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Covariantization of nonlinear elasticity

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Abstract. In this paper we make a connection between covariant elasticity based on covariance of energy balance and Lagrangian field theory of elasticity with two background metrics. We use Kuchař's idea of reparametrization of field theories and make elasticity generally covariant by introducing a "covariance field", which is a time-independent spatial diffeomorphism. We define a modified action for parameterized elasticity and show that the Doyle-Ericksen formula and spatial homogeneity of the Lagrangian density are among its Euler–Lagrange equations.

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1. Introduction

In the geometric field theory of classical elasticity [6,9,10], one introduces two background metric fields, one for the material manifold and one for the ambient space manifold. In the classical theory of nonlinear elasticity and in the absence of defects, these background metrics are given geometric objects with no dynamics, and in this sense, not all fields are on the same footing. These metrics are "absolute" in the sense of Anderson [1] and "structural fields" in the sense of Post [8]. It should also be emphasized that the material and ambient space manifolds are, in general, genuinely different. We should mention that there are concrete examples for which the material metric is a dynamic field, for example, in geometric formulation of growth mechanics [11], thermoelasticity [7], and dislocation mechanics [12].

There are two parallel approaches for geometric formulation of elasticity with no clear explicit connection between them. These are the following: (1) Postulating energy balance and its invariance under arbitrary time-dependent spatial diffeomorphisms (covariance) give all the known balance laws of elasticity and the Doyle-Ericksen formula all covariantly. (2) Lagrangian field theory of elasticity can be formulated geometrically. Here one assumes existence of a Lagrangian density that, in addition to the standard fields, explicitly depends on the metrics of the material and ambient space manifolds. Hamilton's principle of least action then gives the Euler–Lagrange (EL) equations. The only known connection between these two approaches is through Noether's theorem [10]; covariance of Lagrangian density results in the Doyle-Ericksen formula and homogeneity of the Lagrangian density in both spatial and material settings.

There have been attempts in the literature in making field theories with background metrics generally covariant [2,3]. Recently, [4] and [5] extended Kuchař's idea to multisymplectic field theories. The basic idea is to consider two separate copies of space-time, one with a fixed metric and one with a pulled-back metric induced by a diffeomorphism η between the two copies of the space-time. The diffeomorphism η is considered a field by which the space-time metric is reduced to a mere geometric object. Then, Hamilton's action principle for a modified action with η as an extra field gives the standard EL equations, and some vacuous EL equations stating that the stress-energy-momentum tensor is divergence free. In other words, the covariance field η makes the resulting field theory generally covariant and has vacuous EL equations. Our motivation in this paper is to make a connection between covariant balance laws resulting from covariance of energy balance and the Lagrangian field theory of elasticity. We generalize Kuchař's parametrization idea to elasticity and show how one can make elasticity spatially covariant.

This paper is structured as follows. In Sect. 2, we first briefly review geometric elasticity, the Lagrangian field theory of elasticity, covariance of energy balance, and the role of background metric. In Sect. 3, following Kuchař's idea of parametrization of field theories, we parametrize elasticity by introducing a "covariance field" that makes the background metric dynamic and obtain its EL equations. Conclusions are given in Sect. 4.

2. The background metric in geometric elasticity

Let us assume that reference configuration is a Riemannian manifold $(\mathcal{B},\mathbf{G})$ and that the body deforms in a Riemannian ambient space $(\mathcal{S},\mathbf{g})$. Motion is a one-parameter family of maps $\varphi_t:\mathcal{B}\to\mathcal{S}$, where t is time. Let us denote local coordinates on $\mathcal{B}$ and $\mathcal{S}$ by $\{X^A\}$ and $\{x^a\}$, respectively. For a fixed $t,\varphi_t(\mathbf{X})=\varphi(\mathbf{X},t)$, where $\mathbf{X}$ is position of material points in the undeformed configuration $\mathcal{B}$. The material velocity is the map $\mathbf{V}_t:\mathcal{B}\to T_{\varphi(\mathbf{X})}\mathcal{S}$ given by $\mathbf{V}_t(\mathbf{X})=\frac{\partial\varphi(\mathbf{X},t)}{\partial t}$. The material acceleration is defined by $\mathbf{A}_t(\mathbf{X})=\frac{\partial\mathbf{V}_t(\mathbf{X})}{\partial t}$. In components $A^a=\frac{\partial V^a}{\partial t}+\gamma^a_{bc}V^bV^c$, where γ^a_{bc} is the Christoffel symbol of the local coordinate chart $\{x^a\}$. Deformation gradient is the tangent map of φ and is denoted by $\mathbf{F}=T\varphi$. Thus, at each point $\mathbf{X}\in\mathcal{B}$, it is a linear map

