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Nonlinear Elasticity in a Deforming Ambient Space

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Abstract In this paper, we formulate a nonlinear elasticity theory in which the ambient space is evolving. For a continuum moving in an evolving ambient space, we model time dependency of the metric by a time-dependent embedding of the ambient space in a larger manifold with a fixed background metric. We derive both the tangential and the normal governing equations. We then reduce the standard energy balance written in the larger ambient space to that in the evolving ambient space. We consider quasi-static deformations of the ambient space and show that a quasi-static deformation of the ambient space results in stresses, in general. We linearize the nonlinear theory about a reference motion and show that variation of the spatial metric corresponds to an effective field of body forces.

Keywords Geometric mechanics · Nonlinear elasticity · Deforming ambient space

Mathematics Subject Classification 74Axx · 74Bxx · 74Lxx

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1 Introduction

In the geometric theory of elasticity, an elastic body is represented by a *material manifold* $\mathcal{B}$, which defines the natural, stress-free state of the body. The body moves in an ambient space, which in turn is represented by a *spatial manifold* $\mathcal{S}$. The motion of the body is described by a time-dependent configuration map $\varphi_t : \mathcal{B} \to \mathcal{S}$ from the material manifold to the spatial manifold.

The manifolds $\mathcal{B}$ and $\mathcal{S}$ are not simply differential manifolds, but have further geometric structures that allow one to measure the amount of stretch in the body for a given configuration. While more general geometric structures involving torsion and non-metricity are used in the nonlinear mechanics of defects (see Yavari and Goriely 2012a, b, c, 2014 for recent work), in this paper, we restrict our attention to Riemannian manifolds. We assume that both $\mathcal{B}$ and $\mathcal{S}$ are Riemannian, with metric tensors $\boldsymbol{G}$ and $\boldsymbol{g}$, and the associated Levi–Civita connections $\nabla^{\boldsymbol{G}}$ and $\nabla^{\boldsymbol{g}}$, respectively. For a given configuration, these Riemannian structures allow one to evaluate the spatial distances between the points of the body, and distances given by the material metric. A discrepancy between these two types of distances signifies a strain in the body from its natural, stress-free state and hence results in stresses. The material and spatial distances can be unequal even for a body at rest, without any external forces applied—this is the case of residual stresses. This viewpoint has been explored for thermal stresses and for growing bodies by Ozakin and Yavari (2010), Sadik and Yavari (2015), Yavari (2010), and Sadik et al. (2016). Certain non-elastic deformations of a material can be described by a time-dependent material metric $\boldsymbol{G}_t$; for the case of thermal expansion, the material metric has been related to the (possibly evolving) temperature distribution in Ozakin and Yavari (2010); Sadik and Yavari (2015).

In this paper, we take the material metric as fixed, but consider an evolving spatial metric, $\boldsymbol{g}_t$. In the geometric field theory of elasticity (Marsden and Hughes 1983; Simo and Marsden 1984b; Ciarlet 2005; Yavari et al. 2006), the spatial metric is introduced as a fixed background geometry. Likewise, in the classical theory of nonlinear elasticity, this background metric is a given geometric object with no dynamics; $\boldsymbol{h}$ is "absolute" in the sense of Anderson (1967) and is a "structural field" in the sense of Post (1997). Our motivation to study the effects of a time-dependent spatial geometry stems both from the hope of gaining a deeper understanding of the structure of the classical theory[1] and from possible applications involving the analysis of elastic bodies constrained to move on curved, dynamical surfaces. In order to have a sense of what to expect from such a theory, let us consider a simple, two-dimensional example.

Figure 1 shows a thin elastic strip constrained to move on the surface of a torus, which we treat as the ambient space, $\mathcal{S}$. We assume that there is no friction between the torus and the strip—the latter moves freely on the torus, but cannot move away from it.[2] Suppose that the torus is expanding in time in a predetermined manner,

[1] The generalization of a theory obtained by relaxing certain standard assumptions (in this case, the staticity of $\boldsymbol{g}_t$), commonly results in a deeper understanding of the original theory. Examples of this include the geometric notions of stress and traction obtained by allowing the spatial metric to be non-Euclidean.

[2] It may be helpful to imagine the strip as moving between two tori of infinitesimally different sizes, so that the strip is constrained from both sides.

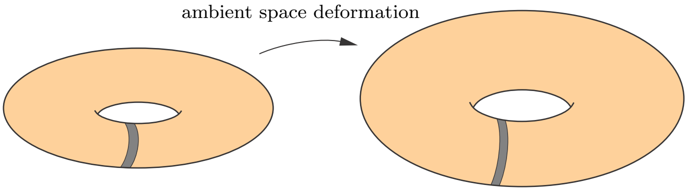

Fig. 1 A thin strip embedded in a toroidal ambient space is stretched when the ambient space deforms

and consider the motion of the strip in this dynamic ambient space. Our aim is to investigate this dynamics in terms of the intrinsic geometry of $\mathcal{S}$ and forces that "live" on this surface, to the extent possible. It is evident that the strip will stretch and thus will store elastic energy as the torus expands, even though it does not see any sources of external forces in the ambient space it observes. Thus, the energy balance written inside the torus (without any reference to the surrounding Euclidean $\mathbb{R}^3$) will suggest a non-conservation of energy. In this example, we know the missing piece in the energy balance; it is simply the work done by the normal forces needed to expand the torus with the strip on it. Our aim is to obtain general equations describing the motion of an elastic body moving in such an evolving ambient space and investigate such issues as energy balance and Lagrangian mechanics.

Implicit in this discussion of Fig. 1 is the fact that the ambient space $\mathcal{S}$ is considered as an embedded submanifold of $\mathbb{R}^3$, instead of simply as a Riemannian manifold in its own right. While it is tempting to consider the dynamics of a body in an ambient space which is described purely intrinsically in terms of the time-dependent metric $\boldsymbol{g}_t$, we will see that one needs to consider the motion in terms of a time-dependent embedding in a larger, static space. For one, it will be possible to identify the missing part in the intrinsic energy balance as work done by/on the outside forces transforming the ambient space $\mathcal{S}$, as the discussion above suggests. In addition, the dynamics in a time-dependent submanifold also results in fictitious forces that cannot be obtained purely from the intrinsic geometry—forces that depend explicitly on the embedding. We will identify the effects of the intrinsic and the extrinsic geometries of $\mathcal{S}$ below.

The notion of a time-dependent ambient space also shows up in the theory of general relativity. While there are connections between the theory of elasticity developed in this paper and general relativistic continuum mechanics, we will leave the discussion of these issues to a future communication, focusing on the non-relativistic case exemplified by the case of Fig. 1 in this paper.

This paper is organized as follows. In Sect. 2, we first formulate a Lagrangian field theory of elasticity when the ambient space has a time-dependent geometry. We obtain both the intrinsic (tangential) and extrinsic (normal) governing equations of motion. We then show that for an elastic body moving in an evolving ambient space, energy balance must be modified and obtain a modified energy balance when the spatial metric is time dependent. We do this by considering a time-dependent embedding of the ambient space in a larger manifold and writing the standard energy balance in the larger manifold. We reduce the energy balance to that written by an observer in the evolving ambient space. In Sect. 3, we look at quasi-static deformations of the

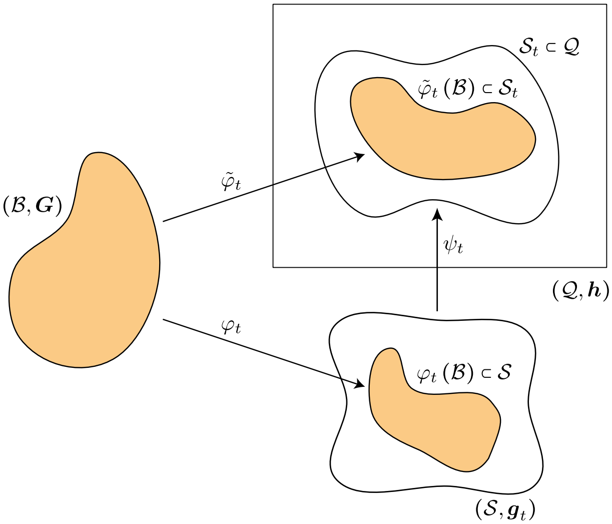

Fig. 2 Motion of an elastic body in an evolving ambient space

ambient space and the corresponding induced stresses. We linearize the governing equations and show that a quasi-static deformation of the ambient space is equivalent to an effective body force in a fixed ambient space.

2 Motion of an Elastic Body in an Evolving Ambient Space

In this section, we study the motion of an elastic body moving in an evolving (time-dependent) ambient space. We will derive both tangential and normal governing equations of motion, balance of mass, and energy balance.

2.1 Lagrangian Field Theory of Elasticity in an Evolving Ambient Space

We identify the reference configuration of an elastic body with a Riemannian manifold $(\mathcal{B}, \boldsymbol{G})$ and let the body deform in a time-dependent ambient space $\mathcal{S}_t$, which is evolving in a Euclidean space $(\mathcal{Q}, \boldsymbol{h})$ of higher dimension. The evolution of the ambient space $\mathcal{S}_t$ is given by a time-dependent embedding $\psi_t : \mathcal{S} \to \mathcal{Q}$, for some abstract manifold $\mathcal{S}$, such that $\psi_t(\mathcal{S}) = \mathcal{S}_t$, and the evolving metric of the ambient space $\mathcal{S}$ is given as the induced metric by that of $\mathcal{Q}$, i.e., $\boldsymbol{g}_t := \psi_t^* \boldsymbol{h}$, which means that ψ_t is an isometric embedding,[3] see Fig. 2. We denote inner products of vectors with respect to the metrics $\boldsymbol{h}$ and $\boldsymbol{g}_t$ by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{h}}$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{g}_t}$, respectively. We denote the local coordinates on $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{Q}$ by $\{X^A\}$, $\{x^a\}$, and $\{\chi^\alpha\}$, respectively. Let $\dim \mathcal{S}_t = n$, and $\dim \mathcal{Q} = n + k = m$. Let $\{\boldsymbol{\eta}_i^t\}_{i=1,\ldots,k}$ be a smooth orthonormal basis for $\mathfrak{X}^\perp(\mathcal{S}_t)$,

[3] Note that for a given t, such an isometric embedding always exists for $\dim \mathcal{Q}$ large enough, by Nash (1956)'s embedding theorem.

the set of vector fields normal to $\mathcal{S}_t$. Let $\{\chi^\alpha\}$ be a local coordinate chart for $\mathcal{Q}$ such that at any point of $\mathcal{S}_t$, $\{\chi^1,\ldots,\chi^n\}$ is a local coordinate chart for $\mathcal{S}_t$ and such that the unit normal vector field $\boldsymbol{\eta}_i^t$, for $i\in\{1,\ldots,k\}$ is tangent to the coordinate curve χ^{n+i}, for $i\in\{1,\ldots,k\}$. Recall, as discussed in Appendix 1, that every vector field $\boldsymbol{u}$ on $\mathcal{Q}$ along $\mathcal{S}_t$ can be written as $\boldsymbol{u}=\boldsymbol{u}_\parallel+\sum_{i=1}^k u_\perp^i\boldsymbol{\eta}_i^t$, where $\boldsymbol{u}_\parallel$ is the part of the vector $\boldsymbol{u}$ tangential to $\mathcal{S}_t$ and $u_\perp^i=u^{n+i}$ for $i\in\{1,\ldots,k\}$. Also, recall that for $i,j\in\{1,\ldots,k\}$, and $\alpha\in\{1,\ldots,n+k\}$, we have $\left\langle\!\left\langle\boldsymbol{\eta}_i^t,\boldsymbol{\eta}_j^t\right\rangle\!\right\rangle_{\boldsymbol{h}}=\delta_{ij}$, $\langle\!\langle\boldsymbol{\eta}_i^t,\boldsymbol{u}_\parallel\rangle\!\rangle_{\boldsymbol{h}}=0$, and $h_{\alpha(n+i)}=\delta_{\alpha(n+i)}$. For $i\in\{1,\ldots,k\}$, we denote the ith second fundamental form of $\mathcal{S}_t$ along the unit normal $\boldsymbol{\eta}_i^t$ by $\boldsymbol{\kappa}_i^t$ and let $\boldsymbol{k}_i^t=\psi_t^*\boldsymbol{\kappa}_i^t$. For $i,j\in\{1,\ldots,k\}$, we denote the normal fundamental 1-form of $\mathcal{S}_t$ relative to the unit normals $\boldsymbol{\eta}_i^t$ and $\boldsymbol{\eta}_j^t$ in this order by $\boldsymbol{\omega}_{ij}^t$ and let $\boldsymbol{o}_{ij}^t=\psi_t^*\boldsymbol{\omega}_{ij}^t$.[4] We define a motion of $(\mathcal{B},\boldsymbol{G})$ in $\left(\mathcal{S}_t,\boldsymbol{h}|_{\mathcal{S}_t}\right)$ as a one-parameter family of maps $\tilde{\varphi}_t:\mathcal{B}\to\mathcal{S}_t$, where t is time. This is equivalent to a motion of $(\mathcal{B},\boldsymbol{G})$ in $\left(\mathcal{S},\boldsymbol{g}_t\right)$, i.e., $\varphi_t:\mathcal{B}\to\mathcal{S}$, such that $\varphi_t=\psi_t^{-1}\circ\tilde{\varphi}_t$. We let $\tilde{\varphi}(X,t):=\tilde{\varphi}_t(X)$, $\varphi(X,t):=\varphi_t(X)$, and $\psi(x,t):=\psi_t(x)$. Let $\{\tilde{\partial}_\alpha^t\}_{\alpha=1,\ldots,n}$ and $\{\partial_a^t\}_{a=1,\ldots,n}$ denote local coordinate bases for $\mathcal{S}_t$ and $\mathcal{S}$, respectively.

In order to describe the dynamics of the motion of $\mathcal{B}$, the Lagrangian field theory should be formulated with respect to the fixed space $\mathcal{Q}$. For an elastic material, the Lagrangian density $\mathcal{L}$ can be written as[5]

$$\mathcal{L}=\mathcal{L}(\boldsymbol{X},\tilde{\varphi},\dot{\tilde{\varphi}},\tilde{\boldsymbol{F}},\boldsymbol{G},\boldsymbol{h}),$$

where $\tilde{\boldsymbol{F}}=T\tilde{\varphi}_t=\psi_{t*}\boldsymbol{F}$ and $\boldsymbol{F}=T\varphi_t$ are the deformation gradients of $\tilde{\varphi}_t$ and φ_t, respectively. We assume that the Lagrangian density can be written as

$$\mathcal{L}=\frac{1}{2}\rho_0\langle\!\langle\boldsymbol{\Upsilon},\boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}}-\rho_0 W(\boldsymbol{X},\tilde{\boldsymbol{F}},\boldsymbol{G},\boldsymbol{h}),\tag{2.1}$$

where ρ_0 is the material mass density, $\boldsymbol{\Upsilon}:=\dot{\tilde{\varphi}}=\psi_{t*}\boldsymbol{V}+\boldsymbol{\zeta}\circ\varphi_t$ is the material velocity vector field of $\tilde{\varphi}$, $\boldsymbol{V}$ is the material velocity vector field of φ, $\boldsymbol{\zeta}=\partial\psi/\partial t$ is the velocity of a given, fixed point $x\in\mathcal{S}$ as it moves in $\mathcal{Q}$, and $W=W(\boldsymbol{X},\tilde{\boldsymbol{F}},\boldsymbol{G},\boldsymbol{h})$ is the elastic energy density (energy function).

Remark 2.1 Note that since $\boldsymbol{g}_t:=\psi_t^*\boldsymbol{h}$, i.e., ψ_t is an isometry between $\left(\mathcal{S},\boldsymbol{g}_t\right)$ and $(\mathcal{S}_t,\boldsymbol{h})$, by objectivity (the isometry ψ_t can be interpreted as a change of observer), the dependence of the elastic energy on $\tilde{\boldsymbol{F}}=\psi_{t*}\boldsymbol{F}$ reduces to a dependence on $\boldsymbol{F}$ only. It should also depend on $\boldsymbol{G}$ and $\boldsymbol{g}_t$ (instead of $\boldsymbol{h}$) so that one can get a scalar out

[4] Recall that the order matters since $\boldsymbol{\omega}_{ij}^t=-\boldsymbol{\omega}_{ji}^t$. See Appendix 1 for more details and the definitions of both the second and the normal fundamental forms.

[5] Note that although the Lagrangian theory is formulated with respect to $\mathcal{Q}$, the density is defined with respect to the volume element of $\mathcal{B}$, i.e., $\mathcal{L}$ is an n-dimensional density, not an m-dimensional one.

of $\boldsymbol{F}$. Hence, we have[6]

$$W(\boldsymbol{X}, \tilde{\boldsymbol{F}}, \boldsymbol{G}, \boldsymbol{h}) = W(\boldsymbol{X}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t). \tag{2.2}$$

For a continuum under a body force field $\boldsymbol{\beta}$ (not necessarily conservative), the Lagrange-d'Alembert principle states that (Marsden and Ratiu 2003)

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L} \mathrm{d}V \mathrm{d}t + \int_{t_0}^{t_1} \int_{\mathcal{B}} \rho_0 \boldsymbol{\beta}^\flat \cdot \delta\tilde{\varphi} \mathrm{d}V \mathrm{d}t = 0, \tag{2.3}$$

where $^\flat$ denotes the flat operator for lowering tensor indices, $\boldsymbol{\beta}$ denotes body force per unit mass, and $\mathrm{d}V$ denotes the volume element for $\mathcal{B}$. Note that $\boldsymbol{\beta}$ is not necessarily tangent to $\mathcal{S}_t$ and we write it as $\boldsymbol{\beta} = \boldsymbol{\beta}_\| + \sum_{i=1}^k B_\perp^i \boldsymbol{\eta}_i^t$, where $\boldsymbol{\beta}_\|$ is the part of $\boldsymbol{\beta}$ tangent to $\mathcal{S}_t$ and $B_\perp^i$, for $i \in \{1, \ldots, k\}$, is its component along the ith normal $\boldsymbol{\eta}_i^t$. The action is defined on the material manifold $(\mathcal{B}, \boldsymbol{G})$ as

$$S(\tilde{\varphi}) = \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L}(\boldsymbol{X}, \tilde{\varphi}, \dot{\tilde{\varphi}}, \tilde{\boldsymbol{F}}, \boldsymbol{G}, \boldsymbol{h}) \mathrm{d}V(\boldsymbol{X}) \mathrm{d}t, \tag{2.4}$$

where $\mathrm{d}V(\boldsymbol{X})$ is the Riemannian volume element on $\mathcal{B}$. For the assumed Lagrangian (2.1), we have $S = S_T + S_W$, where

$$S_T = \int_{t_0}^{t_1} \int_{\mathcal{B}} \frac{1}{2} \rho_0 \langle\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \rangle\rangle_{\boldsymbol{h}} \mathrm{d}V(\boldsymbol{X}) \mathrm{d}t,$$
$$S_W = -\int_{t_0}^{t_1} \int_{\mathcal{B}} \rho_0 W(\boldsymbol{X}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t) \mathrm{d}V(\boldsymbol{X}) \mathrm{d}t.$$

In order to take variations of the action (2.4), we consider a variation field $\tilde{\varphi}_\epsilon$ of $\tilde{\varphi}$ such that $\tilde{\varphi}_0 = \tilde{\varphi}$ and define its variation as

$$\delta\tilde{\varphi} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \tilde{\varphi}_\epsilon.$$

First, we look at the resulting variations of the kinetic energy

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \frac{1}{2} \langle\langle \boldsymbol{\Upsilon}_\epsilon, \boldsymbol{\Upsilon}_\epsilon \rangle\rangle_{\boldsymbol{h}} = \left\langle\!\!\left\langle D_\epsilon^{\boldsymbol{h}} \boldsymbol{\Upsilon}_\epsilon, \boldsymbol{\Upsilon}_\epsilon \right\rangle\!\!\right\rangle_{\boldsymbol{h}},$$

where $D_\epsilon^{\boldsymbol{h}}$ denotes the covariant derivative along the curve $\epsilon \mapsto \tilde{\varphi}_\epsilon(X, t)$ for fixed X and t. Using the symmetry lemma, we have $D_\epsilon^{\boldsymbol{h}} \boldsymbol{\zeta}_\epsilon = D_t^{\boldsymbol{h}} \delta\tilde{\varphi}$, where $D_t^{\boldsymbol{h}}$ denotes

[6] Another way to see this is by looking at the elastic energy as a function of the right Cauchy–Green tensor, i.e., $W = \tilde{W}(\boldsymbol{X}, \tilde{\boldsymbol{C}}, \boldsymbol{G})$. First, we see that since $\boldsymbol{g}_t := \psi_t^* \boldsymbol{h}$, then $(\psi_t \circ \varphi_t)^* \boldsymbol{h} = \varphi_t^* \psi_t^* \boldsymbol{h} = \varphi_t^* \boldsymbol{g}_t$, i.e., the right Cauchy–Green tensors $\boldsymbol{C}$ of φ_t and $\tilde{\boldsymbol{C}}$ of $\tilde{\varphi}_t$ are equal. If we denote $\boldsymbol{f} := T\psi_t$, we write in components $\tilde{C}_{AB} = f^\alpha{}_a F^a{}_A f^\beta{}_b F^b{}_B h_{\alpha\beta} = F^a{}_A F^b{}_B f^\alpha{}_a f^\beta{}_b h_{\alpha\beta} = F^a{}_A F^b{}_B g_{ab} = C_{AB}$. Therefore, $W = \tilde{W}(\boldsymbol{X}, \tilde{\boldsymbol{C}}, \boldsymbol{G}) = \tilde{W}(\boldsymbol{X}, \boldsymbol{C}, \boldsymbol{G})$, that is, the elastic energy does not depend on the embedding ψ_t.

