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On geometric discretization of elasticity

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This paper presents a geometric discretization of elasticity when the ambient space is Euclidean. This theory is built on ideas from algebraic topology, exterior calculus, and the recent developments of discrete exterior calculus. We first review some geometric ideas in continuum mechanics and show how constitutive equations of linearized elasticity, similar to those of electromagnetism, can be written in terms of a material Hodge star operator. In the discrete theory presented in this paper, instead of referring to continuum quantities, we postulate the existence of some discrete scalar-valued and vector-valued primal and dual differential forms on a discretized solid, which is assumed to be a triangulated domain. We find the discrete governing equations by requiring energy balance invariance under time-dependent rigid translations and rotations of the ambient space. There are several subtle differences between the discrete and continuous theories. For example, power of tractions in the discrete theory is written on a layer of cells with a nonzero volume. We obtain the compatibility equations of this discrete theory using tools from algebraic topology. We study a discrete Cosserat medium and obtain its governing equations. Finally, we study the geometric structure of linearized elasticity and write its governing equations in a matrix form. We show that, in addition to constitutive equations, balance of angular momentum is also metric dependent; all the other governing equations are topological. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2830977]

I. INTRODUCTION AND MOTIVATION

The main motivation of this paper is to pave the road for developing geometric discretization schemes for elasticity. To our best knowledge, to date, there is no systematic geometrization of discrete elasticity. In the case of electromagnetism, this is an easier task as all the fields can be expressed by (scalar-valued) differential forms.[12,9,34,18] In other words, the classical theory of exterior calculus is all that one needs to geometrize electromagnetism. It turns out that a complete geometrization of nonlinear elasticity is much more difficult as one encounters various types of tensor and two-point tensor fields. A complete geometrization of nonlinear elasticity should use bundle-valued differential forms. This has been done recently for stress by Kanso *et al.*[22] In this paper, we show how discrete elasticity can be geometrized when the ambient space is Euclidean. Among the interesting things are the constitutive and compatibility equations. We obtain the discrete governing field equations by postulating an energy balance and its invariance under time-dependent rigid translations and rotations of the ambient space.

To date, the traditional way of solving mechanics problems numerically has been the following. Balance laws are postulated for an arbitrary sub-body $\mathcal{P}$ of a (finite) body $\mathcal{B}$. These laws are usually written in integral forms. The localization of the integral balance laws gives the differential (pointwise) governing equations. These governing equations are then discretized using different techniques such as finite element method, finite volume method, etc. In the end, a set of discrete

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governing equations is solved for some discrete quantities that are defined on a discretization of $\mathcal{B}$, which is called the finite element mesh in the finite element method, for example. The following diagram summarizes this process schematically.

$$\begin{array}{ccc} \mathcal{B} \text{ in Equilibrium} & \xrightarrow{\text{Balance Laws}} & \text{Integral Balance Laws} \\ {\scriptstyle \text{Geometric Discretization}}\downarrow & & \downarrow{\scriptstyle \text{Localization}} \\ \text{Discrete Governing Equations} & \xleftarrow{\text{Discretization of BVP}} & \text{Governing Differential Equatic} \end{array}$$

What we have in mind is different. Our idea is shown in the above commutative diagram. Instead of going through the unnecessary step of localization, it would be desirable to postulate the balance laws for the discretized system and obtain the discrete governing equations directly.

The idea of geometric discretization of field theories is not new. In different field theories, it has been realized for quite some time that a more natural discretization of governing equations is to start directly from a discrete system without any reference to the continuum. This of course would be the only choice for systems that are intrinsically discrete, e.g., molecular systems, etc. As an example of a recent attempt in geometric discretization of field theories, we can mention the so-called "finite formulation" method proposed by Tonti.[35] The important thing to note is that before trying to construct a geometric discrete theory for any field theory, one first needs to have a complete understanding of the geometry of the field theory itself. In the case of nonlinear elasticity, this geometric structure is very rich and the existing results from geometric discretization of electromagnetism, for example, cannot be directly and naively used.

Balance laws in continuum mechanics are usually written in terms of conservation of some physical quantities (such as mass, linear momentum, angular momentum, etc.). They all have the following generic form:

$$\frac{d}{dt}\int_{\mathcal{P}} \xi dV = \int_{\partial\mathcal{P}} \eta dA, \tag{1.1}$$

where $\mathcal{P}$ is a sub-body and ξ and η are volume and surface densities of some physical quantities. Tonti[36,37] demonstrated that for a body considered as a cell complex, balance laws can be written as

$$\delta(f) = g, \tag{1.2}$$

where f is a k-cochain representing the flux (a quantity associated with the boundary of each cell) and g is a $(k+1)$-cochain representing the production of the physical quantity inside each cell. However, this is just a formal statement and may not be that trivial to express geometrically for a field theory like elasticity.

A natural tool for geometric formulation of balance laws, or more specifically for scalar balance laws, is exterior calculus. An interesting character of exterior calculus is its generality and coordinate independence. For example, Stokes' theorem compactly represents all the integral theorems of vector calculus.[15] One of the main goals of developing a geometric discrete theory of elasticity is to put all the existing computational techniques in one abstract setting. This rationalization of computational mechanics will be theoretically interesting for its own sake. In addition to this, having such an abstract framework would enable one to develop new numerical schemes that have the required features for a specific application. It will also be useful to be able to separate the topological and metric-dependent quantities and operators, etc., something that is completely clear in the case of electromagnetism. In the continuum setting, nonlinear elasticity has a fairly well-developed geometric formulation. It is seen that unlike electromagnetism, in nonlinear elasticity, one has to work with tensor and two-point tensor fields. This means that elasticity cannot be rewritten geometrically simply in terms of differential forms; it turns out that one needs to consider bundle-valued forms.[22] We should emphasize that convergence issues are not discussed in this paper.

This paper is structured as follows. In Sec. II, in order to make the paper self-contained, we give a brief review of algebraic topology, exterior calculus, and recent developments in discrete exterior calculus. We also extend some of the existing results to the case of discrete bundle-valued forms. We then review geometric continuum mechanics in Sec. III. We first review the geometric formulation of classical electromagnetism and its geometric discretization. Linear elasticity is studied as a geometric linearization of nonlinear elasticity about a given configuration. Part of this section is new. In particular, we show that, similar to those of electromagnetism, constitutive equations of linearized elasticity can be written in terms of a material Hodge star operator. We then formulate linear elasticity in terms of vector-valued and covector-valued differential forms. In Sec. IV, we present a discrete theory of elasticity based on geometric ideas and with no reference to continuum quantities. Conclusions are given in Sec. V.

II. ALGEBRAIC TOPOLOGY, EXTERIOR CALCULUS, BUNDLE-VALUED DIFFERENTIAL FORMS, AND DISCRETE EXTERIOR CALCULUS

Here, we review some basic concepts from algebraic topology. We follow Munkres,[28] Hatcher,[20] Lee,[23] and Hirani.[21]

The main goal of topology is to classify spaces up to homeomorphisms. In general, deciding whether two given spaces are homeomorphic is not easy. Several topological invariants are defined in topology and for a given topological invariant, a necessary condition for two topological spaces to be homeomorphic is to have the same topological invariants. Algebraic topology is a branch of topology that studies algebraic topological invariants (e.g., homotopy, homology and cohomology groups, etc.).

Given a geometrically independent set $\{v_0,\ldots,v_n\}$ in $\mathbb{R}^N$,[1] the n-simplex σ^n is defined as

$$\sigma^n := \left\{ x \in \mathbb{R}^N : x = \sum_{i=0}^{n} t_i v_i, t_i \geqslant 0 \ \forall\, i, \sum_{i=0}^{n} t_i = 1 \right\}. \tag{2.1}$$

The numbers t_i are called the barycentric coordinates of x with respect to $\{v_0,\ldots,v_n\}$. The points $v_0,\ldots,v_n$ are called vertices of σ^n and $\dim \sigma^n = n$. Any simplex spanned by a proper subset of $\{v_0,\ldots,v_n\}$ is called a face of σ^n. If σ is a face of σ^n, we show this by $\sigma \prec \sigma^n$ (or $\sigma^n \succ \sigma$). The smallest affine subspace of $\mathbb{R}^N$ containing σ is called the plane of σ and is denoted $P(\sigma)$. A simplicial complex K in $\mathbb{R}^N$ is a finite collection of simplices (in $\mathbb{R}^N$) such that

- K contains all faces of every simplex;
- intersection of any two simplices of K is a face of both of them.

L, a subcollection of K, is called a subcomplex of K if it contains all faces of its elements. The p-skeleton of K is the collection of all simplices of K of dimension at most p and is denoted $K^{(p)}$. For example, in a truss structure, $K^{(0)}$ is the set of nodes and $K^{(1)}$ is the set of members of the truss.

An Abelian group G is a free group if it has a basis, i.e., if there exists a family $\{g_\alpha\}_{\alpha \in I} \subset G$ (for some index set I) such that every $g \in G$ has the following unique representation:

$$g = \sum_{\alpha} n_\alpha g_\alpha \quad \text{(finite sum)}, \quad \forall\, g \in G, \quad n_\alpha \in \mathbb{Z}. \tag{2.2}$$

For a simplex σ, two orderings of its vertices are equivalent if one is an even permutation of the other. By definition, a zero simplex has only one orientation. For simplices with positive dimension, there are two possible orientations. Suppose that $\{v_0,\ldots,v_n\}$ is a geometrically independent set in $\mathbb{R}^N$. The oriented simplex σ^p spanned by these points is denoted

[1] This means that $\{v_1 - v_0,\ldots,v_n - v_0\}$ is a linearly idependent set of vectors in $\mathbb{R}^N$.

$$\sigma^p = [v_0, \ldots, v_n]. \tag{2.3}$$

Let $\sigma^p = [v_0, \ldots, v_p]$ be an oriented simplex ($p \geqslant 1$). Each $(p-1)$-face of σ^p can be given an orientation called the induced orientation. This orientation is defined by

$$\sigma^{p-1} = (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_p], \tag{2.4}$$

where the hat on v_i means omit the ith vertex.

Given two simplices σ and σ' embedded in $\mathbb{R}^N$ if they share a face of order $n-1$, the two will have the same orientations if they induce opposite orientations on the shared face. If σ and σ' have the same orientations, we write $\operatorname{sgn}(\sigma, \sigma') = +1$; otherwise, we write $\operatorname{sgn}(\sigma, \sigma') = -1$.

A p-chain on a simplicial complex K is a function c from the set of oriented p-simplices of K to $\mathbb{Z}$ such that

- $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex;
- $c(\sigma) \neq 0$ for finitely many p-simplices.

p-chains are added by adding their values. The group of p-chains of K (with this binary operation) is denoted $C_p(K)$. If $p < 0$ or $p > \dim(K)$, $C_p(K)$ is the trivial group by definition. The elementary chain c corresponding to an oriented simplex σ is the function defined as

$$c(\tau) := \begin{cases} 1 & \text{if } \tau = \sigma \\ -1 & \text{if } \tau = \sigma' \text{ (opposite orientation of } \sigma) \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

It can be shown that $C_p(K)$ is a free Abelian group. A basis for this group is obtained by giving each p-simplex an orientation and using the corresponding elementary chains as a basis. Boundary operator is a homomorphism $\partial_p : C_p(K) \to C_{p-1}(K)$ defined on each oriented simplex $\sigma = [v_0, \ldots, v_n]$ as

$$\partial_p(\sigma) = \partial_p[v_0, \ldots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n], \tag{2.6}$$

where $\hat{v}_i$ means that the vertex v_i is removed from the array. Note that for $p < 0$ or $p > \dim(K)$, ∂_p is the trivial homomorphism. Note also that $\partial_{p-1} \circ \partial_p = 0$ (boundary of a boundary is empty).

The group of p-cycles $Z_p(K)$ is the kernel of $\partial_p : C_p(K) \to C_{p-1}(K)$. The group of p-boundaries $B_p(K)$ is the image of $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$. Note that the boundary of a $(p+1)$-chain is always a p-cycle. However, a p-cycle need not be boundary of a $(p+1)$-chain, in general. The pth homology group of K, $H_p(K)$ is the following quotient group:

$$H_p(K) = Z_p(K) / B_p(K). \tag{2.7}$$

Rank of the group $H_p(K)$ is called the pth Betti number $b_p(K)$. For example, intuitively, the first Betti number is the maximum number of cuts that can be made without dividing the space into two pieces. We will see in Sec. IV that this number will play an important role in compatibility equations for a two-dimensional discretized solid.

