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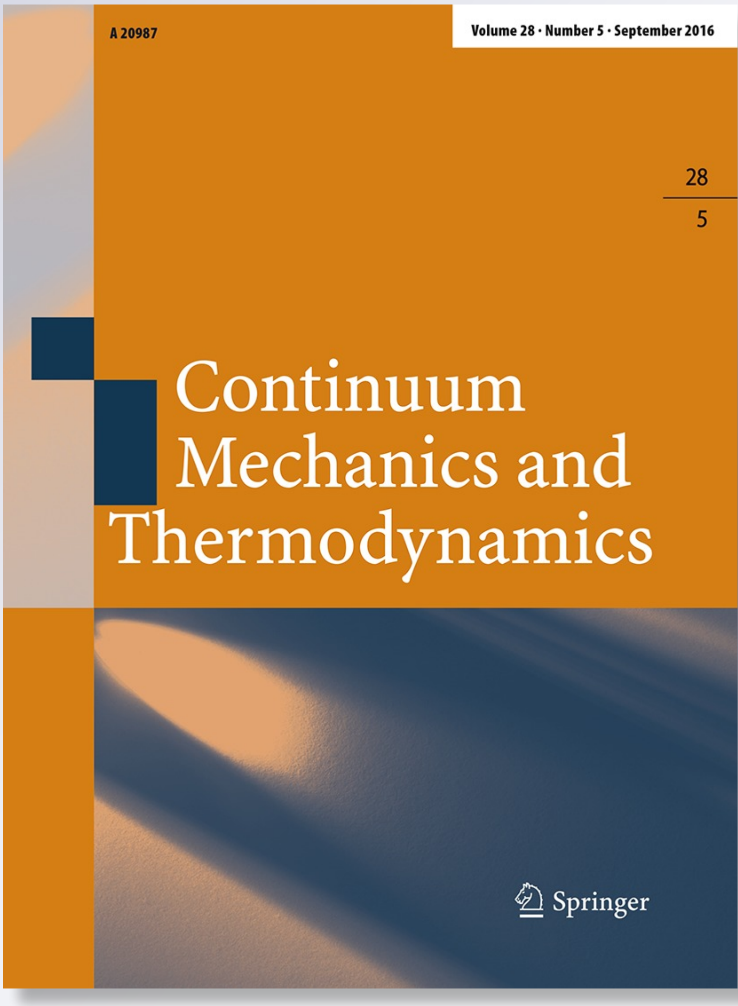

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ORIGINAL ARTICLE

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The weak compatibility equations of nonlinear elasticity and the insufficiency of the Hadamard jump condition for non-simply connected bodies

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Abstract We derive the compatibility equations of L^2 displacement gradients on non-simply connected bodies. These compatibility equations are useful for non-smooth strains such as those associated with deformations of multi-phase materials. As an application of these compatibility equations, we study some configurations of different phases around a hole and show that, in general, the classical Hadamard jump condition is not a sufficient compatibility condition.

Keywords Compatibility equations · Nonlinear elasticity · Hadamard jump condition · Non-simply connected bodies

1 Introduction

If the displacement field of a deforming body is of H^1-class, then the associated L^2-strain is not continuous, in general. For example, consider the following two important classes of problems:

- *Non-convex bodies*: It is well-known that the standard regularity theorem for solutions of elliptic boundary-value problems holds on a strongly Lipschitz body if either its boundary is of $C^{1,1}$-class or the body is convex, for example see [1, Theorems 2.2.2.3 and 3.2.1.2]. Consequently, in the L^2-setting, solutions of the (linear) elasticity problem on non-convex Lipschitz bodies (such as bodies with reentrant corners, cracks, or voids) are merely of H^1-class, in general.
- *Multi-phase materials*: Strains associated with deformations of multi-phase bodies[1] can be discontinuous on interfaces between different phases. This means that these strains do not necessarily belong to the H^1-space.

For such L^2-strains, the compatibility equations for smooth strains should be appropriately modified. In [2, Sect. 3.1], we studied the effects of Dirichlet boundary conditions on the strain compatibility problem of nonlinear elasticity. We wrote the compatibility equations of smooth strains on domains with smooth boundaries

[1] By a phase in a body we mean a subdomain with a uniform deformation gradient. In this sense, we consider different variants of martensite in a twinned crystal of martensite as different phases.

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by using the L^2-inner product. In this paper, we extend those compatibility equations to L^2-strains on strongly Lipschitz bodies. For deriving the compatibility equations in [2], we assumed that displacement boundary conditions are imposed only on boundary-less compact subdomains of the boundary. Here, we also allow boundary conditions to be imposed on compact subdomains of the boundary with non-empty boundaries.

The main results of this paper are derived in Sect. 2 (Theorems 2, 4) by using a Hodge-type orthogonal decomposition for second-order tensors. As an application of these compatibility equations, we study some configurations of different phases around holes in Sect. 3. These examples show that, generally speaking, the classical Hadamard jump condition is not a sufficient compatibility condition in the presence of holes. However, we will observe that in some symmetric cases, the Hadamard condition may be sufficient as well.

Notation Throughout this paper, $\mathcal{B}$ is an open subset of $\mathbb{R}^n$, $n = 2, 3$, which is connected, bounded, and strongly Lipschitz, that is, the boundary $\partial\mathcal{B}$ of $\mathcal{B}$ is locally the graph of a Lipschitz continuous function, and $\mathcal{B}$ lies only on one side of $\partial\mathcal{B}$. The closure of $\mathcal{B}$ is denoted by $\bar{\mathcal{B}}$. We assume that $\partial\mathcal{B} = \bar{\mathcal{S}}_1 \cup \bar{\mathcal{S}}_2$, where the disjoint open subsets $\mathcal{S}_1, \mathcal{S}_2 \subset \partial\mathcal{B}$ are admissible patches in the sense of [3, p. 2051]. Roughly speaking, this means that $\partial\mathcal{S}_1$ and $\partial\mathcal{S}_2$ are either empty or locally graphs of Lipschitz continuous functions with $\mathcal{S}_1$ and $\mathcal{S}_2$ being on the opposite sides of the common boundaries. The Cartesian coordinates, the standard orthonormal basis, and the standard inner product of $\mathbb{R}^n$ are denoted by $\{\mathrm{X}^I\}$, $\{\mathbf{E}_I\}$, and $\langle\!\langle , \rangle\!\rangle$, respectively. For any nonnegative integer s, the Sobolev spaces of H^s $\left(L^2 := H^0\right)$ vector and $\binom{2}{0}$-tensors fields are denoted by $H^s(T\mathcal{B})$ and $H^s(\otimes^2 T\mathcal{B})$, respectively. Note that the Cartesian components of these H^s-fields belong to the standard Sobolev space $H^s(\mathcal{B})$. The partial derivative $\partial f/\partial \mathrm{X}^I$ is denoted by $f_{,I}$. For any $\boldsymbol{U}, \boldsymbol{Z} \in L^2(T\mathcal{B})$ and $\boldsymbol{R}, \boldsymbol{T} \in L^2(\otimes^2 T\mathcal{B})$, the L^2 inner products are defined as $\langle\!\langle \boldsymbol{U}, \boldsymbol{Z} \rangle\!\rangle_{L^2} := \sum_I \langle\!\langle U^I, Z^I \rangle\!\rangle_{L^2}$, and $\langle\!\langle \boldsymbol{R}, \boldsymbol{T} \rangle\!\rangle_{L^2} := \sum_{I,J} \langle\!\langle R^{IJ}, T^{IJ} \rangle\!\rangle_{L^2}$. The summation convention is assumed on repeated indices.

2 The compatibility equations of L^2 displacement gradients

Recall that the gradient of a vector field $\boldsymbol{Y}$ is a $\binom{2}{0}$-tensor field given by $(\mathbf{grad}\,\boldsymbol{Y})^{IJ} := Y^I{}_{,J}$. By interpreting "$, J$" as a weak derivative, one can define the gradient of H^1 vector fields. Now, consider the following compatibility problem for L^2 displacement gradients with prescribed boundary displacements on $\mathcal{S}_1$:

Given a $\binom{2}{0}$-tensor field $\boldsymbol{K} \in L^2(\otimes^2 T\mathcal{B})$ and an arbitrary vector field $\overline{\boldsymbol{U}} \in H^1(T\mathcal{B})$, find a displacement field $\boldsymbol{U} \in H^1(T\mathcal{B})$ such that

$$\begin{aligned} \mathbf{grad}\,\boldsymbol{U} &= \boldsymbol{K}, \quad \text{on } \mathcal{B}, \\ \boldsymbol{U} &= \overline{\boldsymbol{U}}, \quad \text{on } \mathcal{S}_1. \end{aligned} \tag{2.1}$$