the covariant derivative along the curve $t \mapsto \tilde{\varphi}(X,t)$ for fixed X. Therefore, we can write

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{1}{2}\langle\langle \boldsymbol{\Upsilon}_\epsilon, \boldsymbol{\Upsilon}_\epsilon\rangle\rangle_{\boldsymbol{h}} = \left\langle\!\!\left\langle D_t^{\boldsymbol{h}}\delta\tilde{\varphi}, \boldsymbol{\Upsilon}\right\rangle\!\!\right\rangle_{\boldsymbol{h}} = \frac{\mathrm{d}}{\mathrm{d}t}\langle\langle \delta\tilde{\varphi}, \boldsymbol{\Upsilon}\rangle\rangle_{\boldsymbol{h}} - \left\langle\!\!\left\langle \delta\tilde{\varphi}, D_t^{\boldsymbol{h}}\boldsymbol{\Upsilon}\right\rangle\!\!\right\rangle_{\boldsymbol{h}}.$$

Assuming that the variation of $\tilde{\varphi}$ is fixed at t_0 and t_1, i.e., $\delta\tilde{\varphi}(t_0) = \delta\tilde{\varphi}(t_1) = 0$, the first term on the right-hand side does not contribute to the action. We decompose the velocity $\boldsymbol{\Upsilon}$ into tangent and normal components as $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_\parallel + \boldsymbol{\Upsilon}_\perp$, where $\boldsymbol{\Upsilon}_\parallel = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta}_\parallel \circ \varphi_t$ and $\boldsymbol{\Upsilon}_\perp = \sum_{i=1}^k \zeta_\perp^i \boldsymbol{\eta}_i^t$, such that $\boldsymbol{\zeta}$ is written in terms of its tangent and normal components as $\boldsymbol{\zeta} = \boldsymbol{\zeta}_\parallel + \sum_{i=1}^k \zeta_\perp^i \boldsymbol{\eta}_i^t$. We denote the acceleration in $\mathcal{Q}$ by $\boldsymbol{\Gamma} = D_t^{\boldsymbol{h}}\boldsymbol{\Upsilon}$ and decompose it into tangent and normal components with respect to $\mathcal{S}_t$ as $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_\parallel + \sum_{i=1}^k \Gamma_\perp^i \boldsymbol{\eta}_i^t$. We denote by $\boldsymbol{A} = \psi_t^*\boldsymbol{\Gamma}_\parallel$ the intrinsic acceleration of $\mathcal{S}$. Therefore, the variation of the kinetic energy is calculated as

$$\begin{aligned}\delta\left(\frac{1}{2}\rho_0\langle\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}\rangle\rangle_{\boldsymbol{h}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle\langle \delta\tilde{\varphi}, \rho_0\boldsymbol{\Upsilon}\rangle\rangle_{\boldsymbol{h}} - \langle\!\langle \delta\tilde{\varphi}_\parallel, \rho_0\boldsymbol{\Gamma}_\parallel\rangle\!\rangle_{\boldsymbol{h}} - \rho_0\sum_{i=n+1}^m \Gamma_\perp^i\delta\tilde{\varphi}_\perp^i \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle\langle \delta\tilde{\varphi}, \rho_0\boldsymbol{\Upsilon}\rangle\rangle_{\boldsymbol{h}} - \langle\!\langle \psi^*\delta\tilde{\varphi}_\parallel, \rho_0\boldsymbol{A}\rangle\!\rangle_{\boldsymbol{g}_t} - \rho_0\sum_{i=n+1}^m \Gamma_\perp^i\delta\tilde{\varphi}_\perp^i,\end{aligned}$$

where $\delta\tilde{\varphi}_\parallel$ is the part of $\delta\tilde{\varphi}$ tangent to $\mathcal{S}_t$ and $\delta\tilde{\varphi}_\perp^i$ is its component along $\boldsymbol{\eta}_i^t$, for $i \in \{1,\ldots,k\}$. Assuming that the variation of $\tilde{\varphi}$ is fixed on the boundary, i.e., $\delta\tilde{\varphi}|_{\partial\varphi(\mathcal{B})} = 0$, we obtain

$$\delta S_T = -\int_{t_0}^{t_1}\int_{\mathcal{B}}\left(\langle\!\langle \psi^*\delta\tilde{\varphi}_\parallel, \rho_0\boldsymbol{A}\rangle\!\rangle_{\boldsymbol{g}_t} + \rho_0\sum_{i,j=n+1}^m \Gamma_\perp^i\delta\tilde{\varphi}^j\delta_{ij}\right)\mathrm{d}V(\boldsymbol{X})\mathrm{d}t. \tag{2.5}$$

Next we compute the components of the acceleration.

Proposition 2.1 *The tangent and normal accelerations are given by*

$$\begin{aligned}\boldsymbol{A} =& D_t^{\boldsymbol{g}_t}(\boldsymbol{V}+\boldsymbol{Z}) + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{Z} + 2\sum_{i=1}^k \zeta_\perp^i \boldsymbol{g}_t^\sharp\cdot\boldsymbol{k}_i^t\cdot(\boldsymbol{V}+\boldsymbol{Z}) \\ &-\sum_{i=1}^k \zeta_\perp^i\left(d\zeta_\perp^i\right)^\sharp - \sum_{i,j=1}^k \zeta_\perp^i\zeta_\perp^j\boldsymbol{o}_{ij}^{t\sharp} \\ =& D_t^{\boldsymbol{g}_t}(\boldsymbol{V}+\boldsymbol{Z}) - \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}\right]^{\mathsf{T}}\cdot(\boldsymbol{V}+\boldsymbol{Z}) + \boldsymbol{g}_t^\sharp\cdot\frac{\partial\boldsymbol{g}_t}{\partial t}\cdot(\boldsymbol{V}+\boldsymbol{Z}) \\ &-\sum_{i=1}^k \zeta_\perp^i\left(d\zeta_\perp^i\right)^\sharp - \sum_{i,j=1}^k \zeta_\perp^i\zeta_\perp^j\boldsymbol{o}_{ij}^{t\sharp},\end{aligned} \tag{2.6a}$$

$$\Gamma_\perp^i = \frac{d\zeta_\perp^i}{dt} + d\zeta_\perp^i \cdot (\boldsymbol{V}+\boldsymbol{Z}) - \boldsymbol{k}_i^t(\boldsymbol{V}+\boldsymbol{Z},\boldsymbol{V}+\boldsymbol{Z}) - 2\sum_{j=1}^k \zeta_\perp^j \boldsymbol{o}_{ij}^t \cdot (\boldsymbol{V}+\boldsymbol{Z})$$
$$+ \sum_{j,l=1}^k \zeta_\perp^j \zeta_\perp^l \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}_j^t} \boldsymbol{\eta}^t{}_l, \boldsymbol{\eta}_i^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}}, \tag{2.6b}$$

where $i = 1, \ldots, k$, d denotes the exterior derivative on $\mathcal{S}$, i.e., $d\zeta_\perp^i = \sum_{a=1}^n \frac{\partial \zeta_\perp^i}{\partial x^a} dx^a$, $D_t^{\boldsymbol{g}_t}$ denotes the covariant derivative along the curve $t \mapsto \varphi(X,t)$ for fixed X, $\boldsymbol{Z} := \left(\psi_t^* \boldsymbol{\zeta}_\parallel\right) \circ \varphi_t$ is the tangent part of the velocity $\boldsymbol{\zeta}$, and $^\mathsf{T}$ denotes the transpose operator with respect to the metric $\boldsymbol{g}_t$.

Remark 2.2 Before we proceed to the proof, let us first look at some particular cases. If we assume that the evolution of the ambient space is transversal, i.e., $\boldsymbol{Z} = \boldsymbol{0}$, then (2.6) reduces to

$$\boldsymbol{A} = D_t^{\boldsymbol{g}_t}\boldsymbol{V} + 2\sum_{i=1}^k \zeta_\perp^i \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}_i^t \cdot \boldsymbol{V} - \sum_{i=1}^k \zeta_\perp^i \left(\mathrm{d}\zeta_\perp^i\right)^\sharp - \sum_{i,j=1}^k \zeta_\perp^i \zeta_\perp^j \boldsymbol{o}_{ij}^{t\sharp}$$
$$= D_t^{\boldsymbol{g}_t}\boldsymbol{V} + \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{V} - \sum_{i=1}^k \zeta_\perp^i \left(\mathrm{d}\zeta_\perp^i\right)^\sharp - \sum_{i,j=1}^k \zeta_\perp^i \zeta_\perp^j \boldsymbol{o}_{ij}^{t\sharp}, \tag{2.7a}$$

$$\Gamma_\perp^i = \frac{\mathrm{d}\zeta_\perp^i}{\mathrm{d}t} + \mathrm{d}\zeta_\perp^i \cdot \boldsymbol{V} - \boldsymbol{k}_i^t(\boldsymbol{V},\boldsymbol{V}) - 2\sum_{j=1}^k \zeta_\perp^j \boldsymbol{o}_{ij}^t \cdot \boldsymbol{V} + \sum_{j,l=1}^k \zeta_\perp^j \zeta_\perp^l \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}_j^t} \boldsymbol{\eta}^t{}_l, \boldsymbol{\eta}_i^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}}, \tag{2.7b}$$

where $i = 1, \ldots, k$. If we assume that $\mathcal{S}_t$ is a hypersurface in $\mathcal{Q}$, i.e., the co-dimension is $k = 1$, the normal fundamental 1-forms reduce to a vanishing 1-form $\boldsymbol{o}_{11}^t = \boldsymbol{0}$, and $\sum_{j,l=1}^k \zeta_\perp^j \zeta_\perp^l \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}_j^t} \boldsymbol{\eta}^t{}_l, \boldsymbol{\eta}_i^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}} = \zeta_\perp \zeta_\perp \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}^t} \boldsymbol{\eta}^t, \boldsymbol{\eta}^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}} = \boldsymbol{0}$, since $\langle\!\langle \boldsymbol{\eta}^t, \boldsymbol{\eta}^t \rangle\!\rangle_{\boldsymbol{h}} = 1$. Therefore, (2.6) reduces to

$$\boldsymbol{A} = D_t^{\boldsymbol{g}_t}(\boldsymbol{V}+\boldsymbol{Z}) + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{Z} + 2\zeta_\perp \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}^t \cdot (\boldsymbol{V}+\boldsymbol{Z}) - \zeta_\perp (\mathrm{d}\zeta_\perp)^\sharp$$
$$= D_t^{\boldsymbol{g}_t}(\boldsymbol{V}+\boldsymbol{Z}) - \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}\right]^\mathsf{T} \cdot (\boldsymbol{V}+\boldsymbol{Z}) + \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot (\boldsymbol{V}+\boldsymbol{Z}) - \zeta_\perp (\mathrm{d}\zeta_\perp)^\sharp, \tag{2.8a}$$

$$\Gamma_\perp = \frac{\mathrm{d}\zeta_\perp}{\mathrm{d}t} + \mathrm{d}\zeta_\perp \cdot (\boldsymbol{V}+\boldsymbol{Z}) - \boldsymbol{k}^t(\boldsymbol{V}+\boldsymbol{Z},\boldsymbol{V}+\boldsymbol{Z}). \tag{2.8b}$$

Finally, if we assume that $\mathcal{S}_t$ is a hypersurface evolving transversally in $\mathcal{Q}$, i.e., $k = 1$ and $\boldsymbol{Z} = \boldsymbol{0}$, then Eqs. (2.8) reduce to

$$\boldsymbol{A} = D_t^{\boldsymbol{g}_t}\boldsymbol{V} + 2\zeta_\perp \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}^t \cdot \boldsymbol{V} - \zeta_\perp (\mathrm{d}\zeta_\perp)^\sharp$$
$$= D_t^{\boldsymbol{g}_t}\boldsymbol{V} + \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{V} - \zeta_\perp (\mathrm{d}\zeta_\perp)^\sharp, \tag{2.9a}$$

$$\mathit{\Gamma}_\perp = \frac{\mathrm{d}\zeta_\perp}{\mathrm{d}t} + \mathrm{d}\zeta_\perp \cdot \boldsymbol{V} - \boldsymbol{k}^t\left(\boldsymbol{V},\boldsymbol{V}\right). \tag{2.9b}$$

Proof [7] First, we observe that contrary to motions for which $k = 0$, as in the case of 3D elasticity, $\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}$ cannot be defined unambiguously, in general, for motions in an evolving ambient space when $k \geq 1$. For calculating $\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}$ at $\chi \in \tilde{\varphi}_t(\mathcal{B})$ for a fixed time t, one needs values of $\boldsymbol{\Upsilon}$ along a curve $\gamma : (-\epsilon,\epsilon) \to \mathcal{Q}$ with $\gamma(0) = \chi$ and $\gamma'(0) = \boldsymbol{\Upsilon}$. However, in general, we may have $\boldsymbol{\Upsilon} \notin T_{\tilde{\varphi}_t(X)}\tilde{\varphi}_t(\mathcal{B})$ in which case, such a curve γ needs to leave the space $\tilde{\varphi}_t(\mathcal{B})$ but $\boldsymbol{\Upsilon}$ is only defined on $\tilde{\varphi}_t(\mathcal{B})$. Thus, it would not, in general, be possible to compute $\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}}\boldsymbol{\Upsilon}$ unless we can define an extension of $\boldsymbol{\Upsilon}$ to a neighborhood in $\mathcal{Q}$. Note that it is always possible to compute $\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}_\parallel}\boldsymbol{\Upsilon}$; the problem arises in computing $\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}_\perp}\boldsymbol{\Upsilon}$. The spatial velocity at a fixed time t is a vector field on $\tilde{\varphi}_t(\mathcal{B})$ defined as $\boldsymbol{v}(\chi,t) := \boldsymbol{\Upsilon}(\tilde{\varphi}_t^{-1}(\chi),t)$. Note that for each t, $\tilde{\varphi}_t : \mathcal{B} \to \mathcal{Q}$ is a smooth embedding. However, $\tilde{\varphi} : \mathcal{B}\times\mathbb{R} \to \mathcal{Q}$ is not even an immersion, in general. To see this, let $\{X^A\}$ and $\{\chi^a\}$ be local coordinate charts for $\mathcal{B}$ and $\mathcal{Q}$, respectively. The expression of $T_{(X,t)}\tilde{\varphi}$ in these local coordinate charts, for $n = 2$ and $k = 1$, reads (Sadik et al. 2016)

$$T_{(X,t)}\tilde{\varphi} = \begin{pmatrix} \frac{\partial\tilde{\varphi}^1}{\partial X^1} & \frac{\partial\tilde{\varphi}^1}{\partial X^2} & \frac{\partial\tilde{\varphi}^1}{\partial t} \\ \frac{\partial\tilde{\varphi}^2}{\partial X^1} & \frac{\partial\tilde{\varphi}^2}{\partial X^2} & \frac{\partial\tilde{\varphi}^2}{\partial t} \\ \frac{\partial\tilde{\varphi}^3}{\partial X^1} & \frac{\partial\tilde{\varphi}^3}{\partial X^2} & \frac{\partial\tilde{\varphi}^3}{\partial t} \end{pmatrix}.$$

Clearly, if $\boldsymbol{\Upsilon}(X,t) = \boldsymbol{0}$ (i.e., $\partial\tilde{\varphi}^\alpha/\partial t = 0$), or $\tilde{\varphi}$ is an in-plane motion (i.e., in some coordinate chart for $\mathcal{Q}$ such that $\partial_3 = \boldsymbol{n}$ on $\tilde{\varphi}_t(\mathcal{B})$ we have $\tilde{\varphi}^3 = 0$), then $T_{(X,t)}\tilde{\varphi}$ is, in general, not injective. If, however, $T_{(X,t)}\tilde{\varphi}$ is injective, then the implicit function theorem tells us that $\tilde{\varphi}$ is a local diffeomorphism at (X,t). In particular, one obtains a local vector field $\boldsymbol{\mathcal{V}}$ on $\mathcal{Q}$ in a neighborhood of $\tilde{\varphi}(X,t)$ such that $\boldsymbol{\mathcal{V}}(\tilde{\varphi}(X,t)) = \boldsymbol{\Upsilon}(X,t) = \boldsymbol{v}(\tilde{\varphi}(X,t),t)$, and we can define the material acceleration as

$$\boldsymbol{\Gamma}(X,t) = D^{\boldsymbol{h}}_{\tilde{\varphi}_X}\boldsymbol{\Upsilon}_X := \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}(\tilde{\varphi}(X,t)).$$

Recall that we chose $\{\chi^\alpha\}_{\alpha=1,\ldots,n+k}$ to be a local coordinate chart for $\mathcal{Q}$ such that at any point of $\mathcal{S}_t$, $\{\chi^1,\ldots,\chi^n\}$ is a local coordinate chart for $\mathcal{S}_t$ and such that the ith unit normal vector field $\boldsymbol{\eta}^t_i$ for $i \in \{1,\ldots,k\}$ is tangent to the coordinate curve χ^{n+i}. In this coordinate chart, we recall that $h_{\alpha(n+i)} = \left\langle\!\left\langle \tilde{\partial}^t_\alpha, \boldsymbol{\eta}^t_i \right\rangle\!\right\rangle_{\boldsymbol{h}} = \delta_{\alpha(n+i)}$, which will be used frequently in the following computations. Note that when $T_{(X,t)}\tilde{\varphi}$ is injective, the set of chosen unit normal vector fields $\{\boldsymbol{\eta}^t{}_i\}_{i=1,\ldots,k}$ is well defined on a neighborhood of $\tilde{\varphi}(X,t)$ in $\mathcal{Q}$. Hence, one can decompose $\boldsymbol{\mathcal{V}}$ into tangent and normal components with respect to $\{\boldsymbol{\eta}^t{}_i\}_{i=1,\ldots,k}$ as $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_\parallel + \boldsymbol{\mathcal{V}}_\perp$. One then writes

$$\boldsymbol{\Gamma}(X,t) = \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}(\boldsymbol{\mathcal{V}}_\parallel + \boldsymbol{\mathcal{V}}_\perp) = \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\parallel + \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_\perp.$$

[7] This computation is a generalization for a higher codimension of the acceleration computation in Sadik et al. (2016). We should mention that for this calculation we benefited from a discussion with Fabio Sozio.

Since the Levi–Civita connection is by definition torsion-free, one can write

$$\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel} = \left[\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}}_{\parallel}\right] + \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_{\parallel}}\boldsymbol{\mathcal{V}} = \left[\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}}_{\parallel}\right] + \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_{\parallel}}\boldsymbol{\mathcal{V}}_{\parallel} + \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_{\parallel}}\boldsymbol{\mathcal{V}}_{\perp}.$$

Note that since $\mathcal{V} = \mathcal{V}(\tilde{\varphi}(X,t))$ does not explicitly depend on time, one can write $\left[\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{V}}_{\parallel}\right] = \boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel}$, which is tangent to $\mathcal{S}_t$, where $\boldsymbol{L}$ denotes the Lie derivative.[8] Note that

$$\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_{\parallel}}\boldsymbol{\mathcal{V}}_{\parallel} = \psi_{t*}\nabla^{\boldsymbol{g}_t}_{\psi_t^*\boldsymbol{\Upsilon}_{\parallel}}\psi_t^*\boldsymbol{\Upsilon}_{\parallel} - \sum_{i=1}^{k}\boldsymbol{\kappa}_i^t\left(\boldsymbol{\Upsilon}_{\parallel}, \boldsymbol{\Upsilon}_{\parallel}\right)\boldsymbol{\eta}_i^t,$$

$$\begin{aligned}\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_{\parallel}}\boldsymbol{\mathcal{V}}_{\perp} &= \sum_{i=1}^{k}\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}_{\parallel}}\left(\Upsilon_{\perp}^i\boldsymbol{\eta}_i^t\right) = \sum_{i=1}^{k}\left(\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}_{\parallel}}\Upsilon_{\perp}^i\right)\boldsymbol{\eta}_i^t + \sum_{i=1}^{k}\Upsilon_{\perp}^i\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}_{\parallel}}\boldsymbol{\eta}_i^t\\ &= \sum_{i=1}^{k}\left(\tilde{d}\Upsilon_{\perp}^i\cdot\boldsymbol{\Upsilon}_{\parallel}\right)\boldsymbol{\eta}_i^t + \sum_{i=1}^{k}\Upsilon_{\perp}^i\boldsymbol{h}^{\sharp}\cdot\boldsymbol{\kappa}_i^t\cdot\boldsymbol{\Upsilon}_{\parallel} + \sum_{i,j=1}^{k}\Upsilon_{\perp}^i\left(\boldsymbol{\omega}_{ij}^t\cdot\boldsymbol{\Upsilon}_{\parallel}\right)\boldsymbol{\eta}_j^t,\end{aligned}$$

where $\tilde{d}$ denotes the exterior derivative operator on $\mathcal{S}_t$,[9] and where we have used, following (4.3), that for $i \in \{1,\ldots,k\}$[10]

$$\nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}_{\parallel}}\boldsymbol{\eta}_i^t = \boldsymbol{h}^{\sharp}\cdot\boldsymbol{\kappa}_i^t\cdot\boldsymbol{\Upsilon}_{\parallel} + \sum_{j=1}^{k}\left(\boldsymbol{\omega}_{ij}^t\cdot\boldsymbol{\Upsilon}_{\parallel}\right)\boldsymbol{\eta}_j^t.$$

Let us now compute $\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\perp}$. We consider an arbitrary vector field $\boldsymbol{U}$ in $\mathcal{Q}$ such that $\boldsymbol{U}$ is tangent to $\mathcal{S}_t$ in a neighborhood of $\tilde{\varphi}(X,t)$, i.e., $\langle\!\langle\boldsymbol{\mathcal{V}}_{\perp}, \boldsymbol{U}\rangle\!\rangle_{\boldsymbol{h}} = 0$. Hence, $\langle\!\langle\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\perp}, \boldsymbol{U}\rangle\!\rangle_{\boldsymbol{h}} = -\langle\!\langle\boldsymbol{\mathcal{V}}_{\perp}, \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U}\rangle\!\rangle_{\boldsymbol{h}}$. However, at $\tilde{\varphi}(X,t)$, we have

$$\begin{aligned}\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{U} &= [\boldsymbol{\mathcal{V}}, \boldsymbol{U}] + \nabla^{\boldsymbol{h}}_{\boldsymbol{U}}\boldsymbol{\mathcal{V}} = [\boldsymbol{\mathcal{V}}, \boldsymbol{U}] + \nabla^{\boldsymbol{h}}_{\boldsymbol{U}}\boldsymbol{\mathcal{V}}_{\parallel} + \nabla^{\boldsymbol{h}}_{\boldsymbol{U}}\boldsymbol{\mathcal{V}}_{\perp}\\ &= [\boldsymbol{\mathcal{V}}, \boldsymbol{U}] + \psi_{t*}\nabla^{\boldsymbol{g}_t}_{\psi_t^*\boldsymbol{U}}\psi_t^*\boldsymbol{\Upsilon}_{\parallel} - \sum_{i=1}^{k}\boldsymbol{\kappa}_i^t\left(\boldsymbol{\Upsilon}_{\parallel}, \boldsymbol{U}\right)\boldsymbol{\eta}_i^t + \sum_{i=1}^{k}\left(\tilde{d}\Upsilon_{\perp}^i\cdot\boldsymbol{U}\right)\boldsymbol{\eta}_i^t\\ &\quad + \sum_{i=1}^{k}\Upsilon_{\perp}^i\nabla^{\boldsymbol{h}}_{\boldsymbol{U}}\boldsymbol{\eta}_i^t.\end{aligned}$$

[8] The Lie derivative along the vector field $\boldsymbol{\mathcal{V}}$ is defined as $\boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=s}\left[\left(\tilde{\varphi}_t\circ\tilde{\varphi}_s^{-1}\right)^*\boldsymbol{\mathcal{V}}_{\parallel}\right]$, where $\tilde{\varphi}_t\circ\tilde{\varphi}_s^{-1}$ is the flow of $\boldsymbol{\mathcal{V}}$.