The following sequence is called a chain complex and is denoted by $\mathcal{C}$:

$$0 \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{p+2}} C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0. \tag{2.8}$$

Given two Abelian groups G and H, the set $\operatorname{Hom}(G, H)$ of all homomorphisms of G into H is an Abelian group if homomorphisms are added by adding their values in H.

Given a simplicial complex K and an Abelian group G, the group of p-cochains of K with coefficients in G is the group

$$C^p(K;G) \coloneqq \mathrm{Hom}(C_p(K),G). \tag{2.9}$$

The coboundary operator δ is defined to be the dual of $\partial: C_{p+1}(K) \to C_p(K)$, i.e.,

$$\delta: C^p(K;G) \to C^{p+1}(K;G). \tag{2.10}$$

If c^p is a p-cochain and c_p is a p-chain, $\langle c^p, c_p\rangle = c^p(c_p)$ denotes the value of c^p on c_p. In this notation, coboundary operator is defined as

$$\langle \delta c^p, c_{p+1}\rangle = \langle c^p, \partial c_{p+1}\rangle. \tag{2.11}$$

The following sequence is called the cochain complex induced by the coboundary operator:

$$0 \xleftarrow{\delta^n} C^n(K) \xleftarrow{\delta^{n-1}} \cdots \xleftarrow{\delta^{p+1}} C^{p+1}(K) \xleftarrow{\delta^p} C^p(K) \xleftarrow{\delta^{p-1}} C^{p-1}(K) \xleftarrow{\delta^{p-2}} \cdots \xleftarrow{\delta^0} C^0(K) \leftarrow 0. \tag{2.12}$$

The unit k-ball is the following subset of $\mathbb{R}^k$:

$$\{\mathbf{x} \in \mathbb{R}^k : |\mathbf{x}| \leq 1\}. \tag{2.13}$$

A k-cell is a set homeomorphic to the unit k-ball.

Equation (2.11) is known as Stokes' theorem for cochains. It is a well-known fact that Stokes' theorem is metric-free and can be thought of as a connection between topology and analysis (see Refs. 26 and 19). There is also a close connection between cochains and differential forms. Roughly speaking, cochains are discrete analogs of differential forms.

A differential k-form can, in principle, represent a physical quantity. Integrating the differential form on a k-manifold gives the "amount" of the physical quantity in the k-manifold. For example, mass density is a 3-form and integrating it on a 3-manifold (a sub-body) gives the mass of the sub-body.[27]

Consider a differential k-form α defined on a complex K. The associated k-cochain c^k is defined by

$$\int_{c_k} \alpha = \langle c^k, c_k\rangle, \quad c_k \in K. \tag{2.14}$$

Now, the discrete and continuous Stokes' theorems can be related through the following diagram (see Ref. 8):

$$\begin{array}{ccc} k\text{-cochain } c^k & \xrightarrow{\delta} & (k+1)\text{-cochain } c^{k+1} \\ \int_{c_k} \uparrow & & \uparrow \int_{c_{k+1}} \\ k\text{-form } \alpha & \xleftarrow{\mathrm{d}} & (k+1)\text{-form } d\alpha \end{array}$$

A. Discrete exterior calculus (DEC)

In this subsection, we review some recent developments of discrete exterior calculus[22,11] (see also Beauće and Sen[2] and Wilson[39]). A (locally or globally) discretized solid resembles a simplicial (or cell) complex and therefore it would be natural to use the concepts of algebraic topology for such a discretized system. It would also be natural to think about using chains and cochains and try to express the discrete fields on a discretized solid body in terms of cochains.

Primary and secondary complexes. Here, it is assumed that we are given a global triangulation of the domain and that the triangulation is a simplicial complex. It would be a natural question to ask whether this is always possible for an arbitrary domain. The answer is no but we restrict ourselves to cases where this is possible. A classical result is that differentiable manifolds have triangulations.[7] Thus, in this paper, it is assumed that a Riemannian manifold (M,g) is discretized

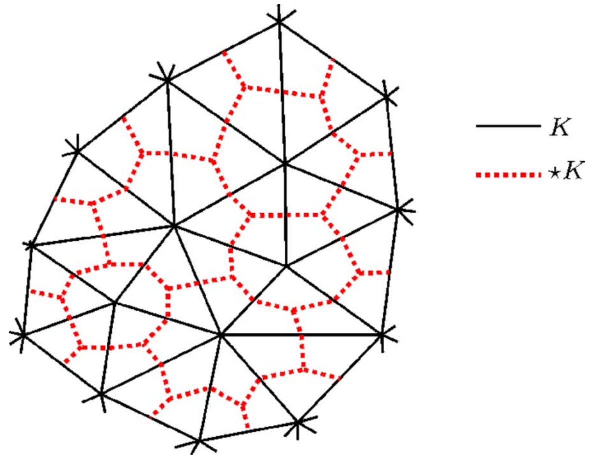

FIG. 1. (Color online) A cell complex K and its barycentric dual $\star K$.

by a simplicial complex K and that K is embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$ (usually $N=2$ or 3). The simplicial complex K is called the primary complex.

In order to be able to develop a discrete exterior calculus, it is necessary to define a dual cell for a given simplicial complex. One motivation for defining a dual complex is the following. Suppose that a 0-form is defined on a simplicial complex K embedded in $\mathbb{R}^2$. As will be explained later, this discrete form assigns a real number to each 0-simplex σ^0. This discrete 0-form is isomorphic (through a discrete Hodge star operator) to a discrete 2-form which assigns a real number to 2-simplices of a complex $\star K$. The dual complex $\star K$ is defined such that its 2-cells are in a one-to-one correspondence with the 0-simplices of K. There are different possibilities for a dual complex; barycentric and circumcentric duals are two examples. It should be noted that duality is problem dependent and one should try to understand the underlying physics of the problem before developing a discrete exterior calculus. In other words, dual of a simplicial complex is not unique and depending on the problem different duals can be defined. In this sense, there is no unique DEC on simplicial complexes.

The circumcenter $c(\sigma^p)$ of a p-simplex σ^p is the unique point in the p-dimensional affine space containing σ^p that is equidistant from all the $p+1$ nodes of σ^p.[2] Circumcentric subdivision of a simplicial complex is the collection of all simplices $[c(\sigma_1),\ldots,c(\sigma_n)]$, where $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_n$. Given a p-simplex σ^p in K, circumcentric duality operator acts on σ^p and gives an $(n-p)$-cell defined by

$$\star \sigma^p = \sum_{\sigma^p \prec \sigma^{p+1} \prec \cdots \sigma^n} \varepsilon(\sigma^p,\sigma^{p+1},\ldots,\sigma^n)[c(\sigma^p),c(\sigma^{p+1}),\ldots,c(\sigma^n)], \tag{2.15}$$

where $\varepsilon(\sigma^p,\sigma^{p+1},\ldots,\sigma^n)=1$ or -1 to ensure that the orientation of $\star\sigma^p$ is consistent with those of σ^p and the volume form of the ambient space $\mathbb{R}^n$.[11] Circumcentric dual has some nice properties but is not suitable for a generic simplicial complex. In this paper, we do not use a specific dual and what follows applies to any well-defined dual complex.

Figure 1 shows a two-dimensional simplicial complex and its barycentric dual. Note that a boundary dual 2-cell does not include any half primal edges.[21]

Discrete differential forms. In the continuous case, a k-form is integrated on a k-manifold. A discrete k-form is defined by the values it associates to each k-simplex of a simplicial complex K. This means that, for example, a discrete 0-form ϕ is completely defined on a simplicial (or nonsimplicial) complex K if its value on each 0-simplex of K is given.

[2] Note that not every simplicial complex has a well-defined circumcentric dual.[32]

A primal discrete p-form α is an element of $\mathrm{Hom}(C_p(K;\mathbb{Z}),\mathbb{R})$. This means that a discrete p-form is a cochain. We make this space an Abelian group; two homomorphisms are added by adding their values in the additive group $\mathbb{R}$. The space of discrete p-forms is denoted by $\Omega_d^p(K)$. Thus

$$\Omega_d^p(K) \coloneqq C^p(K;\mathbb{R}) = \mathrm{Hom}(C_p(K),\mathbb{R}). \tag{2.16}$$

It is possible to define discrete differential forms on a dual complex $\star K$ similar to what was done for K. The space of dual discrete differential p-forms is denoted by $\Omega_d^p(\star K)$.

The discrete exterior derivative $\mathbf{d}^p:\Omega_d^p(K)\to\Omega_d^{p+1}(K)$ is by definition the same as the coboundary operator δ^p. The motivation behind this definition is that $\mathbf{d}^p$ maps a discrete p-form to a discrete $(p+1)$-form. A discrete $(p+1)$-form associates a real number to each $(p+1)$-simplex of K. This is why $\mathbf{d}^p$ should be related to δ^p. We choose to take $\mathbf{d}^p=\delta^p$. This definition is consistent with the smooth theory. Note that

$$\langle \mathbf{d}\circ\mathbf{d}\alpha,\sigma\rangle = \langle \mathbf{d}\alpha,\partial\sigma\rangle = \langle \alpha,\partial\partial\sigma\rangle = \langle \alpha,0\rangle = 0. \tag{2.17}$$

Thus, $\mathbf{d}\circ\mathbf{d}=\mathbf{0}$. This operation is natural with respect to restrictions because it is defined locally on each simplex.

Consider two simplicial complexes K and S. Let $\varphi:K^{(0)}\to S^{(0)}$ be a vertex map. Assume that whenever 0-simplices $v_0,\ldots,v_n\in K$ span an n-simplex of K, the 0-simplices $\varphi(v_0),\ldots,\varphi(v_n)$ are vertices of an n-simplex of S. Then, φ can be extended to a continuous map $\tilde{\varphi}:|K|\to|S|$ such that

$$x=\sum_{i=0}^{n} t_i v_i \Rightarrow \tilde{\varphi}(x)=\sum_{i=0}^{n} t_i\varphi(v_i), \tag{2.18}$$

where $|K|$ and $|S|$ are polytopes of K and S, respectively.[28] The map $\tilde{\varphi}$ is called the linear simplicial map induced by the vertex map φ. If φ is a bijection, $\tilde{\varphi}$ is called a simplicial homeomorphism of K with S. $\tilde{\varphi}$ can be thought of as a discrete deformation mapping if one thinks of K and S as discretizations of a body in its reference and current configurations. Now, it is shown that, similar to the smooth theory, the discrete exterior derivative is natural with respect to discrete pullbacks. Suppose that $\tilde{\varphi}:|K|\to|S|$ is a simplicial homeomorphism and $\alpha\in\Omega_d^p(S)$ and $\sigma^{p+1}\in K$. Then

$$\langle \varphi^*(d\alpha),\sigma^{p+1}\rangle = \langle d\alpha,\varphi(\sigma^{p+1})\rangle = \langle \alpha,\partial\varphi(\sigma^{p+1})\rangle = \langle \alpha,\varphi(\partial\sigma^{p+1})\rangle = \langle \varphi^*(\alpha),\partial\sigma^{p+1}\rangle = \langle d\varphi^*(\alpha),\sigma^{p+1}\rangle.$$

□

For a smooth n-manifold, Hodge star operator is the unique isomorphism (because there is a metric on the manifold) between k- and $(n-k)$-forms. This suggests that $*\alpha^p$ should be defined on $\star\sigma^p$ as $\star\sigma^p$ has dimension $n-p$. Discrete Hodge star operator is the mapping $*:C^p(K)\to C^{n-p}(\star K)$ defined as

$$\frac{1}{|\star\sigma^p|_{n-p}}\langle *\alpha^p,\star\sigma^p\rangle = \frac{1}{|\sigma^p|_p}\langle \alpha^p,\sigma^p\rangle, \tag{2.19}$$

where $|\sigma^p|_p$ is the volume of σ^p. Similarly, $|\star\sigma^p|_{n-p}$ is the volume of $\star\sigma^p$. Note that σ^p is short for "the chain with all weights null, except the one on σ^p, which is 1."

Similar to electromagnetism, where metric properties show up in a material Hodge star operator, we expect to be able to represent the linear stress-strain relations in linear elasticity by a material Hodge star. It will be shown in the sequel that this is indeed possible.

Discrete vector fields. One can define two types of discrete vector fields: primal and dual discrete vector fields. For a flat simplicial complex, [3] a discrete primal vector field assigns a vector to each 0-simplex of K, i.e.,

[3] An n-dimensional simplicial complex is flat if all its simplices lie in the same affine n-subspace of $\mathbb{R}^N$, $0\leq n\leq N$.