$$\mathbf{F}(\mathbf{X}):T_{\mathbf{X}}\mathcal{B}\to T_{\varphi(\mathbf{X})}\mathcal{S}. \tag{2.1}$$

Components of $\mathbf{F}$ are $F^a{}_A(\mathbf{X})=\frac{\partial\varphi^a}{\partial X^A}(\mathbf{X})$. Suppose $\mathcal{B}$ and $\mathcal{S}$ are Riemannian manifolds with inner products $\langle\!\langle,\rangle\!\rangle_{\mathbf{G}}$ and $\langle\!\langle,\rangle\!\rangle_{\mathbf{g}}$ based at $\mathbf{X}\in\mathcal{B}$ and $\mathbf{x}\in\mathcal{S}$, respectively. Transpose of $\mathbf{F}$ is defined by

$$\mathbf{F}^{\mathsf{T}}:T_{\mathbf{x}}\mathcal{S}\to T_{\mathbf{X}}\mathcal{B},\quad \langle\!\langle\mathbf{FV},\mathbf{v}\rangle\!\rangle_{\mathbf{g}}=\langle\!\langle\mathbf{V},\mathbf{F}^{\mathsf{T}}\mathbf{v}\rangle\!\rangle_{\mathbf{G}}\quad\forall\mathbf{V}\in T_{\mathbf{X}}\mathcal{B},\ \mathbf{v}\in T_{\mathbf{x}}\mathcal{S}. \tag{2.2}$$

In components $(F^{\mathsf{T}}(\mathbf{X}))^A{}_a=g_{ab}(\mathbf{x})F^b{}_B(\mathbf{X})G^{AB}(\mathbf{X})$. The right Cauchy-Green deformation tensor is defined by $\mathbf{C}(\mathbf{X})=\mathbf{F}(\mathbf{X})^{\mathsf{T}}\mathbf{F}(\mathbf{X})$, where $\mathbf{g}$ and $\mathbf{G}$ are metric tensors on $\mathcal{S}$ and $\mathcal{B}$, respectively. In components $C^A_B=(F^{\mathsf{T}})^A{}_aF^a{}_B$. One can show that $\mathbf{C}^\flat=\varphi^*(\mathbf{g})=\mathbf{F}^*\mathbf{g}\mathbf{F}$, that is, $C_{AB}=(g_{ab}\circ\varphi)F^a{}_AF^b{}_B$.

Lagrangian field theory of elasticity. In elasticity one assumes existence of a Lagrangian density $\mathcal{L}$ [6] such that[1]

$$\mathcal{L}=\mathcal{L}(\mathbf{X},\mathbf{G},\varphi,\dot{\varphi},\mathbf{F},\mathbf{g}), \tag{2.3}$$

where $\mathbf{F}=T\varphi$ is the so-called deformation gradient. Action is defined on the material manifold $(\mathcal{B},\mathbf{G})$ as

$$S=\int_{t_0}^{t_1}\int_{\mathcal{B}}\mathcal{L}\,\mathrm{d}V\mathrm{d}t, \tag{2.4}$$

where $\mathrm{d}V=\mathrm{d}V(\mathbf{X})$ is the Riemannian volume element on $\mathcal{B}$. Hamilton's principle of least action states that $\delta S=\mathbf{d}S\cdot\delta\varphi=0$. This gives the following Euler–Lagrange equations that are equivalent to balance of linear momentum [10].

$$\frac{\partial\mathcal{L}}{\partial\varphi^a}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}^a}-\left(\frac{\partial\mathcal{L}}{\partial F^a{}_A}\right)_{|A}-\frac{\partial\mathcal{L}}{\partial F^b{}_A}F^c{}_A\gamma^b_{ac}+2\frac{\partial\mathcal{L}}{\partial g_{cd}}\,g_{bd}\gamma^b_{ac}=0. \tag{2.5}$$