[9] For a function f defined on $\mathcal{S}_t$, we write $\tilde{d}f = \sum_{\alpha=1}^{n}\frac{\partial f}{\partial\chi^{\alpha}}d\chi^{\alpha}$.

[10] Recall that the normal fundamental 1-forms are defined for $i, j \in \{1,\ldots,k\}$ as $\boldsymbol{\omega}_{ij}^t\cdot\boldsymbol{w} = \left\langle\!\!\left\langle\nabla^{\boldsymbol{h}}_{\boldsymbol{w}}\boldsymbol{\eta}_i^t, \boldsymbol{\eta}_j^t\right\rangle\!\!\right\rangle_{\boldsymbol{h}}$ for any vector $\boldsymbol{w}$ tangent to $\mathcal{S}_t$.

Thus, we have[11]

$$\left\langle\!\left\langle \boldsymbol{\mathcal{V}}_{\perp}, \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}} \boldsymbol{U} \right\rangle\!\right\rangle_{\boldsymbol{h}} = -\sum_{i=1}^{k} \Upsilon^i_{\perp} \boldsymbol{\kappa}^t_i(\boldsymbol{\Upsilon}_{\parallel}, \boldsymbol{U}) + \sum_{i=1}^{k} \Upsilon^i_{\perp} \left(\tilde{d}\Upsilon^i_{\perp} \cdot \boldsymbol{U}\right) + \sum_{i,j=1}^{k} \Upsilon^i_{\perp} \Upsilon^j_{\perp} \left(\boldsymbol{\omega}^t_{ij} \cdot \boldsymbol{U}\right),$$

where we recall that $\left(\boldsymbol{\omega}^t_{ij} \cdot \boldsymbol{U}\right) = \left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{U}} \boldsymbol{\eta}^t_i, \boldsymbol{\eta}^t_j \right\rangle\!\right\rangle_{\boldsymbol{h}}$. Therefore, it follows from $\langle\!\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}} \boldsymbol{\mathcal{V}}_{\perp}, \boldsymbol{U} \rangle\!\rangle_{\boldsymbol{h}} = -\langle\!\langle \boldsymbol{\mathcal{V}}_{\perp}, \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}} \boldsymbol{U} \rangle\!\rangle_{\boldsymbol{h}}$ and by arbitrariness of $\boldsymbol{U}$ that

$$\left(\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}} \boldsymbol{\mathcal{V}}_{\perp}\right)_{\parallel} = \sum_{i=1}^{k} \Upsilon^i_{\perp} \boldsymbol{h}^{\sharp} \cdot \boldsymbol{\kappa}^t_i \cdot \boldsymbol{\Upsilon}_{\parallel} - \sum_{i=1}^{k} \Upsilon^i_{\perp} \left(\tilde{d}\Upsilon^i_{\perp}\right)^{\sharp} - \sum_{i,j=1}^{k} \Upsilon^i_{\perp} \Upsilon^j_{\perp} \boldsymbol{\omega}^{t\sharp}_{ij}.$$

On the other hand, we have

$$\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}} \boldsymbol{\mathcal{V}}_{\perp} = \sum_{i=1}^{k} \nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}} \left(\mathcal{V}^i_{\perp} \boldsymbol{\eta}^t_i\right) = \sum_{i=1}^{k} \frac{\mathrm{d}\mathcal{V}^i_{\perp}}{\mathrm{d}t} \boldsymbol{\eta}^t_i + \sum_{i=1}^{k} \mathcal{V}^i_{\perp} \left(\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}_{\parallel}} \boldsymbol{\eta}^t{}_i + \sum_{j=1}^{k} \mathcal{V}^j_{\perp} \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}^t_j} \boldsymbol{\eta}^t{}_i\right).$$

Then, it follows that at $\tilde{\varphi}(X, t)$, one can write

$$\begin{aligned}\left(\nabla^{\boldsymbol{h}}_{\boldsymbol{\mathcal{V}}} \boldsymbol{\mathcal{V}}_{\perp}\right)_{\perp} &= \sum_{i=1}^{k} \frac{\mathrm{d}\Upsilon^i_{\perp}}{\mathrm{d}t} \boldsymbol{\eta}^t_i + \sum_{i,j=1}^{k} \Upsilon^i_{\perp} \left(\boldsymbol{\omega}^t_{ij} \cdot \boldsymbol{\Upsilon}_{\parallel}\right) \boldsymbol{\eta}^t{}_j \\ &\quad + \sum_{i,j,l=1}^{k} \Upsilon^i_{\perp} \Upsilon^j_{\perp} \left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}^t_j} \boldsymbol{\eta}^t{}_i, \boldsymbol{\eta}^t_l \right\rangle\!\right\rangle_{\boldsymbol{h}} \boldsymbol{\eta}^t_l \\ &= \sum_{i=1}^{k} \left[\frac{\mathrm{d}\Upsilon^i_{\perp}}{\mathrm{d}t} + \sum_{j=1}^{k} \Upsilon^j_{\perp} \left(\boldsymbol{\omega}^t_{ji} \cdot \boldsymbol{\Upsilon}_{\parallel}\right) + \sum_{j,l=1}^{k} \Upsilon^j_{\perp} \Upsilon^l_{\perp} \left\langle\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}^t_j} \boldsymbol{\eta}^t{}_l, \boldsymbol{\eta}^t_i \right\rangle\!\right\rangle_{\boldsymbol{h}}\right] \boldsymbol{\eta}^t{}_i.\end{aligned}$$

Finally, the tangent and normal components of the acceleration vector read

$$\begin{aligned}\boldsymbol{\Gamma}_{\parallel} &= \boldsymbol{L}_{\boldsymbol{\mathcal{V}}} \boldsymbol{\mathcal{V}}_{\parallel} + \psi_{t*} \nabla^{\boldsymbol{g}_t}_{\psi^*_t \boldsymbol{\Upsilon}_{\parallel}} \psi^*_t \boldsymbol{\Upsilon}_{\parallel} + 2\sum_{i=1}^{k} \Upsilon^i_{\perp} \boldsymbol{h}^{\sharp} \cdot \boldsymbol{\kappa}^t_i \cdot \boldsymbol{\Upsilon}_{\parallel} - \sum_{i=1}^{k} \Upsilon^i_{\perp} \left(\tilde{d}\Upsilon^i_{\perp}\right)^{\sharp} \\ &\quad - \sum_{i,j=1}^{k} \Upsilon^i_{\perp} \Upsilon^j_{\perp} \boldsymbol{\omega}^{t\sharp}_{ij},\end{aligned} \tag{2.10a}$$

[11] Note that since the vector $\boldsymbol{U}$ is tangent to $\mathcal{S}_t$ at $\tilde{\varphi}(X, t)$, the vector $[\boldsymbol{\mathcal{V}}, \boldsymbol{U}] = \boldsymbol{L}_{\boldsymbol{\mathcal{V}}} \boldsymbol{U}$ is tangent to $\mathcal{S}_t$ as well.

$$\boldsymbol{\Gamma}_{\perp} = \sum_{i=1}^{k} \left[\frac{\mathrm{d}\Upsilon_{\perp}^{i}}{\mathrm{d}t} + \tilde{\mathrm{d}}\Upsilon_{\perp}^{i} \cdot \boldsymbol{\mathcal{V}}_{\parallel} - \boldsymbol{\kappa}_{i}^{t}\left(\boldsymbol{\Upsilon}_{\parallel}, \boldsymbol{\Upsilon}_{\parallel}\right) + 2\sum_{j=1}^{k} \Upsilon_{\perp}^{j}\left(\boldsymbol{\omega}_{ji}^{t} \cdot \boldsymbol{\Upsilon}_{\parallel}\right) + \sum_{j,l=1}^{k} \Upsilon_{\perp}^{j}\Upsilon_{\perp}^{l} \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}_{j}^{t}} \boldsymbol{\eta}^{t}{}_{l}, \boldsymbol{\eta}_{i}^{t} \right\rangle\!\!\right\rangle_{\boldsymbol{h}} \right] \boldsymbol{\eta}_{i}^{t}. \tag{2.10b}$$

We recall that $\boldsymbol{\Upsilon} = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta} \circ \varphi_t$, $\boldsymbol{Z} := \psi_t^*\boldsymbol{\zeta}_{\parallel} \circ \varphi_t$, $\boldsymbol{A} := \psi_t^*\boldsymbol{\Gamma}_{\parallel}$, $\boldsymbol{k}_i^t = \psi_t^*\boldsymbol{\kappa}_i^t$, and $\boldsymbol{o}_{ij}^t = \psi_t^*\boldsymbol{\omega}_{ij}^t$. However, following (Marsden and Hughes 1983, Theorem 6.19, p. 101), and recalling that $\boldsymbol{\mathcal{V}} = \boldsymbol{\Upsilon} = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta}$, one can write

$$\psi_t^*\boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel} = \boldsymbol{L}_{\boldsymbol{V}}\psi_t^*\boldsymbol{\Upsilon}_{\parallel} = \boldsymbol{L}_{\boldsymbol{V}}\left(\boldsymbol{V}+\boldsymbol{Z}\right) = \frac{\partial}{\partial t}\left(V^a + Z^a\right)\partial_a + \left[\boldsymbol{V}, \boldsymbol{V}+\boldsymbol{Z}\right].$$

Note that since the connection is torsion-free, it follows that $[\boldsymbol{V}, \boldsymbol{V}+\boldsymbol{Z}] = \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}}(\boldsymbol{V}+\boldsymbol{Z}) - \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{V}$. Denoting by $D_t^{\boldsymbol{g}_t}$ the covariant derivative along the curve $t \to \varphi_t(X)$, one can write

$$D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) = \frac{\partial}{\partial t}\left(V^a + Z^a\right)\partial_a + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}}\left(\boldsymbol{V}+\boldsymbol{Z}\right).$$

Therefore, one concludes that

$$\psi_t^*\boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel} = D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) - \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{V}.$$

We also have

$$\nabla^{\boldsymbol{g}_t}_{\psi_t^*\boldsymbol{\Upsilon}_{\parallel}}\psi_t^*\boldsymbol{\Upsilon}_{\parallel} = \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\left(\boldsymbol{V}+\boldsymbol{Z}\right).$$

Hence

$$\psi_t^*\boldsymbol{L}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}_{\parallel} + \nabla^{\boldsymbol{g}_t}_{\psi_t^*\boldsymbol{\Upsilon}_{\parallel}}\psi_t^*\boldsymbol{\Upsilon}_{\parallel} = \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}}\boldsymbol{Z} - \nabla^{\boldsymbol{g}_t}_{\boldsymbol{Z}}\boldsymbol{V} + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\left(\boldsymbol{V}+\boldsymbol{Z}\right) = D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{Z}.$$

Therefore, denoting by d the exterior derivative on $\mathcal{S}$,[12] one can rewrite (2.10) as

$$\boldsymbol{A} = D_t^{\boldsymbol{g}_t}\left(\boldsymbol{V}+\boldsymbol{Z}\right) + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{Z} + 2\sum_{i=1}^{k}\zeta_{\perp}^{i}\boldsymbol{g}_t^{\sharp}\cdot\boldsymbol{k}_i^t\cdot\left(\boldsymbol{V}+\boldsymbol{Z}\right) - \sum_{i=1}^{k}\zeta_{\perp}^{i}\left(\mathrm{d}\zeta_{\perp}^{i}\right)^{\sharp} - \sum_{i,j=1}^{k}\zeta_{\perp}^{i}\zeta_{\perp}^{j}\boldsymbol{o}_{ij}^{t\sharp}, \tag{2.11a}$$

$$\varGamma_{\perp}^{i} = \frac{\mathrm{d}\zeta_{\perp}^{i}}{\mathrm{d}t} + \mathrm{d}\zeta_{\perp}^{i}\cdot\left(\boldsymbol{V}+\boldsymbol{Z}\right) - \boldsymbol{k}_i^t\left(\boldsymbol{V}+\boldsymbol{Z}, \boldsymbol{V}+\boldsymbol{Z}\right) - 2\sum_{j=1}^{k}\zeta_{\perp}^{j}\boldsymbol{o}_{ij}^{t}\cdot\left(\boldsymbol{V}+\boldsymbol{Z}\right) + \sum_{j,l=1}^{k}\zeta_{\perp}^{j}\zeta_{\perp}^{l}\left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{\eta}_{j}^{t}}\boldsymbol{\eta}^{t}{}_{l}, \boldsymbol{\eta}_{i}^{t}\right\rangle\!\!\right\rangle_{\boldsymbol{h}}, \tag{2.11b}$$

[12] For a function f defined on $\mathcal{S}$, we write $\mathrm{d}f = \sum_{a=1}^{n}\frac{\partial f}{\partial x^a}\mathrm{d}x^a$.

where $i = 1, \dots, k$ and $\mathrm{d}\zeta_\perp^i = \sum_{a=1}^n \frac{\partial \zeta_\perp^i}{\partial x^a} dx^a$. □

Let us next turn to the variation of the elastic energy, which is calculated as

$$\delta W = \frac{\partial W}{\partial \tilde{\boldsymbol{F}}} : \boldsymbol{L}_{\delta\tilde{\varphi}} \tilde{\boldsymbol{F}} + \frac{\partial W}{\partial \boldsymbol{h}} : \boldsymbol{L}_{\delta\tilde{\varphi}} \boldsymbol{h}.$$

However, note that for an arbitrary time-independent material vector field $\boldsymbol{U}$, one has

$$\boldsymbol{L}_{\delta\tilde{\varphi}} \tilde{\boldsymbol{F}} \boldsymbol{U} = \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon} \left(\tilde{\varphi}_\epsilon \circ \tilde{\varphi}_s^{-1} \right)^* \tilde{\boldsymbol{F}} \boldsymbol{U} \right]_{s=\epsilon} = \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon} \tilde{\varphi}_{s*} \tilde{\varphi}_\epsilon^* \tilde{\varphi}_{\epsilon*} \boldsymbol{U} \right]_{s=\epsilon} = \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon} \tilde{\varphi}_{s*} \boldsymbol{U} \right]_{s=\epsilon} = \boldsymbol{0}. \tag{2.12}$$

Thus, $\boldsymbol{L}_{\delta\tilde{\varphi}} \tilde{\boldsymbol{F}} = \boldsymbol{0}$. We also obtain, by using (4.11) and similarly to (4.12), that

$$\boldsymbol{L}_{\delta\tilde{\varphi}} \boldsymbol{h} = \mathfrak{L}_{\delta\tilde{\varphi}} \boldsymbol{h} = \psi_* \left(\mathfrak{L}_{\psi^* \delta\tilde{\varphi}_\parallel} \boldsymbol{g}_t + 2 \sum_{i=1}^k \delta\tilde{\varphi}_\perp^i \boldsymbol{k}_i^t \right), \tag{2.13}$$

where $\delta\tilde{\varphi}_\parallel$ is the part of $\delta\tilde{\varphi}$ tangent to $\psi_t(\mathcal{S})$, $\delta\tilde{\varphi}_\perp^i$, for $i \in \{1, \dots, k\}$, is its component along the unit normal $\boldsymbol{\eta}_i^t$, and $\mathfrak{L}$ denotes the autonomous Lie derivative. Therefore, recalling (2.2), it follows that

$$\delta W = \frac{\partial W}{\partial \boldsymbol{h}} : \left[\psi_* \left(\mathfrak{L}_{\psi^* \delta\tilde{\varphi}_\parallel} \boldsymbol{g}_t + 2 \sum_{i=1}^k \delta\tilde{\varphi}_\perp^i \boldsymbol{k}_i^t \right) \right] = \frac{\partial W}{\partial \boldsymbol{g}} : \left(\mathfrak{L}_{\psi^* \delta\tilde{\varphi}_\parallel} \boldsymbol{g}_t + 2 \sum_{i=1}^k \delta\tilde{\varphi}_\perp^i \boldsymbol{k}_i^t \right). \tag{2.14}$$

Let us first assume that the variations of $\tilde{\varphi}$ are tangent to $\mathcal{S}_t$, i.e., $\delta\tilde{\varphi}_\perp^i = 0, \forall i \in \{1, \dots, k\}$. Therefore, the variation of the action associated with the elastic energy reads

$$\delta S_W = - \int_{t_0}^{t_1} \int_{\mathcal{B}} \rho_0 \frac{\partial W}{\partial \boldsymbol{g}} : \mathfrak{L}_{\psi^* \delta\tilde{\varphi}_\parallel} \boldsymbol{g}_t \mathrm{d}V \mathrm{d}t.$$

Note, however, that $\mathfrak{L}_{\psi^* \delta\tilde{\varphi}_\parallel} \boldsymbol{g}_t = \nabla^{\boldsymbol{g}_t} \psi^* \delta\tilde{\varphi}_\parallel^\flat + \left[\nabla^{\boldsymbol{g}_t} \psi^* \delta\tilde{\varphi}_\parallel^\flat \right]^\mathsf{T}$. Hence, by symmetry of $\boldsymbol{g}_t$, one can write

$$\begin{aligned}
\delta S_W &= - \int_{t_0}^{t_1} \int_{\mathcal{B}} 2\rho_0 \frac{\partial W}{\partial \boldsymbol{g}} : \nabla^{\boldsymbol{g}_t} \psi^* \delta\tilde{\varphi}_\parallel^\flat \mathrm{d}V \mathrm{d}t \\
&= - \int_{t_0}^{t_1} \int_{\varphi_t(\mathcal{B})} 2\rho \frac{\partial W}{\partial \boldsymbol{g}} : \nabla^{\boldsymbol{g}_t} \psi^* \delta\tilde{\varphi}_\parallel^\flat \mathrm{d}v \mathrm{d}t \\
&= - \int_{t_0}^{t_1} \int_{\varphi_t(\mathcal{B})} 2\rho \frac{\partial W}{\partial g_{ab}} \left(\psi^* \delta\tilde{\varphi}_\parallel \right)_{a|b} \mathrm{d}v \mathrm{d}t \\
&= - \int_{t_0}^{t_1} \int_{\varphi_t(\mathcal{B})} \mathrm{div}_{\boldsymbol{g}_t} \left(2\rho \frac{\partial W}{\partial \boldsymbol{g}} \cdot \psi^* \delta\tilde{\varphi}_\parallel^\flat \right) \mathrm{d}v \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle \mathrm{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right), \psi^*\delta\tilde{\varphi}_{\parallel}\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} \mathrm{d}v\mathrm{d}t \\
&= -\int_{t_0}^{t_1}\int_{\partial\varphi_t(\mathcal{B})}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\cdot\psi^*\delta\tilde{\varphi}_{\parallel}^{\flat}\right)\cdot\mathbf{n}\mathrm{d}a\mathrm{d}t \\
&\quad+\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle \mathrm{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right), \psi^*\delta\tilde{\varphi}_{\parallel}\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} \mathrm{d}v\mathrm{d}t,
\end{aligned}
$$

where ρ is the mass density in $\mathcal{S}$, $\mathrm{div}_{\boldsymbol{g}_t}$ (surface divergence) denotes the divergence operator in $(\mathcal{S}, \boldsymbol{g}_t)$, and $\mathbf{n}$ is the unit normal vector to $\partial\varphi_t(\mathcal{B})$ in $\mathcal{S}$. Therefore, assuming that the value of $\tilde{\varphi}$ is fixed on the boundary, i.e., $\delta\tilde{\varphi}|_{\partial\varphi(\mathcal{B})} = 0$, one obtains

$$
\delta S_W = \int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle \mathrm{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right), \psi^*\delta\tilde{\varphi}_{\parallel}\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} \mathrm{d}v\mathrm{d}t. \tag{2.15}
$$