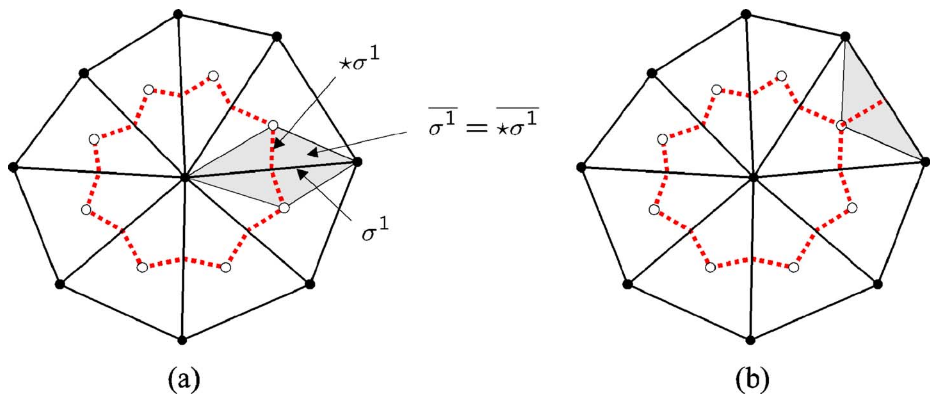

FIG. 2. (Color online) Support volume of a 1-simplex in a two-dimensional simplicial complex. (a) An interior support volume and (b) a boundary support volume.

$$\mathbf{X}: K^{(0)} \to \mathbb{R}^N,$$

$$\sigma^0 \mapsto \mathbf{X}(\sigma^0).$$

In other words, a discrete primal vector field is a primal vector-valued 0-cochain. A discrete dual vector field is a map from dual n-simplices to $\mathbb{R}^N$ such that

$$\mathbf{X}(\star\sigma^n) \in P(\sigma^n), \quad \forall\ \sigma^n \in K. \tag{2.20}$$

Note that a discrete primal vector field is an $\mathbb{R}^N$-valued discrete primal 0-form and similarly a discrete dual vector field is an $\mathbb{R}^N$-valued discrete 0-form.

Discrete primal vector fields. Consider a simplicial complex K embedded in $\mathbb{R}^N$. A primal discrete vector field $\mathbf{X}$ on K is a mapping $\mathbf{X}: K^{(0)} \to \mathbb{R}^N$, where $K^{(0)}$ is the 0-dimensional subcomplex of K, which consists of 0-simplices of K. The value of $\mathbf{X}$ on each primal p-simplex is tangent to the simplex. The space of primal discrete vector fields on K is denoted by $\mathfrak{X}_d(K)$. Note that here we have implicitly assumed that a primal vector field is an $\mathbb{R}^N$-valued primal 0-form, i.e., the 0-form takes its values in the same linear space $\mathbb{R}^N$.

Discrete dual vector fields. Suppose that K is a flat simplicial complex (a simplicial complex of dimension $0 \leqslant n \leqslant N$ embedded in $\mathbb{R}^N$). A dual discrete vector field $\mathbf{X}$ is a map $\mathbf{X}: (\star K)^{(n)} \to \mathbb{R}^N$, where $(\star K)^{(n)}$ is the subcomplex of cells dual to primal nodes. The space of dual discrete vector fields is denoted by $\mathfrak{X}_d(\star K)$. Note that here it has implicitly been assumed that a dual vector field is an $\mathbb{R}^N$-valued dual 0-form, i.e., the 0-form takes its values in the same linear space $\mathbb{R}^N$.

As will be shown in the sequel, having a discrete theory of bundle-valued forms, discrete primal and dual vector fields would naturally be special examples, i.e., they are bundle-valued 0-forms. Several operators in the continuous theory, e.g., flat, sharp, divergence, etc., can be defined in the discrete setting (see Ref. 21 for details).

Support volume of a simplex σ^k in an n-dimensional complex K is the convex hull of the geometric union of σ^k and $\star\sigma^k$, i.e., the discrete n-volume

$$\overline{\sigma^k} = \overline{\star\sigma^k} = \text{convex hull}(\sigma^k, \star\sigma^k) \cap |K|. \tag{2.21}$$

Figure 2(a) shows a 1-simplex in a two-dimensional complex K and its support volume. Figure 2(b) shows a boundary 1-simplex and its support volume.

Hirani[21] defined primal-primal and dual-dual wedge products as follows. For the sake of clarity, we illustrate the definitions for 1-forms. The primal-primal wedge product for $\alpha, \beta \in \Omega_d^1(K)$ is defined as

$$\langle \alpha \wedge \beta, \sigma^2 \rangle = \frac{1}{2!} \sum_{\tau \in S_3} \operatorname{sgn}(\tau) \frac{|\sigma^2 \cap \star v_{\tau(1)}|}{|\sigma^2|} \alpha \smile \beta(\tau(\sigma^2)), \tag{2.22}$$

where S_3 is the permutation group and $\tau(\sigma^2)=\tau([v_0,v_1,v_2])=[v_{\tau(0)},v_{\tau(1)},v_{\tau(2)}]$ and

$$\alpha \smile \beta(\tau(\sigma^2)) = \langle \alpha, [v_{\tau(0)}, v_{\tau(1)}] \rangle \langle \beta, [v_{\tau(1)}, v_{\tau(2)}] \rangle. \tag{2.23}$$

More specifically, $S_3=\{(123),(132),(213),(231),(312),(321)\}$. Therefore, given a 2-cell $\sigma^2 =[v_0,v_1,v_2]$,

$$\begin{aligned}\langle \alpha \wedge \beta, [v_0,v_1,v_2] \rangle = {} & \frac{|\sigma^2 \cap \star v_0|}{|\sigma^2|}\{\alpha([v_2,v_0])\beta([v_0,v_1]) - \beta([v_2,v_0])\alpha([v_0,v_1])\} \\ & + \frac{|\sigma^2 \cap \star v_1|}{|\sigma^2|}\{\alpha([v_0,v_1])\beta([v_1,v_2]) - \beta([v_0,v_1])\alpha([v_1,v_2])\} \\ & + \frac{|\sigma^2 \cap \star v_2|}{|\sigma^2|}\{\alpha([v_1,v_2])\beta([v_2,v_0]) - \beta([v_1,v_2])\alpha([v_2,v_0])\}.\end{aligned} \tag{2.24}$$

The dual-dual wedge product is defined similarly.[21]

In physical applications, there may be different ways for defining wedge products. Any definition should be physically motivated. For elasticity applications, we need to define a wedge product for a k-form and an $(n-k)$-dual form. Note that what we really need in elasticity is a wedge product for a vector-valued k-form and a covector-valued $(n-k)$-dual form. However, having a wedge product for a k-form and a dual $(n-k)$-form, defining it for vector-valued forms will be straightforward. Given a primal discrete k-form $\alpha \in \Omega_d^k(K)$ and a dual discrete $(n-k)$-form $\beta \in \Omega_d^{n-k}(\star K)$, the discrete primal-dual wedge product

$$\wedge : \Omega_d^k(K) \times \Omega_d^{(n-k)}(\star K) \to \overline{\Omega_d^k}(K) \tag{2.25}$$

is defined by the evaluation on a support volume as follows:

$$\langle \alpha \wedge \beta, \overline{\sigma^k} \rangle = \alpha(\sigma^k)\beta(\star\sigma^k). \tag{2.26}$$

B. Discrete bundle-valued differential forms

The motivation of the DEC developed in Ref. 21 was to find systematic numerical schemes for field theories. In this paper, for studying this problem for elasticity, we need to extend some concepts presented in Ref. 21 for discrete bundle-valued forms. Similar to discrete forms that are real-valued linear functionals on the space of chains, discrete bundle-valued forms are $\mathcal{F}$-valued linear functionals on the space of chains, where $\mathcal{F}$ is a bundle (see Ref. 29 for physical examples of bundles.).

A discrete p-bundle in a complex K is $\mathcal{F}=K^{(p)}\otimes \mathbb{V}$, i.e., each fiber on a p-cell σ^p is a linear space $\mathbb{V}$. Discrete $\mathcal{F}$-valued forms are maps from k-chains to $\mathcal{F}$. These are required to be homomorphisms into the linear space $\mathbb{V}$. Thus, a discrete $\mathcal{F}$-valued k-form $\mathbb{A}$ is an element of $\mathrm{Hom}(C_k(K),\mathcal{F})$, the space of $\mathcal{F}$-valued cochains, i.e.,

$$\Omega_d^k(K,\mathcal{F}) \coloneqq \mathrm{Hom}(C_k(K),\mathcal{F}). \tag{2.27}$$

Given a k-chain $\Sigma_i a_i c_i^k$, $a_i \in \mathbb{Z}$, we have

$$\mathbb{A}\left(\sum_i a_i c_i^k\right) = (\mathbf{e}_a \otimes \mathbb{A}^a)\left(\sum_i a_i c_i^k\right) = \mathbf{e}_a \otimes \sum_i a_i \mathbb{A}^a(c_i^k), \tag{2.28}$$

where $\mathbb{A}=\mathbf{e}_a \otimes \mathbb{A}^a$ and $\{\mathbf{e}_a\}$ is a basis for $\mathbb{V}$ and $\mathbb{A}^a$ are k-forms. For $\mathbb{A},\mathbb{B} \in \Omega_d^k(K,\mathcal{F})$ and $a,b \in \mathbb{R}$ and c^k a k-chain,

$$(a\mathbb{A}+b\mathbb{B})(c)=a\mathbb{A}(c)+b\mathbb{B}(c). \tag{2.29}$$

The natural pairing of an $\mathcal{F}$-valued p-form $\mathbb{A}$ and a p-chain c^p is defined as

$$\langle \mathbb{A}, c^p\rangle = \mathbb{A}(c^p). \tag{2.30}$$

Note that because $\mathbb{V}$ is a finite-dimensional linear space, it can be identified with its dual. Discrete Hodge star operator for $\mathcal{F}$-valued discrete forms is a linear mapping $*:\mathcal{F}\otimes\Omega_d^k(K,\mathcal{F})\to\mathcal{F}\otimes\Omega_d^{n-k}(\star K,\mathcal{F})$, which locally maps the form part to its dual, i.e.,

$$\frac{1}{|\sigma^k|}\langle \mathbb{A},\sigma^k\rangle = \frac{1}{|\star\sigma^k|}\langle *\mathbb{A},\star\sigma^k\rangle, \qquad \forall\ \sigma^k \in K. \tag{2.31}$$

For a vector-valued p-form $\mathbb{t}$ with a coordinate representation

$$\mathbb{t}=\mathbf{e}_a\otimes \mathbb{t}^a, \tag{2.32}$$

the discrete covariant exterior derivative has the following form:

$$\mathbb{d}\mathbb{t}=\mathbf{e}_a\otimes \mathbf{d}t^a, \tag{2.33}$$

which is a $\mathbb{V}$-valued $(p+1)$-form. This shows that when all the bundle-valued forms take values in the same linear space $\mathbb{V}$, the discrete covariant exterior derivative can be defined very similarly to the discrete exterior derivative. Suppose that $\mathbb{a}$ is a $\mathbb{V}$-valued p-form, then

$$\langle \mathbb{d}\mathbb{a}, c_{p+1}\rangle = \langle \mathbb{a},\partial c_{p+1}\rangle, \qquad \forall\ c_{p+1}\in C_{p+1}(K), \tag{2.34}$$

where both sides are elements of $\mathbb{V}$.

A primal-dual wedge product. Given a $\mathbb{V}$-valued k-form α and a $\mathbb{V}$-valued $(n-k)$-dual form β, their wedge product is defined as

$$\langle \alpha\dot{\wedge}\beta,\overline{\sigma^k}\rangle = \langle\langle \alpha(\sigma^k),\beta(\star\sigma^k)\rangle\rangle, \tag{2.35}$$

where $\langle\langle .,.\rangle\rangle$ is an inner product on $\mathbb{V}$.

III. GEOMETRIC CONTINUUM MECHANICS

In the engineering literature, traditionally continuum mechanics has been formulated in $\mathbb{R}^n$. However, much geometric information is lost by restricting oneself to the rigid structure of the Euclidean space. Putting continuum mechanics and, in particular, nonlinear elasticity in the proper geometric framework[27] is not just a matter of mathematical authenticity; it has been observed recently that geometry can lead to nontrivial developments.[40,25] Here, we build on the geometric formulation of continuum mechanics, which was developed in Refs. 27, 33, and 40 and references therein.

