We assume that the body is elastic, and hence our material manifold is embedded in the Euclidean ambient space [37–39]. We choose Cartesian coordinates for $\mathcal{B}$ and hence $G_{AB} = \delta_{AB}$. Note that the system of differential equations (2.19) is identical to that obtained in [29]. Given a path γ connecting $X_0, X \in \mathcal{B}$ and parametrized by $s \in I$, we have the following system

[15] We are grateful to Amit Acharya for a discussion on $\mathbf{B}$-compatibility.

of linear ODEs governing the rotation tensor[16]

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{R} = \mathbf{R}\mathbf{K}, \tag{2.24}$$

where

$$K^C{}_A(s) = \Omega^C{}_{AB}(s)\dot{X}^B(s). \tag{2.25}$$

Thus, for each a

$$\frac{\mathrm{d}R^a{}_A}{\mathrm{d}s} - \Omega^C{}_{AB}R^a{}_C\dot{X}^B(s) = 0. \tag{2.26}$$

This is the equation of parallel transport of $R^a{}_A$ along the curve γ when $\mathcal{B}$ is equipped with the connection $\boldsymbol{\Omega}$. Let us assume that $\mathbf{R}(0) = \mathbf{R}_0$. We see that rotation tensor at s is the parallel transport of $\mathbf{R}_0$. Integrability conditions of (2.26), that is, path independence of the integral modulo homotopies fixing the endpoints are equivalent to vanishing of the curvature tensor of $\mathbf{C}$ [26].

The solution of (2.24) can be written in terms of product integration,[17] which was introduced by VITO VOLTERRA [34] (see also [14,31]). Solution of (2.24) in terms of a product integral reads

$$\mathbf{R}(s) = \mathbf{R}_0 \prod_0^s(\gamma)\, e^{\mathbf{K}(\xi)\,\mathrm{d}\xi}, \tag{2.27}$$

where $\mathbf{R}_0$ is an orthogonal tensor assumed to be given and $\prod_0^s(\gamma)\, e^{\mathbf{K}(\xi)\,\mathrm{d}\xi}$ is the product integral of $\mathbf{K}$ along the path γ from 0 to s. The product integral has the following properties [14]:

(i) $\prod(1)\, e^{\mathbf{K}(\xi)\,\mathrm{d}\xi} = \mathbf{I}$, where 1 is the identity loop.

(ii) $\prod(\gamma^{-1})\, \mathrm{e}^{\mathbf{K}(\xi)\,\mathrm{d}\xi} = \left(\prod(\gamma)\, \mathrm{e}^{\mathbf{K}(\xi)\,\mathrm{d}\xi}\right)^{-1}$, for any path γ (not necessarily closed).

(iii) $\prod(\gamma_1.\gamma_2)\, \mathrm{e}^{\mathbf{K}(\xi)\,\mathrm{d}\xi} = \prod(\gamma_2)\, \mathrm{e}^{\mathbf{K}(\xi)\,\mathrm{d}\xi} \prod(\gamma_1)\, \mathrm{e}^{\mathbf{K}(\xi)\,\mathrm{d}\xi}$, for arbitrary paths γ_1 and γ_2.

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16 From (2.24) we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{R}^{\mathsf{T}} = \mathbf{K}^{\mathsf{T}}\mathbf{R}^{\mathsf{T}}. \tag{2.22}$$

Differentiating $\mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$, we obtain

$$\mathbf{0} = \frac{\mathrm{d}\mathbf{R}^{\mathsf{T}}}{\mathrm{d}s}\mathbf{R} + \mathbf{R}^{\mathsf{T}}\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} = \mathbf{K}^{\mathsf{T}}\mathbf{R}^{\mathsf{T}}\mathbf{R} + \mathbf{R}^{\mathsf{T}}\mathbf{R}\mathbf{K} = \mathbf{K}^{\mathsf{T}} + \mathbf{K}. \tag{2.23}$$

In components, this reads $K^A{}_B + K^B{}_A = 0$, that is $\mathbf{K}$ is skew-symmetric.