When non-conservative forces are present the governing equations can be obtained using the Lagrange-d'Alembert principle. Denoting the non-conservative force by $\mathbf{f}$, the above EL equations are modified to

[1] To make a scalar out of the vector field $\dot{\varphi}$ and the two-point tensor $\mathbf{F},\mathcal{L}$ has to explicitly depend on both $\mathbf{G}$ and $\mathbf{g}$.

read

$$\frac{\partial \mathcal{L}}{\partial \varphi^a} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}^a} - \left(\frac{\partial \mathcal{L}}{\partial F^a{}_A}\right)_{|A} - \frac{\partial \mathcal{L}}{\partial F^b{}_A} F^c{}_A \gamma^b_{ac} + 2\frac{\partial \mathcal{L}}{\partial g_{cd}}\, g_{bd}\gamma^b_{ac} + f_a = 0. \tag{2.6}$$

Remark 2.1. *Note that the metric* $\mathbf{g}$ *cannot be variational because if it is assumed that* $\mathbf{g}$ *is variational, then the corresponding EL equation would be* $\frac{\partial \mathcal{L}}{\partial \mathbf{g}} = \mathbf{0}$, *which cannot be the case as was explained in the previous footnote.*

Covariance of energy balance. Another approach in deriving the balance laws of elasticity is to first postulate an energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \mathbf{V}, \mathbf{V}\rangle\!\rangle_{\mathbf{g}}\right) \mathrm{d}V = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \mathbf{B}, \mathbf{V}\rangle\!\rangle_{\mathbf{g}} + R\right) \mathrm{d}V + \int_{\partial\mathcal{U}} \left(\langle\!\langle \mathbf{T}, \mathbf{V}\rangle\!\rangle_{\mathbf{g}} + H\right) \mathrm{d}A, \tag{2.7}$$

where $E = E(\mathbf{X}, \mathsf{N}, \mathbf{G}, \mathbf{F}, \mathbf{g}\circ\varphi)$ is the material internal energy density, $\mathsf{N}, \rho_0, \mathbf{B}, \mathbf{T}, R$, and H are specific entropy, material mass density, body force per unit undeformed mass, traction vector, heat supply, and heat flux, respectively. Then, one postulates that energy balance is covariant, that is, it is invariant under an arbitrary time-dependent spatial change of frame $\xi_t : \mathcal{S} \to \mathcal{S}$, that is, [6]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \rho_0' \left(E' + \frac{1}{2}\langle\!\langle \mathbf{V}', \mathbf{V}'\rangle\!\rangle_{\mathbf{g}'}\right) \mathrm{d}V = \int_{\mathcal{U}} \rho_0' \left(\langle\!\langle \mathbf{B}', \mathbf{V}'\rangle\!\rangle_{\mathbf{g}'} + R'\right) \mathrm{d}V + \int_{\partial\mathcal{U}} \left(\langle\!\langle \mathbf{T}', \mathbf{V}'\rangle\!\rangle_{\mathbf{g}'} + H'\right) \mathrm{d}A. \tag{2.8}$$

It can be shown that the following are necessary and sufficient for covariance of energy balance [10]

$$\frac{\partial \rho_0}{\partial t} = 0, \tag{2.9}$$

$$\mathrm{Div}\,\mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}, \tag{2.10}$$

$$2\rho_0 \frac{\partial E}{\partial \mathbf{g}\circ\varphi} = \boldsymbol{\tau}, \tag{2.11}$$

$$\boldsymbol{\tau}^{\mathsf{T}} = \boldsymbol{\tau}, \tag{2.12}$$

where $\mathbf{P}$ is the first Piola–Kirchhoff stress and $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ is the Kirchhoff stress.