The special case $\mathcal{S}_1 = \varnothing$ is the classical compatibility problem for L^2 displacement gradients. In order to write the necessary and sufficient integrability conditions for the compatibility problem (2.1), we need to briefly review some more notations and results. The divergence of a smooth $\binom{2}{0}$-tensor field $\boldsymbol{T}$ is defined as $(\mathbf{div}\,\boldsymbol{T})^I := T^{IJ}{}_{,J}$. In 3D, one can define the operator $(\mathbf{curl}^{\mathsf{T}}\boldsymbol{T})^{IJ} := \varepsilon_{JKL} T^{IL}{}_{,K}$, where ε_{IKL} is the standard permutation symbol. In 2D, we have the operators $(\boldsymbol{\varkappa}(\boldsymbol{T}))^I := T^{I2}{}_{,1} - T^{I1}{}_{,2}$, and $(\mathbf{s}(\boldsymbol{Y}))^{IJ} := \delta^{1J} Y^I{}_{,2} - \delta^{2J} Y^I{}_{,1}$, with δ^{IJ} being the Kronecker delta. Clearly, $\mathbf{s}$ can be defined for H^1 vector fields as well. The other operators can be extended in the distributional sense as follows: For $\boldsymbol{T} \in L^2(\otimes^2 T\mathcal{B})$, $\mathbf{curl}^{\mathsf{T}}\boldsymbol{T}$, $\boldsymbol{\varkappa}(\boldsymbol{T})$, and $\mathbf{div}\,\boldsymbol{T}$ are defined such that

$$\begin{aligned} \langle\!\langle \boldsymbol{T}, \mathbf{curl}^{\mathsf{T}}\boldsymbol{Q} \rangle\!\rangle_{L^2} &= \langle\!\langle \mathbf{curl}^{\mathsf{T}}\boldsymbol{T}, \boldsymbol{Q} \rangle\!\rangle_{L^2}, \quad \forall \boldsymbol{Q} \in C_0^\infty(\otimes^2 T\mathcal{B}), \\ \langle\!\langle \boldsymbol{T}, \mathbf{s}(\boldsymbol{W}) \rangle\!\rangle_{L^2} &= \langle\!\langle \boldsymbol{\varkappa}(\boldsymbol{T}), \boldsymbol{W} \rangle\!\rangle_{L^2}, \quad \forall \boldsymbol{W} \in C_0^\infty(T\mathcal{B}), \\ \langle\!\langle \boldsymbol{T}, \mathbf{grad}\,\boldsymbol{W} \rangle\!\rangle_{L^2} &= -\langle\!\langle \mathbf{div}\,\boldsymbol{T}, \boldsymbol{W} \rangle\!\rangle_{L^2}, \quad \forall \boldsymbol{W} \in C_0^\infty(T\mathcal{B}), \end{aligned}$$

where $C_0^\infty(T\mathcal{B})$ and $C_0^\infty(\otimes^2 T\mathcal{B})$ are the spaces of C^∞ vector and tensor fields with compact supports in $\mathcal{B}$. Next, consider the following subspaces of L^2 $\binom{2}{0}$-tensor fields: $H^{\mathbf{c}}(\mathcal{B}) := \{\boldsymbol{T} \in L^2 : \mathbf{curl}^{\mathsf{T}}\boldsymbol{T} \in L^2\}$, $H^{\boldsymbol{\varkappa}}(\mathcal{B}) := \{\boldsymbol{T} \in L^2 : \boldsymbol{\varkappa}(\boldsymbol{T}) \in L^2\}$, $H^{\mathbf{d}}(\mathcal{B}) := \{\boldsymbol{T} \in L^2 : \mathbf{div}\,\boldsymbol{T} \in L^2\}$. Then, one can define the operators $\mathbf{curl}^{\mathsf{T}} : H^{\mathbf{c}}(\mathcal{B}) \to H^{\mathbf{d}}(\mathcal{B})$, $\boldsymbol{\varkappa} : H^{\boldsymbol{\varkappa}}(\mathcal{B}) \to L^2(T\mathcal{B})$, $\mathbf{div} : H^{\mathbf{d}}(\mathcal{B}) \to L^2(T\mathcal{B})$, $\mathbf{s} : H^1(T\mathcal{B}) \to H^{\mathbf{d}}(\mathcal{B})$, and $\mathbf{grad} : H^1(T\mathcal{B}) \to \mathcal{V}$, where $\mathcal{V}$ is either $H^{\mathbf{c}}(\mathcal{B})$ or $H^{\boldsymbol{\varkappa}}(\mathcal{B})$. One can show that

$$\mathbf{curl}^{\mathsf{T}} \circ \mathbf{grad} = 0, \quad \mathbf{div} \circ \mathbf{curl}^{\mathsf{T}} = 0,$$
$$\boldsymbol{\varkappa} \circ \mathbf{grad} = 0, \qquad \mathbf{div} \circ \mathbf{s} = 0.$$

As was discussed in [4,5], the above relations give rise to some Hilbert complexes, which are isomorphic to the weak $\mathbb{R}^n$-valued de Rham complex. This means that any result for L^2 differential forms and the de Rham complex has a counterpart for L^2 vector and $\binom{2}{0}$-tensor fields.

Let $H^1(T\mathcal{B}, \mathcal{S}_1)$ be the space of H^1 vector fields that vanish on $\mathcal{S}_1$ in the trace sense. The $H^{\mathbf{c}}$, $H^{\boldsymbol{\varkappa}}$, and $H^{\mathbf{d}}$ spaces contain the H^1 space, and therefore, the restriction of their elements to $\partial\mathcal{B}$ is not well-defined, in general. Nonetheless, by using the notions of normal and tangent parts of differential forms discussed in [3, Sect. 3], it is possible to define some boundary tractions for these spaces. More specifically, for a vector field $\boldsymbol{Z}$ on $\partial\mathcal{B}$, let $\boldsymbol{T}(\boldsymbol{Z}) := T^{IJ} Z^J \mathbf{E}_I$, and let $\boldsymbol{N}$ be the unit outward normal vector field of $\partial\mathcal{B}$, which is defined almost everywhere on $\partial\mathcal{B}$. Then, for any $\boldsymbol{T} \in H^{\mathbf{d}}(\mathcal{B})$, the normal traction $\boldsymbol{T}(\boldsymbol{N})$ is well-defined. For any $\boldsymbol{T}$ in $H^{\mathbf{c}}$ and $H^{\boldsymbol{\varkappa}}$ and $\boldsymbol{Y} \| \partial\mathcal{B}$, the tangent traction $\boldsymbol{T}(\boldsymbol{Y})$ is well-defined.[2] Let $H^{\mathbf{d}}(\mathcal{B}, \mathcal{S}_1)$ be the space of $H^{\mathbf{d}}$ tensor fields with zero tractions on $\mathcal{S}_1$. Similarly, let $H^{\mathbf{c}}(\mathcal{B}, \mathcal{S}_1)$ and $H^{\boldsymbol{\varkappa}}(\mathcal{B}, \mathcal{S}_1)$ be the space of $H^{\mathbf{c}}$ and $H^{\boldsymbol{\varkappa}}$ tensor fields with zero tangent tractions on $\mathcal{S}_1$. Then, the weak de Rham complex for differential forms with zero tangent parts on an admissible patch introduced in [6, Sect. 5.2] implies that $\mathbf{grad}\big(H^1(T\mathcal{B}, \mathcal{S}_1)\big)$ is a subset of $H^{\mathbf{c}}(\mathcal{B}, \mathcal{S}_1)$ and $H^{\boldsymbol{\varkappa}}(\mathcal{B}, \mathcal{S}_1)$. Moreover, $\mathbf{curl}^{\mathsf{T}}\big(H^{\mathbf{c}}(\mathcal{B}, \mathcal{S}_1)\big)$ and $\mathbf{s}\big(H^1(T\mathcal{B}, \mathcal{S}_1)\big)$ are subsets of $H^{\mathbf{d}}(\mathcal{B}, \mathcal{S}_1)$. The Green's formula for differential forms derived in [3, Theorem 3.4] allows one to write the following Green's formulas: For arbitrary $\boldsymbol{Y} \in H^1(T\mathcal{B})$, $\boldsymbol{T} \in H^{\mathbf{d}}(\mathcal{B})$, $\boldsymbol{R} \in H^{\boldsymbol{\varkappa}}(\mathcal{B})$, and $\boldsymbol{Q}, \boldsymbol{S} \in H^{\mathbf{c}}(\mathcal{B})$, we have

$$\langle\!\langle \mathbf{grad}\, \boldsymbol{Y}, \boldsymbol{T} \rangle\!\rangle_{L^2} = -\langle\!\langle \boldsymbol{Y}, \mathbf{div}\, \boldsymbol{T} \rangle\!\rangle_{L^2} + \int_{\partial\mathcal{B}} \langle\!\langle \boldsymbol{Y}, \boldsymbol{T}(\boldsymbol{N}) \rangle\!\rangle \mathrm{d}A, \tag{2.2a}$$

$$\langle\!\langle \boldsymbol{\varkappa}(\boldsymbol{R}), \boldsymbol{Y} \rangle\!\rangle_{L^2} = \langle\!\langle \boldsymbol{R}, \mathbf{s}(\boldsymbol{Y}) \rangle\!\rangle_{L^2} + \int_{\partial\mathcal{B}} \langle\!\langle \boldsymbol{Y}, \boldsymbol{R}(\boldsymbol{t}_\partial) \rangle\!\rangle \mathrm{d}s, \tag{2.2b}$$

$$\langle\!\langle \mathbf{curl}^{\mathsf{T}} \boldsymbol{Q}, \boldsymbol{S} \rangle\!\rangle_{L^2} = \langle\!\langle \boldsymbol{Q}, \mathbf{curl}^{\mathsf{T}} \boldsymbol{S} \rangle\!\rangle_{L^2} + \sum_{I=1}^{3} \int_{\partial\mathcal{B}} \langle\!\langle \overrightarrow{\boldsymbol{Q}}_I \times \overrightarrow{\boldsymbol{S}}_I, \boldsymbol{N} \rangle\!\rangle \mathrm{d}A, \tag{2.2c}$$

where $\boldsymbol{t}_\partial$ is the oriented unit vector field along $\partial\mathcal{B}$, $\overrightarrow{\boldsymbol{Q}}_I := Q^{IJ}\mathbf{E}_J$ is the vector field in the Ith row of $\boldsymbol{Q}$, and $\times$ is the standard cross product of vector fields in $\mathbb{R}^3$. Note that the boundary terms in the above formulas are interpreted as duality parings in the sense of [3, Proposition 3.3].