Hence, by (2.5) and (2.15), the Lagrange-d'Alembert principle (2.3) reads

$$
\begin{aligned}
&\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle -\rho\boldsymbol{A} + \mathrm{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right), \psi^*\delta\tilde{\varphi}_{\parallel}\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} \mathrm{d}v\mathrm{d}t \\
&\quad+\int_{t_0}^{t_1}\int_{\varphi_t(\mathcal{B})}\left\langle\!\!\left\langle \rho\boldsymbol{B}, \psi^*\delta\tilde{\varphi}_{\parallel}\right\rangle\!\!\right\rangle_{\boldsymbol{g}_t} \mathrm{d}v\mathrm{d}t = 0,
\end{aligned}
$$

where $\boldsymbol{B} = \psi^*\boldsymbol{\beta}_{\parallel}$. Therefore, by arbitrariness of $\delta\tilde{\varphi}_{\parallel}$, one obtains the following tangent Euler–Lagrange equations

$$
\mathrm{div}_{\boldsymbol{g}_t}\left(2\rho\frac{\partial W}{\partial \boldsymbol{g}}\right) + \rho\boldsymbol{B} = \rho\boldsymbol{A}. \tag{2.16}
$$

In terms of the Cauchy stress tensor $\boldsymbol{\sigma} = 2\rho\frac{\partial W}{\partial \boldsymbol{g}}$, we have[13]

$$
\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma} + \rho\boldsymbol{B} = \rho\boldsymbol{A}. \tag{2.17}
$$

In the particular case, when $\mathcal{S}_t$ is a hyperspace evolving transversally in $\mathcal{Q}$, i.e., $k = 1$ and $\boldsymbol{Z} = \boldsymbol{0}$, the tangent Euler–Lagrange equations read

$$
\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma} + \rho\boldsymbol{B} = \rho D_t^{\boldsymbol{g}_t}\boldsymbol{V} + 2\rho\zeta_{\perp}\boldsymbol{g}_t^{\sharp}\cdot\boldsymbol{k}^t\cdot\boldsymbol{V} - \rho\zeta_{\perp}(\mathrm{d}\zeta_{\perp})^{\sharp}. \tag{2.18}
$$

Equivalently, in terms of the rate of change in the spatial metric one has (c.f. (4.9))

$$
\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma} + \rho\boldsymbol{B} = \rho D_t^{\boldsymbol{g}_t}\boldsymbol{V} + \rho\boldsymbol{g}_t^{\sharp}\cdot\frac{\partial\boldsymbol{g}_t}{\partial t}\cdot\boldsymbol{V} - \rho\zeta_{\perp}(\mathrm{d}\zeta_{\perp})^{\sharp}. \tag{2.19}
$$

13 An alternate proof of this result for the special case of a transversal embedding is given in Appendix 2.

Now, we assume that the variations of $\tilde{\varphi}$ are normal to $\mathcal{S}_t$, i.e., $\delta\tilde{\varphi}_{\parallel} = 0$. Using (2.5) and (2.14), we obtain from (2.3), by arbitrariness of $\delta\tilde{\varphi}^i_{\perp}$, the following normal Euler–Lagrange equations

$$- 2\rho_0 \frac{\partial W}{\partial \boldsymbol{g}} : \boldsymbol{k}^t_i + \rho_0 B^i_{\perp} = \rho_0 \Gamma^i_{\perp}, \quad i = 1, \ldots, k. \tag{2.20}$$

In terms of the Cauchy stress, one has

$$- \boldsymbol{\sigma} : \boldsymbol{k}^t_i + \rho B^i_{\perp} = \rho \Gamma^i_{\perp}, \quad i = 1, \ldots, k. \tag{2.21}$$

Remark 2.3 Equation (2.18) is identical to the tangential component of Scriven (1960)'s Eq. (27). However, we believe that the expression of the acceleration he wrote before his Eq. (16) should be corrected to include the extra terms depending on the second fundamental form and the gradient of the embedding normal velocity as can be seen in (2.18). If one neglects the inertial terms, Eq. (2.18) is identical to Arroyo and DeSimone (2009)'s Eq. (4). However, it is not identical to their Eq. (3) in the presence of inertial effects. For them, acceleration reads

$$\frac{\partial \boldsymbol{V}}{\partial t} + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}} \boldsymbol{V} + V_n H \boldsymbol{V},$$

where $H = g^{ab} k_{ab}$ is twice the mean curvature. Note that, when $\boldsymbol{Z} = \boldsymbol{0}$ and $k = 1$, their $\boldsymbol{k}$ is our $-\boldsymbol{k}^t$, $V_n := \zeta_{\perp}$, and their acceleration should be corrected to read (in their notation)[14]

$$\frac{\partial \boldsymbol{V}}{\partial t} + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}} \boldsymbol{V} - 2V_n \boldsymbol{k} \cdot \boldsymbol{V} - V_n \left(\mathrm{d}V_n\right)^{\sharp} .$$

We also note that in the case of a 2D shell embedded as a hypersurface in $\mathbb{R}^3$, (2.21) is identical to the normal component of Scriven (1960)'s Eq. (27), although Scriven (1960) did not write down the expression of the extrinsic acceleration. Ignoring the inertial terms, Eq. (2.21) is identical to Arroyo and DeSimone (2009)'s Eq. (5).

2.2 Conservation of Mass for Motion in an Evolving Ambient Space

Locally, conservation of mass is equivalent to (Simo and Marsden 1984a)

$$\rho(\boldsymbol{x}, t) J(\boldsymbol{X}, t) = \rho_0(\boldsymbol{X}),$$

[14] We communicated with A. DeSimone and M. Arroyo, and they kindly confirmed the mistake in their acceleration. They indicated that they followed the *master balance law* of Marsden and Hughes (1983, p. 129). In Appendix 2, we show this derivation by using the *master balance law* and demonstrate that the results are identical to those we obtain using Hamilton's principle.

where $J(\boldsymbol{X},t)=\sqrt{\frac{\det \boldsymbol{g}_t}{\det \boldsymbol{G}}}\det \boldsymbol{F}$ is the Jacobian of deformation mapping φ.[15] Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho J)=0.$$

Note that

$$\dot{J}=\left(\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{V}\right)J+\frac{1}{2}J\,\mathrm{tr}\left(\frac{\partial \boldsymbol{g}_t}{\partial t}\right),$$

where the superposed dot denotes total time differentiation, i.e., $\dot{J}=\frac{\mathrm{d}J}{\mathrm{d}t}$. Therefore[16]

$$\dot{\rho}+\rho\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{V}+\frac{1}{2}\rho\,\mathrm{tr}\left(\frac{\partial \boldsymbol{g}_t}{\partial t}\right)=0. \tag{2.22}$$

Note that even if $\boldsymbol{V}=\boldsymbol{0}$, ρ is time dependent. Therefore, in the case of a 2D shell transversally embedded in $\mathbb{R}^3$—recalling Lemma 3.2, which in this case reads $\frac{\partial \boldsymbol{g}_t}{\partial t}=2\zeta_\perp \boldsymbol{k}^t+\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat+\left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}^\flat\right]^{\mathsf{T}}$—(2.22) can be written as

$$\dot{\rho}+\rho\,\mathrm{div}_{\boldsymbol{g}_t}(\boldsymbol{V}+\boldsymbol{Z})+\rho\zeta_\perp H=0, \tag{2.23}$$

where $H=\mathrm{tr}\,\boldsymbol{k}^t$ is twice the mean curvature. Equation (2.23) is identical to the conservation of mass for shells appearing in Scriven 1960, Eq. (21), Marsden and Hughes 1983, p. 92, and Arroyo and DeSimone 2009, (Eq. 1). Note that, if we look at the spatial mass density form $\boldsymbol{\rho}:=\rho\mathrm{d}v$, (2.22) reads

$$\boldsymbol{L}_{\boldsymbol{V}}\boldsymbol{\rho}=0. \tag{2.24}$$

Equivalently, one can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(\mathcal{U})}\rho\mathrm{d}v=\int_{\varphi_t(\mathcal{U})}\boldsymbol{L}_{\boldsymbol{V}}(\rho\mathrm{d}v)=\int_{\varphi_t(\mathcal{U})}\left[\boldsymbol{L}_{\boldsymbol{V}}\rho\,\mathrm{d}v+\rho\boldsymbol{L}_{\boldsymbol{V}}(\mathrm{d}v)\right]=0. \tag{2.25}$$

We know that

$$\boldsymbol{L}_{\boldsymbol{V}}(\mathrm{d}v)=\mathfrak{L}_{\boldsymbol{V}}(\mathrm{d}v)+\frac{\partial}{\partial t}(\mathrm{d}v)=\left[\mathrm{div}\,\boldsymbol{V}+\frac{1}{2}\mathrm{tr}\left(\frac{\partial \boldsymbol{g}_t}{\partial t}\right)\right]\mathrm{d}v. \tag{2.26}$$

Substituting (2.26) into (2.25) and localizing gives (2.22), which is the local form of conservation of mass.

[15] Note that the Jacobian of the deformation $\tilde{\varphi}$ is equal to that of φ, i.e., $\sqrt{\frac{\det \boldsymbol{h}}{\det \boldsymbol{G}}}\det \tilde{\boldsymbol{F}}=\sqrt{\frac{\det \boldsymbol{g}_t}{\det \boldsymbol{G}}}\det \boldsymbol{F}$, which follows from $\boldsymbol{g}_t:=\psi_t^*\boldsymbol{h}$.

[16] Note that there is a typo in the corresponding equation in Marsden and Hughes (1983), p. 92.

2.3 Energy Balance in Nonlinear Elasticity in an Evolving Ambient Space

Let us consider an elastic body deforming in an evolving ambient space. We are interested in making an explicit connection between the deformation of the elastic body embedded in this ambient space and that in an ambient space with a dynamic metric. Let the ambient space $\mathcal{S}$ move in a larger (fixed) manifold $\mathcal{Q}$, i.e., $\psi_t : \mathcal{S} \to \mathcal{Q}$. The fixed background metric in $(\mathcal{Q}, \boldsymbol{h})$ induces a time-dependent metric on $\mathcal{S}$, i.e., $\boldsymbol{g}_t = \psi_t^* \boldsymbol{h}$. Energy balance can be easily written in $\mathcal{Q}$, but we are interested to see how it is written for an observer in $\mathcal{S}$. When the metric $\boldsymbol{g}$ of $\mathcal{S}$ is fixed, the standard material balance of energy for a given subset $\mathcal{U} \subset \mathcal{B}$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2} \langle\!\langle \boldsymbol{V}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}} \right) \mathrm{d}V = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \boldsymbol{B}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}} + R \right) \mathrm{d}V + \int_{\partial\mathcal{U}} \left(\langle\!\langle \boldsymbol{T}, \boldsymbol{V} \rangle\!\rangle_{\boldsymbol{g}} + H \right) \mathrm{d}A, \tag{2.27}$$

where E, R, H, and $\boldsymbol{T}$ are the internal energy function per unit mass, the heat supply per unit mass, the heat flux per unit undeformed area, and the boundary traction vector per unit undeformed area, respectively. Note that $E = E(X, \mathsf{N}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g})$, where N is the specific entropy. However, here the ambient space is evolving in time and the energy balance should be modified to accommodate this time dependency. First, let us look at an example to motivate our discussion.

Example 2.1 Suppose the ambient space is a two-dimensional sphere of radius R that shrinks/expands in time. Then, whatever elastic material lives on this sphere will be compressed/stretched over time. As a simple case, assume that the material manifold is also a sphere, with radius equal to the initial radius of the ambient sphere. Assume that the deformation map φ is constant over time, as the metric evolves. This means that there will be an increase in elastic energy over time, not accounted for in terms of the work done by external forces—since there are no external forces.

Let the ambient metric, as a function of time be $g_{ij}(\theta, \phi, t) = f(t) g_{ij}^{\text{sphere}}(R)(\theta, \phi)$, where t is time, $f(t)$ is some function of time (the shrinkage/expansion factor) such that $f(t) > 0,\ f(t_0) = 1$, and $\boldsymbol{g}_t^{\text{sphere}}(R)$ is the metric of the 2-sphere with radius R. Note that this is a uniform rescaling of the metric. Then, let the material manifold be just $G_{IJ}(\Theta, \Phi) = G_{IJ}^{\text{sphere}}(R)(\Theta, \Phi)$, and let the deformation map simply send Θ to θ and Φ to ϕ at all times. Therefore, even though the material "is not moving" in terms of the coordinates ϕ and θ (a given material point sits at the same ϕ and θ at all times), it is shrinking/expanding. Note that $\Psi = \Psi(\boldsymbol{X}, \boldsymbol{C})$, where $C_{AB} = F^a{}_A F^b{}_B g_{ab} f(t)$. Thus, even if $F^a{}_B = \delta^a{}_A$, we have $C_{AB} = \delta^a{}_A \delta^b{}_B g_{ab} f(t)$. This means that Ψ explicitly depends on $f(t)$ and hence there is stored elastic energy coming from the changes in the ambient space metric.

To visualize the time dependency of the metric of the ambient space, let us embed the initial sphere of radius $r = R$ in the Euclidean space $\mathbb{R}^3$. We then assume that the ambient space moves in the Euclidean space, i.e., there is a map $\psi_t : S^2(r) \to \mathbb{R}^3$. Explicitly this can be written in the spherical coordinates as

$(\tilde{r}, \tilde{\theta}, \tilde{\phi}) = \psi_t(r, \theta, \phi) = (k(t)r, \theta, \phi)$ with $k(t) > 0$. Note that deformation mapping is the inclusion map, i.e., $(\theta, \phi) = \varphi_t(\Theta, \Phi) = (\Theta, \Phi)$. The metric of the Euclidean space in spherical coordinates reads $\boldsymbol{h} = \mathrm{diag}\left(1, \tilde{r}^2, \tilde{r}^2 \sin^2\theta\right)$. Now the map ψ_t induces a metric $\boldsymbol{g}_t = \psi_t^* \boldsymbol{h}$ on the ambient space that has the following representation: $\boldsymbol{g}_t = \mathrm{diag}\left(k(t)^2 r^2, k(t)^2 r^2 \sin^2\theta\right)$. It is seen that $f(t) = k(t)^2$, i.e., when expanding the ambient space by k, all the square distances in the ambient space with the time-dependent metric are multiplied by $f = k^2$ as expected. It is seen that time dependency of the ambient space metric can be visualized using a time-dependent embedding in a larger space with a fixed background metric (see Yavari (2010) for a similar discussion). In the following, we look at this in the general case of an arbitrary deformable body.

Next, we prove the following proposition for an arbitrary deformable body:

Proposition 2.2 *Energy balance for a deformable body moving in an evolving ambient space is given by*

$$\frac{d}{dt}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\langle \boldsymbol{V}, \boldsymbol{V}\rangle\rangle_{\boldsymbol{g}_t}\right) dV = \int_{\mathcal{U}} \rho_0 \Bigg[\langle\langle \boldsymbol{B} + \boldsymbol{F}_{\mathit{fic}}, \boldsymbol{V}\rangle\rangle_{\boldsymbol{g}_t} + \left(\frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t} + \frac{1}{2}\langle\langle \boldsymbol{V}, \boldsymbol{V}\rangle\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}\right) + R\Bigg] dV + \int_{\partial\mathcal{U}} \left(\langle\langle \boldsymbol{T}, \boldsymbol{V}\rangle\rangle_{\boldsymbol{g}_t} + H\right) dA, \tag{2.28}$$

where we recall that $E = E(\boldsymbol{X}, \mathsf{N}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t)$[17] *is the material internal energy density per unit mass and $\mathsf{N}, R, H, \boldsymbol{T}$, and $\boldsymbol{B}$ are the specific entropy per unit mass, the heat supply per unit mass, the heat flux per unit undeformed area, the boundary traction vector per unit undeformed area, and the tangent body force per unit mass, respectively. We also recall that $\boldsymbol{V}$ is the velocity of φ_t and $\boldsymbol{Z} = \psi_t^* \boldsymbol{\zeta}_\parallel$ is the tangent velocity of the embedding ψ_t. $\boldsymbol{F}_{\mathit{fic}}$ denotes the fictitious body force due to the evolution of $\mathcal{S}_t$ and reads*

$$\begin{aligned}\boldsymbol{F}_{\mathit{fic}} &= -\left(\boldsymbol{A} - D_t^{\boldsymbol{g}_t}\boldsymbol{V}\right) = -D_t^{\boldsymbol{g}_t}\boldsymbol{Z} - \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}+\boldsymbol{Z}}\boldsymbol{Z} - 2\sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}_i^t \cdot (\boldsymbol{V} + \boldsymbol{Z}) \\ &\quad + \sum_{i=1}^{k} \zeta_\perp^i \left(d\zeta_\perp^i\right)^\sharp + \sum_{i,j=1}^{k} \zeta_\perp^i \zeta_\perp^j \boldsymbol{o}_{ij}^{t\sharp} \\ &= -D_t^{\boldsymbol{g}_t}\boldsymbol{Z} + \left[\nabla^{\boldsymbol{g}_t}\boldsymbol{Z}\right]^{\mathsf{T}} \cdot (\boldsymbol{V} + \boldsymbol{Z}) - \boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot (\boldsymbol{V} + \boldsymbol{Z}) + \sum_{i=1}^{k} \zeta_\perp^i \left(d\zeta_\perp^i\right)^\sharp \\ &\quad + \sum_{i,j=1}^{k} \zeta_\perp^i \zeta_\perp^j \boldsymbol{o}_{ij}^{t\sharp}.\end{aligned} \tag{2.29}$$

[17] Similar to the discussion of Remark 2.1, we can conclude that $E(\boldsymbol{X}, \mathsf{N}, \psi_* \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{h}) = E(\boldsymbol{X}, \mathsf{N}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g}_t)$.

Note that in the particular case of a transversal evolution of the ambient space as a hypersurface in $\mathcal{Q}$, *i.e.,* $\mathbf{Z} = \mathbf{0}$ *and* $k = 1$, *the fictitious body force reduces to*

$$\boldsymbol{F}_{fic} = -2\zeta_\perp \boldsymbol{g}_t^\sharp \cdot \boldsymbol{k}^t \cdot \boldsymbol{V} + \zeta_\perp (d\zeta_\perp)^\sharp = -\boldsymbol{g}_t^\sharp \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{V} + \zeta_\perp (d\zeta_\perp)^\sharp . \tag{2.30}$$

Proof For an observer in $\mathcal{Q}$, energy balance for a subbody $\mathcal{U} \subset \mathcal{B}$ is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}}\right) \mathrm{d}V = \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\!\left\langle \tilde{\boldsymbol{\beta}}, \boldsymbol{\Upsilon}\right\rangle\!\!\right\rangle_{\boldsymbol{h}} + R\right) \mathrm{d}V + \int_{\partial\mathcal{U}} \left(\left\langle\!\!\left\langle \tilde{\boldsymbol{T}}, \boldsymbol{\Upsilon}\right\rangle\!\!\right\rangle_{\boldsymbol{h}} + H\right) \mathrm{d}A.$$

Body force can be decomposed into tangent and normal components with respect to $\mathcal{S}_t$ as $\boldsymbol{\beta} = \boldsymbol{\beta}_\parallel + \sum_{i=1}^k B_\perp^i \boldsymbol{\eta}_i^t$. Note that the traction vector is tangent to $\mathcal{S}_t$. We denote $\boldsymbol{B} = \psi_t^* \boldsymbol{\beta}_\parallel$, and $\boldsymbol{T} = \psi_t^* \tilde{\boldsymbol{T}}$. Recalling that $\boldsymbol{\Upsilon} = \psi_{t*}\boldsymbol{V} + \boldsymbol{\zeta} \circ \varphi_t$ and $\boldsymbol{Z} = \psi_t^* \boldsymbol{\zeta}_\parallel$, the energy balance is simplified to read

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}}\right) \mathrm{d}V = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \boldsymbol{B}, \boldsymbol{V} + \boldsymbol{Z}\rangle\!\rangle_{\boldsymbol{g}_t} + \sum_{i=1}^k B_\perp^i \zeta_\perp^i + R\right) \mathrm{d}V + \int_{\partial\mathcal{U}} \left(\langle\!\langle \boldsymbol{T}, \boldsymbol{V} + \boldsymbol{Z}\rangle\!\rangle_{\boldsymbol{g}_t} + H\right) \mathrm{d}A. \tag{2.31}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\langle\!\langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}} = \left\langle\!\!\left\langle D_t^{\boldsymbol{h}}\boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}\right\rangle\!\!\right\rangle_{\boldsymbol{h}} = \langle\!\langle \boldsymbol{\Gamma}, \boldsymbol{\Upsilon}\rangle\!\rangle_{\boldsymbol{h}} = \langle\!\langle \boldsymbol{A}, \boldsymbol{V} + \boldsymbol{Z}\rangle\!\rangle_{\boldsymbol{g}_t} + \sum_{i=1}^k \zeta_\perp^i \Gamma_\perp^i, \tag{2.32}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \boldsymbol{L}_{\boldsymbol{\Upsilon}} E = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \tilde{\boldsymbol{F}}} : \boldsymbol{L}_{\boldsymbol{\Upsilon}} \tilde{\boldsymbol{F}} + \frac{\partial E}{\partial \boldsymbol{h}} : \boldsymbol{L}_{\boldsymbol{\Upsilon}} \boldsymbol{h}.$$

Similar to (2.12), we see that $\boldsymbol{L}_{\boldsymbol{\Upsilon}} \tilde{\boldsymbol{F}} = \mathbf{0}$. Note that[18]

$$\boldsymbol{L}_{\boldsymbol{\Upsilon}} \boldsymbol{h} = \boldsymbol{L}_{\psi_* \boldsymbol{V}} \boldsymbol{h} + \boldsymbol{L}_{\boldsymbol{\zeta}} \boldsymbol{h} = \mathfrak{L}_{\psi_* \boldsymbol{V}} \boldsymbol{h} + \psi_{t*}\frac{\partial \boldsymbol{g}_t}{\partial t} = \psi_* \left(\mathfrak{L}_{(\boldsymbol{V}+\boldsymbol{Z})} \boldsymbol{g}_t + \frac{\partial \boldsymbol{g}_t}{\partial t}\right) = \psi_* \boldsymbol{L}_{\boldsymbol{V}} \boldsymbol{g}_t,$$

where we used (4.10) to write $\boldsymbol{L}_{\boldsymbol{\zeta}} \boldsymbol{h} = \psi_{t*}\frac{\partial \boldsymbol{g}_t}{\partial t}$. Therefore, it follows that in $\mathcal{Q}$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \boldsymbol{h}} : \boldsymbol{L}_{\boldsymbol{\Upsilon}} \boldsymbol{h} = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \boldsymbol{g}} : \boldsymbol{L}_{\boldsymbol{V}} \boldsymbol{g}_t. \tag{2.33}$$

[18] An alternate proof for this result can be found in Marsden and Hughes (1983), p. 101.