A. Geometry of Maxwell's equations

Before developing a theory of discrete elasticity, it would be helpful to look at Maxwell's equations and their geometry. Many works have been done so far in understanding the geometry of Maxwell's equations for both continuous and discretized systems (see Refs. 12, 9, 34, 4, and 18 and references therein). Maxwell's equations are all vectorial and in that sense simpler than equations of continuum mechanics. Understanding Maxwell's equations and their geometric characteristics would help one in developing a theory of discrete elasticity.

There has been known for a long time that Maxwell's equations can be written in the language of exterior calculus. In this framework, it is seen that Maxwell's equations are metric independent and all the metric information shows up in the constitutive equations in the form of generalized material-dependent Hodge star operators. Having this reformulation, one can have a theory of electromagnetism for discretized bodies without any reference to the continuum formulation. This is Tonti's idea of "finite formulation" of electromagnetism. There have been recent efforts in extending this idea to linear elasticity, e.g., the so-called "cell method." As we will see in the sequel, cell method is not a geometric discretization.

Historically, Maxwell's equations are expressed using the vector calculus language. They are the following system of partial differential equations:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + J_E, \tag{3.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.3}$$

$$\nabla \cdot \mathbf{D} = \rho_E, \tag{3.4}$$

where **H** and **E** are magnetic and electric field intensity vectors, **D** and **B** are electric and magnetic flux density vectors, J_E is the electric current density (scalar), and ρ_E is the volumetric electric charge density (scalar).

Theory of differential forms is an alternative mathematical language for describing classical electromagnetic theory. Writing Maxwell's equations in terms of differential forms enables one to clearly see the geometric features of the electromagnetic field theory. When Maxwell's equations are written in the form explained above, the metric independence of the equations cannot be seen as the topological and metric structures are unnecessarily intertwined. Maxwell's equations have the following representation when written in the language of differential forms:

$$\mathbf{d}H = \frac{\partial D}{\partial t} + J_E, \tag{3.5}$$

$$\mathbf{d}E = -\frac{\partial B}{\partial t}, \tag{3.6}$$

$$\mathbf{d}B = 0, \tag{3.7}$$

$$\mathbf{d}D = \rho_E, \tag{3.8}$$

where H and E are magnetic and electric field intensity 1-forms, D and B are electric and magnetic flux density 2-forms, J_E is the electric current density 2-form, and ρ_E is the electric charge density 3-form. Note that the exterior derivative operator **d** is metric independent and the above equations always have this form no matter what the metric is. This is in contrast with vector calculus, where in different coordinate systems, a given operator (like grad, div, or curl) has different forms. It should be noted that continuum Maxwell's equations written in terms of differential forms are invariant under diffeomorphisms; the same equations for a lattice written in terms of discrete differential forms are invariant under homeomorphisms,[34] for example.

In this formulation, any metric dependency appears only in constitutive equations of the medium and is represented by Hodge star operators. In Maxwell's equations, constitutive equations relate the 1-forms E and H to the 2-forms D and B, respectively, and have the following forms:

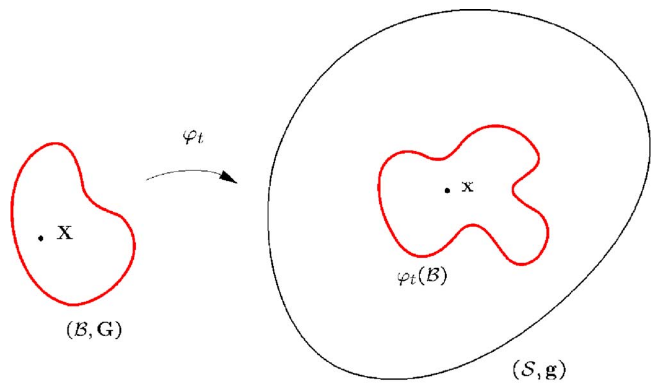

FIG. 3. (Color online) Deformation of a continuum is represented by a map between two Riemannian manifolds.

$$D = *_E E, \quad B = *_H H, \tag{3.9}$$

where $*_E$ and $*_H$ both depend on the medium and the metric. Hodge star operator is metric dependent and changing the metric, representation of constitutive equations will change, in general. One should note that this geometric representation is possible only for linear constitutive equations; any nonlinearity in constitutive equations would need a different representation.

A comment is in order here. Having written Maxwell's equations and constitutive equations geometrically in the forms (3.5)–(3.9), one can directly discretize them by replacing the continuous quantities and operators by their discrete counterparts. As we will see shortly, this is not the case in elasticity; one cannot simply work with a formal discretization of the geometric field equations. We will use an energy balance invariance argument to derive the discrete field equations.

B. Nonlinear elasticity

In this section, we briefly review geometric continuum mechanics. For more details, the reader is referred to Refs. 27 and 40. In continuum mechanics, deformation is thought of as a mapping between two configurations. Geometrically, we think of deformation as a map between two Riemannian manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$, which are called reference space and ambient space manifolds, respectively (see Fig. 3). Configuration manifold of deformations $\mathcal{C}$ is the set of all mappings $\varphi: \mathcal{B} \to \mathcal{S}$ and a motion $\varphi_t: \mathcal{B} \to \mathcal{S}$ is a curve in $\mathcal{C}$. For the so-called simple materials, deformations can be locally studied by looking at the map between tangent spaces of $\mathcal{B}$ and $\mathcal{S}$ at points $\mathbf{X} \in \mathcal{B}$ and $\varphi_t(\mathbf{X})$, respectively. The so-called deformation gradient is the tangent map $T\varphi_t: T_{\mathbf{X}}\mathcal{B} \to T_{\mathbf{x}}\mathcal{S}$ and has the following local representation:

$$\mathbf{F} = \frac{\partial \varphi^a}{\partial X^A} \mathbf{e}_a \otimes dX^A. \tag{3.10}$$

Deformation gradient can be thought of as a vector-valued 1-form, i.e., $\mathbf{F} \in T\mathcal{S} \otimes \Omega^1(\mathcal{B})$. The right Cauchy-Green strain tensor $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F}$ is a map from $T_{\mathbf{X}}\mathcal{B}$ to itself and has the following local representation:

$$\mathbf{C} = C^A{}_B \mathbf{E}_A \otimes dX^B \tag{3.11}$$

and is a vector-valued 1-form, i.e., $\mathbf{C} \in T\mathcal{B} \otimes \Omega^1(\mathcal{B})$. It can be shown that $\mathbf{C}$ is related to the pullback of the metric of the ambient space, i.e.,

$$\mathbf{C}^{\flat} = \varphi_t^{*}\mathbf{g}. \tag{3.12}$$

Note also that one can think of deformation mapping as a vector-valued 0-form, i.e., $\varphi \in T\mathcal{S} \otimes \Omega^0(\mathcal{B})$.

C. Geometric character of stress

In Ref. 22, it is shown that stress can be thought of as a covector-valued 2-form and all the governing equations can be written in terms of bundle-valued forms, covariant exterior derivative, and some other operators. One observation there is that unlike Maxwell's equations, balance of linear momentum, for example, is metric dependent as the covariant exterior derivative is a metric-dependent operator. More precisely, covariant exterior derivative explicitly depends on the metric of the bundle. Building on ideas presented in Refs. 27 and 22, here we study the geometric structure of linear and nonlinear elasticities. Then, based on the geometric developments in the continuum case, we will define discrete stress and strain for a discretized solid in the next section. We then look at this geometric theory for the special case where the ambient space is Euclidean, i.e., the case where all stresses take values in the same linear space.

In classical continuum mechanics, one starts by assuming the existence of a traction vector field $\mathbf{t}=\mathbf{t}(t,\mathbf{x},\mathbf{n})$. This means that given two surfaces passing through a point $\mathbf{x}$ with unit normal $\mathbf{n}$, traction is the same on both surfaces. Writing balance of linear momentum for an arbitrary sub-body, one obtains two things: (i) Cauchy's theorem that says $\mathbf{t}$ is linear in $\mathbf{n}$, i.e., there exists a second-order tensor $\boldsymbol{\sigma}$ such that $\mathbf{t}(\mathbf{n})=\langle\langle\boldsymbol{\sigma},\mathbf{n}\rangle\rangle$, and (ii) local form of balance of linear momentum. Having $\mathbf{t}(\mathbf{n})=\langle\langle\boldsymbol{\sigma},\mathbf{n}\rangle\rangle$, it is clear that $\mathbf{t}(-\mathbf{n})=-\mathbf{t}(\mathbf{n})$. This is not surprising as a surface by itself does not mean anything in this context; a surface is meaningful as the local boundary of a sub-body. As a simple example, note that it does not make sense to say that force in a spring is f. Instead, one may say that force exerted by an external agency on the spring is f and hence the force exerted by the spring on the external agency is $-f$.

Now, suppose that one starts with a more geometric point of view and assumes the existence of a stress form, i.e., a covector-valued differential form that associates a force to a given surface. Again, a surface by itself is irrelevant, i.e., one needs an oriented surface. Stress being a covector-valued differential form, changing the orientation of a given surface, the covector associated to it changes sign automatically.

It turns out that the right geometric machinery for continuum mechanics is the calculus of bundle-valued forms.[22] A bundle-valued differential form is a generalization of standard differential forms in which an n-form is an element of $\mathbb{F}\otimes\Omega^n(\mathcal{S})$, where $\mathbb{F}$ is a bundle, which for us could be $T\mathcal{S}$ or $T^*\mathcal{S}$. In the case of stress, intuitively, we expect it to be a covector-valued 2-form, i.e., stress at $\mathbf{x}\in\mathcal{S}$ is an element of $T^*_{\mathbf{x}}\mathcal{S}\otimes\Omega^2(\mathcal{S})$. This means that the stress form $\mathbb{t}$ has the following local representation:

$$\mathbb{t} = dx^a \otimes \mathbb{t}_a, \tag{3.13}$$

where $\mathbb{t}_a$ are 2-forms. In Ref. 22, it is shown that

$$\mathbb{t} = *_2\boldsymbol{\sigma}, \tag{3.14}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress. This means that

$$\mathbb{t} = \sigma_{ab}dx^a \otimes (*dx^b). \tag{3.15}$$

Here, $*_2$ means that the usual Hodge star operator acts on the area form of the stress form, i.e., on the second form. Assuming the existence of stress form, Cauchy stress is expressed as

$$\boldsymbol{\sigma} = -*_2\mathbb{t}, \tag{3.16}$$

i.e.,

$$\sigma_{ab}dx^a \otimes dx^b = -dx^a \otimes (*\mathbb{t}_a). \tag{3.17}$$

One can define the generalized wedge products $\dot{\wedge}$ and $\overset{\otimes}{\wedge}$ as, for example, $\dot{\wedge}: T\mathcal{S}\otimes\Omega^k(\mathcal{S})\times T^*\mathcal{S}\otimes\Omega^l(\mathcal{S})\to\Omega^{k+l}(\mathcal{S})$ and $\overset{\otimes}{\wedge}: T\mathcal{S}\otimes\Omega^k(\mathcal{S})\times T\mathcal{S}\otimes\Omega^l(\mathcal{S})\to T\mathcal{S}\otimes T\mathcal{S}\otimes\Omega^{k+l}(\mathcal{S})$, i.e.,

$$(\mathbf{u}\otimes\alpha)\dot{\wedge}(\beta\otimes\eta)=\langle\mathbf{u},\beta\rangle\alpha\wedge\eta, \tag{3.18}$$

$$(\mathbf{u}\otimes\alpha)\overset{\otimes}{\wedge}(\mathbf{v}\otimes\beta)=(\mathbf{u}\otimes\mathbf{v})\alpha\wedge\beta, \tag{3.19}$$

where $\mathbf{u}$ and $\mathbf{v}$ are vectors and α, β, and η are 1-forms. Exterior covariant derivative of $T^*\mathcal{S}$-valued $(k-1)$-forms can be defined as $\mathbb{d}: T^*\mathcal{S}\otimes\Omega^{k-1}\to T^*\mathcal{S}\otimes\Omega^k$ such that

$$\langle\mathbf{u},\mathbb{d}\mathbb{T}\rangle=\mathbf{d}(\langle\mathbf{u},\mathbb{T}\rangle)-\nabla\mathbf{u}\dot{\wedge}\mathbb{T},\qquad\forall\,\mathbf{u}\in T_{\mathbf{x}}\mathcal{S}. \tag{3.20}$$

Similarly for $T\mathcal{S}$-valued $(k-1)$-forms, it can be defined as $\mathbb{d}: T\mathcal{S}\otimes\Omega^{k-1}\to T\mathcal{S}\otimes\Omega^k$ such that

$$\langle\alpha,\mathbb{d}\mathbb{T}\rangle=\mathbf{d}(\langle\alpha,\mathbb{T}\rangle)-\nabla\alpha\dot{\wedge}\mathbb{T},\qquad\forall\,\alpha\in T^*_{\mathbf{x}}\mathcal{S}. \tag{3.21}$$

Here, ∇ is the covariant derivative of vector fields (or 1-forms) induced from the metric $\mathbf{g}$. It would be instructive to have the covariant exterior derivative in component form. For a $T\mathcal{S}$-valued $(k-1)$-form $\mathbb{T}$, one has

$$\mathbb{d}\mathbb{T}=\mathbf{e}_a\otimes(\mathbf{d}\mathbb{T}^a+\gamma^a_{bc}dx^b\wedge\mathbb{T}^c), \tag{3.22}$$

and for a $T^*\mathcal{S}$-valued $(k-1)$-form $\mathbb{T}$,

$$\mathbb{d}\mathbb{T}=dx^a\otimes(\mathbf{d}\mathbb{T}_a-\gamma^c_{ab}dx^b\wedge\mathbb{T}_c). \tag{3.23}$$

It is seen that the covariant exterior derivative always depends on the metric of the bundle. For $\mathbb{R}^n$-valued forms defined on an arbitrary manifold, one has

$$\mathbb{d}\mathbb{T}=dx^a\otimes\mathbf{d}\mathbb{T}_a. \tag{3.24}$$

In this case because the bundle has a trivial metric, one can say that the covariant exterior derivative is metric independent. This would be useful for later applications when the ambient space is Euclidean.