By using the Hodge–Morrey decomposition for L^2 differential forms on strongly Lipschitz domains (e.g., see [7, Theorem 6.3]), one can write the following L^2-orthogonal decomposition for any L^2 $\binom{2}{0}$-tensor field on a 3D body:

$$\boldsymbol{T} = \mathbf{grad}\, \boldsymbol{Y}_{\boldsymbol{T}} + \mathbf{curl}^{\mathsf{T}} \boldsymbol{Q}_{\boldsymbol{T}} + \boldsymbol{H}_{\boldsymbol{T}}, \tag{2.3a}$$

where $\boldsymbol{Y}_{\boldsymbol{T}} \in H^1(T\mathcal{B}, \partial\mathcal{B})$, $\boldsymbol{Q}_{\boldsymbol{T}} \in H^{\mathbf{c}}(\mathcal{B}, \partial\mathcal{B})$, and $\boldsymbol{H}_{\boldsymbol{T}} \in \mathcal{H}^{\otimes}(\mathcal{B}) := \ker \mathbf{div} \cap \ker \mathbf{curl}^{\mathsf{T}}$. The 2D analogue of the above decomposition reads

$$\boldsymbol{T} = \mathbf{grad}\, \boldsymbol{Y}_{\boldsymbol{T}} + \mathbf{s}(\boldsymbol{W}_{\boldsymbol{T}}) + \boldsymbol{H}_{\boldsymbol{T}}, \tag{2.3b}$$

with $\boldsymbol{Y}_{\boldsymbol{T}}, \boldsymbol{W}_{\boldsymbol{T}} \in H^1(T\mathcal{B}, \partial\mathcal{B})$ and $\boldsymbol{H}_{\boldsymbol{T}} \in \mathcal{H}^{\otimes}(\mathcal{B}) := \ker \mathbf{div} \cap \ker \boldsymbol{\varkappa}$. Corollary 5.4 of [6] suggests that the infinite-dimensional harmonic space $\mathcal{H}^{\otimes}(\mathcal{B})$ admits the following finite-dimensional subspace:

$$\mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) := \mathcal{H}^{\otimes}(\mathcal{B}) \cap H^{\mathbf{c}}(\mathcal{B}, \mathcal{S}_1) \cap H^{\mathbf{d}}(\mathcal{B}, \mathcal{S}_2),$$

where $H^{\mathbf{c}}(\mathcal{B}, \mathcal{S}_1)$ is replaced with $H^{\boldsymbol{\varkappa}}(\mathcal{B}, \mathcal{S}_1)$ in 2D. We have

$$\dim \mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) := n\, b_1(\bar{\mathcal{B}}, \mathcal{S}_1),$$

where the purely topological property $b_1(\bar{\mathcal{B}}, \mathcal{S}_1)$ is the first relative Betti number of the pair $(\bar{\mathcal{B}}, \mathcal{S}_1)$. The 2D and 3D tensor Laplacian for $\binom{2}{0}$-tensor fields are, respectively, defined as

$$\mathscr{L}(\boldsymbol{T}) := \mathbf{s} \circ \boldsymbol{\varkappa}(\boldsymbol{T}) - \mathbf{grad} \circ \mathbf{div}\, \boldsymbol{T},$$
$$\mathscr{L}(\boldsymbol{T}) := \mathbf{curl}^{\mathsf{T}} \circ \mathbf{curl}^{\mathsf{T}}\, \boldsymbol{T} - \mathbf{grad} \circ \mathbf{div}\, \boldsymbol{T}.$$

[2] These normal and tangent tractions are defined as certain distributions such that one recovers Green's formula, see [3, Proposition 3.3 and Theorem 3.4].

Now, consider the following boundary-value problem for the tensor Laplacian.

Given $\boldsymbol{Q} \in L^2(\otimes^2 T\mathcal{B})$*, find* $\boldsymbol{T} \in L^2(\otimes^2 T\mathcal{B})$ *such that* $\mathscr{L}(\boldsymbol{T}) = \boldsymbol{Q}$*, and depending on* $\dim \mathcal{B}$*,* $\boldsymbol{T}$ *satisfies one of the following boundary conditions:*

$$\left.\begin{array}{ll}\boldsymbol{T} \in H^{\boldsymbol{\varkappa}}(\mathcal{B}, \mathcal{S}_1), & \mathbf{div}\, \boldsymbol{T} \in H^1(T\mathcal{B}, \mathcal{S}_1),\\ \boldsymbol{T} \in H^{\mathbf{d}}(\mathcal{B}, \mathcal{S}_2), & \boldsymbol{\varkappa}(\boldsymbol{T}) \in H^1(T\mathcal{B}, \mathcal{S}_2),\end{array}\right\} \text{2D B.C.}$$
$$\left.\begin{array}{ll}\boldsymbol{T} \in H^{\mathbf{c}}(\mathcal{B}, \mathcal{S}_1), & \mathbf{div}\, \boldsymbol{T} \in H^1(T\mathcal{B}, \mathcal{S}_1),\\ \boldsymbol{T} \in H^{\mathbf{d}}(\mathcal{B}, \mathcal{S}_2), & \mathbf{curl}^{\mathsf{T}} \boldsymbol{T} \in H^{\mathbf{c}}(\mathcal{B}, \mathcal{S}_2).\end{array}\right\} \text{3D B.C.} \tag{2.4}$$

Since $\mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$ is finite-dimensional, it is a closed subspace of $L^2(\otimes^2 T\mathcal{B})$ and one can write the following decomposition

$$L^2(\otimes^2 T\mathcal{B}) = \mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) \oplus \mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^{\perp},$$

where $\mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^{\perp}$ is the orthogonal complement of $\mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$. By using the integrability condition of the Hodge Laplacian with mixed boundary conditions discussed in [6, Theorem 6.1], one concludes that the problem (2.4) admits a solution if and only if $\boldsymbol{Q} \in \mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^{\perp}$. Moreover, $\mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$ is the space of solutions of (2.4) for $\boldsymbol{Q} = 0$. Consequently, we can define the Green's operator

$$\mathscr{G} : \mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^{\perp} \to \mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^{\perp},$$

where for any $\boldsymbol{Q} \in \mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^{\perp}$, $\mathscr{G}(\boldsymbol{Q})$ is defined to be the unique solution of (2.4) that belongs to $\mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^{\perp}$. The Green's operator allows one to write a Friedrichs-type decomposition for the harmonic space $\mathcal{H}^{\otimes}(\mathcal{B})$. More specifically, let

$$\begin{aligned}
\mathcal{H}^{\otimes}_{\mathbf{g}}(\mathcal{B}, \mathcal{S}_1) &:= \mathcal{H}^{\otimes}(\mathcal{B}) \cap \mathbf{grad}\big(H^1(T\mathcal{B}, \mathcal{S}_1)\big),\\
\mathcal{H}^{\otimes}_{\mathbf{s}}(\mathcal{B}, \mathcal{S}_2) &:= \mathcal{H}^{\otimes}(\mathcal{B}) \cap \mathbf{s}\big(H^1(T\mathcal{B}, \mathcal{S}_2)\big),\\
\mathcal{H}^{\otimes}_{\mathbf{c}}(\mathcal{B}, \mathcal{S}_2) &:= \mathcal{H}^{\otimes}(\mathcal{B}) \cap \mathbf{curl}^{\mathsf{T}}\big(H^{\mathbf{c}}(\mathcal{B}, \mathcal{S}_2)\big),\\
\mathcal{H}^{\otimes}_{0}(\mathcal{B}, \mathcal{S}_2) &:= \mathcal{H}^{\otimes}(\mathcal{B}) \cap H^{\mathbf{d}}(\mathcal{B}, \mathcal{S}_2).
\end{aligned}$$

Note that $\mathcal{H}^{\otimes}_{0}(\mathcal{B}, \mathcal{S}_2)$ is the space of harmonic tensor fields with zero traction vector field on $\mathcal{S}_2$. One concludes that:

Lemma 1 *The harmonic space* $\mathcal{H}^{\otimes}(\mathcal{B})$ *admits the following* L^2*-orthogonal decomposition*

$$\mathcal{H}^{\otimes}(\mathcal{B}) = \mathcal{H}^{\otimes}_{0}(\mathcal{B}, \mathcal{S}_2) \oplus \mathcal{H}^{\otimes}_{\mathbf{g}}(\mathcal{B}, \mathcal{S}_1). \tag{2.5}$$

Furthermore, $\mathcal{H}^{\otimes}_{0}(\mathcal{B}, \mathcal{S}_2)$ *can be orthogonally decomposed as* $\mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) \oplus \mathcal{W}$*, where depending on* $\dim \mathcal{B}$*,* $\mathcal{W}$ *is either* $\mathcal{H}^{\otimes}_{\mathbf{s}}(\mathcal{B}, \mathcal{S}_2)$ *or* $\mathcal{H}^{\otimes}_{\mathbf{c}}(\mathcal{B}, \mathcal{S}_2)$*.*

Proof As the proof is similar to that of the Friedrichs-type decomposition for differential forms derived in [2, Theorem 1], we only mention the main steps of the proof. The orthogonality of the components of the above decompositions follows from Green's formulas (2.2). Any $\boldsymbol{H} \in \mathcal{H}^{\otimes}(\mathcal{B})$ can be uniquely decomposed as $\bar{\boldsymbol{H}} + \boldsymbol{H}^{\perp}$, where $\bar{\boldsymbol{H}} \in \mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$ and $\boldsymbol{H}^{\perp}$ belongs to the orthonormal complement of $\mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$ in $\mathcal{H}^{\otimes}(\mathcal{B})$. In 3D, $\boldsymbol{H}^{\perp}$ can be uniquely decomposed as

$$\boldsymbol{H}^{\perp} = \mathbf{curl}^{\mathsf{T}} \circ \mathbf{curl}^{\mathsf{T}} \mathscr{G}(\boldsymbol{H}^{\perp}) - \mathbf{grad} \circ \mathbf{div}\, \mathscr{G}\big(\widetilde{\boldsymbol{H}}\big), \tag{2.6}$$

where $\widetilde{\boldsymbol{H}} := \boldsymbol{H}^{\perp} - \mathbf{curl}^{\mathsf{T}} \circ \mathbf{curl}^{\mathsf{T}} \mathscr{G}(\boldsymbol{H}^{\perp})$. Consequently, any $\boldsymbol{H} \in \mathcal{H}^{\otimes}_{0}(\mathcal{B}, \mathcal{S}_2)$ can be decomposed as $\bar{\boldsymbol{H}} + \mathbf{curl}^{\mathsf{T}} \circ \mathbf{curl}^{\mathsf{T}} \mathscr{G}(\boldsymbol{H}^{\perp})$, which gives the decomposition for $\mathcal{H}^{\otimes}_{0}(\mathcal{B}, \mathcal{S}_2)$. The corresponding decomposition in 2D can be derived analogously. The decomposition (2.5) then follows from this decomposition together with (2.6). □

Now, we are in a position to write the necessary and sufficient integrability conditions for the compatibility problem (2.1). Our main tools are the Hodge–Morrey decompositions (2.3) and the Friedrichs-type decomposition (2.5).