An observer in $\mathcal{S}$ writes the energy balance as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2}\langle\langle \boldsymbol{V}, \boldsymbol{V}\rangle\rangle_{\boldsymbol{g}_t}\right) \mathrm{d}V = \int_{\mathcal{U}} \rho_0 \left(\langle\langle \boldsymbol{B}, \boldsymbol{V}\rangle\rangle_{\boldsymbol{g}_t} + \Xi + R\right) \mathrm{d}V + \int_{\partial\mathcal{U}} \left(\langle\langle \boldsymbol{T}, \boldsymbol{V}\rangle\rangle_{\boldsymbol{g}_t} + H\right) \mathrm{d}A, \tag{2.34}$$

where $\Xi = 0$ if the ambient space metric is fixed. Note that in $(\mathcal{S}, \boldsymbol{g}_t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\langle\langle \boldsymbol{V}, \boldsymbol{V}\rangle\rangle_{\boldsymbol{g}_t} = \left\langle\!\left\langle D_t^{\boldsymbol{g}_t}\boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \frac{1}{2}\langle\langle \boldsymbol{V}, \boldsymbol{V}\rangle\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}, \quad \text{and} \quad \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial \mathsf{N}}\dot{\mathsf{N}} + \frac{\partial E}{\partial \boldsymbol{g}} : \boldsymbol{L}_{\boldsymbol{V}}\boldsymbol{g}_t. \tag{2.35}$$

Let us now find Ξ. Subtracting (2.34) from (2.31) and using (2.32), (2.33), and (2.35), one obtains

$$\int_{\mathcal{U}} \left(\rho_0 \langle\langle \boldsymbol{A}, \boldsymbol{Z}\rangle\rangle_{\boldsymbol{g}_t} + \rho_0 \left\langle\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t}\boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} - \frac{1}{2}\rho_0 \langle\langle \boldsymbol{V}, \boldsymbol{V}\rangle\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} + \rho_0 \sum_{i=1}^{k} \zeta_\perp^i \Gamma_\perp^i\right) \mathrm{d}V$$
$$= \int_{\mathcal{U}} \left(\rho_0 \langle\langle \boldsymbol{B}, \boldsymbol{Z}\rangle\rangle_{\boldsymbol{g}_t} + \rho_0 \sum_{i=1}^{k} B_\perp^i \zeta_\perp^i - \Xi\right) \mathrm{d}V + \int_{\partial\mathcal{U}} \langle\langle \boldsymbol{T}, \boldsymbol{Z}\rangle\rangle_{\boldsymbol{g}_t} \mathrm{d}A.$$

Note that $2\rho\frac{\partial E}{\partial \boldsymbol{g}} = 2\rho\frac{\partial W}{\partial \boldsymbol{g}} = \boldsymbol{\sigma}$, and $\frac{\partial E}{\partial \boldsymbol{g}} : \boldsymbol{L}_{\boldsymbol{Z}}\boldsymbol{g}_t = 2\frac{\partial E}{\partial \boldsymbol{g}} : \nabla^{\boldsymbol{g}_t}\boldsymbol{Z}$. Therefore, by using (2.16) and (2.20) we have

$$\int_{\mathcal{U}} \left(\rho_0 \left\langle\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t}\boldsymbol{V}, \boldsymbol{V}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} - \frac{1}{2}\rho_0 \langle\langle \boldsymbol{V}, \boldsymbol{V}\rangle\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}\right) \mathrm{d}V = \int_{\partial\mathcal{U}} \langle\langle \boldsymbol{T}, \boldsymbol{Z}\rangle\rangle_{\boldsymbol{g}_t} \mathrm{d}A$$
$$+ \int_{\mathcal{U}} \left(\sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{\sigma} : \boldsymbol{k}_i^t - J\left\langle\!\left\langle \mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma}, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} - \Xi\right) \mathrm{d}V.$$

Also, note that

$$\begin{aligned}
\int_{\partial\mathcal{U}} \langle\langle \boldsymbol{T}, \boldsymbol{Z}\rangle\rangle_{\boldsymbol{g}_t} \mathrm{d}A &= \int_{\partial\varphi_t(\mathcal{U})} \left\langle\!\left\langle \boldsymbol{\sigma}\cdot\boldsymbol{\eta}^t, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t} \mathrm{d}a \\
&= \int_{\varphi_t(\mathcal{U})} \mathrm{div}_{\boldsymbol{g}_t} \langle\langle \boldsymbol{\sigma}, \boldsymbol{Z}\rangle\rangle_{\boldsymbol{g}_t} \mathrm{d}v \\
&= \int_{\varphi_t(\mathcal{U})} \left(\boldsymbol{\sigma} : \nabla^{\boldsymbol{g}_t}\boldsymbol{Z} + \left\langle\!\left\langle \mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma}, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t}\right) \mathrm{d}v \\
&= \int_{\mathcal{U}} \left(\frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{L}_{\boldsymbol{Z}}\boldsymbol{g}_t + J\left\langle\!\left\langle \mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{\sigma}, \boldsymbol{Z}\right\rangle\!\right\rangle_{\boldsymbol{g}_t}\right) \mathrm{d}V.
\end{aligned}$$

Thus

$$\int_{\mathcal{U}} \left(\rho_0 \left\langle\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t} \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} - \frac{1}{2} \rho_0 \left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} \right) \mathrm{d}V$$
$$= \int_{\mathcal{U}} \left(\frac{1}{2} \boldsymbol{\sigma} : \left(\mathbf{L}_{\boldsymbol{Z}} \boldsymbol{g}_t + 2 \sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{k}_i^t \right) - \Xi \right) \mathrm{d}V.$$

However, since $\boldsymbol{\sigma} = 2\rho \frac{\partial W}{\partial \boldsymbol{g}} = 2\rho \frac{\partial E}{\partial \boldsymbol{g}}$, and $\frac{\partial \boldsymbol{g}_t}{\partial t} = 2 \sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{k}_i^t + \mathfrak{L}_{\boldsymbol{Z}} \boldsymbol{g}_t$, it follows that

$$\Xi = \frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t} - \rho_0 \left\langle\!\left\langle \boldsymbol{A} - D_t^{\boldsymbol{g}_t} \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \frac{1}{2} \rho_0 \left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}.$$

Therefore, the balance of energy in $(\mathcal{S}, \boldsymbol{g}_t)$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2} \left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} \right) \mathrm{d}V$$
$$= \int_{\mathcal{U}} \rho_0 \left(\left\langle\!\left\langle \boldsymbol{B} + \boldsymbol{F}_{\mathrm{fic}}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t} + \frac{1}{2} \left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} + R \right) \mathrm{d}V$$
$$+ \int_{\partial \mathcal{U}} \left(\left\langle\!\left\langle \boldsymbol{T}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + H \right) \mathrm{d}A,$$

where the fictitious body force $\boldsymbol{F}_{\mathrm{fic}}$ is given in (2.29). If the evolution of the ambient space as a hyperspace in $\mathcal{Q}$ is transversal, i.e., $\boldsymbol{Z} = \boldsymbol{0}$ and $k = 1$, the fictitious body force reduces to (2.30). □

Remark 2.4 Note that the non-classical extra terms appearing in the energy balance (2.28) can be written as

$$\frac{\partial E}{\partial \boldsymbol{g}} : \frac{\partial \boldsymbol{g}_t}{\partial t} + \frac{1}{2} \left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} = \frac{\partial}{\partial \boldsymbol{g}} \left(E + \frac{1}{2} \left\langle\!\left\langle \boldsymbol{V}, \boldsymbol{V} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} \right) : \frac{\partial \boldsymbol{g}_t}{\partial t},$$

so that this term cancels out the contribution of the rate of change in the energy (internal + kinetic) due to the evolving ambient space metric appearing on the left-hand side of (2.28).

3 Quasi-Static Deformations of the Ambient Space Metric

Let us consider a spatial metric that depends on a position-dependent parameter $\omega(\boldsymbol{x})$, e.g., $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x}, \omega(\boldsymbol{x}))$. In other words, given an initial metric $\boldsymbol{g}_0$, we quasi-statically deform the ambient space manifold. As an application, we can think of a situation when a thin sheet of metal is compressed between two identical and evolving surfaces to make different curved sheets, e.g., some pieces of an automobile body. As an example, one can start with a rescaling of the spatial metric, i.e.,

$$\boldsymbol{g}(\boldsymbol{x}, \omega(\boldsymbol{x})) = \mathrm{e}^{2\omega(\boldsymbol{x})} \boldsymbol{g}_0(\boldsymbol{x}).$$

Note that Jacobian J of deformation in the new ambient space is related to the Jacobian with respect to the old ambient space J_0 as follows:

$$J = \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}} \det \boldsymbol{F} = \mathrm{e}^{\frac{n\omega(\boldsymbol{x})}{2}} J_0,$$

where $\dim \mathcal{S} = n$. Having an equilibrium configuration, replacing $\boldsymbol{g}_0$ by its rescaled version, the equilibrium configuration will change, in general. The following two examples show the effect of a rescaling of the spatial metric on equilibrium configuration and the corresponding stresses.

3.1 Examples of Elastic Bodies in Evolving Ambient Spaces and the Induced Stresses

We consider two examples in this section. In the first example, a disk of radius R_o is embedded in a two-dimensional ambient space that is initially flat. We first show that the disk remains stress-free when the metric of the ambient space is rescaled homogeneously. Then we calculate the stresses in the case of an ambient space inclusion (the ambient space metric is uniformly rescaled inside a finite disk and is left unchanged outside the disk). In the second example, we consider a spherical cap embedded in a spherical ambient space. We uniformly rescale the spatial spherical metric (equivalently changing the radius of the sphere) and calculate the resulting stresses. In both examples, we assume an incompressible and isotropic solid. For such solids, the energy function depends on the first and second principal invariants of the right Cauchy–Green strain $\mathbf{C}$ (or the left Cauchy–Green strain $\mathbf{b}$, also known as the Finger deformation tensor), i.e., $W = W(I_1, I_2)$ (Ogden 1997). Note that for an incompressible solid $I_3 = J^2 = 1$. The Finger deformation tensor $\boldsymbol{b}$ has components $b^{ab} = F^a{}_A F^b{}_B G^{AB}$. For an incompressible isotropic solid, the Cauchy stress has the following representation (Doyle and Ericksen 1956; Truesdell and Noll 2004)

$$\boldsymbol{\sigma} = \left(-p + 2I_2 \frac{\partial W}{\partial I_2}\right) \boldsymbol{g}^\sharp + 2\frac{\partial W}{\partial I_1}\boldsymbol{b}^\sharp - 2\frac{\partial W}{\partial I_2}\boldsymbol{b}^{-1}, \tag{3.1}$$

where p is the Lagrange multiplier corresponding to the incompressibility constraint $J = 1$. Note that $\boldsymbol{b}^{-1} = \boldsymbol{c}$ has components $c^{ab} = \left(F^{-1}\right)^A{}_m \left(F^{-1}\right)^B{}_n G_{AB}\, g^{am} g^{bn}$.

Example 3.1 **(Disk in a flat 2D plane)** Let us consider a two-dimensional disk of initial radius R_o made of an incompressible and isotropic material in an initially flat two-dimensional spatial manifold. We would like to calculate the stresses that occur in the new equilibrium configuration after we change the spatial metric in a radially symmetric way. In spatial polar coordinates (r, θ), the spatial metric is assumed to be:

$$\boldsymbol{g} = \begin{pmatrix} \mathrm{e}^{2\omega(r)} & 0 \\ 0 & r^2\mathrm{e}^{2\omega(r)} \end{pmatrix}.$$

The nonzero connection coefficients for $\boldsymbol{g}$ are:

$$\gamma^r_{rr} = \omega'(r), \quad \gamma^r_{\theta\theta} = -r(1 + r\omega'(r)), \quad \gamma^\theta_{\theta r} = \frac{1}{r} + \omega'(r). \tag{3.2}$$

In material polar coordinates (R, Θ), the material metric reads

$$\boldsymbol{G} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}.$$

We look for solutions of the form $\varphi(R, \Theta) = (r, \theta) = (r(R), \Theta)$. Thus, $\boldsymbol{F} = \operatorname{diag}\left(r'(R), 1\right)$. This gives the Jacobian as

$$J = \mathrm{e}^{2\omega(r)} \frac{r'(R) r}{R}.$$

Therefore, incompressibility ($J = 1$) gives the following ODE for $r(R)$

$$r(R) r'(R) \mathrm{e}^{2\omega(r(R))} = R. \tag{3.3}$$

To fix rigid body translations, we assume that $r(0) = 0$. Hence, $r(R)$ satisfies the following integral equation.

$$r(R) = \left(\int_0^R 2\xi \mathrm{e}^{-2\omega(r(\xi))} d\xi \right)^{1/2}. \tag{3.4}$$

Note that the Finger tensor reads

$$\boldsymbol{b} = \begin{pmatrix} \frac{\mathrm{e}^{-4\omega} R^2}{r^2} & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix},$$

and hence $I_1 = \frac{r^2 \mathrm{e}^{2\omega}}{R^2} + \frac{R^2 \mathrm{e}^{-2\omega}}{r^2}$ and $I_2 = 1$. Therefore, we obtain from (3.1) the nonzero stress components as

$$\begin{aligned} \sigma^{rr} &= -\mathrm{e}^{-2\omega} p + \frac{\mathrm{e}^{-4\omega} R^2}{r^2} \alpha + \left(\mathrm{e}^{-2\omega} - \frac{r^2}{R^2} \right) \beta, \quad \text{and} \\ \sigma^{\theta\theta} &= -\frac{\mathrm{e}^{-2\omega}}{r^2} p + \frac{1}{R^2} \alpha + \frac{\mathrm{e}^{-2\omega}}{r^2} \left(1 - \frac{\mathrm{e}^{-2\omega} R^2}{r^2} \right) \beta, \end{aligned} \tag{3.5}$$

where $p(R)$ is the unknown Lagrange multiplier and

$$\alpha(R) = 2 \frac{\partial W(I_1, I_2)}{\partial I_1}, \quad \beta(R) = 2 \frac{\partial W(I_1, I_2)}{\partial I_2}.$$

In terms of the Cauchy stress tensor, the only non-trivial equilibrium equation is ${\sigma^{ra}}_{|a} = 0$, which, by using (3.2), reads

$$\frac{r(R)\mathrm{e}^{2\omega(r(R))}}{R}\sigma^{rr}{}_{,R} + \left(\frac{1}{r} + 3\omega'\right)\sigma^{rr} - r\left(1 + r\omega'\right)\sigma^{\theta\theta} = 0. \tag{3.6}$$

By using (3.5), the equilibrium equation (3.6) reduces to

$$p' = \beta' - \frac{R^3\mathrm{e}^{-4\omega}(\alpha-\beta)\left(r\omega'+1\right)}{r^4} + \frac{r^2\mathrm{e}^{2\omega}\left(2\beta - R\beta'\right)}{R^3} + \frac{R\mathrm{e}^{-2\omega}\left(R\alpha' + 2\alpha\right)}{r^2} - \frac{(\alpha+3\beta)\left(r\omega'+1\right)}{R}. \tag{3.7}$$

Note that because of the flat geometry of the ambient space, the second fundamental form of the ambient plane in $\mathcal{Q} = \mathbb{R}^3$ is zero, and hence the extrinsic equilibrium equation (2.21) gives a zero extrinsic body force B^n on the disk.

Uniform scaling of the metric Let us consider the particular case of a uniform (homogeneous) scaling of the ambient space metric, i.e., $\omega(r) = \omega_o$. It follows from (3.3) that $r(R) = R\mathrm{e}^{-\omega_o}$, and hence we get from (3.7) that $p' = \alpha'$. Assuming zero traction at the boundary, i.e., $\sigma^{rr}(R_o) = 0$, we find that $p = \alpha$. Thus, it follows that $\sigma^{rr}(R) = \sigma^{\theta\theta}(R) = 0$, i.e., the disk remains stress-free when the ambient space is uniformly expanded/contracted.

Remark 3.1 Since the components of the deformation gradient do not have a coordinate-independent meaning, in a given pair of bases in the material and spatial spaces, $\boldsymbol{F}$ can be the identity matrix, but that does not mean that there is no stretch in the material. Likewise, $\boldsymbol{F}$ can be a position-dependent, non-identity matrix, and that does not mean that there is any stretch. What matters is a coordinate-independent measure of stretch. One such measure is $\boldsymbol{G} - \varphi^*\boldsymbol{g}$. If this vanishes, we can claim that there is no stretch (strain), according to this particular (and appropriate for shell-like situations) definition of strain.

In this example, we chose to represent the stretch in the ambient space (assuming uniform expansion) as a change in $\boldsymbol{g}$. The new equilibrium configuration map was given by a "rescaled" version of the original configuration map (by "rescaled" we mean rescaled in the Euclidean coordinate representation), and the final state was still stress (and strain)-free. $\boldsymbol{G} - \varphi^*\boldsymbol{g}$ will still be zero, with both φ and $\boldsymbol{g}$ having changed during the process. In short, as $\boldsymbol{g}$ changes, as a result the equilibrium mapping φ changes, and likewise $\boldsymbol{F}$ changes, but the pullback of $\boldsymbol{g}$ stays the same for the set of equilibrium configurations during the process.

An ambient space inclusion We assume that the spatial metric is uniformly scaled inside a disk of radius r_i and is left unchanged elsewhere, i.e.,

$$\omega(r) = \begin{cases} \omega_o & \text{if } \; 0 \leq r \leq r_i, \\ 0 & \text{if } \; r_i < r. \end{cases} \tag{3.8}$$

Motivated by Eshelby's inclusion problem Eshelby (1957) (see Yavari and Goriely 2013 for a discussion on finite eigenstrains and a brief history of the problem), we call this an *ambient space inclusion*. We may think of this as a defect in the ambient space. It follows from (3.4) that

$$r(R) = \begin{cases} R\mathrm{e}^{-\omega_o} & 0 \le R \le r_i \mathrm{e}^{\omega_o}, \\ \left[R^2 + r_i^2\left(1 - \mathrm{e}^{2\omega_o}\right)\right]^{1/2} & r_i \mathrm{e}^{\omega_o} < R \le R_o. \end{cases} \tag{3.9}$$

Remark 3.2 Note that if $R_o \le r_i \mathrm{e}^{\omega_o}$, we have $r(R) = R\mathrm{e}^{-\omega_o}$. Hence, it follows, similar to the previous case where the ambient metric is uniformly scaled, that the disk remains stress-free. We assume in the remainder of this example that $R_o > r_i \mathrm{e}^{\omega_o}$.