Remark 2.2. *Note that for energy balance to be covariant, in addition to the standard balance laws, a nontrivial relation, that is, the Doyle-Ericksen formula must hold.*

3. Covariantization of elasticity

We consider a time-independent spatial change of frame $\eta : \mathcal{S} \to \mathcal{S}$ as our covariance field (see Fig. 1)[2]. Let us define $\widetilde{\mathcal{L}}$ as

$$\begin{aligned}\widetilde{\mathcal{L}}(\mathbf{X}, \mathbf{G}, \varphi, \dot{\varphi}, T\varphi, \eta, \dot{\eta}, T\eta) &:= \mathcal{L}\left(\mathbf{X}, \mathbf{G}, \eta\circ\varphi, \dot{\overline{\eta\circ\varphi}}, T\eta\cdot T\varphi, \eta_*\mathbf{g}\right)\\ &= \mathcal{L}\left(\mathbf{X}, \mathbf{G}, \eta\circ\varphi, \partial\eta/\partial t + T\eta\cdot\dot{\varphi}, T\eta\cdot T\varphi, \eta_*\mathbf{g}\right).\end{aligned} \tag{3.1}$$

Note that in components

$$\widetilde{g}_{\alpha\beta} := (\eta_*\mathbf{g})_{\alpha\beta} = \frac{\partial x^a}{\partial \eta^\alpha}\frac{\partial x^b}{\partial \eta^\beta}\, g_{ab}\circ\varphi. \tag{3.2}$$

[2] If η is time dependent, one of the Euler–Lagrange equations would be $\partial\mathcal{L}/\partial\dot{\varphi} = 0$, which is not physical unless the problem is static. Note that in Noether's theorem, one considers a time-independent vector field and its flow [6,10].

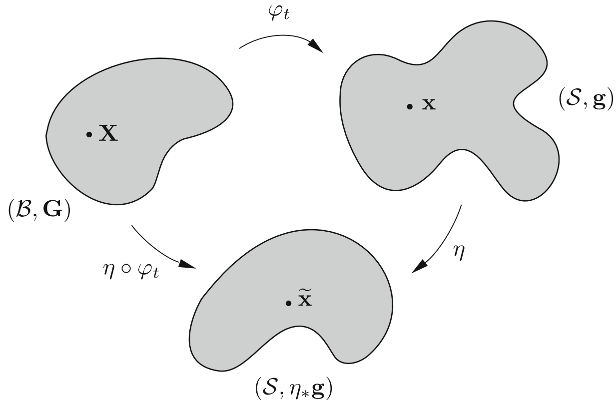

Fig. 1. Covariance field for nonlinear elasticity

A modified action $\widetilde{S}$ is defined as

$$\widetilde{S}=\int_{t_0}^{t_1}\int_{\mathcal{B}}\widetilde{\mathcal{L}}(\mathbf{X},\mathbf{G},\varphi,\dot{\varphi},T\varphi,\eta,\dot{\eta},T\eta)\ \mathrm{d}V\mathrm{d}t. \tag{3.3}$$

Next, we obtain the φ and η-variations of the modified action.

φ-Variation. φ-variation of action is written as

$$\delta_\varphi\widetilde{S}=\int_{t_0}^{t_1}\int_{\mathcal{B}}\left(\frac{\partial\widetilde{\mathcal{L}}}{\partial\varphi^a}\delta\varphi^a+\frac{\partial\widetilde{\mathcal{L}}}{\partial\dot{\varphi}^a}\delta\dot{\varphi}^a+\frac{\partial\widetilde{\mathcal{L}}}{\partial F^a{}_A}\delta F^a{}_A\right)\mathrm{d}V\mathrm{d}t. \tag{3.4}$$

or

$$\delta_\varphi\widetilde{S}=\int_{t_0}^{t_1}\int_{\mathcal{B}}\left[\frac{\partial\widetilde{\mathcal{L}}}{\partial\varphi^a}-\frac{d}{dt}\left(\frac{\partial\widetilde{\mathcal{L}}}{\partial\dot{\varphi}^a}\right)-\left(\frac{\partial\widetilde{\mathcal{L}}}{\partial F^a{}_A}\right)_{|A}-\frac{\partial\widetilde{\mathcal{L}}}{\partial F^c{}_A}F^b{}_A\gamma^c_{ab}\right]\delta\varphi^a\ \mathrm{d}V\mathrm{d}t. \tag{3.5}$$