Theorem 2 (The compatibility equations for L^2 displacement gradients) *The following conditions are both necessary and sufficient for the existence of a solution to* (2.1)*:*

$$2D\begin{cases}\varkappa(\boldsymbol{K})=0, & (2.7a)\\ \langle\langle \boldsymbol{K},\boldsymbol{H}\rangle\rangle_{L^2}=\int_{\mathcal{S}_1}\langle\langle \overline{\boldsymbol{U}},\boldsymbol{H}(\boldsymbol{N})\rangle\rangle \mathrm{d}s,\ \forall \boldsymbol{H}\in\mathcal{H}_0^{\otimes}(\mathcal{B},\mathcal{S}_2), & (2.7b)\end{cases}$$

$$3D\begin{cases}\mathbf{curl}^{\mathsf{T}}\boldsymbol{K}=0, & (2.8a)\\ \langle\langle \boldsymbol{K},\boldsymbol{H}\rangle\rangle_{L^2}=\int_{\mathcal{S}_1}\langle\langle \overline{\boldsymbol{U}},\boldsymbol{H}(\boldsymbol{N})\rangle\rangle \mathrm{d}A,\ \forall \boldsymbol{H}\in\mathcal{H}_0^{\otimes}(\mathcal{B},\mathcal{S}_2). & (2.8b)\end{cases}$$

Proof As the proofs in 2D and 3D are similar, we only prove the 3D case. The necessity of (2.8) simply follows from the fact that $\mathbf{curl}^{\mathsf{T}}\circ\mathbf{grad}=0$, and Green's formula (2.2a). Conversely, suppose the conditions (2.8) hold. Then, the Hodge–Morrey decomposition (2.3a), the decomposition (2.5), and (2.8a) allow one to write the decomposition $\boldsymbol{K}=\mathbf{grad}\,\boldsymbol{Y}_{\boldsymbol{K}}+\widehat{\boldsymbol{H}}_{\boldsymbol{K}}+\mathbf{grad}\,\boldsymbol{Z}_{\boldsymbol{K}}$, with $\boldsymbol{Y}_{\boldsymbol{K}}\in H^1(T\mathcal{B},\partial\mathcal{B})$, $\widehat{\boldsymbol{H}}_{\boldsymbol{K}}\in\mathcal{H}_0^{\otimes}(\mathcal{B},\mathcal{S}_2)$, and $\boldsymbol{Z}_{\boldsymbol{K}}\in\mathcal{H}_{\mathbf{g}}^{\otimes}(\mathcal{B},\mathcal{S}_1)$. Similarly, $\boldsymbol{R}:=\mathbf{grad}\,\overline{\boldsymbol{U}}$ admits the decomposition $\boldsymbol{R}=\mathbf{grad}\,\boldsymbol{Y}_{\boldsymbol{R}}+\widehat{\boldsymbol{H}}_{\boldsymbol{R}}+\mathbf{grad}\,\boldsymbol{Z}_{\boldsymbol{R}}$. Define

$$\boldsymbol{U}:=\boldsymbol{Y}_{\boldsymbol{K}}+\boldsymbol{Z}_{\boldsymbol{K}}+\overline{\boldsymbol{U}}-\boldsymbol{Y}_{\boldsymbol{R}}-\boldsymbol{Z}_{\boldsymbol{R}}. \tag{2.9}$$

Then, $\boldsymbol{U}|_{\mathcal{S}_1}=\overline{\boldsymbol{U}}|_{\mathcal{S}_1}$, and $\mathbf{grad}\,\boldsymbol{U}=\boldsymbol{K}+\widehat{\boldsymbol{H}}_{\boldsymbol{R}}-\widehat{\boldsymbol{H}}_{\boldsymbol{K}}$. Note that $\widehat{\boldsymbol{H}}_{\boldsymbol{R}}-\widehat{\boldsymbol{H}}_{\boldsymbol{K}}\in\mathcal{H}_0^{\otimes}(\mathcal{B},\mathcal{S}_2)$. On the other hand, $\widehat{\boldsymbol{H}}_{\boldsymbol{R}}-\widehat{\boldsymbol{H}}_{\boldsymbol{K}}$ is also normal to $\mathcal{H}_0^{\otimes}(\mathcal{B},\mathcal{S}_2)$ since for any $\boldsymbol{H}\in\mathcal{H}_0^{\otimes}(\mathcal{B},\mathcal{S}_2)$, the condition (2.8b) implies that

$$\langle\langle\widehat{\boldsymbol{H}}_{\boldsymbol{R}}-\widehat{\boldsymbol{H}}_{\boldsymbol{K}},\boldsymbol{H}\rangle\rangle_{L^2}=\int_{\mathcal{S}_1}\langle\langle\boldsymbol{U},\boldsymbol{H}(\boldsymbol{N})\rangle\rangle\mathrm{d}A-\langle\langle\boldsymbol{K},\boldsymbol{H}\rangle\rangle_{L^2}=0.$$

Therefore, $\widehat{\boldsymbol{H}}_{\boldsymbol{R}}-\widehat{\boldsymbol{H}}_{\boldsymbol{K}}=0$, and $\boldsymbol{U}$ defined in (2.9) is a solution of (2.1). □

Remark 3 Note that the condition (2.8a) implicitly implies that $\boldsymbol{K}$ must belong to $H^{\mathbf{c}}(\mathcal{B})$. The Helmholtz–Weyl decompositions for differential forms on strongly Lipschitz domains derived in [7, Theorem 6.2] allow one to write the following decomposition for 3D bodies:

$$L^2(\otimes^2T\mathcal{B})=\ker\mathbf{curl}^{\mathsf{T}}\oplus\mathbf{curl}^{\mathsf{T}}(H^{\mathbf{c}}(\mathcal{B},\partial\mathcal{B})).$$

This suggests that (2.8a) is equivalent to $\langle\langle\boldsymbol{K},\mathbf{curl}^{\mathsf{T}}\boldsymbol{T}\rangle\rangle_{L^2}=0,\forall\boldsymbol{T}\in H^{\mathbf{c}}(\mathcal{B},\partial\mathcal{B})$. Similarly, (2.7a) is equivalent to $\langle\langle\boldsymbol{K},\mathbf{s}(\mathbf{Z})\rangle\rangle_{L^2}=0,\forall\mathbf{Z}\in H^1(T\mathcal{B},\partial\mathcal{B})$.

Next, we write an alternative expression of the compatibility equations of Theorem 2, which is more appropriate for some multi-phase bodies. Suppose $\mathcal{B}$ consists of finitely many disjoint, strongly Lipschitz, open subsets $\{\mathcal{B}_i\}$ such that $\bar{\mathcal{B}}=\bigcup_i\bar{\mathcal{B}}_i$, see Fig. 1.[3] Let $\mathcal{I}_{i,j}$ be the $(n-1)$-dimensional interface of two different phases $\mathcal{B}_i$ and $\mathcal{B}_j$. It is straightforward to show that piecewise H^1 fields with respect to $\{\mathcal{B}_i\}$ are of H^1-class on $\mathcal{B}$ if and only if their traces on all $\mathcal{I}_{i,j}$'s are single-valued. This means that a piecewise H^1 displacement field with respect to $\{\mathcal{B}_i\}$ is of H^1-class on $\mathcal{B}$ if and only if different phases do not slip relative to each other along the interfaces. The compatibility equations on a multi-phase body $\mathcal{B}$ that guarantee the existence of a displacement in $H^1(T\mathcal{B})$ can be stated as follows.

Theorem 4 *Suppose $\mathcal{B}$ consists of the subdomains $\{\mathcal{B}_i\}$ and let $\boldsymbol{K}_i:=\boldsymbol{K}|_{\mathcal{B}_i}$. The compatibility equations* (2.8) *are equivalent to:*

$$\mathbf{curl}^{\mathsf{T}}\boldsymbol{K}_i=0, \tag{2.10a}$$
$$\boldsymbol{K}_i(\mathbf{Z})=\boldsymbol{K}_j(\mathbf{Z}),\qquad\forall\mathbf{Z}\|\mathcal{I}_{i,j}, \tag{2.10b}$$
$$\boldsymbol{K}(\mathbf{Z})|_{\mathcal{S}_1}=(\mathbf{grad}\,\overline{\boldsymbol{U}})(\mathbf{Z}),\qquad\forall\mathbf{Z}\|\mathcal{S}_1, \tag{2.10c}$$
$$\sum_i\langle\langle\boldsymbol{K}_i,\boldsymbol{\lambda}_i\rangle\rangle_{L^2}=\int_{\mathcal{S}_1}\langle\langle\overline{\boldsymbol{U}},\boldsymbol{\lambda}(\boldsymbol{N})\rangle\rangle\mathrm{d}A,\quad\forall\boldsymbol{\lambda}\in\mathcal{H}^{\otimes}(\mathcal{B},\mathcal{S}_1,\mathcal{S}_2). \tag{2.10d}$$

In 2D, the condition (2.10a) *is replaced with* $\varkappa(\boldsymbol{K}_i)=0$.

[3] This assumption is not appropriate for all multi-phase bodies, e.g., the fine mixture of phases shown in [8, Fig. 6]. Nonetheless, as we discuss in the next section, it is useful for studying some local arrangements of different phases.