It can be shown that balance of linear momentum in terms of stress form can be written as[4]

$$\mathbb{d}\mathbb{t}+\mathbb{b}\otimes\rho=\mathbb{a}\otimes\rho, \tag{3.25}$$

where $\mathbb{b}$, $\mathbb{a}$, and ρ are the body force form (covector-valued 3-form), the inertial force form, and the density form, respectively. This is the geometric version of div $\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\mathbf{a}$. Balance of angular momentum reads

$$(\alpha\otimes\beta)\dot{\wedge}\mathbb{t}^{\sharp 1}=(\beta\otimes\alpha)\dot{\wedge}\mathbb{t}^{\sharp 1},\qquad\forall\,\alpha,\beta\in\Omega^1(\varphi(\mathcal{B})), \tag{3.26}$$

where $\sharp 1$ is the sharp operator on the covector part. This is the geometric version of $\boldsymbol{\sigma}^{\mathsf{T}}=\boldsymbol{\sigma}$.

D. Linear elasticity as a geometric linearization of nonlinear elasticity

Marsden and Hughes[27] formulated the theory of linear elasticity by linearizing nonlinear elasticity assuming that reference and ambient space manifolds are Riemannian. Here, we review their ideas and obtain some new results (see also Ref. 41 for more details).

We denote by $\mathcal{C}$ the set of all deformation mappings $\varphi:\mathcal{B}\to\mathcal{S}$. We do not discuss boundary conditions but assume that deformation mappings satisfy all the displacement (essential) boundary

[4]This can be derived from the classical balance of linear momentum or from covariance arguments without any reference to the classical formulation.[22]

conditions. One can prove that $\mathcal{C}$ is an infinite-dimensional manifold. Consider $\mathring{\varphi}_t \in \mathcal{C}$, where $\mathring{\varphi}_t$ is a given reference motion. An element of $T_{\mathring{\varphi}_t}\mathcal{C}$ is tangent to a curve $\varphi_{t,s} \in \mathcal{C}$ such that $\varphi_{t,0} = \mathring{\varphi}_t$. This is called variation of the configuration and is denoted $\mathbf{U} = \delta\varphi_t$.

Suppose that $\pi: \mathcal{E} \to \mathcal{C}$ is a vector bundle over $\mathcal{C}$ and let $f: \mathcal{C} \to \mathcal{E}$ be a section of this bundle. Let us assume that $\mathcal{E}$ is equipped with a connection $\boldsymbol{\nabla}$. Linearization of $f(\varphi)$ at $\mathring{\varphi}_t \in \mathcal{C}$ is defined as[27]

$$\mathcal{L}(f;\mathring{\varphi}_t) \coloneqq f(\mathring{\varphi}_t) + \boldsymbol{\nabla} f(\mathring{\varphi}_t)\cdot \mathbf{U}, \quad \mathbf{U} \in T_{\mathring{\varphi}_t}\mathcal{C}, \tag{3.27}$$

where

$$\boldsymbol{\nabla} f(\mathring{\varphi}_t)\cdot \mathbf{U} = \frac{d}{ds}\boldsymbol{\alpha}_s \cdot f(\varphi_{t,s})\big|_{s=0} \tag{3.28}$$

and $\boldsymbol{\alpha}_s$ is the parallel transport of members of $\mathcal{E}_{\varphi_{t,s}}$ to $\mathcal{E}_{\mathring{\varphi}_t}$ along a curve $\varphi_{t,s}$ tangent to $\mathbf{U}$ at $\mathring{\varphi}_t$. In Ref. 27, it is shown that the deformation gradient has the following linearization about $\mathring{\varphi}_t$:

$$\mathcal{L}(\mathbf{F};\mathring{\varphi}) = \mathring{\mathbf{F}} + \boldsymbol{\nabla}\mathbf{U}, \tag{3.29}$$

where $\mathring{\mathbf{F}} = T\mathring{\varphi}_t$. One can think of $\mathbf{F}$ as a vector-valued 1-form with the local representation

$$\mathbf{F} = F^a{}_A \mathbf{e}_a \otimes dX^A. \tag{3.30}$$

Thus,

$$\boldsymbol{\epsilon} \coloneqq \mathcal{L}(\mathbf{F};\mathring{\varphi}) - \mathring{\mathbf{F}} = U^a{}_{|A}\, \mathbf{e}_a \otimes dX^A \tag{3.31}$$

can be thought of as a geometric linearized strain, which is a vector-valued 1-form. Note that nonlinear elasticity can be linearized using the idea of variation of maps too.[41]

Linearization of velocity. Material velocity is linearized as follows:

$$\mathcal{L}(\mathbf{V};\mathring{\varphi}) = \mathring{\mathbf{V}} + \dot{\mathbf{U}}, \tag{3.32}$$

where $\dot{\mathbf{U}}$ is the covariant time derivative of $\mathbf{U}$, i.e.,

$$\dot{U}^a = \frac{\partial U^a}{\partial t} + \gamma^a_{bc}\mathring{V}^b U^c. \tag{3.33}$$

Linearization of acceleration. Material acceleration is linearized as follows:

$$\mathcal{L}(\mathbf{A};\mathring{\varphi}) = \mathring{\mathbf{A}} + \ddot{\mathbf{U}} + \mathbf{R}(\mathring{\mathbf{V}},\mathbf{U},\mathring{\mathbf{V}}), \tag{3.34}$$

where $\mathbf{R}$ is the curvature tensor of $(\mathcal{S},\mathbf{g})$. In components, the linearized acceleration has the following form:

$$\mathring{A}^a + \ddot{U}^a + R^a{}_{bcd}\mathring{V}^b\mathring{V}^d U^c. \tag{3.35}$$

Proof of this is lengthy but straightforward. Note also that this is a generalization of the Jacobi equation.[24]

Marsden and Hughes[27] proved that given a two-point tensor (of arbitrary rank) function of deformation gradient, $\mathbf{H} = \mathbf{H}(\mathbf{F})$, linearization of $\mathbf{H}$ at $\mathring{\varphi}_t$ reads

$$\mathcal{L}(\mathbf{H};\mathring{\varphi}) = \mathring{\mathbf{H}} + \left.\frac{\partial \mathbf{H}}{\partial \mathbf{F}}\right|_{\mathring{\varphi}_t} \cdot \boldsymbol{\nabla}\mathbf{U}. \tag{3.36}$$

This theorem can be directly used in the linearization of many quantities of interest in elasticity.

Linearization of $\mathbf{F}^{\mathsf{T}}$. Transpose of deformation gradient is defined as

$$\langle\langle \mathbf{FW},\mathbf{z}\rangle\rangle_{\mathbf{g}} = \langle\langle \mathbf{W},\mathbf{F}^{\mathsf{T}}\mathbf{z}\rangle\rangle_{\mathbf{G}}, \quad \forall\, \mathbf{W} \in T_{\mathbf{X}}\mathcal{B}, \quad \mathbf{z} \in T_{\mathbf{x}}\mathcal{S}. \tag{3.37}$$

This means that in components

$$(F^{\mathsf{T}})^A{}_a = g_{ab}G^{AB}F^b{}_B. \tag{3.38}$$

Noting that $\mathbf{g}$ is covariantly constant, one can write

$$(\boldsymbol{\alpha}_{t,s}\cdot \mathbf{F}^{\mathsf{T}})^A{}_a = g_{ab}(\varphi_{t,s})G^{AB}(\boldsymbol{\alpha}_{t,s}\cdot \mathbf{F})^b{}_B. \tag{3.39}$$

Differentiating both sides with respect to s and evaluating at $s=0$ yield

$$\left.\frac{d}{ds}\right|_{s=0}(\boldsymbol{\alpha}_{t,s}\cdot \mathbf{F}^{\mathsf{T}})^A{}_a = U^b{}_{|B}\, g_{ab}(\varphi_{t,s})G^{AB} + \mathring{F}^b{}_B G^{AB}\frac{\partial g_{ab}}{\partial x^c}U^c. \tag{3.40}$$

Note that

$$\frac{\partial g_{ab}}{\partial x^c} = g_{ad}\gamma^d_{bc} + g_{bd}\gamma^d_{ac}. \tag{3.41}$$

Thus

$$\mathcal{L}(\mathbf{F}^{\mathsf{T}};\mathring{\varphi}) = \mathring{\mathbf{F}}^{\mathsf{T}} + (\boldsymbol{\nabla}\mathbf{U})^{\mathsf{T}}. \tag{3.42}$$

Lemma 3.1: The right Cauchy-Green strain tensor has the following linearization:

$$\mathcal{L}(\mathbf{C};\mathring{\varphi}_t) = \mathring{\mathbf{C}} + \mathring{\mathbf{F}}^{\mathsf{T}}\,\boldsymbol{\nabla}\,\mathbf{U} + (\boldsymbol{\nabla}\mathbf{U})^{\mathsf{T}}\mathring{\mathbf{F}} \tag{3.43}$$

or in component form

$$\mathcal{L}(\mathbf{C};\mathring{\varphi}_t)_{AB} = \mathring{C}_{AB} + g_{ab}\mathring{F}^a{}_A U^b{}_B + g_{ab}\mathring{F}^b{}_B U^a{}_A. \tag{3.44}$$

Proof: We need to calculate

$$\frac{d}{ds}\mathbf{C}_{t,s}\,(\mathbf{X})\big|_{s=0} = \frac{d}{ds}(\boldsymbol{\alpha}_{t,s}\cdot \mathbf{C})\,(\mathbf{X})\big|_{s=0}. \tag{3.45}$$

Writing this in components and noting that $\mathbf{g}$ is covariantly constant, the lemma is easily proved.

Linearization of conservation of mass. Conservation of mass states that

$$\rho(\varphi_{t,s}(\mathbf{X}))J(\varphi_{t,s}(\mathbf{X})) = \rho_0(\mathbf{X}). \tag{3.46}$$

Thus, linearizing the above relation about $\mathring{\varphi}_t$ reads

$$\mathring{\rho}\mathring{J} + \left.\frac{d}{ds}(\rho(\varphi_{t,s}(\mathbf{X}))J(\varphi_{t,s}(\mathbf{X})))\right|_{s=0} = \rho_0(\mathbf{X}). \tag{3.47}$$

Hence

$$\dot{\overline{\mathring{\rho}}} + \mathring{\rho}[(\mathrm{div}\,\mathbf{u})\circ\mathring{\varphi}] = 0, \tag{3.48}$$

where $\mathbf{u}=\mathbf{U}\circ\mathring{\varphi}^{-1}$ and $\dot{\overline{\mathring{\rho}}}$ is the material time derivative of $\mathring{\rho}$. It is seen that this has the exact same form of the usual conservation of mass if $\mathbf{u}$ is thought of as the spatial velocity of the variation of deformation map.