It can be shown that

$$\frac{\partial\widetilde{\mathcal{L}}}{\partial\varphi^a}=\frac{\partial\mathcal{L}}{\partial\widetilde{\varphi}^\alpha}\frac{\partial\eta^\alpha}{\partial x^a}+2\frac{\partial\mathcal{L}}{\partial\widetilde{g}_{\alpha\beta}}\widetilde{g}_{\beta\mu}\frac{\partial x^m}{\partial\eta^\alpha}\frac{\partial\eta^\mu}{\partial x^d}\gamma^d_{am}, \tag{3.6}$$

$$\frac{\partial\widetilde{\mathcal{L}}}{\partial\dot{\varphi}^a}=\frac{\partial\mathcal{L}}{\partial\dot{\widetilde{\varphi}}^\alpha}\frac{\partial\eta^\alpha}{\partial x^a}, \tag{3.7}$$

$$\frac{\partial\widetilde{\mathcal{L}}}{\partial F^a{}_A}=\frac{\partial\mathcal{L}}{\partial\widetilde{F}^\alpha{}_A}\frac{\partial\eta^\alpha}{\partial x^a}. \tag{3.8}$$

Also

$$\left(\frac{\partial\widetilde{\mathcal{L}}}{\partial F^a{}_A}\right)_{|A}=\left(\frac{\partial\mathcal{L}}{\partial\widetilde{F}^\alpha{}_A}\right)_{|A}\frac{\partial\eta^\alpha}{\partial x^a}. \tag{3.9}$$

We know that the connection coefficients are transformed as follows

$$\gamma^c_{ab}=\frac{\partial x^c}{\partial\eta^\mu}\frac{\partial\eta^\alpha}{\partial x^a}\frac{\partial\eta^\beta}{\partial x^b}\widetilde{\gamma}^\mu_{\alpha\beta}+\frac{\partial^2\eta^\lambda}{\partial x^a\partial x^b}\frac{\partial x^c}{\partial\eta^\lambda}. \tag{3.10}$$

After some lengthy calculations, it can be shown that

$$\delta_\varphi \widetilde{S} = \int_{t_0}^{t_1} \int_{\mathcal{B}} \left\{ \left[\frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^\alpha} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^\alpha} \right) - \left(\frac{\partial \mathcal{L}}{\partial \widetilde{F}^\alpha{}_A} \right)_{|A} - \frac{\partial \mathcal{L}}{\partial \widetilde{F}^\mu{}_A} \widetilde{F}^\beta{}_A \widetilde{\gamma}^\mu_{\alpha\beta} + 2\frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\beta\lambda}} \widetilde{g}^{\beta\mu} \widetilde{\gamma}^\mu_{\alpha\lambda} \right] \frac{\partial \eta^\alpha}{\partial x^a} \delta\varphi^a \right.$$
$$\left. + \left(2\frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\beta\mu}} \widetilde{g}_{\alpha\mu} - \frac{\partial \mathcal{L}}{\partial \widetilde{F}^\alpha{}_A} \widetilde{F}^\beta{}_A \right) \frac{\partial^2 \eta^\alpha}{\partial x^a \partial x^b} \frac{\partial x^b}{\partial \eta^\beta} \delta\varphi^a - \frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^\alpha} \frac{d}{dt} \frac{\partial \eta^\alpha}{\partial x^a} \right\} \mathrm{d}V \mathrm{d}t. \tag{3.11}$$

Note that

$$\frac{d}{dt}\frac{\partial \eta^\alpha}{\partial x^a} = \frac{\partial^2 \eta^\alpha}{\partial x^a \partial x^b} \dot{\varphi}^b. \tag{3.12}$$

Also note that

$$\dot{\widetilde{\varphi}}^{\,\beta} = \frac{\partial \eta^\beta}{\partial x^b}\ \dot{\varphi}^b. \tag{3.13}$$

Hence

$$\dot{\varphi}^b = \frac{\partial x^b}{\partial \eta^\beta} \dot{\widetilde{\varphi}}^{\,\beta}. \tag{3.14}$$