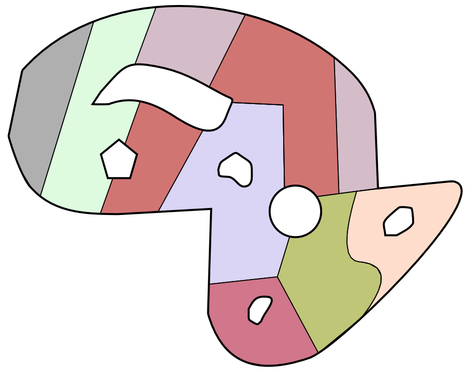


Fig. 1 A body consisting of finitely many strongly Lipschitz subdomains

Proof First, we show that (2.8a) is equivalent to (2.10a) and (2.10b). If (2.8a) holds, then the weak definition of $\mathbf{curl}^{\mathsf{T}}$ suggests that each $\boldsymbol{K}_i$ is also $\mathbf{curl}^{\mathsf{T}}$-free. Using Green's formula (2.2c) and arguments similar to those of [9, Lemma 5.1], one concludes that piecewise $H^{\mathbf{d}}$ tensors with respect to $\{\mathcal{B}_i\}$ belong to $H^{\mathbf{d}}(\mathcal{B})$ if and only if their tangent tractions are single-valued along all the interfaces. These arguments prove the above equivalence. The fact that $\langle\!\langle \boldsymbol{K}, \boldsymbol{\lambda}\rangle\!\rangle_{L^2} = \sum_i \langle\!\langle \boldsymbol{K}_i, \boldsymbol{\lambda}_i\rangle\!\rangle_{L^2}$ implies that (2.10d) is a subcondition of (2.8b), and therefore, if the latter holds so does the former. The condition (2.8b) also suggests that the projections of the $\mathbf{curl}^{\mathsf{T}}$-free tensors $\boldsymbol{K}$ and $\mathbf{grad}\,\overline{\boldsymbol{U}}$ on $\mathcal{H}_0^{\otimes}(\mathcal{B}, \mathcal{S}_2)$ are the same and since the tangent tractions of these tensors on $\mathcal{S}_1$ are determined by these projections, one concludes that (2.8b) also implies (2.10c). Thus, (2.8) $\Rightarrow$ (2.10). For proving the converse, Lemma 1 tells us that it suffices to show that (2.10) results in

$$\langle\!\langle \boldsymbol{K}, \mathbf{curl}^{\mathsf{T}}\boldsymbol{S}\rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \langle\!\langle \overline{\boldsymbol{U}}, (\mathbf{curl}^{\mathsf{T}}\boldsymbol{S})(\boldsymbol{N})\rangle\!\rangle \mathrm{d}A, \quad \forall \mathbf{curl}^{\mathsf{T}}\boldsymbol{S} \in \mathcal{H}_{\mathbf{c}}^{\otimes}(\mathcal{B}, \mathcal{S}_2).$$

By using (2.10c), one can write

$$\begin{aligned}\langle\!\langle \boldsymbol{K}, \mathbf{curl}^{\mathsf{T}}\boldsymbol{S}\rangle\!\rangle_{L^2} &= \sum_{I=1}^{3}\int_{\mathcal{S}_1} \langle\!\langle \overrightarrow{\boldsymbol{S}}_I \times \overrightarrow{\boldsymbol{K}}_I, \boldsymbol{N}\rangle\!\rangle \mathrm{d}A = \sum_{I=1}^{3}\int_{\mathcal{S}_1} \langle\!\langle \overrightarrow{\boldsymbol{S}}_I \times \overrightarrow{(\mathbf{grad}\,\overline{\boldsymbol{U}})_I}, \boldsymbol{N}\rangle\!\rangle \mathrm{d}A \\ &= \langle\!\langle \mathbf{grad}\,\overline{\boldsymbol{U}}, \mathbf{curl}^{\mathsf{T}}\boldsymbol{S}\rangle\!\rangle_{L^2} = \int_{\mathcal{S}_1} \langle\!\langle \overline{\boldsymbol{U}}, (\mathbf{curl}^{\mathsf{T}}\boldsymbol{S})(\boldsymbol{N})\rangle\!\rangle \mathrm{d}A.\end{aligned}$$

Therefore, (2.10) $\Rightarrow$ (2.8). The 2D case can be proved similarly. □

Remark 5 The condition (2.10b) is the classical Hadamard jump condition, e.g., see [8]. This condition is necessary and sufficient for piecewise $H^{\mathbf{c}}$ tensor fields to belong to $H^{\mathbf{c}}(\mathcal{B})$. The conditions (2.10a)–(2.10c) are local in the sense that they depend on the values of displacement gradients on each phase $\mathcal{B}_i$. On the other hand, (2.10d) depends on the global topology of $\mathcal{B}$ and $\mathcal{S}_1$. Note that in practice, the compatibility equations of Theorem 4 are more useful than those of Theorem 2 because unlike the conditions (2.7b) and (2.8b), which are written in terms of the infinite-dimensional space $\mathcal{H}_0^{\otimes}(\mathcal{B}, \mathcal{S}_2)$, the condition (2.10d) involves the finite-dimensional space $\mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$.

3 The compatibility of multi-phase bodies in the presence of holes

As an application of the L^2 compatibility equations that were derived earlier, we next study the compatibility of some local 2D arrangements of different phases around a hole. We begin our discussion by considering circular holes and later, we also consider holes with arbitrary shapes. Assume that $\mathcal{B}$ is an annulus such as the one depicted in Fig. 2. In this case, it is straightforward to show that the space of harmonic tensors $\mathcal{H}^{\otimes}(\mathcal{B}, \varnothing, \partial\mathcal{B})$ in the compatibility condition (2.10d) is two-dimensional with the basis $\{\boldsymbol{Q}_1, \boldsymbol{Q}_2\}$, where

$$\boldsymbol{Q}_1(X,Y) = \begin{bmatrix} -h(X,Y) & f(X,Y) \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{Q}_2(X,Y) = \begin{bmatrix} 0 & 0 \\ -h(X,Y) & f(X,Y) \end{bmatrix}, \tag{3.1}$$

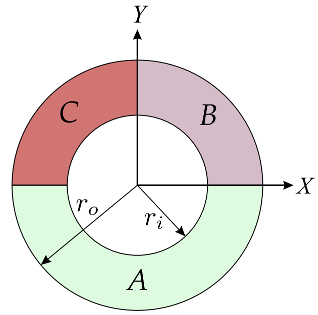


Fig. 2 An annulus composed of three different phases A, B, and C

with

$$h(X,Y)=\frac{Y}{X^2+Y^2},\quad \text{and}\quad f(X,Y)=\frac{X}{X^2+Y^2}.$$

Theorem 4 tells us that if $\mathcal{S}_1=\varnothing$, the Hadamard jump condition is necessary and sufficient for the compatibility of piecewise $\boldsymbol{\varkappa}$-free tensor fields on simply connected bodies. However, as the following example for a "Volterra-type" displacement gradient shows, in the presence of a hole the compatibility condition (2.10d) must be verified as the harmonic space $\mathcal{H}^{\otimes}(\mathcal{B},\varnothing,\partial\mathcal{B})$ is not trivial anymore.

Example 6 Suppose that the annulus $\mathcal{B}$ consists of three subdomains $\mathcal{B}_A$, $\mathcal{B}_B$, and $\mathcal{B}_C$ as shown in Fig. 2. We want to study the compatibility of the L^2 tensor field $\boldsymbol{K}$ defined as

$$\boldsymbol{K}:=\begin{cases}\boldsymbol{Q}_1, & \text{on } \mathcal{B}_A,\\ \boldsymbol{0}, & \text{on } \mathcal{B}_B,\\ \boldsymbol{Q}_2, & \text{on } \mathcal{B}_C.\end{cases}$$

It is straightforward to show that $\boldsymbol{K}$ satisfies (2.10a). Let $\boldsymbol{K}_A:=\boldsymbol{K}|_{\mathcal{B}_A}$. Vectors tangent to the interface between A and B and tangent to the interface between A and C can be written as $[\begin{smallmatrix}a\\0\end{smallmatrix}]$, $\forall a\in\mathbb{R}$, and those tangent to the interface between B and C can be written as $[\begin{smallmatrix}0\\a\end{smallmatrix}]$, $\forall a\in\mathbb{R}$. Note that

$$\begin{aligned}
\boldsymbol{K}_A(X,0)\cdot\begin{bmatrix}a\\0\end{bmatrix}&=\begin{bmatrix}0&1/X\\0&0\end{bmatrix}\cdot\begin{bmatrix}a\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\\
\boldsymbol{K}_C(X,0)\cdot\begin{bmatrix}a\\0\end{bmatrix}&=\begin{bmatrix}0&0\\0&1/X\end{bmatrix}\cdot\begin{bmatrix}a\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\\
\boldsymbol{K}_C(0,Y)\cdot\begin{bmatrix}0\\a\end{bmatrix}&=\begin{bmatrix}0&0\\-1/Y&0\end{bmatrix}\cdot\begin{bmatrix}0\\a\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.
\end{aligned}$$

Therefore, one concludes that $\boldsymbol{K}_A$, $\boldsymbol{K}_B$, and $\boldsymbol{K}_C$ satisfy the Hadamard jump condition (2.10b). The condition (2.10c) is trivial for compatibility without boundary conditions as $\mathcal{S}_1=\varnothing$. Since $\{\boldsymbol{Q}_1,\boldsymbol{Q}_2\}$ is a basis for $\mathcal{H}^{\otimes}(\mathcal{B},\varnothing,\partial\mathcal{B})$, the condition (2.10d) is equivalent to $\langle\!\langle \boldsymbol{K},\boldsymbol{Q}_i\rangle\!\rangle_{L^2}=0$, $i=1,2$. However, note that

$$\begin{aligned}
\langle\!\langle \boldsymbol{K},\boldsymbol{Q}_1\rangle\!\rangle_{L^2}&=\langle\!\langle \boldsymbol{K}_A,(\boldsymbol{Q}_1)|_{\mathcal{B}_A}\rangle\!\rangle_{L^2}=\pi\ln\left(\frac{r_o}{r_i}\right)\neq 0,\\
\langle\!\langle \boldsymbol{K},\boldsymbol{Q}_2\rangle\!\rangle_{L^2}&=\langle\!\langle \boldsymbol{K}_C,(\boldsymbol{Q}_2)|_{\mathcal{B}_C}\rangle\!\rangle_{L^2}=\frac{\pi}{2}\ln\left(\frac{r_o}{r_i}\right)\neq 0,
\end{aligned}$$

where r_i and r_o are, respectively, the radii of the inner and outer boundary circles. Therefore, $\boldsymbol{K}$ does not satisfy (2.10d) and is not compatible. In fact, the L^2-orthogonal decomposition of $\boldsymbol{\varkappa}$-free tensor fields discussed in [4, Sect. 3.1] implies that $\boldsymbol{K}$ can be decomposed as

$$\boldsymbol{K}=\frac{1}{2}\boldsymbol{Q}_1+\frac{1}{4}\boldsymbol{Q}_2+\mathbf{grad}\,\boldsymbol{Y}.$$

The condition (2.10d) tells us that the harmonic component $\frac{1}{2}\boldsymbol{Q}_1+\frac{1}{4}\boldsymbol{Q}_2$ is the obstruction to the compatibility of $\boldsymbol{K}$.