Linearization of balance of linear momentum. Linearized balance of linear momentum reads (Ref. 41)[5]

[5]Note that in Ref. 27, it is implicitly assumed that the curvature tensor is zero.

$$\mathrm{Div}(\mathring{\mathsf{A}} \cdot \nabla \mathbf{U}) + \rho_0 \nabla_{\mathbf{U}} \mathbf{B} = \rho_0 \ddot{\mathbf{U}} + \rho_0 \mathbf{R}(\mathring{\mathbf{V}}, \mathbf{U}, \mathring{\mathbf{V}}), \tag{3.49}$$

where $\mathring{\mathsf{A}}$ is the elasticity tensor.

Linearization of balance of angular momentum. Balance of angular momentum in component form reads

$$P^{aA} F^b{}_A = P^{bA} F^a{}_A. \tag{3.50}$$

This also implies that

$$\mathring{P}^{aA} \mathring{F}^b{}_A = \mathring{P}^{bA} \mathring{F}^a{}_A. \tag{3.51}$$

Linearization of this relation about $\mathring{\varphi}$ reads

$$\mathring{P}^{aA} \mathring{F}^b{}_A + \mathring{P}^{aA} U^b{}_{|A} + (\mathring{\mathsf{A}}^{aA}{}_c{}^B) \mathring{F}^b{}_A U^c{}_{|B} = \mathring{P}^{bA} \mathring{F}^a{}_A + \mathring{P}^{bA} U^a{}_{|A} + (\mathring{\mathsf{A}}^{bA}{}_c{}^B) \mathring{F}^a{}_A U^c{}_{|B}, \tag{3.52}$$

which can be simplified to read

$$\mathring{P}^{aA} U^b{}_{|A} + (\mathring{\mathsf{A}}^{aA}{}_c{}^B) \mathring{F}^b{}_A U^c{}_{|B} = \mathring{P}^{bA} U^a{}_{|A} + (\mathring{\mathsf{A}}^{bA}{}_c{}^B) \mathring{F}^a{}_A U^c{}_{|B}. \tag{3.53}$$

After some simplifications, this can be written in the spatial form as

$$\mathring{\sigma}^{ac} u^b{}_{|c} + \mathring{\mathsf{a}}^{ab}{}_c{}^d u^c{}_{|d} = \mathring{\sigma}^{bc} u^a{}_{|c} + \mathring{\mathsf{a}}^{ba}{}_c{}^d u^c{}_{|d} \tag{3.54}$$

or

$$\mathring{\boldsymbol{\sigma}}{:}\nabla \boldsymbol{u} + \mathring{\mathsf{a}}{:}\nabla \boldsymbol{u} = \nabla \boldsymbol{u}{:}\mathring{\boldsymbol{\sigma}} + \nabla \boldsymbol{u}{:}\mathring{\mathsf{a}}. \tag{3.55}$$

Note that for the "product"[6] of two two-point tensors $\mathbf{A}$ and $\mathbf{B}$, one has

$$\mathcal{L}(\mathbf{AB}; \mathring{\varphi}_t) = \mathring{\mathbf{A}}\mathring{\mathbf{B}} + \left(\left.\frac{\partial \mathbf{A}}{\partial \mathbf{F}}\right|_{\mathring{\varphi}_t} \mathring{\mathbf{B}} + \mathring{\mathbf{A}} \left.\frac{\partial \mathbf{B}}{\partial \mathbf{F}}\right|_{\mathring{\varphi}_t} \right) \cdot \nabla \mathbf{U}. \tag{3.56}$$

Now, let us look at constitutive equations in linearized elasticity from a geometric point of view in terms of a generalized Hodge star operator.

1. Material Hodge star in linear elasticity

We know that linearized strain is defined as[27]

$$\boldsymbol{\epsilon} = \tfrac{1}{2} \mathfrak{L}_{\mathbf{v}} \mathbf{g} \tag{3.57}$$

and the Cauchy stress is given by

$$\boldsymbol{\sigma} = 2\rho \frac{\partial e}{\partial \mathbf{g}}. \tag{3.58}$$

Let us define the linearized strain form $\mathbb{e} = \mathbb{e}^a{}_b \mathbf{e}_a \otimes dx^b$ such that

$$\mathbb{e} \dot{\wedge} \mathbb{t} = (\boldsymbol{\sigma}{:}\boldsymbol{\epsilon}) \mu, \tag{3.59}$$

where μ is the volume form of $(\mathcal{S}, \mathbf{g})$. This requires that

$$\mathbb{e}^a{}_b = g^{ac} \epsilon_{cb}. \tag{3.60}$$

Let us now look at constitutive equations. We know that in linear elasticity[27]

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[6]This includes several possibilities, e.g., tensor product with or without contracting some indices.

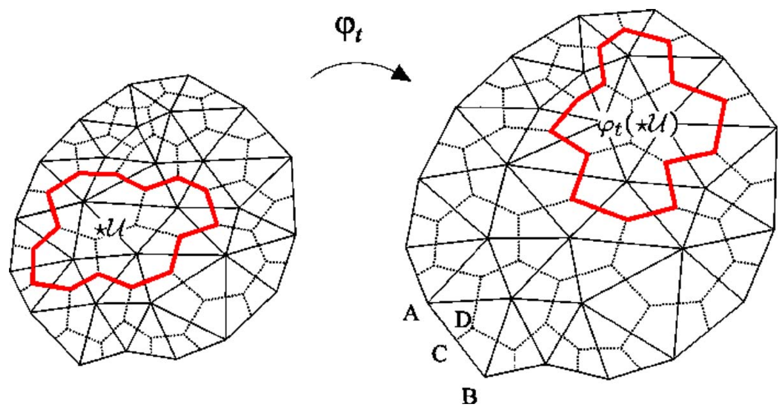

FIG. 4. (Color online) Deformation of a discretized solid. Note that, for the sake of generality, we are considering a generic dual and not necessarily the barycentric dual. *AB* and *CD* are primal and dual boundary 1-cells, respectively.

$$\sigma^{ab} = c^{abcd}\epsilon_{cd} \text{ or } \sigma_{ab} = g_{ak}g_{bl}c^{klcd}\epsilon_{cd}. \tag{3.61}$$

Thus

$$\mathbb{t} = dx^a \otimes [g_{ak}g_{bl}g_{mc}c^{klcd}\mathbb{e}^m{}_d](* dx^b) = *_E\, \mathbb{e}^{\flat 1}, \tag{3.62}$$

where $*_E$ is a material Hodge star and is a linear operator that relates the vector-valued strain 1-form to the covector-valued stress form, i.e., $*_E : T\mathcal{S}\otimes\Omega^1(\mathcal{S}) \to T^*\mathcal{S}\otimes\Omega^{p-1}(\mathcal{S})$, where $p=2$ for two-dimensional (2D) and $p=3$ for three-dimensional (3D) problems. Assuming that

$$*_E(dx^a \otimes dx^b) = K^{ab}{}_{cd}(dx^c \otimes * dx^d), \tag{3.63}$$

it is easy to show that

$$K^{ab}{}_{cd} = g_{ck}g_{dl}c^{klab}. \tag{3.64}$$

It is seen that the material Hodge star explicitly depends on material properties. For an isotropic material, one has

$$c^{abcd} = \mu(g^{ac}g^{bd} + g^{ad}g^{bc}) + \lambda g^{ab}g^{cd}. \tag{3.65}$$

Therefore, in this case

$$K^{ab}{}_{cd} = \mu(\delta^a_c\delta^b_d + \delta^b_c\delta^a_d) + \lambda g_{cd}g^{ab}. \tag{3.66}$$

It is seen that constitutive equations are written in terms of a Hodge star operator that, in addition to the metric, depends on the mechanical properties of the medium as well. It is also seen that unlike electromagnetism, the material Hodge star is not directly related to the usual Hodge star operator, i.e., metric and mechanical effects are, in general, coupled. In other words, it is impossible to expect the following elasticity Hodge star: $*_E(dx^a\otimes dx^b) = *_E(dx^a)\otimes * dx^b$. We observe that constitutive equations of linear elasticity are not as simple as those of electromagnetism.

IV. A DISCRETE THEORY OF ELASTICITY

In this section, we present a discrete theory of elasticity with no reference to the continuous theory. We assume that a discretized solid is embedded in an oriented Euclidean ambient space. Although this is not the most general possibility, similar to the existing developments of DEC, it is a natural starting point for geometrization of discrete elasticity.

Let us assume that a discretized continuum is modeled by a simplicial complex K embedded in an oriented Euclidean ambient space. A discrete deformation mapping is a time-dependent simplicial mapping $\varphi_t : K \to \varphi_t(K)$ (see Fig. 4). Thus

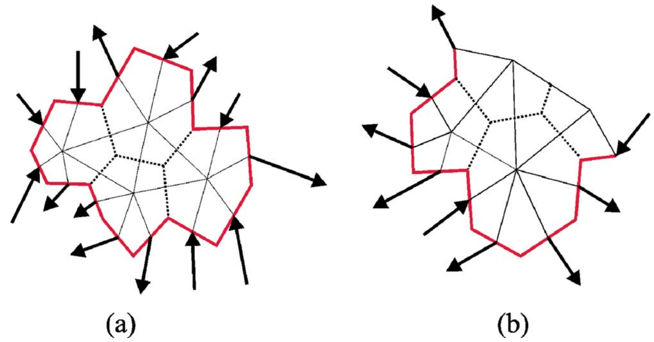

FIG. 5. (Color online) (a) Traction covectors acting on the boundary of a deformed dual subcomplex that does not intersect the boundary of K. (b) Traction covectors acting on the boundary of a deformed dual subcomplex that intersects the boundary of K. Note that for $\star\sigma_t^1$, traction acts at the point $\sigma_t^1 \cap \star\sigma_t^1$.

$$\sigma_i^0(t) = \varphi_t(\sigma_i^0), \quad \forall\ \sigma_i \in K^{(0)}, \tag{4.1}$$

and with a misuse of notation, we identify a zero cell $\sigma_i^0(t)$ with its position vector in the Euclidean ambient space.

Discrete velocity vector field is a discrete primal vector field on K, i.e.,

$$\mathbf{V}_i(t) := \langle \mathbf{V}, \sigma_i^0 \rangle = \dot{\varphi}_t(\sigma_i^0), \quad \forall\ \sigma_i^0 \in K^{(0)}. \tag{4.2}$$

Similarly, on $K_t = \varphi_t(K)$

$$\mathbf{v}_i(t) := \langle \mathbf{v}, \sigma_i^0(t) \rangle = \langle \mathbf{V}, \sigma_i^0 \rangle, \quad \forall\ \sigma_i^0 \in K^{(0)}. \tag{4.3}$$

Traction is a discrete dual $\mathbb{R}^p$-valued $(p-1)$-form $(p=2$ or $3)$.[7] Thus, given a dual $(p-1)$-cell $\star\sigma^1(t)$, traction is $\mathbb{t}(\star\sigma^1(t)) = \langle \mathbb{t}, \star\sigma^1(t) \rangle \in \mathbb{R}^p$. We assume that traction covector acts at the point $\sigma_t^1 \cap \star\sigma_t^1$ (see Fig. 5). Given an orientation for K, $\star K$ would be oriented as well,[28] i.e., each $\star\sigma^0$ is oriented consistently. A dual $(p-1)$-cell shared by two dual p-cells has opposite induced orientations. Thus, traction on $\star\sigma^1 \prec \star\sigma^0$ is $\mathbb{t}(\star\sigma^1)$, while traction on $\star\sigma^1 \prec \star\sigma'^0$ is $-\mathbb{t}(\star\sigma^1)$. Given an orientation to p-cells in an oriented complex K, dual p-cells can be oriented consistently. Discrete stress $\mathbb{t}$ associates a covector to each oriented $\star\sigma^1$. More specifically, given $\star\sigma^1$ as the boundary of $\star\sigma^0$, traction is given by

$$\mathbb{t}(\star\sigma^1 \prec \star\sigma^0) = \epsilon(\star\sigma^1, \star\sigma^0)\langle \mathbb{t}, \star\sigma^1(t) \rangle, \tag{4.4}$$

where

$$\epsilon(\star\sigma^1, \star\sigma^0) := \begin{cases} 1 & \text{if orientation } (\star\sigma^1) = \text{orientation } (\star\sigma^1; \star\sigma^0) \\ -1 & \text{if orientation } (\star\sigma^1) = -\text{ orientation } (\star\sigma^1; \star\sigma^0) \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Here, orientation $(\star\sigma^1; \star\sigma^0)$ is the induced orientation of $\star\sigma^1$ from $\star\sigma^0$. Figure 5(a) shows tractions acting on the boundary of a dual subcomplex that does not intersect the boundary of K. Figure 5(b) shows tractions acting on the boundary of a dual subcomplex that intersects the boundary of K.