17 In the physics literature, this is called path-ordered integration or path-ordered exponential integration.

(iv) If $\mathbf{K}(s_1)\mathbf{K}(s_2) = \mathbf{K}(s_2)\mathbf{K}(s_1),\ \forall s_1, s_2$, then $\prod(\gamma)\, \mathrm{e}^{\mathbf{K}(\xi)\ \mathrm{d}\xi} = \mathrm{e}^{\int_\gamma \mathbf{K}(\xi)\ \mathrm{d}\xi}$.

Note that conditions i), ii), and iii) imply that $\gamma \to \prod(\gamma)\, \mathrm{e}^{\mathbf{K}(\xi)d\xi}$ defines a group representation of $\pi_1(\mathcal{B})$. We know that $\mathbf{K}$ is continuous, hence the product integral can be written in terms of the following uniformly convergent series [14]

$$\prod_0^s(\gamma)\, \mathrm{e}^{\mathbf{K}(s)\ \mathrm{d}s} = \sum_{k=0}^{\infty} \mathbf{J}_k(s;\gamma), \tag{2.28}$$

where

$$\mathbf{J}_0(s;\gamma) = \mathbf{I}, \quad \mathbf{J}_n(s;\gamma) = \int_0^s \mathbf{K}(\tau;\gamma)\mathbf{J}_{n-1}(\tau;\gamma)d\tau, \quad n \geqq 1. \tag{2.29}$$

For $\mathbf{C}$ to be compatible, rotation tensor calculated from (2.27) must be independent of the path γ. This means that

$$\prod_\gamma \mathrm{e}^{\mathbf{K}(s)\ \mathrm{d}s} = \mathbf{I}, \quad \forall \text{ closed path } \gamma. \tag{2.30}$$

A necessary and sufficient condition for (2.30) is the vanishing of the product integral of $\mathbf{K}$ over the generators of the fundamental group $\pi_1(\mathcal{B})$.

Remark 2.6. If $\mathbf{K}(s_1)\mathbf{K}(s_2) = \mathbf{K}(s_2)\mathbf{K}(s_1),\ \forall s_1, s_2$, then, the condition (2.30) is equivalent to

$$\int_0^1 \mathbf{K}(s)\ \mathrm{d}s = \mathbf{0}, \tag{2.31}$$

where $\gamma : [0, 1] \to \mathcal{B}$ is any closed path.

Note that because $\mathbf{K}$ is skew-symmetric, (2.30) has $N(N-1)/2$ independent components. If the path independence of the product integral is guaranteed, then a unique rotation field $\mathbf{R}$, and hence a unique deformation gradient $\mathbf{F} = \mathbf{R}\sqrt{\mathbf{C}}$, is calculated.

In summary, given $\mathbf{C}$, $\mathbf{U} = \sqrt{\mathbf{C}}$ is uniquely determined. Rotation $\mathbf{R}$ is governed by the system of PDEs (2.19). Rotation is determined using a product integral, which is path independent if and only if the curvature tensor of $\mathbf{C}$ vanishes and (2.30) is satisfied over each generator of the first homotopy group $\pi_1(\mathcal{B})$.

As a consequence of the previous discussions we have the following proposition.

Proposition 2.7. *The necessary and sufficient conditions for compatibility of* $\mathbf{C}$ *in* $\mathcal{B}$ *are the following:*

(i) $\boldsymbol{\mathcal{R}}(\mathbf{C}) = \mathbf{0}$ *in* $\mathcal{B}$,
(ii) $\prod_{\gamma_i} e^{\mathbf{K}(s)ds} = \mathbf{I}$, *where* γ_i*'s are generators of* $\pi_1(\mathcal{B})$,
(iii) *for the uniquely calculated deformation gradient* $\mathbf{F} = \mathbf{R}\sqrt{\mathbf{C}}$, *we must have* $\int_{c_i} \mathbf{F}d\mathbf{X} = \mathbf{0},\ i = 1, \ldots, \beta_1(\mathcal{B})$, *where* c_i *are generators of* $H_1(\mathcal{B}; \mathbb{R})$.

It is seen that each generator of the first homotopy group corresponds to $N(N-1)/2$ complementary compatibility equations and each generator of $H_1(\mathcal{B}; \mathbb{R})$ corresponds to N additional complementary compatibility equations.