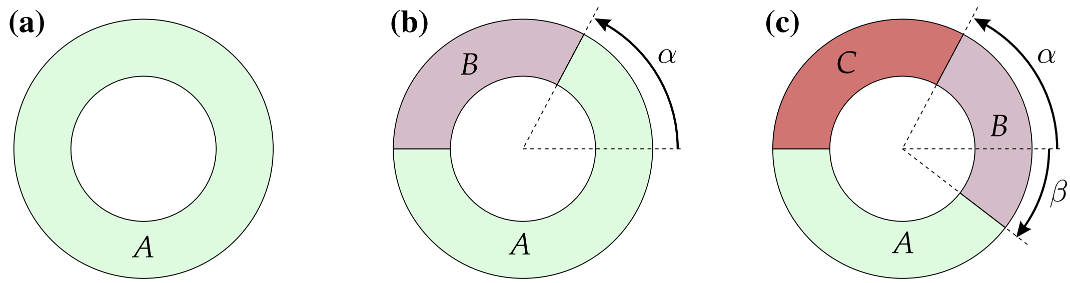


Fig. 3 Some angular arrangements of the phases A, B, and C around a hole. If the displacement gradient of each phase is constant, then the Hadamard jump condition is the necessary and sufficient compatibility condition for all these three arrangements

Solid–solid phase transformations such as martensitic transformations near transformation temperatures can result in complicated arrangements of phases. Each phase in martensitic transformations has a constant deformation gradient called the transformation strain [8]. In the following examples, we study the compatibility of different arrangements of phases with constant displacement gradients around a hole. Phases are indicated by capital letters A, B, etc., and the associated constant displacement gradients belong to $\mathbb{R}^{2\times 2}$ and are denoted by the corresponding bold letters $\mathbf{A}$, $\mathbf{B}$, etc.

Example 7 In this example, we study the angular arrangements of phases around a hole shown in Fig. 3 as follows:

- *Single-phase*: Consider the single-phase annulus depicted in Fig. 3a. Clearly, the conditions (2.10a)–(2.10c) are all trivial in this case. Let A_{ij}, $i, j = 1, 2$, denote the components of $\mathbf{A}$. Using the basis $\{\boldsymbol{Q}_1, \boldsymbol{Q}_2\}$ for $\mathcal{H}^{\otimes}(\mathcal{B}, \varnothing, \partial\mathcal{B})$ introduced in (3.1) and the fact that $\int_{\mathcal{B}} h\, \mathrm{d}A = \int_{\mathcal{B}} f\, \mathrm{d}A = 0$, one concludes that
$$\langle\!\langle \mathbf{A}, \boldsymbol{Q}_i \rangle\!\rangle_{L^2} = -A_{i1} \int_{\mathcal{B}} h\, \mathrm{d}A + A_{i2} \int_{\mathcal{B}} f\, \mathrm{d}A = 0, \quad i = 1, 2,$$
and therefore, (2.10d) holds as well and the single-phase case is always compatible.
- *Two-phase*: Next, consider the two-phase arrangement shown in Fig. 3b, with $0 \le \alpha < \pi$. The Hadamard jump condition tells us that
$$\begin{cases} A_{i1} = B_{i1}, \ i = 1, 2, & \text{if } \alpha = 0, \\ \mathbf{A} = \mathbf{B}, & \text{if } 0 < \alpha < \pi. \end{cases}$$
The condition (2.10d) is equivalent to
$$\sum_{\Upsilon = A, B} \langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{Q}_i|_{\mathcal{B}_\Upsilon} \rangle\!\rangle_{L^2} = (r_o - r_i)\big\{ (A_{i1} - B_{i1})(1 + \cos\alpha) + (A_{i2} - B_{i2}) \sin\alpha \big\} = 0, \quad i = 1, 2.$$
Therefore, if the Hadamard jump condition holds so does the condition (2.10d). Consequently, one concludes that the two-phase case with $\mathbf{A} \neq \mathbf{B}$ is always incompatible if $0 < \alpha < \pi$, and if $\alpha = 0$, it is compatible if and only if $A_{i1} = B_{i1}$, $i = 1, 2$.
- *Three-phase*: Finally, consider the three-phase arrangement depicted in Fig. 3c, with $0 \le \alpha < \pi$ and $0 < \beta < \pi$. The Hadamard jump condition implies that
$$\left.\begin{array}{l} A_{i1} - C_{i1} = 0, \\ (A_{i1} - B_{i1}) \cos\beta - (A_{i2} - B_{i2}) \sin\beta = 0, \\ (B_{i1} - C_{i1}) \cos\alpha + (B_{i2} - C_{i2}) \sin\alpha = 0, \end{array}\right\} \quad i = 1, 2. \tag{3.2}$$
On the other hand, (2.10d) can be written as
$$\sum_{\Upsilon = A, B, C} \langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{Q}_i|_{\mathcal{B}_\Upsilon} \rangle\!\rangle_{L^2} = (r_o - r_i)\Big\{ (A_{i1} - C_{i1}) + (A_{i1} - B_{i1}) \cos\beta - (A_{i2} - B_{i2}) \sin\beta$$
$$+ (B_{i1} - C_{i1}) \cos\alpha + (B_{i2} - C_{i2}) \sin\alpha \Big\} = 0, \quad i = 1, 2.$$
Thus, if the Hadamard jump condition holds so does (2.10d). Therefore, the three-phase arrangement is compatible if and only if the Hadamard jump conditions (3.2) hold.

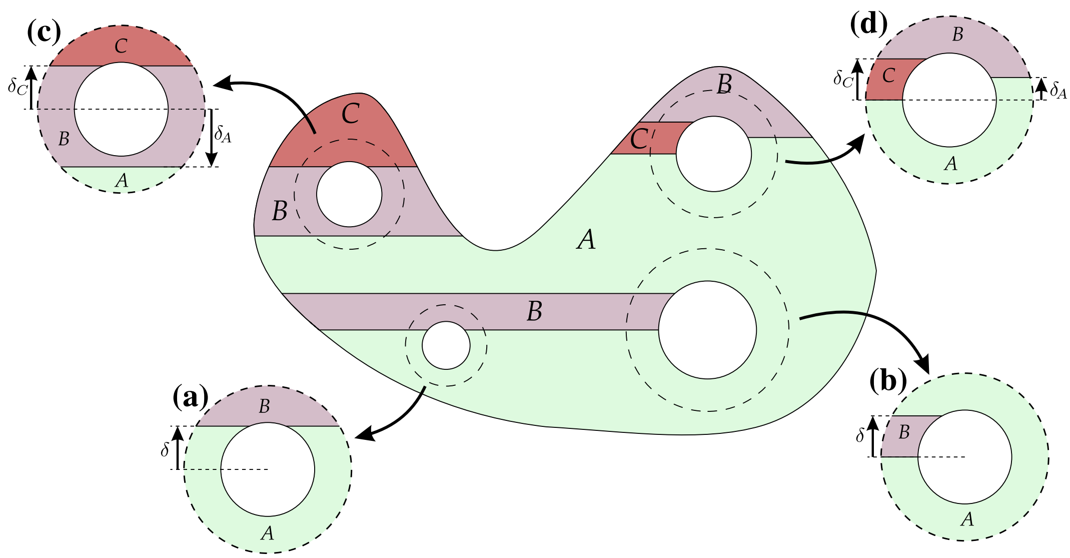


Fig. 4 Holes in a laminated arrangement of three phases A, B, and C. If the displacement gradient of each phase is constant, then the Hadamard jump condition is the necessary and sufficient condition for the compatibility of the symmetric cases (**a**) and (**c**). However, the Hadamard jump condition is not a sufficient compatibility condition for the asymmetric cases (**b**) and (**d**)

Note that for all the angular arrangements of Fig. 3, the condition (2.10d) does not give any additional compatibility equations although it is non-trivial in the sense that $\dim \mathcal{H}^{\otimes}(\mathcal{B}, \varnothing, \partial\mathcal{B}) \neq 0$.

Example 8 In the following, we study the local compatibility of holes in the three-phase laminated arrangement of Fig. 4.