Discrete Piola transform. In the continous case, given a vector field $\mathbf{v}$ on a manifold $\mathcal{N}$ and a map $\varphi: \mathcal{M} \to \mathcal{N}$, Piola transform of $\mathbf{v}$ is defined as

[7] Throughout this paper, $p=2$ corresponds to a 2D discrete problem and $p=3$ corresponds to a 3D discrete problem. We denote by $\#\star\sigma^1$ the number of dual p-cells. Obviously, $\#\star\sigma^1 = \#\sigma^1$.

$$\mathbf{V} = J\varphi^{*}\mathbf{v}, \tag{4.6}$$

where J is the Jacobian of φ. Piola transform is used for the first leg of the Cauchy stress to define the first Piola-Kirchhoff stress with the following property:

$$\langle\langle \mathbf{P},\mathbf{N}\rangle\rangle dA = \langle\langle \boldsymbol{\sigma},\mathbf{n}\rangle\rangle da. \tag{4.7}$$

Discrete Piola-Kirchhoff stress is defined as

$$\langle \mathbb{P},\star\sigma^1\rangle = \frac{|\star\sigma^1(t)|}{|\star\sigma^1|}\langle \mathbb{t},\star\sigma^1(t)\rangle, \qquad \forall\ \star\sigma^1. \tag{4.8}$$

Note that again the traction covector $\langle \mathbb{P},\star\sigma^1\rangle$ acts at the point $\sigma^1_t \cap \star\sigma^1_t$.

Formal geometric discretization of balance of linear momentum (3.25) reads

$$\langle \mathbb{d}\mathbb{t},\star c_p\rangle + \langle \mathbb{b},\star c_p\rangle = \langle \mathbb{a},\star c_p\rangle, \qquad \forall\ \star c_p \in \star K, \tag{4.9}$$

where $p=2,3$ for 2D and 3D problems, respectively. Note that

$$\langle \mathbb{d}\mathbb{t},\star c_p\rangle = dx^a \otimes \langle \mathbf{d}\mathbb{t}_a,\star c_p\rangle = dx^a \otimes \langle \mathbb{t}_a,\partial\star c_p\rangle = \langle \mathbb{t},\partial\star c_p\rangle. \tag{4.10}$$

This formal discretiziation is not useful as one needs to have an explicit form for $\mathbb{d}\mathbb{t}$. We will derive an explicit expression for $\mathbb{d}\mathbb{t}$ using energy balance invariance arguments in the sequel.

Discrete strain $\mathbb{F}$ is a discrete primal vector-valued 1-form. Given a 1-cell σ^1, $\mathbb{F}(\sigma^1)=\varphi_t(\sigma^1)$, where again with a misuse of notation we denote the 1-simplex (in both reference and current configurations) and its position vectors in the Euclidean ambient space both by σ^1 and σ^1_t, respectively. Given φ_t and $\sigma^1=[\sigma^0_1,\sigma^0_2]$

$$\mathbb{F}(\sigma^1) = [\varphi_t(\sigma^0_1),\varphi_t(\sigma^0_1)] = \langle \mathbb{d}\varphi_t,\sigma^1\rangle. \tag{4.11}$$

Discrete displacement field is defined as

$$\boldsymbol{u}(\sigma^0) = \varphi_t(\sigma^0) - \varphi_0(\sigma^0), \qquad \forall\ \sigma^0 \in K. \tag{4.12}$$

Density is a dual p-form in the sense that it associates a scalar to each dual p-cell. Conservation of mass in the continuous case reads

$$\mathbf{L}_{\mathbf{v}}\rho = \frac{\partial\rho}{\partial t} + \mathfrak{L}_{\mathbf{v}}\rho = 0. \tag{4.13}$$

For a time-independent discrete primal vector field $\mathbf{X}$, Desbrun *et al.*[11] defined the discrete autonomous Lie derivative using Bossavit's[5] idea of extrusion as

$$\langle \mathfrak{L}_{\mathbf{X}}\alpha,\sigma^k\rangle = \left.\frac{d}{dt}\right|_{t=0}\langle \alpha,\varphi_t(\sigma^k)\rangle, \tag{4.14}$$

where $\varphi_t(\sigma^k)$ is σ^k carried by the flow of the (time-independent) discrete primal vector field $\mathbf{X}$. In the case of discrete elasticity, one needs to have a way of defining discrete Lie derivatives with respect to time-dependent vector fields as the relevant vector field in elasticity is the velocity field, which is always time dependent.

Let us first review the definition of the nonautonomous Lie derivative. Suppose that $\mathbf{X}$ is a time-dependent vector field on a manifold M. An integral curve $\mathbf{x}(t)$ of $\mathbf{X}$ is a curve in M such that

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{X}(\mathbf{x}(t),t). \tag{4.15}$$

The flow generated by $\mathbf{X}$ is a map $\psi:\mathbb{R}\times\mathbb{R}\times M \to M$ such that for any s and $\mathbf{x}\in M$, $t\mapsto\psi_{t,s}(\mathbf{x})$ is an integral curve of $\mathbf{X}$ and $\psi_{s,s}(\mathbf{x})=\mathbf{x}$. Now, given another vector field $\mathbf{Y}$

$$\mathbf{L_X Y} = \left(\frac{d}{dt}\psi_{t,s}^{*}\mathbf{Y}_t\right)_{s=t} = \frac{\partial \mathbf{Y}}{\partial t} + \mathfrak{L}_{\mathbf{X}}\mathbf{Y}. \tag{4.16}$$

In the case of motion of a continuum, flow of $\mathbf{v}$ has the form $\psi_{t,s}=\varphi_t \circ \varphi_s^{-1}$.

Let M be an n-dimensional manifold, N a k-dimensional submanifold, and let $\mathbf{X}$ be a time-dependent vector field on M. Fixing time at an instant s, the manifold obtained by sweeping N along the flow of $\mathbf{X}_s$ for time t is called the extrusion of N at time s by $\mathbf{X}$ for time t and is denoted by $E_{t,s}(N)$.

Lemma 4.1: Lie derivative of a k-form α *with respect to a time-dependent vector field* $\mathbf{X}$ *is defined as*

$$\int_{\varphi_t(N)} \mathbf{L_X}\alpha = \frac{d}{dt}\int_{\varphi_t(N)} \alpha. \tag{4.17}$$

Proof:

$$\begin{aligned}\frac{d}{dt}\int_{\varphi_t(N)} \alpha &= \int_N \frac{d}{dt}\varphi_t^{*}\alpha = \int_N \left[\varphi_s^{*}\frac{d}{dt}(\varphi_{s*}\varphi_t^{*}\alpha)\right]_{s=t} = \int_N \varphi_t^{*} \left.\frac{d}{dt}\right|_{s=t}(\psi_{t,s}^{*}\alpha) = \int_N \varphi_t^{*}(\mathbf{L_X}\alpha)\\ &= \int_{\varphi_t(N)} \mathbf{L_X}\alpha . \quad \square\end{aligned} \tag{4.18}$$

This lemma motivates the following definition for the Lie derivative of an arbitrary k-form with respect to a discrete time-dependent vector field $\mathbf{X}$,

$$\langle \mathbf{L_X}\alpha, \sigma^k(t)\rangle = \frac{d}{dt}\langle \alpha, \sigma^k(t)\rangle. \tag{4.19}$$

Discrete conservation of mass. Conservation of mass can be written as

$$\int_{\mathcal{U}} \rho_0 = \int_{\varphi_t(\mathcal{U})} \rho = \int_{\mathcal{U}} \varphi_t^{*}\rho, \quad \forall\, \mathcal{U} \subset \mathcal{B}. \tag{4.20}$$

Thus

$$\varphi_t^{*}\rho = \rho_0. \tag{4.21}$$

Hence

$$\varphi_{s*}\varphi_t^{*}\rho = \psi_{t,s}^{*}\rho = \varphi_{s*}\rho_0. \tag{4.22}$$

Therefore

$$\frac{d}{dt}\psi_{t,s}^{*}\rho = 0. \tag{4.23}$$

Thus, $\mathbf{L_v}\rho=0$ as expected.[40]

Conservation of mass for a discrete system can be written as

$$\langle \rho, \varphi_t(\star\sigma^0)\rangle = \langle \rho_0, \star\sigma^0\rangle, \quad \forall\, \star\sigma^0 \in \star K. \tag{4.24}$$

Thus

$$\frac{d}{dt}\langle \rho, \varphi_t(\star\sigma^0)\rangle = \langle \mathbf{L_v}\rho, \varphi_t(\star\sigma^0)\rangle = 0, \quad \forall\, \star\sigma^0 \in \star K. \tag{4.25}$$

Discrete balance of angular momentum. Formal discretization of balance of angular momentum using the continuous local balance law can formally be written as

$$\langle(\alpha \otimes \beta)\dot{\wedge}\mathfrak{t}^{\#1}, \star c_p\rangle = \langle(\beta \otimes \alpha)\dot{\wedge}\mathfrak{t}^{\#1}, \star c_p\rangle, \quad \forall \star c_p \in \star K, \quad \forall\ \alpha,\beta \in \Omega^1(\varphi_t(\star K)) \tag{4.26}$$

or

$$\langle(\alpha \wedge \beta)\dot{\wedge}\mathfrak{t}^{\#1}, \star c_p\rangle = 0, \quad \forall \star c_p \in \star K. \tag{4.27}$$

However, it is not clear how this can be explicitly written on a dual p-cell. We will use an energy balance invariance argument to obtain the explicit form of this balance law in the sequel.

Remark: It should be mentioned that many of the known physical quantities should be modeled by the so-called twisted differential forms. These were introduced by Weyl[38] (see also Refs. 13, 6, and 1) and are sometimes called pseudoforms or forms of odd type. A twisted form $\tilde{\alpha}$ on a manifold M can be defined in terms of (straight) forms as a pair $\tilde{\alpha}=(\alpha,[M])$, where α is a differential form and $[M]$ is an orientation of M, with the equivalence relation $(\alpha,[M]) \sim (-\alpha, -[M])$. This, in particular, means that changing the orientation of the manifold M, integral of the twisted form $\tilde{\alpha}$ on M would not change. As an example, mass density when integrated on a 3-manifold representing a body will give the total mass of the body. Total mass of the body should be independent of orientation of its representing manifold and hence mass density is a twisted 3-form.

It should be noted that stress is a twisted form. However, if the ambient space is oriented, there is no need to distinguish between forms and twisted forms. There are two important issues here. The first is that if one wants to think of stress as a twisted form, other quantities should be changed accordingly. For example, as was mentioned in the above paragraph, mass density is a twisted form. The second is that in the case of stress, there is a subtle issue; stress is a twisted form not as a consequence of balance of linear momentum, instead as a consequence of the fact that any "density" is a twisted form and stress is in some sense a "density of force."[8]

In the present discrete theory, it is assumed that all the quantities of interest are represented by forms, vector-valued forms, and covector-valued forms. We start with an oriented K complex embedded in an oriented Euclidean ambient space. Defining a dual complex $\star K$, one can orient it consistently. Discrete analogs of what one sees in nonlinear elasticity are scalar, vector, and covector-valued differential forms, defined on the primal or dual cells. In particular, stress is defined on dual $(p-1)$-cells $\star\sigma^1$. However, there is a subtlety here. Unlike discrete strain $\mathbb{F}$ that is defined on primal 1-cells independently of any primal 2-cell, discrete stress associates a covector to $\star\sigma^1$ as the boundary of a dual 2-cell $\star\sigma^0$, i.e., it is only meaningful to define stress on $\star\sigma^1 \prec \star\sigma^0$. Note that we assume that K and $\star K$, and hence all their simplices, have fixed orientations and are embedded in an oriented Euclidean ambient space. Note also that a given $(p-1)$-cell $\star\sigma^1$ is the boundary of two dual p-cells and has opposite orientations induced from them. Now, this implies that $\mathfrak{t}$ associates two opposite covectors to the same dual 1-cell as boundaries of the two dual p-cells sharing $\star\sigma^1$, and this guarantees balance of linear momentum on dual $(p-1)$-cells (a measure zero set).

Up to this point, there is no reason to worry about twisted forms. Let us now consider $\star\sigma^1 \prec \star\sigma^0$. If the orientation of $\star\sigma^0$ is changed, the induced orientation of $\star\sigma^1$ will be reversed too but the stress covector acting on $\star\sigma^1$ should not change. In this sense, stress is a discrete dual twisted $(p-1)$-form. It is easy to show that discrete strain is a (straight) primal 1-form. For similar arguments for the differential form representation of Maxwell's equations, see Ref. 6.