By substituting (3.8) and (3.9) into (3.7), one finds

$$p'(R) = \begin{cases} \alpha'(R) & 0 \le R \le r_i \mathrm{e}^{\omega_o}, \\ f(R) & r_i \mathrm{e}^{\omega_o} < R \le R_o, \end{cases} \tag{3.10}$$

where

$$f(R) = \frac{R\left(R\alpha' + 2\alpha\right)}{R^2 + r_i^2\left(1 - \mathrm{e}^{2\omega_o}\right)} - \frac{R^3(\alpha - \beta)}{\left(R^2 + r_i^2\left(1 - \mathrm{e}^{2\omega_o}\right)\right)^2} + \frac{\left(2\beta - R\beta'\right)\left(R^2 + r_i^2\left(1 - \mathrm{e}^{2\omega_o}\right)\right)}{R^3} - \frac{\alpha + 3\beta}{R} + \beta'. \tag{3.11}$$

Assuming zero traction at the boundary $R = R_0$, we obtain

$$p(R) = \begin{cases} \alpha(R) + \hat{\sigma}_o & 0 \le R \le r_i \mathrm{e}^{\omega_o}, \\ \int_{R_o}^{R} f(\xi)d\xi + \frac{R_o^2}{R_o^2 + r_i^2(1 - \mathrm{e}^{2\omega_o})}\alpha(R_o) + \left(1 - \frac{R_o^2 + r_i^2(1 - \mathrm{e}^{2\omega_o})}{R_o^2}\right)\beta(R_o) & r_i \mathrm{e}^{\omega_o} < R \le R_o, \end{cases} \tag{3.12}$$

where $\hat{\sigma}_o$ is a constant to be determined by enforcing the continuity of the traction vector across the boundary of the ambient space inclusion, i.e., continuity of σ^{rr} across the disk of radius r_i in the deformed configuration. Following (3.5), the nonzero physical components of the Cauchy stress tensor read

$$\hat{\sigma}^{rr} = \begin{cases} \hat{\sigma}_o & 0 \le R \le r_i \mathrm{e}^{\omega_o}, \\ \int_{R}^{R_o} f(\xi)d\xi + \frac{R^2}{R^2 + r_i^2\left(1 - \mathrm{e}^{2\omega_o}\right)}\alpha(R) + \left(1 - \frac{R^2 + r_i^2\left(1 - \mathrm{e}^{2\omega_o}\right)}{R^2}\right)\beta(R) \\ \quad - \frac{R_o^2}{R_o^2 + r_i^2\left(1 - \mathrm{e}^{2\omega_o}\right)}\alpha(R_o) - \left(1 - \frac{R_o^2 + r_i^2\left(1 - \mathrm{e}^{2\omega_o}\right)}{R_o^2}\right)\beta(R_o) & r_i \mathrm{e}^{\omega_o} < R \le R_o, \end{cases} \tag{3.13}$$

$$\hat{\sigma}^{\theta\theta} = \begin{cases} \hat{\sigma}_o & 0 \le R \le r_i \mathrm{e}^{\omega_o}, \\ \hat{\sigma}^{rr} + \left[\frac{R^2 + r_i^2(1 - \mathrm{e}^{2\omega_o})}{R^2} - \frac{R^2}{R^2 + r_i^2(1 - \mathrm{e}^{2\omega_o})}\right](\alpha(R) + \beta(R)) & r_i \mathrm{e}^{\omega_o} < R \le R_o, \end{cases} \tag{3.14}$$

where $\hat{\sigma}_o$ is a constant given by:

$$\begin{aligned}\hat{\sigma}_o = &\int_{r_i \mathrm{e}^{\omega_o}}^{R_o} f(\xi)d\xi + \frac{r_i^2 \mathrm{e}^{2\omega_o}}{r_i^2 \mathrm{e}^{2\omega_o} + r_i^2\left(1-\mathrm{e}^{2\omega_o}\right)}\alpha(r_i \mathrm{e}^{\omega_o}) \\ &+ \left(1 - \frac{r_i^2 \mathrm{e}^{2\omega_o} + r_i^2\left(1-\mathrm{e}^{2\omega_o}\right)}{r_i^2 \mathrm{e}^{2\omega_o}}\right)\beta(r_i \mathrm{e}^{\omega_o}) \\ &- \frac{R_o^2}{R_o^2 + r_i^2\left(1-\mathrm{e}^{2\omega_o}\right)}\alpha(R_o) - \left(1 - \frac{R_o^2 + r_i^2\left(1-\mathrm{e}^{2\omega_o}\right)}{R_o^2}\right)\beta(R_o).\end{aligned} \tag{3.15}$$

Therefore, for an arbitrary incompressible isotropic solid, the stress is hydrostatic inside the ambient space inclusion and is equal to $\hat{\sigma}_o$.

Let us assume the particular case of a disk made of a homogeneous neo-Hookean solid, i.e., $\alpha(R) = \mu$ and $\beta(R) = 0$. Therefore, the nonzero physical components of the Cauchy stress (3.13) and (3.14) simplify to read:

$$\hat{\sigma}^{rr} = \begin{cases} \hat{\sigma}_o & 0 \le R \le r_i \mathrm{e}^{\omega_o}, \\ \frac{\mu}{2}\left[\frac{R^2}{R^2+r_i^2(1-\mathrm{e}^{2\omega_o})} - \frac{R_o^2}{R_o^2+r_i^2(1-\mathrm{e}^{2\omega_o})} - \log\left(\frac{R^2+r_i^2(1-\mathrm{e}^{2\omega_o})}{R_o^2+r_i^2(1-\mathrm{e}^{2\omega_o})}\frac{R_o^2}{R^2}\right)\right] & r_i \mathrm{e}^{\omega_o} < R \le R_o, \end{cases} \tag{3.16}$$

$$\hat{\sigma}^{\theta\theta} = \begin{cases} \hat{\sigma}_o & 0 \le R \le r_i \mathrm{e}^{\omega_o}, \\ \hat{\sigma}^{rr} + \mu\frac{R^2+r_i^2(1-\mathrm{e}^{2\omega_o})}{R^2} - \mu\frac{R^2}{R^2+r_i^2(1-\mathrm{e}^{2\omega_o})} & r_i \mathrm{e}^{\omega_o} < R \le R_o, \end{cases} \tag{3.17}$$

where

$$\begin{aligned}\hat{\sigma}_o = \frac{\mu}{2}\Bigg[&\frac{r_i^2 \mathrm{e}^{2\omega_o}}{r_i^2 \mathrm{e}^{2\omega_o} + r_i^2\left(1-\mathrm{e}^{2\omega_o}\right)} - \frac{R_o^2}{R_o^2 + r_i^2\left(1-\mathrm{e}^{2\omega_o}\right)} \\ &- \log\left(\frac{r_i^2 \mathrm{e}^{2\omega_o} + r_i^2\left(1-\mathrm{e}^{2\omega_o}\right)}{R_o^2 + r_i^2\left(1-\mathrm{e}^{2\omega_o}\right)}\frac{R_o^2}{r_i^2 \mathrm{e}^{2\omega_o}}\right)\Bigg].\end{aligned} \tag{3.18}$$

We plot in Fig. 3 the profile of stresses in a disk of initial radius R_o, due to an ambient space inclusion of radius $r_i = 0.1 R_o$ and $\omega_o = 0.05$. Note that since $\mathrm{e}^{-\omega_o} < 0.1$, we have $r_i < R_o \mathrm{e}^{\omega_o}$. Hence, the ambient space inclusion lies entirely inside the deformed disk of radius $r(R_o)$. We observe that the stress is indeed hydrostatic inside the metric inclusion. We also observe a discontinuity of the circumferential stress while the radial stress is continuous as expected following the continuity of the traction vector. Finally, we observe that the stress is asymptotically vanishing as we move away from the ambient space inclusion.

Example 3.2 **(Spherical cap on a 2D sphere)** Let us consider a two-dimensional spherical cap of angular radius Θ_o lying on a sphere of initial radius R_o. We assume that the spherical cap is made of an incompressible and isotropic material. We would like to calculate the stresses that occur in the new equilibrium configuration after

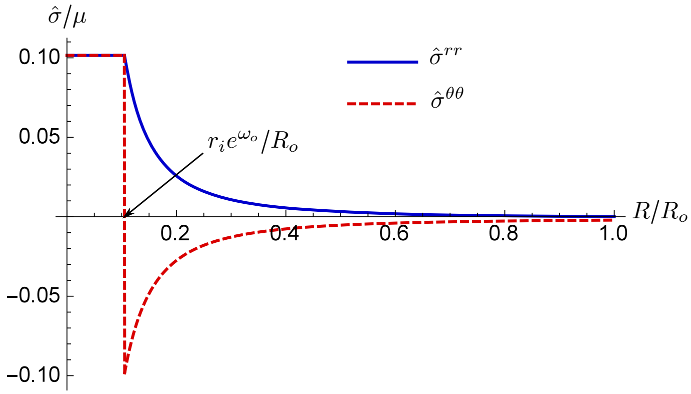

Fig. 3 Stresses in a disk of initial radius R_o , due to an ambient space inclusion of radius $r_i = 0.1R_o$ and $\omega_o = 0.05$

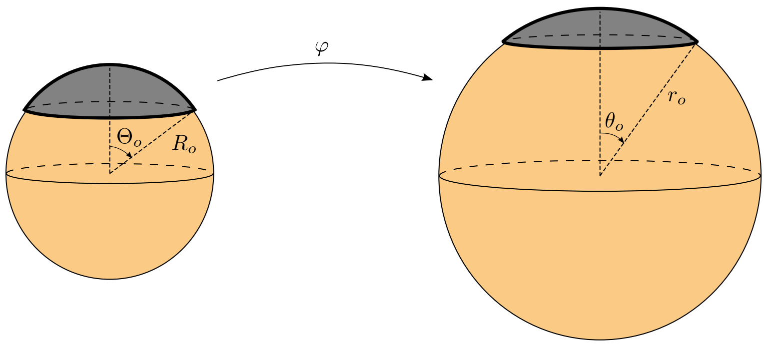

Fig. 4 Deformation of a spherical cap due to a change in the radius of the ambient space

the radius of the ambient sphere is changed to r_o , see Fig. 4. In spatial spherical coordinates (θ, ϕ) , the spatial metric reads

$$\boldsymbol{g} = \begin{pmatrix} r_o^2 & 0 \\ 0 & r_o^2 \sin^2\theta \end{pmatrix}.$$

Note that changing the radius of the sphere from R_o to r_o is equivalent to a uniform scaling of its spatial metric by $\mathrm{e}^{2\omega_o} = \frac{r_o^2}{R_o^2}$. Changing the spatial metric the equilibrium configuration changes. We look for solutions of the form $\varphi(\Theta, \Phi) = (\theta, \phi) = (\theta(\Theta), \Phi)$. Thus, $\boldsymbol{F} = \mathrm{diag}(\theta'(\Theta), 1)$. It follows that the Jacobian reads

$$J = \theta'(\Theta)\frac{r_o^2 \sin[\theta(\Theta)]}{R_o^2 \sin\Theta}.$$

Assuming that the spherical cap is made of an incompressible material, i.e., $J = 1$, fixing rigid body translations by taking $\theta(0) = 0$, and since $0 \leq \theta < \pi$, we find that

$$\theta(\Theta) = \cos^{-1}\left[\frac{r_o^2}{R_o^2}(\cos\Theta - 1) + 1\right]. \tag{3.19}$$

For this deformation, the Finger tensor reads

$$\boldsymbol{b} = \begin{pmatrix} \frac{R_o^2 \sin^2(\Theta)}{r_o^4 \sin^2(\theta)} & 0 \\ 0 & \frac{1}{R_o^2 \sin^2(\Theta)} \end{pmatrix},$$

and hence $I_1 = \frac{R_o^2 \sin^2\Theta}{r_o^2 \sin^2\theta} + \frac{r_o^2 \sin^2\theta}{R_o^2 \sin^2\Theta}$ and $I_2 = 1$. Therefore, we obtain from (3.1) the nonzero stress components as

$$\begin{aligned} \sigma^{\theta\theta} &= -\frac{1}{r_o^2}p + \frac{R_o^2 \sin^2\Theta}{r_o^4 \sin^2\theta}\alpha + \frac{1}{r_o^2}\left(1 - \frac{r_o^2 \sin^2\theta}{R_o^2 \sin^2\Theta}\right)\beta, \\ \sigma^{\phi\phi} &= -\frac{1}{r_o^2 \sin^2\theta}p + \frac{1}{R_o^2 \sin^2\Theta}\alpha + \frac{1}{r_o^2 \sin^2\theta}\left(1 - \frac{R_o^2 \sin^2\Theta}{r_o^2 \sin^2\theta}\right)\beta, \end{aligned} \tag{3.20}$$

where $p(\Theta)$ is the unknown Lagrange multiplier and

$$\alpha(\Theta) = 2\frac{\partial W(I_1, I_2)}{\partial I_1}, \quad \beta(\Theta) = 2\frac{\partial W(I_1, I_2)}{\partial I_2}.$$

Using (3.19), the physical components of stress (3.20) are written as

$$\begin{aligned} \hat{\sigma}^{\theta\theta} &= -p + \frac{r_o^2(\cos\Theta + 1)}{2r_o^2 + R_o^2\cos\Theta - R_o^2}\alpha + \left[1 - \frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{r_o^2(\cos\Theta + 1)}\right]\beta, \\ \hat{\sigma}^{\phi\phi} &= -p + \frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{r_o^2(\cos\Theta + 1)}\alpha + \left[1 - \frac{r_o^2(\cos\Theta + 1)}{2r_o^2 + R_o^2\cos\Theta - R_o^2}\right]\beta. \end{aligned} \tag{3.21}$$

In terms of the Cauchy stress tensor, the only non-trivial intrinsic equilibrium equation is ${\sigma^{\theta a}}_{|a} = 0$, which reads

$$\frac{r_o^2 \sin^2\theta}{R_o^2 \sin^2\Theta}{\sigma^{\theta\theta}}_{,\Theta} + \frac{1}{\tan\theta}\sigma^{\theta\theta} - \sin\theta\cos\theta\sigma^{\phi\phi} = 0. \tag{3.22}$$

By using (3.19) and (3.20), the equilibrium equation (3.22) reduces to

$$\begin{aligned} p' =& -\tan^2\left(\frac{\Theta}{2}\right)\beta' + \frac{r_o^2\alpha'}{R_o^2} - \frac{\tan\left(\frac{\Theta}{2}\right)\left(r_o^2-2R_o^2\right)(\alpha+\beta)}{2r_o^2} \\ &+ \frac{r_o^2\sin(\Theta)(r_o^2-R_o^2)(\beta-\alpha)}{\left(2r_o^2+R_o^2\cos\Theta-R_o^2\right)^2} + \frac{R_o^2\tan^2\left(\frac{\Theta}{2}\right)\beta'}{r_o^2} + \frac{8\sin^4\left(\frac{\Theta}{2}\right)\left(R_o^2-r_o^2\right)\beta}{r_o^2\sin^3\Theta} \\ &+ \frac{4r_o^2\left(R_o^2-r_o^2\right)\alpha' + R_o^2\sin(\Theta)\left(R_o^2-2r_o^2\right)(\alpha+\beta)}{4r_o^2R_o^2+2R_o^4\cos\Theta-2R_o^4}. \end{aligned} \tag{3.23}$$

Note that, unlike the previous example, the spherical cap does not necessarily remain stress-free by a uniform scaling of the spatial metric. If, however, we take $r_o = R_o$, then (3.23) reduces to $p' = \alpha'$, which yields no stress by assuming zero boundary traction at $\Theta = \Theta_o$. Hence, we recover the case of a trivial embedding. Back to the general case when $r_o \neq R_o$, the evolving ambient sphere can be isometrically embedded in $\mathbb{R}^3$, i.e., $\mathcal{Q} = \mathbb{R}^3$, where the second fundamental form of the sphere reads

$$\boldsymbol{k} = \begin{pmatrix} -r_o & 0 \\ 0 & -r_o\sin^2\theta \end{pmatrix}.$$

We only have one extrinsic equilibrium equation (2.21), which gives the normal component of the body force required to balance the stress field in the spherical cap. It is written as

$$B^n = \frac{1}{\rho}\boldsymbol{\sigma}:\boldsymbol{k} = -\frac{\hat{\sigma}^{\theta\theta}+\hat{\sigma}^{\phi\phi}}{r_o\rho}. \tag{3.24}$$

In the following, we explore the particular case when the spherical cap is made of a neo-Hookean solid, i.e., $\alpha(R) = \mu$ and $\beta(R) = 0$. For a neo-Hookean solid, (3.23) reduces to

$$\begin{aligned} p' =& \mu\frac{\tan\left(\frac{\Theta}{2}\right)\left(2R_o^2-r_o^2\right)}{2r_o^2} - \mu\frac{r_o^2\sin(\Theta)(r_o^2-R_o^2)}{\left(2r_o^2+R_o^2\cos(\Theta)-R_o^2\right)^2} \\ &+ \mu\frac{\sin(\Theta)\left(R_o^2-2r_o^2\right)}{4r_o^2+2R_o^2\cos(\Theta)-2R_o^2}. \end{aligned} \tag{3.25}$$

Therefore, assuming zero boundary traction at $\Theta = \Theta_o$, i.e., $\sigma^{\theta\theta}(\Theta_o) = 0$, we find that

$$p(\Theta) = \mu\left[g(\Theta) - g(\Theta_o) + \frac{r_o^2(\cos\Theta_o+1)}{2r_o^2+R_o^2\cos\Theta_o-R_o^2}\right], \tag{3.26}$$

where

$$g(\Theta) = \frac{2r_o^2 - R_o^2}{2R_o^2} \log\left(2r_o^2 + R_o^2 \cos\Theta - R_o^2\right) + \frac{2R_o^2 - r_o^2}{2r_o^2} \log\left[\cos^2\left(\frac{\Theta}{2}\right)\right] - \frac{r_o^2\left(r_o^2 - R_o^2\right)}{R_o^2\left(2r_o^2 + R_o^2 \cos\Theta - R_o^2\right)}. \tag{3.27}$$

Therefore, the stress field (3.21) and the extrinsic body force (3.24) are given by

$$\begin{aligned}
\hat{\sigma}^{\theta\theta}(\Theta) = \mu\Bigg[&\frac{R_o^2 - 2r_o^2}{2R_o^2} \log\left(\frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2}\right) + \frac{r_o^2\cos\Theta}{2r_o^2 + R_o^2\cos\Theta - R_o^2} \\
&- \frac{r_o^2\cos\Theta_o}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2} + \frac{r_o^2 - 2R_o^2}{2r_o^2}\log\left(\frac{1+\cos\Theta_o}{1+\cos\Theta}\right) \\
&- \frac{r_o^4(\cos\Theta - \cos\Theta_o)}{\left(2r_o^2 + R_o^2\cos\Theta - R_o^2\right)\left(2r_o^2 + R_o^2\cos\Theta_o - R_o^2\right)}\Bigg], \\
\hat{\sigma}^{\phi\phi}(\Theta) = \mu\Bigg[&\frac{R_o^2 - 2r_o^2}{2R_o^2} \log\left(\frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2}\right) + \frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{r_o^2(\cos\Theta + 1)} \\
&- \frac{r_o^2(\cos\Theta_o + 1)}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2} + \frac{r_o^2 - 2R_o^2}{2r_o^2}\log\left(\frac{1+\cos\Theta_o}{1+\cos\Theta}\right) \\
&- \frac{r_o^2(r_o^2 - R_o^2)(\cos\Theta - \cos\Theta_o)}{\left(2r_o^2 + R_o^2\cos\Theta - R_o^2\right)\left(2r_o^2 + R_o^2\cos\Theta_o - R_o^2\right)}\Bigg], \\
r_o\rho B^n(\Theta) = \mu\Bigg[&\frac{2r_o^2 - R_o^2}{R_o^2} \log\left(\frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2}\right) + \frac{r_o^2(\cos\Theta_o + 1)}{2r_o^2 + R_o^2\cos\Theta_o - R_o^2} \\
&- \frac{2r_o^2 + R_o^2\cos\Theta - R_o^2}{r_o^2(\cos\Theta + 1)} + \frac{2R_o^2 - r_o^2}{r_o^2}\log\left(\frac{1+\cos\Theta_o}{1+\cos\Theta}\right)\Bigg].
\end{aligned} \tag{3.28}$$

We plot in Fig. 5 the profile of stresses and the extrinsic body force in a spherical cap of angular radius $\Theta_o = \frac{\pi}{4}$ initially lying on a sphere of radius R_o, due to a change in the radius of the ambient sphere to $r_o = 1.5R_o$.

In the limiting case $r_o \to \infty$, which corresponds to flattening the spherical cap, we obtain from (3.28) that the stress field is given by

$$\begin{aligned}
\hat{\sigma}^{\theta\theta}(\Theta) &= -\mu\left[\frac{1}{2}\log\left(\frac{1+\cos\Theta}{1+\cos\Theta_o}\right) + \frac{1}{4}(\cos\Theta - \cos\Theta_o)\right], \\
\hat{\sigma}^{\phi\phi}(\Theta) &= -\mu\left[\frac{1}{2}\log\left(\frac{1+\cos\Theta}{1+\cos\Theta_o}\right) + \frac{3}{4}(\cos\Theta - \cos\Theta_o) - \frac{2}{\cos\Theta + 1} + \frac{\cos\Theta_o + 1}{2}\right].
\end{aligned} \tag{3.29}$$

Note that the extrinsic body force field B^n vanishes when $r_o \to \infty$, since this case corresponds to a flat geometry of the ambient space.

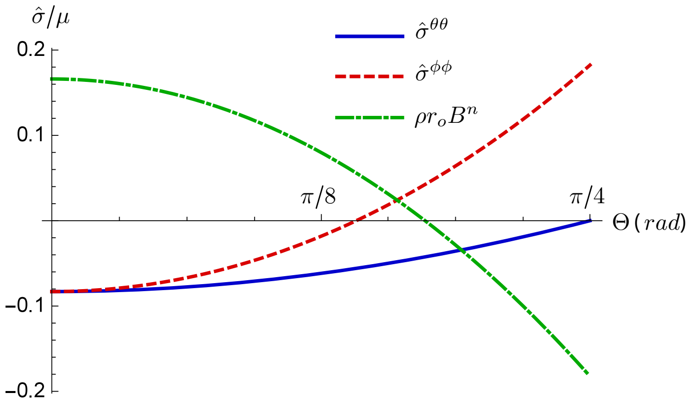

Fig. 5 Stresses and the extrinsic body force in a spherical cap of initial angular radius $\Theta_o = \frac{\pi}{4}$ initially lying on a sphere of radius R_o , due to a change in the radius of the ambient sphere to $r_o = 1.5R_o$

3.2 Elastic Deformations Due to Linear Perturbations of the Ambient Space Metric

In this section, we linearize the governing equations of the nonlinear theory presented in the previous sections about a reference motion. This will shed light on the mechanical effects of a slight deformation of ambient space on the equilibrium configuration of a deformable body.