2.3. Compatibility Equations in Linearized Elasticity

In this section, we derive the necessary and sufficient compatibility equations for linearized strain when $\dim \mathcal{B} = \dim \mathcal{S}$. If the ambient space is Euclidean and the coordinates are Cartesian, the linear strain components read

$$e_{ab} = \frac{1}{2}\left(\frac{\partial u_a}{\partial x^b} + \frac{\partial u_b}{\partial x^a}\right). \tag{2.32}$$

We know that the necessary and sufficient conditions for compatibility in terms of $\mathbf{F}$ are

$$\int_\gamma \mathbf{F}\, \mathrm{d}\mathbf{X} = \mathbf{0}, \tag{2.33}$$

for every loop γ in $\mathcal{B}$. Linearization of (2.33) reads

$$\int_\gamma \nabla\mathbf{U}\, \mathrm{d}\mathbf{X} = \mathbf{0}. \tag{2.34}$$

In components, we have $\int_\gamma u^a{}_{,B} dX^B = 0$, where $\{X^A\}$ and $\{x^a\}$ are coordinate charts for $\mathcal{B}$ and $\mathcal{S}$, respectively. We assume that linearization is about the standard embedding of $\mathcal{B}$ in $\mathbb{R}^N$, that is, $F^a{}_A = \delta^a_A$. Thus, $dX^B = \frac{\partial X^B}{\partial x^b} dx^b = \delta^B_b dx^b$, hence we can write

$$\int_\gamma u_{a,B}\, \mathrm{d}X^B = \int_\gamma u_{a,b}\, \mathrm{d}x^b = \int_\gamma \left(e_{ab} + \omega_{ab}\right)\, \mathrm{d}x^b = 0, \tag{2.35}$$

where $e_{ab} = u_{(a,b)} = \frac{1}{2}\left(u_{a,b} + u_{b,a}\right)$ and $\omega_{ab} = u_{[a,b]} = \frac{1}{2}\left(u_{a,b} - u_{b,a}\right)$ are the linearized strain and rotation tensors, respectively. Note that

$$\int_\gamma \omega_{ab}\, \mathrm{d}x^b = \int_\gamma \left[(x^c\omega_{ac})_{,b} - x^c\omega_{ac,b}\right]\, \mathrm{d}x^b = -\int_\gamma x^c\omega_{ac,b}\, \mathrm{d}x^b. \tag{2.36}$$

Note also that

$$\begin{aligned} \omega_{ac,b} &= \frac{1}{2}\left(u_{a,cb} - u_{c,ab}\right) + \frac{1}{2}\left(u_{b,ac} - u_{b,ac}\right) \\ &= \frac{1}{2}\left(u_{a,bc} + u_{b,ac}\right) - \frac{1}{2}\left(u_{c,abc} + u_{b,ac}\right) \\ &= e_{ab,c} - e_{bc,a}. \end{aligned} \tag{2.37}$$

Given e_{ab}, ω_{ab} is obtained by integrating $\omega_{ab,c} = e_{ac,b} - e_{cb,a}$ along an arbitrary curve. To ensure that the given strain tensor corresponds to a well-defined rotation field over any closed path $\gamma \in \mathcal{B}$, we must have[18]

$$\int_\gamma \left(e_{ac,b} - e_{cb,a}\right)\, \mathrm{d}x^c = 0. \tag{2.38}$$

[18] We benefited from a discussion with James R. Barber on the number of complementary compatibility equations in linearized elasticity when the body is not simply-connected, when he clarified his treatment of compatibility equations in [2].

When γ is null-homotopic, $\gamma = \partial\Omega$ (Ω is the parameter domain of the null-homotopy) and hence

$$\begin{aligned}\int_\gamma \left(e_{ac,b} - e_{cb,a}\right) \,\mathrm{d}x^c &= \int_\Omega \mathrm{d}\left(e_{ac,b} - e_{cb,a}\right) \wedge \,\mathrm{d}x^c \\ &= \int_\Omega \left(e_{ad,bc} + e_{bc,ad} - e_{ac,bd} - e_{bd,ac}\right) (\,\mathrm{d}x^c \wedge \,\mathrm{d}x^d) \\ &= 0,\end{aligned} \tag{2.39}$$

where $\{(\,\mathrm{d}x^c \wedge \,\mathrm{d}x^d)\} = \{\,\mathrm{d}x^c \wedge \,\mathrm{d}x^d\}_{c<d}$ is a basis of 2-forms. It can be shown that (2.39) are equivalent to $\mathsf{Curl}\,\mathsf{Curl}\,\mathbf{e} = \mathbf{0}$, which are the classical bulk compatibility equations [21]. Knowing that the first homology group with real coefficients has the generators $c_i, i = 1, \ldots, \beta_1(\mathcal{B})$, we have the following complementary compatibility equations

$$\int_{c_i} \left(e_{ac,b} - e_{cb,a}\right) \,\mathrm{d}x^c = 0, \quad i = 1, \ldots, \beta_1(\mathcal{B}). \tag{2.40}$$