- *Symmetric two-phase*: For the symmetric two-phase arrangement of Fig. 4a with $0 \leq \delta$, the Hadamard jump condition states that $A_{i1} = B_{i1}$, $i = 1, 2$. The symmetry of the phases with respect to the Y-axis allows one to write the condition (2.10d) as

$$\sum_{\Upsilon=A,B} \langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{Q}_i|_{\mathcal{B}_\Upsilon} \rangle\!\rangle_{L^2} = (A_{i1} - B_{i1}) \int_{\mathcal{B}_B} h \, \mathrm{d}A = 0, \quad i = 1, 2.$$

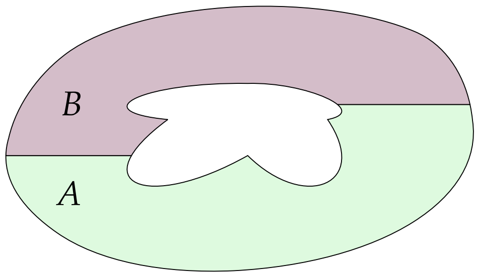
  Therefore, the Hadamard jump condition is the necessary and sufficient condition for the compatibility of the two-phase arrangement of Fig. 4a.
- *Asymmetric two-phase*: Next, consider the asymmetric two-phase arrangement of Fig. 4b, where $0 < \delta \leq r_i$. The Hadamard jump condition states that $A_{i1} = B_{i1}$, $i = 1, 2$, and the condition (2.10d) reads

$$\sum_{\Upsilon=A,B} \langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{Q}_i|_{\mathcal{B}_\Upsilon} \rangle\!\rangle_{L^2} = (A_{i1} - B_{i1}) \int_{\mathcal{B}_B} h \, \mathrm{d}A - (A_{i2} - B_{i2}) \int_{\mathcal{B}_B} f \, \mathrm{d}A = 0, \quad i = 1, 2,$$

  where $\int_{\mathcal{B}_B} h \, \mathrm{d}A$ and $\int_{\mathcal{B}_B} f \, \mathrm{d}A$ are not zero. The above conditions suggest that the asymmetric two-phase arrangement is not compatible if $\mathbf{A} \neq \mathbf{B}$. Note that this conclusion is similar to that of the case in Fig. 3b. However, for the laminated arrangement, the condition (2.10d) and the Hadamard jump condition are independent.
- *Symmetric three-phase*: For the symmetric three-phase arrangement of Fig. 4c, with $0 \leq \delta_A$ and $0 < \delta_C$, it is straightforward to show that the Hadamard jump condition

$$A_{i1} = B_{i1} = C_{i1}, \quad i = 1, 2, \tag{3.3}$$

  is the necessary and sufficient compatibility condition.



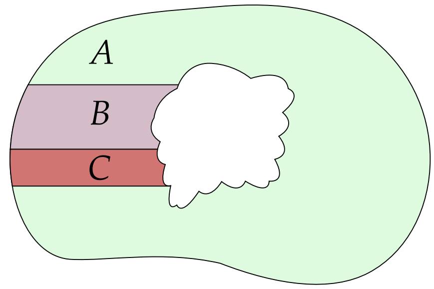


Fig. 5 Laminated arrangements of two and three phases around arbitrary holes

- *Asymmetric three-phase*: For the asymmetric three-phase arrangement shown in Fig. 4d with $0 \leq \delta_A \leq r_i$ and $0 < \delta_C \leq r_i$, the Hadamard jump condition is similar to (3.3) and the condition (2.10d) can be written as

$$\sum_{\Upsilon=A,B,C} \langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{Q}_i|_{\mathcal{B}_\Upsilon} \rangle\!\rangle_{L^2} = (B_{i1} - A_{i1}) \int_{\mathcal{B}_A} h \, \mathrm{d}A + (B_{i1} - C_{i1}) \int_{\mathcal{B}_C} h \, \mathrm{d}A$$
$$+ \left(\ln \frac{r_o}{r_i} \right) \Big\{ (A_{i2} - B_{i2})\delta_A - (C_{i2} - B_{i2})\delta_C \Big\} = 0, \quad i = 1, 2.$$

Therefore, the necessary and sufficient conditions for the compatibility of the three-phase configuration of Fig. 4d are given by

$$\left.\begin{array}{l} A_{i1} = B_{i1} = C_{i1}, \\ (A_{i2} - B_{i2})\delta_A = (C_{i2} - B_{i2})\delta_C, \end{array}\right\} \quad i = 1, 2. \tag{3.4}$$

Note that in this case, similar to the asymmetric case of Fig. 4b, the Hadamard jump condition and the condition (2.10d) are independent. It is seen that the Hadamard jump condition is not sufficient for compatibility.

Example 9 Consider an arbitrary domain containing a hole with an arbitrary shape such as the ones depicted in Fig. 5. For such domains, the harmonic space $\mathcal{H}^{\otimes}(\mathcal{B}, \varnothing, \partial\mathcal{B})$ is two-dimensional and admits a basis $\{\boldsymbol{H}_1, \boldsymbol{H}_2\}$, where

$$\boldsymbol{H}_1(X, Y) = \begin{bmatrix} p(X, Y) & q(X, Y) \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{H}_2(X, Y) = \begin{bmatrix} 0 & 0 \\ p(X, Y) & q(X, Y) \end{bmatrix}.$$

Of course, the explicit forms of the functions p and q depend on the specific shape of $\mathcal{B}$. Nonetheless, one can show that

$$\int_{\mathcal{B}} p \, \mathrm{d}A = \int_{\mathcal{B}} q \, \mathrm{d}A = 0. \tag{3.5}$$

To prove this, let $\boldsymbol{M}_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and $\boldsymbol{M}_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Note that $\boldsymbol{M}_1 = \mathbf{grad} \left[\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right]$, and $\boldsymbol{M}_2 = \mathbf{grad} \left[\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right]$, and since the image of **grad** is L^2-orthogonal to $\mathcal{H}^{\otimes}(\mathcal{B}, \varnothing, \partial\mathcal{B})$ (see [4, Sect. 3.1]), one can write $\langle\!\langle \boldsymbol{H}_i, \boldsymbol{M}_i \rangle\!\rangle_{L^2} = 0, i = 1, 2$, which is equivalent to (3.5). In the following, we use (3.5) to study the compatibility of the configurations of Fig. 5.

- *Two-phase*: For the laminated two-phase arrangement of Fig. 5, the Hadamard jump condition states that $A_{i1} = B_{i1}$, $i = 1, 2$. By using (3.5) and the fact that $\int_{\mathcal{B}} p \, \mathrm{d}A = \int_{\mathcal{B}_A} p \, \mathrm{d}A + \int_{\mathcal{B}_B} p \, \mathrm{d}A$, one can write the condition (2.10d) as

$$(A_{i1} - B_{i1}) \int_{\mathcal{B}_A} p \, \mathrm{d}A + (A_{i2} - B_{i2}) \int_{\mathcal{B}_A} q \, \mathrm{d}A = 0, \ i = 1, 2.$$

Therefore, the compatibility equations are

$$\left.\begin{array}{l} A_{i1} = B_{i1}, \\ (A_{i2} - B_{i2}) \int_{\mathcal{B}_A} q \, \mathrm{d}A = 0, \end{array}\right\} \quad i = 1, 2. \tag{3.6}$$

If $\int_{\mathcal{B}_A} q \, \mathrm{d}A = 0$ such as the symmetric case of Fig. 4(a), then the Hadamard jump condition is the necessary and sufficient compatibility equation. If $\int_{\mathcal{B}_A} q \, \mathrm{d}A \neq 0$ such as the asymmetric case of Fig. 4b, then the compatibility equations imply that $\mathbf{A} = \mathbf{B}$.

- *Three-phase*: For the laminated three-phase arrangement of Fig. 5, the Hadamard jump condition is similar to (3.3) and the relation $\int_{\mathcal{B}_A} p\,\mathrm{d}A + \int_{\mathcal{B}_B} p\,\mathrm{d}A + \int_{\mathcal{B}_C} p\,\mathrm{d}A = 0$ allows one to write (2.10d) as

$$(B_{i1} - A_{i1}) \int_{\mathcal{B}_B} p\,\mathrm{d}A + (C_{i1} - A_{i1}) \int_{\mathcal{B}_C} p\,\mathrm{d}A$$
$$+ (B_{i2} - A_{i2}) \int_{\mathcal{B}_B} q\,\mathrm{d}A + (C_{i2} - A_{i2}) \int_{\mathcal{B}_C} q\,\mathrm{d}A = 0, \; i = 1, 2.$$

Thus, the compatibility equations read

$$\left.\begin{array}{l} A_{i1} = B_{i1} = C_{i1}, \\ (B_{i2} - A_{i2}) \int_{\mathcal{B}_B} q\,\mathrm{d}A + (C_{i2} - A_{i2}) \int_{\mathcal{B}_C} q\,\mathrm{d}A = 0, \end{array}\right\} \quad i = 1, 2. \tag{3.7}$$

If $\int_{\mathcal{B}_B} q\,\mathrm{d}A = \int_{\mathcal{B}_C} q\,\mathrm{d}A = 0$ such as the symmetric arrangement of Fig. 4c, then the Hadamard jump condition is the necessary and sufficient compatibility equation. If $\int_{\mathcal{B}_B} q\,\mathrm{d}A$ and $\int_{\mathcal{B}_C} q\,\mathrm{d}A$ do not vanish, then one should also consider the second compatibility equation in (3.7). The explicit form of this equation depends on the specific geometries of a domain and its different phases.

Note that the relations (3.6) and (3.7) hold for any two-phase and three-phase arrangements of phases regardless of the geometries of phases. However, the Hadamard jump condition and consequently the resulting compatibility equations depend on the specific arrangement of phases with respect to each other. Using arguments similar to those of this example, it is possible to study the compatibility of any arrangement of phases around a hole. However, to obtain the explicit form of the compatibility equations, one also needs the explicit form of a basis of the space of harmonic tensors.

Remark 10 In the case of smooth deformation gradients in a body $\mathcal{B}$, the necessary and sufficient compatibility equations can be stated as

$$\mathbf{curl}^{\mathsf{T}}\mathbf{F} = \mathbf{0}, \quad \int_{c_i} \mathbf{F} \cdot d\mathbf{X} = \mathbf{0}, \quad i = 1, \ldots, \beta_1(\mathcal{B}), \tag{3.8}$$

where $\{c_1, \ldots, c_{\beta_1(\mathcal{B})}\}$ is a set of generators of the first homology group $H_1(\mathcal{B})$ [10]. By using arguments similar to those of [11, Theorem 3], one can show that if $\boldsymbol{F}$ is of C^1-class, then the conditions $(3.8)_2$ and (2.10d) (with $\mathcal{S}_1 = \varnothing$) are equivalent. However, (2.10d) is much more general than $(3.8)_2$ in the sense that the condition $(3.8)_2$ is meaningless for arbitrary L^2 deformation gradients, since the restriction of an L^2 mapping to a 1D curve is not well-defined, in general. One difficulty in using (2.10d) is the calculation of harmonic tensors. This difficulty is not present when using $(3.8)_2$. In this remark, without providing a rigorous proof, we investigate the equivalence of $(3.8)_2$ and (2.10d) for piecewise constant deformation gradients.