Therefore

$$\frac{d}{dt}\frac{\partial \eta^\alpha}{\partial x^a} = \dot{\widetilde{\varphi}}^{\,\beta} \frac{\partial^2 \eta^\alpha}{\partial x^a \partial x^b} \frac{\partial x^b}{\partial \eta^\beta}. \tag{3.15}$$

Substituting (3.15) into (3.11), one obtains

$$\delta_\varphi \widetilde{S} = \int_{t_0}^{t_1} \int_{\mathcal{B}} \left\{ \left[\frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^\alpha} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^\alpha} \right) - \left(\frac{\partial \mathcal{L}}{\partial \widetilde{F}^\alpha{}_A} \right)_{|A} - \frac{\partial \mathcal{L}}{\partial \widetilde{F}^\mu{}_A} \widetilde{F}^\beta{}_A\ \widetilde{\gamma}^\mu_{\alpha\beta} + 2\frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\beta\lambda}} \widetilde{g}^{\beta\mu}\ \widetilde{\gamma}^\mu_{\alpha\lambda} \right] \frac{\partial \eta^\alpha}{\partial x^a} \delta\varphi^a \right.$$
$$\left. + \left(2\frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\beta\mu}} \widetilde{g}_{\alpha\mu} - \frac{\partial \mathcal{L}}{\partial \widetilde{F}^\alpha{}_A} \widetilde{F}^\beta{}_A - \frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^{\,\alpha}} \dot{\widetilde{\varphi}}^{\,\beta} \right) \frac{\partial^2 \eta^\alpha}{\partial x^a \partial x^b} \frac{\partial x^b}{\partial \eta^\beta} \delta\varphi^a \right\} \mathrm{d}V \mathrm{d}t = 0. \tag{3.16}$$

As η is arbitrary, it can be chosen such that $T\eta$ is independent of $\mathbf{x}$. This would imply that one has the following two sets of EL equations:

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^\alpha} \right) - \left(\frac{\partial \mathcal{L}}{\partial \widetilde{F}^\alpha{}_A} \right)_{|A} - \frac{\partial \mathcal{L}}{\partial \widetilde{F}^\mu{}_A} \widetilde{F}^\beta{}_A \widetilde{\gamma}^\mu_{\alpha\beta} + 2\frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\beta\lambda}} \widetilde{g}^{\beta\mu} \widetilde{\gamma}^\mu_{\alpha\lambda} = 0, \tag{3.17}$$

$$2\frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\beta\mu}} \widetilde{g}_{\alpha\mu} - \frac{\partial \mathcal{L}}{\partial \widetilde{F}^\alpha{}_A} \widetilde{F}^\beta{}_A - \frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^\alpha} \dot{\widetilde{\varphi}}^\beta = 0. \tag{3.18}$$

Note that (3.17) is the standard EL equations (2.5) written with respect to $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \eta_* \mathbf{g})$ and (3.18) is the Doyle-Ericksen formula again with respect to $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \eta_* \mathbf{g})$.

***η*-Variation.** η-variation of action is written as

$$\delta_\eta \widetilde{S} = \int_{t_0}^{t_1} \int_{\mathcal{B}} \left(\frac{\partial \widetilde{\mathcal{L}}}{\partial \eta^\alpha} \delta\eta^\alpha + \frac{\partial \widetilde{\mathcal{L}}}{\partial F_\eta{}^a{}_A} \delta F_\eta{}^a{}_A \right) \mathrm{d}V \mathrm{d}t. \tag{3.19}$$

Note that

$$\frac{\partial \widetilde{\mathcal{L}}}{\partial \eta^\alpha} = \frac{\partial \mathcal{L}}{\partial \eta^\alpha \circ \varphi} = \frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^\alpha}. \tag{3.20}$$

After some lengthy manipulations, it can be shown that

$$\frac{\partial \widetilde{\mathcal{L}}}{\partial F_{\eta}{}^{a}{}_{A}} = \left(\frac{\partial \mathcal{L}}{\partial \widetilde{F}^{\alpha}{}_{A}} \widetilde{F}^{\beta}{}_{A} + \frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^{\alpha}} \dot{\widetilde{\varphi}}^{\beta} - 2 \frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\beta\mu}} \widetilde{g}_{\alpha\mu} \right) \frac{\partial x^{a}}{\partial \eta^{\beta}}. \tag{3.21}$$