Geometric linearization of elasticity was first discussed by Marsden and Hughes (1983) and was further developed by Yavari and Ozakin (2008), see also Mazzucato and Rachele (2006). Given a reference motion, we obtain the linearized governing equations with respect to this motion. Suppose a given solid is in a static equilibrium configuration φ in an ambient space with metric $\boldsymbol{g}$. Let $\boldsymbol{g}_\epsilon$ be a one-parameter family of spatial metrics, φ_ϵ be the corresponding equilibrium configuration, and $\boldsymbol{P}_\epsilon$ be the corresponding first Piola–Kirchhoff stress. Let $\epsilon = 0$ describe the reference motion. Now, for a fixed point $\boldsymbol{X}$ in the material manifold, $\varphi_\epsilon(\boldsymbol{X})$ describes a curve in the spatial manifold, and its derivative at $\epsilon = 0$ gives the variation $\delta\varphi$ as a vector $\boldsymbol{U}(\boldsymbol{X})$ at $\varphi(\boldsymbol{X})$:

$$\delta\varphi(X) = \boldsymbol{U}(\boldsymbol{X}) = \left.\frac{\mathrm{d}\varphi_\epsilon(\boldsymbol{X})}{\mathrm{d}\epsilon}\right|_{\epsilon=0}. \tag{3.30}$$

The variation of the ambient space metric is defined as

$$\delta\boldsymbol{g} = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\boldsymbol{g}_\epsilon. \tag{3.31}$$

Now consider, in the absence of body forces, the equilibrium equation Div $\boldsymbol{P} = \boldsymbol{0}$ for the family of spatial metrics parametrized by ϵ:

$$\mathrm{Div}_\epsilon\, \boldsymbol{P}_\epsilon = \boldsymbol{0}. \tag{3.32}$$

Linearization of (3.32) is defined as (Marsden and Hughes 1983; Yavari and Ozakin 2008):

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} (\mathrm{Div}_{\epsilon}\, \boldsymbol{P}_{\epsilon}) = \mathbf{0}. \tag{3.33}$$

Once again, one should note that since the equilibrium configuration is different for each ϵ, $\boldsymbol{P}_{\epsilon}$ is based at different points in the ambient space for different values of ϵ, and in order to calculate the derivative with respect to ϵ, one in general needs to use the connection (parallel transport) in the ambient space. In components, balance of linear momentum reads

$$\frac{\partial P^{aA}(\epsilon)}{\partial X^A} + \Gamma^A_{AB} P^{aB}(\epsilon) + P^{bB}(\epsilon)\gamma(\epsilon)^a_{bc} F(\epsilon)^c{}_A = 0. \tag{3.34}$$

Thus, the linearized balance of linear momentum is written as

$$\begin{aligned}&\frac{\partial}{\partial X^A}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} P^{aA}(\epsilon) + \Gamma^A_{AB}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} P^{aB}(\epsilon) + \left[\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} P^{bB}(\epsilon)\right]\gamma^a_{bc} F^c{}_A \\ &+ P^{bB}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\left[\gamma(\epsilon)^a_{bc}\right] F^c{}_A + P^{bB}\gamma^a_{bc}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\left[F(\epsilon)^c{}_A\right] = 0. \end{aligned} \tag{3.35}$$

Let us consider a one-parameter family of metrics $g_{ab}(\epsilon)$ such that

$$g_{ab}(0) = g_{ab}, \quad \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} g_{ab}(\epsilon) = \delta g_{ab}. \tag{3.36}$$

δg_{ab} is called the metric variation. It can be shown that (Chow et al. 2006)

$$\delta g^{ab} = -g^{am} g^{bn} \delta g_{mn}, \tag{3.37}$$

$$\delta\gamma^a{}_{bc} = \frac{1}{2} g^{ad}\left(\delta g_{cd|b} + \delta g_{bd|c} - \delta g_{bc|d}\right). \tag{3.38}$$

Thus

$$\begin{aligned}\delta\,(\mathrm{Div}\,\boldsymbol{P})^a &= \frac{\partial}{\partial X^A}\delta P^{aA} + \Gamma^A_{AB}\delta P^{aB} + \gamma^a_{bc} F^c{}_A \delta P^{bB} + \delta\gamma^a_{bc} F^c{}_A P^{bB} \\ &\quad + \gamma^a_{bc}\delta F^c{}_A P^{bB} \\ &= \delta P^{aA}{}_{|A} + \frac{1}{2} P^{bB} F^c{}_A g^{ad}\left(\delta g_{cd|b} + \delta g_{bd|c} - \delta g_{bc|d}\right) \\ &\quad + P^{bB}\gamma^a_{bc} U^c{}_{|A}. \end{aligned} \tag{3.39}$$

If the initial equilibrium configuration is stress-free, we have $\delta P^{aA}{}_{|A} = 0$. Note that

$$P^{aA} = g^{ab}\frac{\partial\Psi}{\partial F^b_A}, \tag{3.40}$$

where $\Psi = \Psi(\boldsymbol{X}, \Theta, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{g})$ is the material free energy density and Θ is the absolute temperature. In calculating $\frac{\mathrm{d}P^{aA}(\epsilon)}{\mathrm{d}\epsilon}$, we need to consider the change in $\boldsymbol{F}$ due to the change in the ambient space metric as follows:

$$\frac{\mathrm{d}P^{aA(\epsilon)}}{\mathrm{d}\epsilon} = \frac{\mathrm{d}}{\mathrm{d}\epsilon} g^{ab}(\epsilon) \frac{\partial \Psi}{\partial F^b{}_A} + g^{ab} \frac{\partial^2 \Psi}{\partial F^c{}_C \partial F^b{}_A} \frac{\mathrm{d}F^c{}_C(\epsilon)}{\mathrm{d}\epsilon} + g^{ab} \frac{\partial^2 \Psi}{\partial g_{cd} \partial F^b{}_A} \frac{\mathrm{d}g_{cd}(\epsilon)}{\mathrm{d}\epsilon}. \tag{3.41}$$

Let us define

$$\mathbb{A}^a{}_c{}^{CA} = g^{ab} \frac{\partial^2 \Psi}{\partial F^c{}_C \partial F^b{}_A} \quad \text{and} \quad \mathbb{B}^{acdA} = g^{ab} \frac{\partial^2 \Psi}{\partial g_{cd} \partial F^b{}_A}, \tag{3.42}$$

where the derivatives are evaluated at the reference configuration corresponding to $\epsilon = 0$. Note that $\mathbb{A}$ is the equivalent of the elasticity tensor in classical linear elasticity. Therefore

$$\delta P^{aA}{}_{|A} = -g^{ac} P^{dA} \delta g_{cd} + \mathbb{A}^a{}_c{}^{CA} U^c{}_{|C} + \mathbb{B}^{acdA} \delta g_{cd} = \mathbb{A}^a{}_c{}^{CA} U^c{}_{|C} + \mathbb{B}^{acdA} \delta g_{cd}, \tag{3.43}$$

as the initial configuration is assumed to be stress-free. With these results, the linearized balance of linear momentum (3.33) is simplified to read

$$\left[\mathbb{A}^a{}_c{}^{CA} U^c{}_{|C} + \mathbb{B}^{acdA} \delta g_{cd} \right]_{|A} = 0 \quad \text{or} \quad \operatorname{Div}(\mathbb{A} \cdot \nabla \boldsymbol{U} + \mathbb{B} \cdot \delta \boldsymbol{g}) = \boldsymbol{0}. \tag{3.44}$$

Given $\delta \boldsymbol{g}$, the above equations are the governing equations for the displacement field $\boldsymbol{U}$ that results from this change in spatial metric.

We know that for a body deforming quasi-statically in an ambient space with a fixed background metric, linearized balance of linear momentum reads (Yavari and Ozakin 2008)

$$\left[\mathbb{A}^a{}_c{}^{CA} U^c{}_{|C} \right]_{|A} + \rho_0 (\mathfrak{L} B_0)^a = 0, \tag{3.45}$$

where $\mathfrak{L}\boldsymbol{B}$ is the linearized body force. It is seen that in the absence of (mechanical) body forces and when the ambient space is deformed, one can think of $\operatorname{Div}(\mathbb{B} \cdot \delta \boldsymbol{g})$ as an effective body force. In other words, deformation of the ambient space and the equivalent body force will have the same mechanical effect on the deformable body.

Initially Euclidean Metric Let us assume that the initial metric is Euclidean and is isotropically rescaled, i.e., consider a one-parameter family of spatial metrics of the form $(g_\epsilon(\boldsymbol{x}))_{ab} = \mathrm{e}^{2\omega_\epsilon(\boldsymbol{x})} \delta_{ab}$. Thus, $\delta g_{ab} = 2\delta\omega \delta_{ab}$. In this case, (3.44) is simplified to read

$$\left[\mathbb{A}^a{}_c{}^{CA} U^c{}_{,C} + 2\mathbb{B}^{acdA} \delta_{cd} \delta\omega \right]_{,A} = 0. \tag{3.46}$$

When $\mathbb{A}$ and $\mathbb{B}$ are constants (homogeneous medium), the above equation reads

$$\mathbb{A}^a{}_c{}^{CA} U^c{}_{,CA} + 2\mathbb{B}^{acdA} \delta_{cd} F^b{}_A \delta\omega_{,b} = 0. \tag{3.47}$$

Now if $\delta\omega$ is independent of $\boldsymbol{x}$, one finds that $\boldsymbol{U} = \boldsymbol{c}$ (a constant vector) is a solution, i.e., the body will stay stress-free in the new (Euclidean) ambient space.

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Appendix 1: Geometry of Riemannian Submanifolds

In the following, we tersely review a few elements of the geometry of embedded submanifolds. Here we mainly follow do Carmo (1992), Capovilla and Guven (1995), Spivak (1999), and Kuchař (1976). Let us consider a Riemannian manifold $\mathcal{S}$ embedded in another Riemannian manifold $\mathcal{Q}$ and assume that $\dim \mathcal{S} < \dim \mathcal{Q}$. We consider a time-dependent embedding $\psi_t : \mathcal{S} \to \mathcal{Q}$. The metric $\boldsymbol{h}$ on $\mathcal{Q}$ induces a metric $\boldsymbol{g}_t = \psi_t^* \boldsymbol{h}$ on $\mathcal{S}$ (the first fundamental form). At any given point p of $\mathcal{S}$, the tangent space $T_p\mathcal{S}_t$ has an orthogonal complement $\left(T_p\mathcal{S}_t\right)^{\perp} \subset T\mathcal{Q}$ such that

$$T_p\mathcal{Q} = T_p\mathcal{S}_t \oplus \left(T_p\mathcal{S}_t\right)^{\perp}. \tag{4.1}$$

Note that such a decomposition is smooth in the sense that any smooth vector field $\boldsymbol{u}$ on $\mathcal{S}_t$ can be smoothly decomposed into a vector field $\boldsymbol{u}_{\parallel}$ tangent to $\mathcal{S}_t$ and a vector field $\boldsymbol{u}_{\perp}$ normal to $\mathcal{S}_t$, so that $p \to (\boldsymbol{u}_{\parallel})_p = (\boldsymbol{u}_p)_{\parallel}$ and $p \to (\boldsymbol{u}_{\perp})_p = (\boldsymbol{u}_p)_{\perp}$ are smooth. We write $\boldsymbol{u} = \boldsymbol{u}_{\parallel} + \boldsymbol{u}_{\perp}$. The orientation of $\boldsymbol{\eta}_i^t$, for $i \in \{1, \ldots, k\}$, is chosen such that the orientations of $\mathcal{S}_t$ and $\mathcal{Q}$ are consistent in the sense that the orientation induced from $\mathcal{S}_t$ along with the ordered sequence $\{\boldsymbol{\eta}_i^t\}_{i\in\{1,\ldots,k\}}$ is equivalent to the orientation of $\mathcal{Q}$. Let $\dim \mathcal{S} = n$ and $\dim \mathcal{Q} = n + k = m$. Following the smoothness of the decomposition (4.1), one can take a set of smooth vector fields $\{\boldsymbol{\eta}_i^t\}_{i=1,\ldots,k}$ normal to $\mathcal{S}_t$ such that they form an orthonormal basis for $\mathfrak{X}^{\perp}(\mathcal{S}_t)$, the set of vector fields normal to $\mathcal{S}_t$. Let $\{\chi^{\alpha}\}_{\alpha=1,\ldots,n+k}$ be a local coordinate chart for $\mathcal{Q}$ such that at any point of $\mathcal{S}_t$, $\{\chi^1, \ldots, \chi^n\}$ is a local coordinate chart for $\mathcal{S}_t$, and such that the ith unit normal vector field $\boldsymbol{\eta}_i^t$ for $i \in \{1, \ldots, k\}$ is tangent to the coordinate curve χ^{n+i}. Hence, every vector field $\boldsymbol{u}$ on $\mathcal{Q}$ along $\mathcal{S}_t$ can be written as $\boldsymbol{u} = \boldsymbol{u}_{\parallel} + \sum_{i=1}^{k} u_{\perp}^i \boldsymbol{\eta}_i^t$.[19] Note that, for $i, j \in \{1, \ldots, k\}$, one has $\left\langle\!\left\langle \boldsymbol{\eta}_i^t, \boldsymbol{\eta}_j^t \right\rangle\!\right\rangle_{\boldsymbol{h}} = \delta_{ij}$ and $\left\langle\!\left\langle \boldsymbol{\eta}_i^t, \boldsymbol{u}_{\parallel} \right\rangle\!\right\rangle_{\boldsymbol{h}} = 0$, where the Kronecker delta symbol δ_{ij} is defined as: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Note that at any point of $\mathcal{S}_t$, one has $h_{\alpha(n+i)} = \delta_{\alpha(n+i)}$, for $i \in \{1, \ldots, k\}$ and $\alpha \in \{1, \ldots, n+k\}$. We denote the connection coefficients for the Levi–Civita connections $\nabla^{\boldsymbol{h}}$ and $\nabla^{\boldsymbol{g}_t}$ corresponding to the metrics $\boldsymbol{h}$ and $\boldsymbol{g}_t$ by $\tilde{\gamma}^{\alpha}_{\beta\gamma}$ and γ^{a}_{bc}, respectively. We denote by $D_t^{\boldsymbol{h}}$ and $D_t^{\boldsymbol{g}_t}$ the covariant derivatives along $\tilde{\varphi}_X$ and φ_X, respectively. For a vector field $\boldsymbol{u}$ on $\mathcal{Q}$ along $\mathcal{S}_t$, we write $D_t^{\boldsymbol{h}}\boldsymbol{u} = \frac{\partial u^{\alpha}}{\partial t}\tilde{\partial}_{\alpha}^t + \frac{\partial u_{\perp}^i}{\partial t}\boldsymbol{\eta}_i^t + \nabla^{\boldsymbol{h}}_{\boldsymbol{\Upsilon}}\boldsymbol{u}$ and for a vector field $\boldsymbol{w}$ on $\mathcal{S}$, $D_t^{\boldsymbol{g}_t}\boldsymbol{w} = \frac{\partial w^a}{\partial t}\partial_a^t + \nabla^{\boldsymbol{g}_t}_{\boldsymbol{V}}\boldsymbol{w}$, where $\{\tilde{\partial}_{\alpha}^t\}_{\alpha=1,\ldots,n}$ and $\{\partial_a^t\}_{a=1,\ldots,n}$ denote local coordinate bases for $\mathcal{S}_t$ and $\mathcal{S}$, respectively.

[19] In the local coordinate $\{\chi^{\alpha}\}_{\alpha=1,\ldots,n+k}$, we denote $u_{\perp}^i = u^{n+i}$ for $i \in \{1, \ldots, k\}$.

Note that for vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined on $\mathcal{S}_t$ and $\mathcal{Q}$, respectively, such that $\boldsymbol{Y}$ is everywhere tangent to $\mathcal{S}_t$, $\nabla^{\boldsymbol{g}_t}_{\psi_t^* \boldsymbol{X}} \psi_t^* \boldsymbol{Y} = \psi_t^* (\nabla^{\boldsymbol{h}}_{\boldsymbol{X}} \boldsymbol{Y})_{\|}$.[20] As a corollary, given a curve c in $\mathcal{S}_t$ and $\boldsymbol{X}$ a vector field along c tangent to $\mathcal{S}_t$ everywhere, $D^{\boldsymbol{g}_t}_c \psi_t^* \boldsymbol{X} = (D^{\boldsymbol{h}}_c \boldsymbol{X})_{\|}$. For $i \in \{1, \ldots, k\}$, the ith second fundamental form of $\mathcal{S}_t$ along $\boldsymbol{\eta}_i^t$ is a $\binom{0}{2}$-tensor $\boldsymbol{\kappa}_i^t$ on $\mathcal{S}_t$ defined as (do Carmo 1992; Capovilla and Guven 1995)

$$\boldsymbol{\kappa}_i^t(\boldsymbol{u}, \boldsymbol{w}) = \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{u}} \boldsymbol{\eta}_i^t, \boldsymbol{w} \right\rangle\!\!\right\rangle_{\boldsymbol{h}}, \quad \forall\, \boldsymbol{u}, \boldsymbol{w} \in T_\chi \mathcal{S}_t. \tag{4.2}$$

It is known that $\boldsymbol{\kappa}_i^t$ is a symmetric tensor and can equivalently be written as

$$\boldsymbol{\kappa}_i^t = (\nabla^{\boldsymbol{h}} \boldsymbol{\eta}_i^t)_{\|}^{\flat}, \quad i = 1, \ldots, k.$$

On $\mathcal{S}$, we define, for $i \in \{1, \ldots, k\}$, the ith second fundamental form as $\boldsymbol{k}_i^t = \psi_t^* \boldsymbol{\kappa}_i^t$. For vector fields $\boldsymbol{u}, \boldsymbol{w} \in T_{\boldsymbol{x}} \mathcal{S}$, one can write $\nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w} = \psi_* \nabla^{\boldsymbol{g}_t}_{\boldsymbol{u}} \boldsymbol{w} + \sum_{i=1}^k h^i(\boldsymbol{u}, \boldsymbol{w}) \boldsymbol{\eta}_i^t$, where $h^i(.,.)$ is a bilinear form. Therefore

$$h^i(\boldsymbol{u}, \boldsymbol{w}) = \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w}, \boldsymbol{\eta}_i^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}}, \quad i = 1, \ldots, k.$$

Knowing that $\langle\!\langle \psi_* \boldsymbol{w}, \boldsymbol{\eta}_i^t \rangle\!\rangle_{\boldsymbol{h}} = 0$, one concludes that

$$\left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w}, \boldsymbol{\eta}_i^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}} = - \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \boldsymbol{\eta}_i^t, \psi_* \boldsymbol{w}, \right\rangle\!\!\right\rangle_{\boldsymbol{h}}, \quad i = 1, \ldots, k.$$

Hence

$$h^i(\boldsymbol{u}, \boldsymbol{w}) = - \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \boldsymbol{\eta}_i^t, \psi_* \boldsymbol{w}, \right\rangle\!\!\right\rangle_{\boldsymbol{h}} = -(\nabla^{\boldsymbol{h}} \boldsymbol{\eta}_i^t)^{\flat}(\psi_* \boldsymbol{u}, \psi_* \boldsymbol{w}) = -\boldsymbol{k}_i^t(\boldsymbol{u}, \boldsymbol{w}),$$
$$i = 1, \ldots, k.$$

Therefore, we obtain Gauss's equation

$$\nabla^{\boldsymbol{h}}_{\psi_* \boldsymbol{u}} \psi_* \boldsymbol{w} = \psi_* \nabla^{\boldsymbol{g}_t}_{\boldsymbol{u}} \boldsymbol{w} - \sum_{i=1}^k \boldsymbol{k}_i^t(\boldsymbol{u}, \boldsymbol{w}) \boldsymbol{\eta}_i^t.$$

On the other hand, for $i, j \in \{1, \ldots, k\}$, the projection of $\nabla^{\boldsymbol{h}} \boldsymbol{\eta}_i^t$ along $\boldsymbol{\eta}_j^t$ defines $\boldsymbol{\omega}_{ij}^t$, the normal fundamental 1-form of $\mathcal{S}_t$ relative to the unit normals $\boldsymbol{\eta}_i^t$ and $\boldsymbol{\eta}_j^t$. For any vector $\boldsymbol{w}$ tangent to $\mathcal{S}_t$, the 1-form $\boldsymbol{\omega}_{ij}^t$ is defined by (Capovilla and Guven 1995)

$$\boldsymbol{\omega}_{ij}^t \cdot \boldsymbol{w} = \left\langle\!\!\left\langle \nabla^{\boldsymbol{h}}_{\boldsymbol{w}} \boldsymbol{\eta}_i^t, \boldsymbol{\eta}_j^t \right\rangle\!\!\right\rangle_{\boldsymbol{h}}.$$

[20] The proof given in Spivak (1999) still holds even when the embedding is time dependent. Note that $\nabla^{\boldsymbol{g}_t}$ and $\nabla^{\boldsymbol{h}}$ are the Levi–Civita connections corresponding to $\boldsymbol{g}_t$ and $\boldsymbol{h}$, respectively.