Note that $e_{ac,b} - e_{cb,a}$ is anti-symmetric in (ab), hence each integral has $N(N-1)/2$ independent components ($N = \dim \mathcal{B} = \dim \mathcal{S}$).

Now using (2.37), we have

$$\int_\gamma u_{a,b} \,\mathrm{d}x^b = \int_\gamma \left[e_{ab} - x^c(e_{ab,c} - e_{bc,a})\right] \,\mathrm{d}x^b = 0. \tag{2.41}$$

This is called the Cesàro path integral [8]. Let us define the Cesàro tensor $\mathsf{C}_{ab} = e_{ab} - x^c(e_{ab,c} - e_{bc,a})$. Suppose γ is null-homotopic and hence $\gamma = \partial\Omega$ (Ω is the parameter domain of the null-homotopy). Thus, using Stokes' theorem

$$\begin{aligned}\int_\gamma \mathsf{C}_{ab} \,\mathrm{d}x^b &= \int_\Omega d\mathsf{C}_{ab} \wedge \,\mathrm{d}x^b \\ &= \int_\Omega \mathsf{C}_{ab,c} dx^c \wedge \,\mathrm{d}x^b \\ &= \int_\Omega \left[e_{bc,a} - x^d\left(e_{ab,cd} - e_{bd,ac}\right)\right] \,\mathrm{d}x^c \wedge \,\mathrm{d}x^b.\end{aligned} \tag{2.42}$$

Note that because of symmetry of strain, $e_{bc,a} \,\mathrm{d}x^c \wedge \,\mathrm{d}x^b = 0$, hence

$$\begin{aligned}\int_\gamma u_{a,b} \,\mathrm{d}x^b &= \int_\Omega x^d\left(e_{bd,ac} - e_{ab,cd}\right) \,\mathrm{d}x^c \wedge \,\mathrm{d}x^b \\ &= \int_\Omega x^d\left(e_{ab,cd} + e_{cd,ab} - e_{ac,bd} - e_{bd,ac}\right) (\,\mathrm{d}x^b \wedge \,\mathrm{d}x^c) \\ &= 0.\end{aligned} \tag{2.43}$$

It can be shown that (2.43) are equivalent to $\mathsf{Curl}\,\mathsf{Curl}\,\mathbf{e} = \mathbf{0}$, which are the classical bulk compatibility equations [21]. Thus, we have proved the following proposition.

Proposition 2.8. *The necessary and sufficient conditions for compatibility of linearized strain* $\mathbf{e}$ *in* $\mathcal{B}$ *are the following:*

(i) $\mathsf{Curl\,Curl}\,\mathbf{e}=\mathbf{0}$ *in* $\mathcal{B}$,
(ii) *for each generator of* $H_1(\mathcal{B};\mathbb{R})$

$$\int_{c_i}\left[e_{ab}-x^c(e_{ab,c}-e_{bc,a})\right]\,dx^b=0,\quad i=1,\ldots,\beta_1(\mathcal{B}),\tag{2.44}$$

$$\int_{c_i}\left(e_{ac,b}-e_{cb,a}\right)\,dx^c=0,\quad i=1,\ldots,\beta_1(\mathcal{B}).\tag{2.45}$$