Suppose $\mathcal{B}$ contains a hole and also consists of n phases with constant deformation gradients $\mathbf{F}^{(i)}$, $i = 1, \ldots, n$. In this case, any curve γ enclosing the hole is a generator of the first homology group. One then has $\int_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{X} = \sum_{i=1}^n \int_{\gamma_i} \mathbf{F}^{(i)} \cdot \mathrm{d}\mathbf{X}$, where $\gamma = \bigcup_{i=1}^n \gamma_i$, and γ_i lies entirely in $\bar{\mathcal{B}}_i$. It is straightforward to see that each path integral makes sense when the Hadamard jump condition is satisfied and that $\int_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{X}$ is independent of γ up to homology classes. Therefore, the integral on the left side of $(3.8)_2$ is well-defined for piecewise constant deformation gradients. Note that the above discussion does not imply that (3.8) is sufficient for the compatibility of piecewise constant deformation gradients. We conjuncture that (3.8) together with the Hadamard jump condition are also sufficient for the compatibility of piecewise C^1 deformation gradients.

For all the previous examples, it is straightforward to check that the compatibility equations resulting from (2.10d) and $(3.8)_2$ are identical. One advantage of $(3.8)_2$ is that it is easier to use for holes with arbitrary shapes. For example, we calculate $(3.8)_2$ for triangular and square holes.

- The first example is a rhomboidal hole surrounded by four phases with deformation gradients $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ (see Fig. 6a). Denoting the unit vectors in the X and Y directions by $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively, the Hadamard condition implies that

$$(\mathbf{A} - \mathbf{B}) \cdot \mathbf{E}_2 = (\mathbf{B} - \mathbf{C}) \cdot \mathbf{E}_1 = (\mathbf{C} - \mathbf{D}) \cdot \mathbf{E}_2 = (\mathbf{D} - \mathbf{A}) \cdot \mathbf{E}_1 = \mathbf{0}. \tag{3.9}$$

The condition $(3.8)_2$ for the path shown in Fig. 6a implies that

$$\mathbf{A} \cdot (-\mathbf{E}_1 + \mathbf{E}_2) + \mathbf{B} \cdot (-\mathbf{E}_1 - \mathbf{E}_2) + \mathbf{C} \cdot (\mathbf{E}_1 - \mathbf{E}_2) + \mathbf{D} \cdot (\mathbf{E}_1 + \mathbf{E}_2) = \mathbf{0}. \tag{3.10}$$

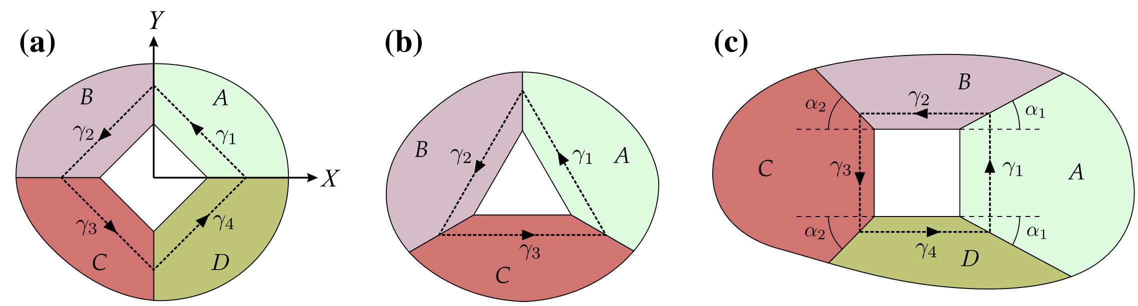


Fig. 6 Some arrangements of the phases A, B, C, and D around square and triangular holes. The condition $(3.8)_2$ suggests that the Hadamard jump condition is the necessary and sufficient compatibility condition for (**a**) and (**b**), while it is only a necessary compatibility condition for the case (**c**)

Note that (3.10) follows from (3.9), i.e., in this example, the condition $(3.8)_2$ suggests that the Hadamard condition is a sufficient compatibility condition.

- The second example is an equilateral triangular hole surrounded by three phases with deformation gradients **A**, **B**, and **C** (see Fig. 6b). The Hadamard jump condition implies that

$$(\mathbf{A}-\mathbf{B})\cdot\mathbf{E}_2=(\mathbf{B}-\mathbf{C})\cdot\left(\frac{\sqrt{3}}{2}\mathbf{E}_1+\frac{1}{2}\mathbf{E}_2\right)=(\mathbf{C}-\mathbf{A})\cdot\left(\frac{\sqrt{3}}{2}\mathbf{E}_1-\frac{1}{2}\mathbf{E}_2\right)=\mathbf{0}. \tag{3.11}$$

The condition $(3.8)_2$ for the path shown in Fig. 6b implies that

$$\mathbf{A}\cdot\left(-\frac{1}{2}\mathbf{E}_1+\frac{\sqrt{3}}{2}\mathbf{E}_2\right)+\mathbf{B}\cdot\left(-\frac{1}{2}\mathbf{E}_1-\frac{\sqrt{3}}{2}\mathbf{E}_2\right)+\mathbf{C}\cdot\mathbf{E}_1=\mathbf{0}. \tag{3.12}$$

Note that (3.12) follows from (3.11), i.e., in this example, the condition $(3.8)_2$ suggests that the Hadamard condition is a sufficient compatibility condition.

- In the third example, we generalize the first example in the following sense: Consider a square hole surrounded by four phases with uniform deformation gradients **A**, **B**, **C**, and **D** (see Fig. 6c). The Hadamard condition implies that

$$\begin{aligned}&(\mathbf{A}-\mathbf{B})\cdot(\cos\alpha_1\mathbf{E}_1+\sin\alpha_1\mathbf{E}_2)=(\mathbf{D}-\mathbf{A})\cdot(\cos\alpha_1\mathbf{E}_1-\sin\alpha_1\mathbf{E}_2)=\mathbf{0},\\&(\mathbf{B}-\mathbf{C})\cdot(-\cos\alpha_2\mathbf{E}_1+\sin\alpha_2\mathbf{E}_2)=(\mathbf{C}-\mathbf{D})\cdot(\cos\alpha_2\mathbf{E}_1+\sin\alpha_2\mathbf{E}_2)=\mathbf{0}.\end{aligned} \tag{3.13}$$

Let us assume that $0<\alpha_1,\alpha_2<\pi/2$. The condition $(3.8)_2$ for the path shown in Fig. 6c implies that

$$\ell_2\,(\mathbf{D}-\mathbf{B})\cdot\mathbf{E}_1+\ell_1\,(\mathbf{A}-\mathbf{C})\cdot\mathbf{E}_2=\mathbf{0}, \tag{3.14}$$

where $\ell_1=|\gamma_1|=|\gamma_3|$ and $\ell_2=|\gamma_2|=|\gamma_4|$. As the path integral in $(3.8)_2$ is path independent after enforcing the Hadamard condition, one can shrink the path γ to the hole and in this limit $\ell_1=\ell_2$. Therefore,

$$(\mathbf{D}-\mathbf{B})\cdot\mathbf{E}_1+(\mathbf{A}-\mathbf{C})\cdot\mathbf{E}_2=\mathbf{0}. \tag{3.15}$$

Note that (3.15) is, in general, independent from (3.13). An exception is when $\alpha_1=\alpha_2=\pi/4$, which was studied in the first example. In this case, Hadamard condition is the sufficient compatibility equation. However, in general, even when $\alpha_1=\alpha_2$, the condition $(3.8)_2$ suggests that the Hadamard jump condition is not sufficient for compatibility.

Thus, we observe that checking $(3.8)_2$ can be easier than (2.10d). Here, we should emphasize again that we have not proven that (3.8) is sufficient for the compatibility of piecewise constant deformation gradients.

4 Discussion

We derived the weak compatibility equations for L^2 displacement gradients (or equivalently, L^2 deformation gradients) on strongly Lipschitz bodies. In Theorem 4, we wrote an alternative expression for the weak compatibility equations that are more useful for multi-phase bodies. As an application of the weak compatibility equations, we studied the compatibility of some 2D configurations of different phases around a hole. We showed that in the presence of holes, the classical Hadamard jump condition is no longer a sufficient compatibility condition, in general; one needs to consider the condition (2.10d) as well. This condition depends on the topology of bodies and is non-trivial for non-simply connected bodies.

For angular arrangements of phases with constant deformation gradients around a circular hole, we showed that although the condition (2.10d) is not trivial, it follows from the Hadamard jump condition, and therefore, the latter is still the necessary and sufficient compatibility condition even in the presence of a hole. However, for asymmetric arrangements of phases around a hole such as some laminated arrangements around a circular hole and a four-phase arrangement around a square hole, we showed that the condition (2.10d) results in compatibility equations that are independent from the Hadamard condition. In particular, we derived the compatibility equations (3.4) for asymmetric laminated arrangement of Fig. 4 and the compatibility equations (3.13) and (3.15) for the four-phase arrangement of Fig. 6c. These equations provide a relation between the geometry and the deformation gradients of compatible arrangements of phases.

Finally, we should mention that including boundary conditions in the compatibility equations of Theorem 4 can be useful for studying compatibility in the presence of *rigid* inclusions. For rigid inclusions, one can assume that the displacement is zero on $\mathcal{S}_1$, where $\mathcal{S}_1$ is the boundary of the inclusions. Note that if $\mathcal{S}_1 \neq \varnothing$, then the condition (2.10c) is non-trivial. Also note that the condition (2.10d) depends on the topological properties of both $\mathcal{B}$ and $\mathcal{S}_1$. For example, if $\mathcal{B}$ is an annulus, then $\dim \mathcal{H}^{\otimes}(\mathcal{B}, \varnothing, \partial\mathcal{B}) = 2$. If $\mathcal{S}_1$ is chosen to be the inner circle of $\partial\mathcal{B}$, then $\dim \mathcal{H}^{\otimes}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) = 0$, and therefore, the condition (2.10d) is trivial although $\mathcal{B}$ is not simply connected.

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