Note that, for $i, j \in \{1, \ldots, k\}$, the normal fundamental 1-form $\boldsymbol{\omega}^t_{ij}$ is such that $\boldsymbol{\omega}^t_{ij} = -\boldsymbol{\omega}^t_{ji}$. On $\mathcal{S}$, one defines the normal fundamental 1-forms, for $i, j \in \{1, \ldots, k\}$, as $\boldsymbol{o}^t_{ij} = \psi^*_t \boldsymbol{\omega}^t_{ij}$. Note that, for a tangent vector field $\boldsymbol{w}$ on $\mathcal{S}_t$, one can write the following[21]

$$\nabla^{\boldsymbol{h}}_{\boldsymbol{w}} \boldsymbol{\eta}^t_i = \boldsymbol{h}^\sharp \cdot \boldsymbol{\kappa}^t_i \cdot \boldsymbol{w} + \sum_{j=1}^{k} \left(\boldsymbol{\omega}^t_{ij} \cdot \boldsymbol{w} \right) \boldsymbol{\eta}^t_j. \tag{4.3}$$

One needs to be careful in calculating time derivatives in $\left(\mathcal{S}, \boldsymbol{g}_t\right)$, since the induced metric $\boldsymbol{g}_t$ itself depends on time. In particular, when calculating the derivative of the inner product $\langle\langle \boldsymbol{u}, \boldsymbol{w} \rangle\rangle_{\boldsymbol{g}_t}$ of two vector fields $\boldsymbol{u}$ and $\boldsymbol{w}$ along a time-parametrized curve c, the usual formula

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle\langle \boldsymbol{u}, \boldsymbol{w} \rangle\rangle_{\boldsymbol{g}_t} = \left\langle\!\left\langle D^{\boldsymbol{g}_t}_t \boldsymbol{u}, \boldsymbol{w} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \left\langle\!\left\langle \boldsymbol{u}, D^{\boldsymbol{g}_t}_t \boldsymbol{w} \right\rangle\!\right\rangle_{\boldsymbol{g}_t}, \tag{4.4}$$

is no longer valid when the metric $\boldsymbol{g}_t$ is t dependent. One instead has[22]

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle\langle \boldsymbol{u}, \boldsymbol{w} \rangle\rangle_{\boldsymbol{g}_t} = \left\langle\!\left\langle D^{\boldsymbol{g}_t}_t \boldsymbol{u}, \boldsymbol{w} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \left\langle\!\left\langle \boldsymbol{u}, D^{\boldsymbol{g}_t}_t \boldsymbol{w} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \langle\langle \boldsymbol{u}, \boldsymbol{w} \rangle\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}}, \tag{4.5}$$

where

$$\langle\langle \boldsymbol{u}, \boldsymbol{w} \rangle\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} = u^a v^b \frac{\partial g_{t\,ab}}{\partial t}.$$

This can be written in terms of the inner product with respect to $\boldsymbol{g}_t$ as

$$\langle\langle \boldsymbol{u}, \boldsymbol{w} \rangle\rangle_{\frac{\partial \boldsymbol{g}_t}{\partial t}} = \left\langle\!\!\left\langle \boldsymbol{u}, \boldsymbol{g}^\sharp_t \cdot \frac{\partial \boldsymbol{g}}{\partial t} \cdot \boldsymbol{w} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t},$$

where $\boldsymbol{g}^\sharp_t$ denotes the "inverse metric," with components g^{ab}_t. Therefore[23]

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle\langle \boldsymbol{u}, \boldsymbol{w} \rangle\rangle_{\boldsymbol{g}_t} = \left\langle\!\left\langle D^{\boldsymbol{g}_t}_t \boldsymbol{u}, \boldsymbol{w} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \left\langle\!\left\langle \boldsymbol{u}, D^{\boldsymbol{g}_t}_t \boldsymbol{w} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \left\langle\!\!\left\langle \boldsymbol{u}, \boldsymbol{g}^\sharp_t \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{w} \right\rangle\!\!\right\rangle_{\boldsymbol{g}_t}. \tag{4.7}$$

[21] Recall that in the chosen coordinate chart $\{\chi^\alpha\}_{\alpha=1,\ldots,n+k}$, one has $h_{\alpha(n+i)} = \left\langle\!\left\langle \tilde{\partial}^t_\alpha, \boldsymbol{\eta}^t_i \right\rangle\!\right\rangle_{\boldsymbol{h}} = \delta_{\alpha(n+i)}$.

[22] Note that $D^{\boldsymbol{g}_t}_t \boldsymbol{g}_t = \frac{\partial \boldsymbol{g}_t}{\partial t}$.

[23] It is also possible to define an alternative covariant time derivative, $\tilde{D}^{\boldsymbol{g}_t}_t$, so that an identity analogous to (4.4) holds. If we let

$$(\tilde{D}^{\boldsymbol{g}_t}_t \boldsymbol{u})^a = \frac{du^a}{dt} + \gamma^a_{cd} u^d \frac{dx^c}{dt} + \frac{1}{2} g^{ab} \frac{\partial g_{bc}}{\partial t} u^c, \tag{4.6}$$

one readily verifies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle\langle \boldsymbol{u}, \boldsymbol{w} \rangle\rangle_{\boldsymbol{g}_t} = \left\langle\!\left\langle \tilde{D}^{\boldsymbol{g}_t}_t \boldsymbol{u}, \boldsymbol{w} \right\rangle\!\right\rangle_{\boldsymbol{g}_t} + \left\langle\!\left\langle \boldsymbol{u}, \tilde{D}^{\boldsymbol{g}_t}_t \boldsymbol{w} \right\rangle\!\right\rangle_{\boldsymbol{g}_t}.$$

See Thiffeault (2001) for a discussion on this alternative covariant time derivative.

Using the Levi–Civita connection for the metric $\boldsymbol{g}_t$ to calculate covariant derivatives, the symmetry lemma of classical Riemann geometry (Lee 1997; Nishikawa 2002) still holds.[24]

Lemma 3.1 *For a Riemannian manifold with a time-dependent metric* $\boldsymbol{g}_t$,

$$D_\epsilon^{\boldsymbol{g}_t}\frac{\partial c(t,\epsilon)}{\partial t} = D_t^{\boldsymbol{g}_t}\frac{\partial c(t,\epsilon)}{\partial \epsilon}.$$

The velocity of the time-dependent embedding ψ_t is defined as

$$\boldsymbol{\zeta} = \frac{\partial \psi(t,\boldsymbol{x})}{\partial t} = \boldsymbol{\zeta}_\parallel + \sum_{i=1}^{k}\zeta_\perp^i\boldsymbol{\eta}_i^t,$$

where $\boldsymbol{\zeta}_\parallel$ is the tangential velocity of the embedding. We also define $\boldsymbol{Z} := \psi_t^*\boldsymbol{\zeta}_\parallel \circ \varphi_t$.

Lemma 3.2 *For an arbitrary embedding* ψ_t , *the following relation holds*

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = \mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t + 2\sum_{i=1}^{k}\zeta_\perp^i\boldsymbol{k}_i^t, \tag{4.8}$$

where $\mathfrak{L}$ *denotes the autonomous Lie derivative.*[25] *For a transversal embedding, i.e., when* $\boldsymbol{Z} = \boldsymbol{0}$, (4.8) *reduces to*

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = 2\sum_{i=1}^{k}\zeta_\perp^i\boldsymbol{k}_i^t. \tag{4.9}$$

Proof First, we note that

$$\begin{aligned}\boldsymbol{L}_\zeta\boldsymbol{h} &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t\circ\psi_s^{-1}\right)^*\boldsymbol{h}\right]_{s=t} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\psi_{s*}\psi_t^*\boldsymbol{h}\right]_{s=t} = \left[\frac{\mathrm{d}}{\mathrm{d}t}\psi_{s*}\boldsymbol{g}_t\right]_{s=t}\\ &= \psi_{t*}\left[D_t^{\boldsymbol{g}_t}\boldsymbol{g}_t\right]_{s=t} = \psi_{t*}\frac{\partial \boldsymbol{g}_t}{\partial t}.\end{aligned} \tag{4.10}$$

On the other hand, we also have

$$\boldsymbol{L}_\zeta\boldsymbol{h} = \mathfrak{L}_\zeta\boldsymbol{h} = \mathfrak{L}_{\zeta_\parallel}\boldsymbol{h} + \sum_{i=1}^{k}\zeta_\perp^i\mathfrak{L}_{\eta_i^t}\boldsymbol{h}.$$

[24] Note that if we were to use the alternative covariant derivative (4.6), this formula would need to be modified.

[25] The autonomous Lie derivative $\mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t$ is defined by holding the explicit time dependence of $\boldsymbol{g}_t$ fixed, i.e., $\mathfrak{L}_{\boldsymbol{Z}}\boldsymbol{g}_t = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{t=s}\left[\left(\psi_t\circ\psi_s^{-1}\right)^*\boldsymbol{g}_s\right]$.

However, for $i \in \{1, \ldots, k\}$ one has

$$\left(\mathfrak{L}_{\boldsymbol{\eta}_i^t} \boldsymbol{h}\right)_{\alpha\beta} = (\boldsymbol{\eta}_i^t)_{\alpha|\beta} + (\boldsymbol{\eta}_i^t)_{\beta|\alpha} = 2\kappa_{(i)\alpha\beta}. \tag{4.11}$$

We observe that $\mathfrak{L}_{\boldsymbol{\zeta}_\parallel} \boldsymbol{h} = \mathfrak{L}_{\psi_{t*}\boldsymbol{Z}} \psi_{t*} \boldsymbol{g}_t$, and following (Marsden and Hughes 1983, p. 98), we have $\mathfrak{L}_{\psi_{t*}\boldsymbol{Z}} \psi_{t*} \boldsymbol{g}_t = \psi_{t*} \mathfrak{L}_{\boldsymbol{Z}} \boldsymbol{g}_t$. Thus

$$\boldsymbol{L}_{\zeta} \boldsymbol{h} = \psi_{t*} \left(\mathfrak{L}_{\boldsymbol{Z}} \boldsymbol{g}_t + 2 \sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{k}_i^t \right). \tag{4.12}$$

Finally, it follows from (4.10) and (4.12) that

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = \mathfrak{L}_{\boldsymbol{Z}} \boldsymbol{g}_t + 2 \sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{k}_i^t.$$

□

Appendix 2: An Alternative Derivation of the Tangent Balance of Linear Momentum

In this section, we provide an alternate proof for the tangential balance of linear momentum in the particular case of a transversal evolution of the ambient space. This derivation is a generalized version, for arbitrary co-dimension $k = \dim \mathcal{Q} - \dim \mathcal{S}_t$, of a theorem appearing in Marsden and Hughes (1983), p. 129. The generalized version can be stated as follows (see § 2.1 and Fig. 2 to recall the notation):

Theorem 3.1 *Assume that given scalar functions a and b, and a vector field* $\boldsymbol{c}$ *satisfy the following master balance law for any open set* $\mathcal{U}$ *with* C^1 *piecewise boundary:*

$$\frac{d}{dt} \int_{\varphi_t(\mathcal{U})} a dv = \int_{\varphi_t(\mathcal{U})} b dv + \int_{\partial\varphi_t(\mathcal{U})} \langle\langle \boldsymbol{c}, \mathbf{n} \rangle\rangle_{\boldsymbol{g}_t} da, \tag{5.1}$$

where $\mathbf{n}$ *is the unit normal vector to* $\partial\varphi_t(\mathcal{U})$ *in* $\mathcal{S}$. *Localization of* (5.1) *gives one*

$$\frac{da}{dt} + a \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{v} + a \sum_{i=1}^{k} \zeta_\perp^i \operatorname{tr}\left(\boldsymbol{k}_i^t\right) = b + \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{c}, \tag{5.2}$$

where we recall that $\boldsymbol{v}$ *is the velocity field of* φ_t *and* $\boldsymbol{\zeta} = \sum_{i=1}^{k} \zeta_\perp^i \boldsymbol{\eta}_i^t$ *is the velocity field of* ψ_t *(we have* $\boldsymbol{\zeta}_\parallel = \boldsymbol{0}$ *since we are assuming transversal evolution).*[26]

[26] Note that what Marsden and Hughes (1983) denote by $\boldsymbol{v}$ is the equivalent of $\boldsymbol{\Upsilon}$ in our notation, so that their $\boldsymbol{v}_\parallel$ corresponds to $\boldsymbol{v}$ (recall that $z = \psi_t^* \boldsymbol{\zeta}_\parallel = \boldsymbol{0}$) and their v_n would be $\zeta_\perp$ in the particular case when $\mathcal{S}_t$ is a hypersurface in $\mathcal{Q}$.

Proof Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(\mathcal{U})} a\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}} aJ\mathrm{d}V = \int_{\mathcal{U}} \frac{\mathrm{d}}{\mathrm{d}t}(aJ)\mathrm{d}V = \int_{\mathcal{U}} \left(\frac{\mathrm{d}a}{\mathrm{d}t}J + a\frac{\mathrm{d}J}{\mathrm{d}t}\right)\mathrm{d}V.$$

However

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \left(\mathrm{div}_{\boldsymbol{g}_t}\,\boldsymbol{v}\right)J + \frac{1}{2}J\,\mathrm{tr}\left(\frac{\partial \boldsymbol{g}_t}{\partial t}\right),$$

and following (4.8), one has

$$\frac{\partial \boldsymbol{g}_t}{\partial t} = 2\sum_{i=1}^{k}\zeta_\perp^i \boldsymbol{k}_i^t.$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(\mathcal{U})} a\mathrm{d}v &= \int_{\mathcal{U}}\left(\frac{\mathrm{d}a}{\mathrm{d}t} + a\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v} + a\sum_{i=1}^{k}\zeta_\perp^i\,\mathrm{tr}\left(\boldsymbol{k}_i^t\right)\right)J\mathrm{d}V \\ &= \int_{\varphi_t(\mathcal{U})}\left(\frac{\mathrm{d}a}{\mathrm{d}t} + a\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v} + a\sum_{i=1}^{k}\zeta_\perp^i\,\mathrm{tr}\left(\boldsymbol{k}_i^t\right)\right)\mathrm{d}v.\end{aligned} \tag{5.3}$$

On the other hand, by using Stokes' theorem, one can write

$$\int_{\partial\varphi_t(\mathcal{U})} \langle\!\langle \boldsymbol{c}, \mathbf{n}\rangle\!\rangle_{\boldsymbol{g}_t}\, da = \int_{\varphi_t(\mathcal{U})} \mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{c}\,\mathrm{d}v. \tag{5.4}$$

Therefore, by using (5.3) and (5.4), (5.1) transforms to

$$\int_{\varphi_t(\mathcal{U})}\left(\frac{\mathrm{d}a}{\mathrm{d}t} + a\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v} + a\sum_{i=1}^{k}\zeta_\perp^i\,\mathrm{tr}\left(\boldsymbol{k}_i^t\right)\right)\mathrm{d}v = \int_{\varphi_t(\mathcal{U})}\left(b + \mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{c}\right)\mathrm{d}v.$$

Thus, by arbitrariness of the subset $\mathcal{U}$, one finds that

$$\frac{\mathrm{d}a}{\mathrm{d}t} + a\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v} + a\sum_{i=1}^{k}\zeta_\perp^i\,\mathrm{tr}\left(\boldsymbol{k}_i^t\right) = b + \mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{c}. \tag{5.5}$$

□

First, the localized conservation of mass is derived. In integral form, conservation of mass reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(\mathcal{U})} \rho\mathrm{d}v = 0.$$

Hence, using the above theorem ($a=\rho$, $b=0$, and $c=0$), the conservation of mass in localized form reads[27]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v}+\rho\sum_{i=1}^{k}\zeta_{\perp}^{i}\,\mathrm{tr}\left(\boldsymbol{k}_i^t\right)=0. \tag{5.6}$$

Next, we look at the balance of linear momentum, which in integral form reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(\mathcal{U})}\rho\boldsymbol{v}\mathrm{d}v=\int_{\varphi_t(\mathcal{U})}\rho\boldsymbol{B}\mathrm{d}v+\int_{\partial\varphi_t(\mathcal{U})}\boldsymbol{\sigma}\cdot\mathbf{n}\mathrm{d}a, \tag{5.7}$$

where $\boldsymbol{B}$ is the body force per unit mass, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, and $\mathbf{n}$ is the unit normal vector to $\partial\varphi_t(\mathcal{U})$.

Remark 3.3 Note that (5.7) makes sense only when the ambient space $\mathcal{S}_t$ is endowed with a linear structure. In a general manifold, integrating a vector field does not make sense. Therefore, the proof given in this appendix is only valid for linear ambient spaces. However, the resulting localized tangential balance of linear momentum (5.8) still holds in the case of a general manifold as shown in § 2.1 using a Lagrangian field theory, see Eq. (2.17).

In order to use the above theorem, we contract the balance of linear momentum (5.7) with an arbitrary time-independent covariantly constant vector field $\boldsymbol{u}$ tangent to $\varphi_t(\mathcal{U})$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(\mathcal{U})}\langle\langle\rho\boldsymbol{v},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}\,\mathrm{d}v=\int_{\varphi_t(\mathcal{U})}\langle\langle\rho\boldsymbol{B},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}\,\mathrm{d}v+\int_{\partial\varphi_t(\mathcal{U})}\langle\langle\boldsymbol{\sigma}\cdot\mathbf{n},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}\,\mathrm{d}a.$$

We can then use the above theorem for $a=\langle\langle\rho\boldsymbol{v},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}$, $b=\langle\langle\rho\boldsymbol{B},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}$, and $\boldsymbol{c}=\boldsymbol{\sigma}\cdot\boldsymbol{u}$. Hence, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle\rho\boldsymbol{v},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}+\langle\langle\rho\boldsymbol{v},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}\,\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v}+\langle\langle\rho\boldsymbol{v},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}\,v_n\mathrm{tr}\boldsymbol{k}=\langle\langle\rho\boldsymbol{B},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}+\mathrm{div}_{\boldsymbol{g}_t}(\boldsymbol{\sigma}\cdot\boldsymbol{u}).$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle\rho\boldsymbol{v},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}=\frac{\mathrm{d}\rho}{\mathrm{d}t}\langle\langle\boldsymbol{v},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}+\rho\left\langle\!\!\left\langle D_t^{\boldsymbol{h}}\boldsymbol{v},\boldsymbol{u}\right\rangle\!\!\right\rangle_{\boldsymbol{h}},$$

where $D_t^{\boldsymbol{h}}$ denotes the covariant time derivative with respect to the metric $\boldsymbol{h}$. Therefore, it follows that

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\mathrm{div}_{\boldsymbol{g}_t}\boldsymbol{v}+\rho v_n\mathrm{tr}\boldsymbol{k}\right)\langle\langle\boldsymbol{v},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}+\rho\left\langle\!\!\left\langle D_t^{\boldsymbol{h}}\boldsymbol{v},\boldsymbol{u}\right\rangle\!\!\right\rangle_{\boldsymbol{h}}=\langle\langle\rho\boldsymbol{B},\boldsymbol{u}\rangle\rangle_{\boldsymbol{h}}+\mathrm{div}_{\boldsymbol{g}_t}(\boldsymbol{\sigma}\cdot\boldsymbol{u}).$$

[27] Note that (5.6) is equivalent (2.22) since by Lemma 3.2, we have $\frac{\partial\boldsymbol{g}_t}{\partial t}=2\sum_{i=1}^{k}\zeta_{\perp}^{i}\boldsymbol{k}_i^t$.

The first term vanishes following the conservation of mass (5.6). Thus, by arbitrariness of $\boldsymbol{u}$ one concludes that

$$\rho \left(D_t^{\boldsymbol{h}} \boldsymbol{v} \right)_{\parallel} = \rho \boldsymbol{B} + \operatorname{div}_{\boldsymbol{g}_t} \boldsymbol{\sigma} . \tag{5.8}$$

Note that $\left(D_t^{\boldsymbol{h}} \boldsymbol{v} \right)_{\parallel}$ is different from $D_t^{\boldsymbol{g}} \boldsymbol{v}$. In fact, we can write following Proposition 2.1 that

$$\begin{aligned} \left(D_t^{\boldsymbol{h}} \boldsymbol{v} \right)_{\parallel} &= D_t^{\boldsymbol{g}_t} \boldsymbol{v} + 2 \sum_{i=1}^{k} \zeta_{\perp}^{i} \boldsymbol{g}_t^{\sharp} \cdot \boldsymbol{k}_i^t \cdot \boldsymbol{v} - \sum_{i=1}^{k} \zeta_{\perp}^{i} \left(\mathrm{d}\zeta_{\perp}^{i} \right)^{\sharp} - \sum_{i,j=1}^{k} \zeta_{\perp}^{i} \zeta_{\perp}^{j} \boldsymbol{o}_{ij}^{t\sharp} \\ &= D_t^{\boldsymbol{g}_t} \boldsymbol{v} + \boldsymbol{g}_t^{\sharp} \cdot \frac{\partial \boldsymbol{g}_t}{\partial t} \cdot \boldsymbol{v} - \sum_{i=1}^{k} \zeta_{\perp}^{i} \left(\mathrm{d}\zeta_{\perp}^{i} \right)^{\sharp} - \sum_{i,j=1}^{k} \zeta_{\perp}^{i} \zeta_{\perp}^{j} \boldsymbol{o}_{ij}^{t\sharp}. \end{aligned} \tag{5.9}$$

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