Example 2.9. Let us consider the Saint Venant's torsion problem. One considers a cylindrical bar parallel to the x^3-axis with an arbitrary cross section Ω with n holes with boundaries c_i (the bar has the same cross section everywhere). In Saint-Venant's semi-inverse solution the following displacements are assumed: $u_1=-\vartheta x^2x^3$, $u_2=\vartheta x^1x^3$, $u_3=\vartheta\psi(x^1,x^2)$, where ϑ is the rate of twist (twist per unit length) and $\psi(x^1,x^2)$ is the warping function. The only nonzero strains are $e_{13}=\frac{1}{2}\vartheta(\psi_{,1}-x^2)$ and $e_{23}=\frac{1}{2}\vartheta(\psi_{,2}+x^1)$. The equilibrium equations are trivially satisfied if stresses $\sigma_{13}=2Ge_{13}$ and $\sigma_{23}=2Ge_{23}$(G is the shear modulus) are expressed in terms of the Prandtl stress function $\phi(x^1,x^2)$ as $\sigma_{13}=\phi_{,2}$ and $\sigma_{23}=-\phi_{,1}$. Eliminating ψ from $\sigma_{13}=G\vartheta(\psi_{,1}-x^2)$ and $\sigma_{23}=G\vartheta(\psi_{,2}+x^1)$ yields $\nabla^2\phi=-2G\vartheta$ (the bulk compatibility equation). The traction-free boundary conditions imply that ϕ is constant on each connected component of $\partial\Omega$ (the boundary of Ω). We assume that $\phi=0$ on the outer boundary of Ω and equal to ϕ_i on the boundary of the ith hole.

We assume that the cross section Ω is normal to the bar axis, and for all its points $x^3=a$ (we can consider any other cross section, of course). From (2.44) we obtain

$$\int_{c_i}\left(e_{13,1}\,\mathrm{d}x^1+e_{23,1}\,\mathrm{d}x^2\right)=0,\tag{2.46}$$

$$\int_{c_i}\left(e_{13,2}\,\mathrm{d}x^1+e_{23,2}\,\mathrm{d}x^2\right)=0,\tag{2.47}$$

$$\int_{c_i}\left[(e_{13}-x^1e_{13,1}-x^2e_{13,2})\,\mathrm{d}x^1+(e_{23}-x^1e_{23,1}-x^2e_{23,2})\,\mathrm{d}x^2\right]=0.\tag{2.48}$$

It can be easily shown that when the above three equations are satisfied, (2.45) would be trivially satisfied, that is, each hole has only three complementary compatibility equations. In terms of stresses, complementary compatibility equations are identical to (2.46), (2.47), and (2.48) when strains are replaced by their corresponding stresses. Note that

$$\sigma_{13,1}\,\mathrm{d}x^1+\sigma_{23,1}\,\mathrm{d}x^2=\mathrm{d}(\phi_{,2})-\nabla^2\phi\;\mathrm{d}x^2=\mathrm{d}(\phi_{,2})+2G\vartheta\,\mathrm{d}x^2,\tag{2.49}$$

$$\sigma_{13,2}\,\mathrm{d}x^1+\sigma_{23,2}\,\mathrm{d}x^2=-\,\mathrm{d}(\phi_{,1})+\nabla^2\phi\;\mathrm{d}x^1=\mathrm{d}(\phi_{,1})-2G\vartheta\,\mathrm{d}x^1.\tag{2.50}$$

This means that the first two complementary compatibility equations (2.46) and (2.47) are trivially satisfied. Note also that

$$\begin{aligned}&(\sigma_{13}-x^1\sigma_{13,1}-x^2\sigma_{13,2})\,\mathrm{d}x^1+(\sigma_{23}-x^1\sigma_{23,1}-x^2\sigma_{23,2})\,\mathrm{d}x^2\\&=2\left(\phi_{,2}\,\mathrm{d}x^1-\phi_{,1}\,\mathrm{d}x^2\right)+2G\vartheta\left(x^2\,\mathrm{d}x^1-x^1\,\mathrm{d}x^2\right)\\&\quad+\mathrm{d}\left(-x^1\phi_{,2}+x^2\phi_{,1}\right).\end{aligned}\tag{2.51}$$

Thus (2.48) gives us

$$\int_{c_i} \left(\phi_{,2}\,\mathrm{d}x^1 - \phi_{,1}\,\mathrm{d}x^2\right) + 2G\vartheta \int_{c_i} \vartheta \left(x^2\,\mathrm{d}x^1 - x^1\,\mathrm{d}x^2\right) = 0. \tag{2.52}$$

Therefore

$$\int_{c_i} \nabla\phi \cdot \hat{\mathbf{n}}\,\mathrm{d}s = -2G\vartheta A_i, \tag{2.53}$$

where $\hat{\mathbf{n}}$ is the unit normal vector to the boundary of the ith hole and A_i is the area of the ith hole.