Therefore, the η-variation of the modified action reads

$$\delta_{\eta} \widetilde{S} = \int_{t_0}^{t_1} \int_{\mathcal{B}} \left[\frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^{\alpha}} \delta \eta^{\alpha} - \left(2 \frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\beta\mu}} \widetilde{g}_{\alpha\mu} - \frac{\partial \mathcal{L}}{\partial \widetilde{F}^{\alpha}{}_{A}} \widetilde{F}^{\beta}{}_{A} - \frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^{\alpha}} \dot{\widetilde{\varphi}}^{\beta} \right) \frac{\partial x^{a}}{\partial \eta^{\beta}} \delta F_{\eta}{}^{a}{}_{A} \right] \mathrm{d}V \mathrm{d}t. \tag{3.22}$$

Hence, from $\delta_{\eta} \widetilde{S} = 0$ and (3.18), we obtain

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^{\alpha}} = 0. \tag{3.23}$$

In summary, we have proved the following proposition.

Proposition 3.1. *Given a Lagrangian density $\mathcal{L}$, define an auxiliary Lagrangian density $\widetilde{\mathcal{L}}$ by (3.1) and its corresponding action $\widetilde{S}$ as in (3.3). Hamilton's principle of least action for $\widetilde{S}$, that is, $\delta \widetilde{S} = \mathbf{d}\widetilde{S} \cdot (\delta\varphi, \delta\eta) = 0$ gives the following Euler–Lagrange equations (written with respect to $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \eta_{*}\mathbf{g})$):*

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^{\alpha}} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^{\alpha}} \right) - \left(\frac{\partial \mathcal{L}}{\partial \widetilde{F}^{\alpha}{}_{A}} \right)_{|A} - \frac{\partial \mathcal{L}}{\partial \widetilde{F}^{\mu}{}_{A}} \widetilde{F}^{\beta}{}_{A} \widetilde{\gamma}^{\mu}_{\alpha\beta} + 2 \frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\beta\lambda}} \widetilde{g}^{\beta\mu} \widetilde{\gamma}^{\mu}_{\alpha\lambda} = 0, \tag{3.24}$$

$$2 \frac{\partial \mathcal{L}}{\partial \widetilde{g}_{\beta\mu}} \widetilde{g}_{\alpha\mu} - \frac{\partial \mathcal{L}}{\partial \widetilde{F}^{\alpha}{}_{A}} \widetilde{F}^{\beta}{}_{A} - \frac{\partial \mathcal{L}}{\partial \dot{\widetilde{\varphi}}^{\alpha}} \dot{\widetilde{\varphi}}^{\beta} = 0, \tag{3.25}$$

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\varphi}^{\alpha}} = 0. \tag{3.26}$$

In other words, stationarity of the modified action $\widetilde{S}$ gives the standard EL equations, the Doyle-Ericksen formula, and spatial homogeneity of the Lagrangian density $\mathcal{L}$.

Remark 3.2. *Note that this result is similar to what was obtained in [10], where using Noether's theorem, it was shown that spatial covariance of the Lagrangian density leads to the Doyle-Erciksen formula and spatial homogeneity of the Lagrangian density.*

Remark 3.3. *There are subtle differences between spatial and material manifolds. The spatial manifold is homogenous and isotropic. Material manifold—where the body is stress free—is inhomogenous, in general, and has a nontrivial geometry [11,12]. In other words, material metric is a dynamic field, and for this reason, we do not discuss a material version of covariantization.*

4. Concluding remarks

In this paper we studied the problem of covariance of the field theory of elasticity. First, we observed that the non-dynamic nature of the spatial metric prevents the field theory of elasticity to be generally covariant. We extended Kuchař's idea of parametrization of field theories to elasticity by defining a spatial covariance field to be a time-independent spatial diffeomorphism. We then defined a modified action that, in addition to depending on the standard fields, depends on the covariance field as well. We showed that the Euler–Lagrange equations of the modified field theory, in addition to the standard EL equations, contain spatial homogeneity of the Lagrangian density and the Doyle-Ericksen formula.

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