Remark 2.10. Note that for each c_i, (2.44) and (2.45) give N and $N(N-1)/2$ complementary compatibility equations, respectively. Thus, each c_i has $N(N+1)$ complementary compatibility equations. This is obviously consistent with Weingarten's theorem [36], which says that if a body is cut along a surface, the jump in the displacement field is a rigid-body motion ($N(N+1)/2$ degrees of freedom). See Love [21] for a detailed discussion (Love calls homotopic paths, "reconcilable circuits" and a null-homotopic path, an "evanescible circuit".). Zubov [40] and Casey [7] showed that this theorem holds for finite strains as well. We should also mention that the discussion in [30] regarding sufficient compatibility equations in linear elasticity is flawed, as they missed those complementary compatibility conditions that guarantee existence of a well-defined field of rotations, that is (2.45).

Remark 2.11. Relative homology groups were introduced by Lefschetz [20].[19] Lefschetz duality tells us that for a compact n-manifold M, $H_c^{n-p}(M) \cong H_p(M, \partial M)$. From de Rham's theorem, $H_{n-p}(M) \cong H_c^p(M\backslash\partial M)$. Therefore, $H_{n-p}(M) \cong H_p(M, \partial M)$. Thus, $\beta_{n-p}(M) = \beta_p(M, \partial M)$. Let us now restrict ourselves to embedded 3-submanifolds of $\mathbb{R}^3$, which model our three-dimensional deformable bodies [6]. $H_1(M)$ is generated by equivalence classes of oriented loops; two loops are in the same equivalence class if their "difference" is the boundary of an oriented surface in $M.H_1(M, \partial M)$ is generated by the equivalence class of oriented paths with end points on ∂M; two paths are equivalent if their "difference" (augmented by paths on ∂M, if necessary) is the boundary of an oriented surface in M. From Poincaré's duality we know that $H_2(M) \cong H_1(M, \partial M)$. Define $M^c = \mathbb{R}^3\backslash M$. Alexander's duality tells us that $H_1(M) \cong H_1(M^c)$. Let $\Sigma_1, \ldots, \Sigma_k$ be a family of cross-sectional surfaces in M with boundaries on ∂M such that they generate $H_2(M, \partial M)$. As an example, consider the two-hole solid torus shown in Fig. 2, for which $k = 2$. Let γ_1, γ_2 (loops in the interior of M) be generators of $H_1(M)$ chosen such that intersection number of c_i with Σ_j is δ_{ij}. One can make these loops disjoint. If one pushes the boundaries of Σ_1, Σ_2 a bit into M^c, one obtains the loops Γ_1, Γ_2, which generate $H_1(M^c)$.

Love [21] in Article 156 writes: "Now suppose the multiply-connected region to be reduced to a simply-connected one by means of a system of barriers." Note that

[19] It is interesting that the first academic degree of Solomon Lefschetz—one of the most influential algebraic topologists—was in mechanical engineering.

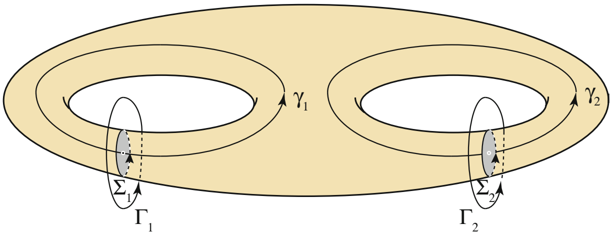

Fig. 2. A two-hole solid torus M (a genus two handlebody). γ_1 and γ_2 are generators of $H_1(M)$. Γ_1 and Γ_2 are generators of $H_1(\mathbb{R}^3\backslash M)$

a "barrier" Ω in a three-dimensional body $\mathcal{B}$ is a generator of $H_2(\mathcal{B}, \partial\mathcal{B}) \cong H_1(\mathcal{B})$, and in a two-dimensional body it is a generator of $H_2(\mathcal{B}, \partial\mathcal{B}) \cong H_1(\mathcal{B})$.

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