A. Energy balance for a discretized solid

Balance of energy for a subset $\varphi_t(\mathcal{U}) \subset \varphi_t(\mathcal{B})$ of a deformed continuum body $\varphi_t(\mathcal{B})$ reads

[8]Frankel[16] mentioned that stress is a pseudoform.

$$\frac{d}{dt}\left(\int_{\varphi_t(\mathcal{U})} \tfrac{1}{2}\langle\langle \mathbf{v},\mathbf{v}\rangle\rangle\rho + \int_{\varphi_t(\mathcal{U})} e\rho\right) = \int_{\varphi_t(\mathcal{U})} (\langle \mathbf{v},\mathbb{b}\rangle\rho + r\rho) + \int_{\varphi_t(\partial\mathcal{U})} (\langle \mathbf{v},\mathbb{t}\rangle + h), \tag{4.28}$$

where $\mathbf{v}$, e, r, and h are spatial velocity, the internal energy function per unit mass, the heat supply 3-form per unit mass, and the heat flux 2-form, respectively. Note that $\langle\langle,\rangle\rangle$ is the inner product induced from the metric of the ambient space. In a discretized solid, internal energy density is a p-form defined on support volumes. Kinetic energy density κ is a dual p-form. The only nontrivial part of the energy balance is the power of tractions. In the discrete case, velocity is a dual vector field, while traction is a dual $(p-1)$-form. We assume that given a dual p-cell $\star\sigma^0$, tractions on faces of $\star\sigma^0$ are paired with velocity in the dual p-cell. This will be explained in more detail in the sequel.

For writing balance of energy, one needs to make sure that all the contributions to power are taken into account in a consistent geometric form. Body force is a discrete $\mathbb{V}$-valued dual p-form ($p=2$ or 3) and velocity is a discrete $\mathbb{V}$-valued primal 0-form. Therefore, power of body forces is defined on each dual p-form as

$$\langle \mathbb{b}\dot{\wedge}\mathbf{v},\star\sigma^0(t)\rangle = \langle\langle\, \langle \mathbb{b},\star\sigma^0(t)\rangle,\langle \mathbf{v},\sigma^0(t)\rangle\,\rangle\rangle. \tag{4.29}$$

Power of tractions is a dual p-form, defined on each dual p-cell as

$$\langle \mathbb{t}\dot{\wedge}\mathbf{v},\star\sigma^0(t)\rangle = \sum_{\sigma^1>\sigma^0} \epsilon(\star\sigma^1,\star\sigma^0)\langle\langle\, \langle \mathbb{t},\star\sigma^1(t)\rangle,\langle \mathbf{v},\sigma^0(t)\rangle\,\rangle\rangle. \tag{4.30}$$

This means that power in each dual p-cell is defined as a weighted sum of powers of each traction on the boundary of the dual p-cell. A comment is in order here. Traction covectors and velocities act at different points as $\mathbf{v}$ is defined on σ_t^0, while $\mathbb{t}$ acts at the point $\sigma_t^1\cap\star\sigma_t^1$. Given a 0-simplex σ^0, each traction covector acting at $\star\sigma^1\prec\star\sigma^0$ is parallel transported to σ^0. Doing so, one needs to carry a moment with each transported traction. However, because in this theory there are no independent rotations, these transported moments do not contribute to power and this justifies (4.30).

Internal energy is defined on support volumes [see Fig. 6(c)]. Given a dual subcomplex $\star\mathcal{U}_t\subset\star K_t$, internal energy is written as[9]

$$\sum_{\star\sigma^0\in\star\mathcal{U}_t}\ \sum_{\substack{\star\sigma^1\prec\star\sigma^0\\ \star\sigma^1\notin\partial\star\mathcal{U}}} \langle\rho,\star\sigma^0(t)\rangle\frac{|\overline{\star\sigma^1(t)}\cap\star\sigma^0(t)|}{|\star\sigma^0(t)|}\langle e,\overline{\star\sigma^1(t)}\rangle. \tag{4.31}$$

Note that internal energy is defined only for internal support volumes.[10] Kinetic energy is a dual p-form and is defined on each dual p-cell as

$$\langle\kappa,\star\sigma_t^0\rangle = \tfrac{1}{2}\langle\rho,\star\sigma^0(t)\rangle\langle\langle\, \langle \mathbf{v},\sigma^0(t)\rangle,\langle \mathbf{v},\sigma^0(t)\rangle\,\rangle\rangle. \tag{4.32}$$

We assume that heat supply r is a dual p-form and heat flux h is a dual $(p-1)$-form.

[9]Note that this holds for those dual subcomplexes that intersect ∂K as well.

[10]This is the discrete analog of what one sees in the continuous case with the difference that in the discrete theory physical boundary is a layer of support volumes with a nonzero p-volume.

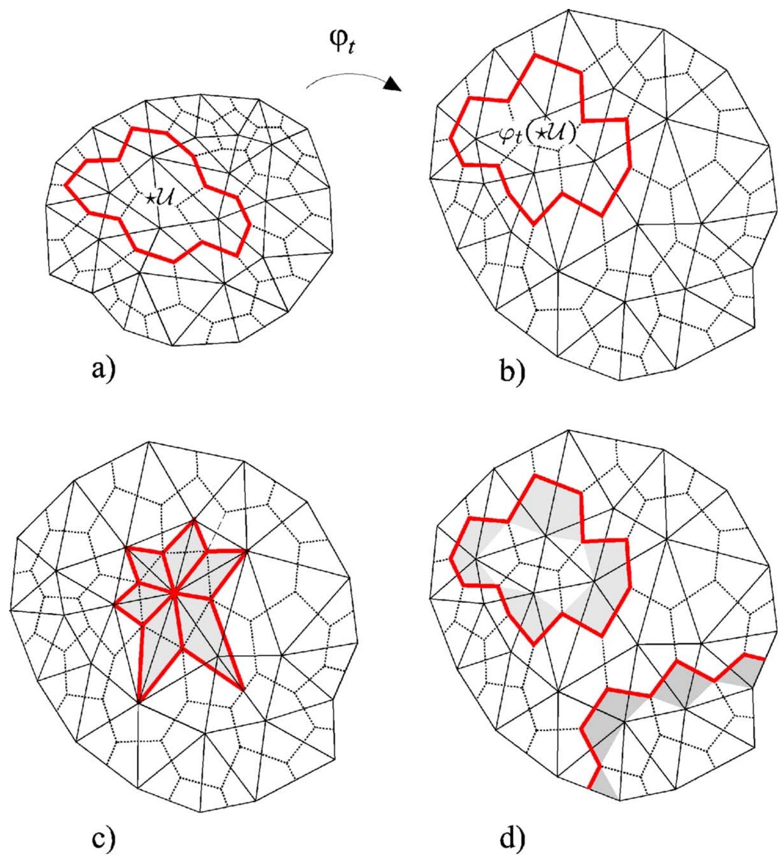

FIG. 6. (Color online) Deformation of a discretized solid. (a) Reference configuration. $\star\mathcal{U}$ is a dual subcomplex. (b) Deformed configuration. The dual subcomplex $\star\mathcal{U}$ is mapped to $\varphi_t(\star\mathcal{U})$. (c) A subset of the deformed configuration where internal energy is defined. This is a collection of support volumes. (d) Internal and boundary dual deformed subcomplexes. The shaded regions are where power of tractions is defined and the unshaded interior regions (collection of support volumes) are where internal energy is defined.

Balance of energy for $\star\mathcal{U}_t \subset \star K_t$ is now written as

$$\frac{d}{dt}\sum_{\star\sigma^0\in\star\mathcal{U}}\sum_{\substack{\star\sigma^1\prec\star\sigma^0\\ \star\sigma^1\notin\partial\star\mathcal{U}}}\langle\rho,\star\sigma^0(t)\rangle\frac{|\overline{\star\sigma^1(t)}\cap\star\sigma^0(t)|}{|\star\sigma^0(t)|}\langle e,\overline{\star\sigma^1(t)}\rangle+\frac{d}{dt}\sum_{\star\sigma^0\in\star\mathcal{U}}\frac{1}{2}\langle\rho,\star\sigma^0(t)\rangle$$
$$\times\langle\langle\,\langle\mathbf{v},\sigma^0(t)\rangle,\langle\mathbf{v},\sigma^0(t)\rangle\,\rangle\rangle=\sum_{\star\sigma^0\in\star\mathcal{U}}\langle\rho,\star\sigma^0(t)\rangle[\langle\langle\,\langle\mathbb{b},\star\sigma^0(t)\rangle,\langle\mathbf{v},\sigma^0(t)\rangle\,\rangle\rangle+\langle r,\star\sigma^0(t)\rangle]$$
$$+\sum_{\star\sigma^1\in\star\mathcal{U}}[\epsilon(\star\sigma^1,\star\sigma^0)\langle\langle\,\langle\mathbb{t},\star\sigma^1(t)\rangle,\langle\mathbf{v},\sigma^0(t)\rangle\,\rangle\rangle+\langle h,\star\sigma^1(t)\rangle], \tag{4.33}$$

where in the last sum $\sigma^0\in\mathcal{U}$, $\sigma^0\prec\sigma^1$, and $\overline{\star\sigma^1(t)}$ is the corresponding support volume. Balance of energy can be rewritten as

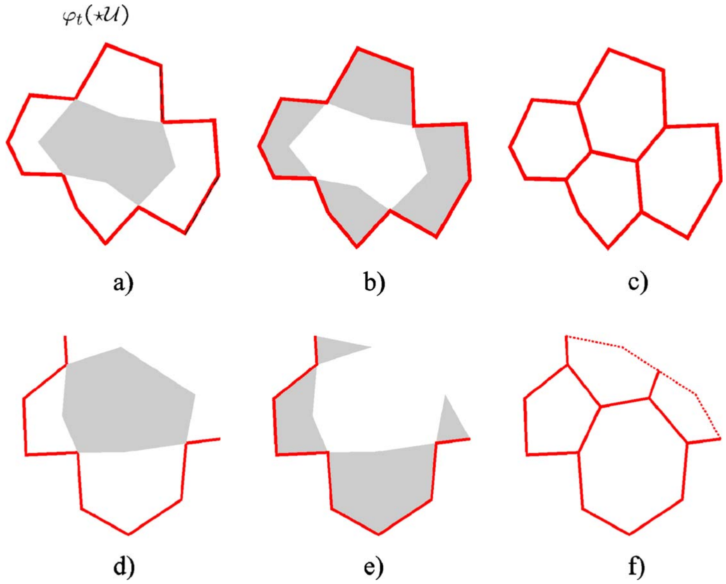

FIG. 7. (Color online) (a) Interior part of a deformed internal dual subcomplex where internal energy is defined. (b) Boundary part of the deformed subcomplex where power of tractions is defined. (c) Dual cells are where power of body forces and kinetic energy are defined. In (d), (e), and (f), the same things are shown for a dual subcomplex that intersects ∂K.

$$\frac{d}{dt}\sum_{\star\sigma^0\in\star\mathcal{U}}\sum_{\substack{\star\sigma^1\prec\star\sigma^0\\ \star\sigma^1\notin\partial\star\mathcal{U}}}\rho(\star\sigma_t^0)\bar{e}(\overline{\star\sigma^1(t)})+\frac{d}{dt}\sum_{\star\sigma^0\in\star\mathcal{U}}\frac{1}{2}\rho(\star\sigma^0(t))\mathbf{v}(\sigma^0(t))\cdot\mathbf{v}(\sigma^0(t))$$
$$=\sum_{\star\sigma^0\in\star\mathcal{U}}\rho(\star\sigma_t^0)[\mathbb{b}(\star\sigma^0(t))\cdot\mathbf{v}(\sigma^0(t))+r(\star\sigma^0(t))]$$
$$+\sum_{\star\sigma^1\in\star\mathcal{U}}[\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t))\cdot\mathbf{v}(\sigma^0(t))+\langle h,\star\sigma^1(t)\rangle], \tag{4.34}$$

where

$$\bar{e}(\overline{\star\sigma^1(t)})=\frac{|\overline{\star\sigma^1(t)}\cap\star\sigma^0(t)|}{|\star\sigma^0(t)|}\langle e,\overline{\star\sigma^1(t)}\rangle. \tag{4.35}$$

Figures 7(a)–7(c) show an internal deformed subcomplex $\star\mathcal{U}_t=\varphi_t(\mathcal{U})$ and its partitioning into interior and boundary parts, where internal energy and traction power are defined, respectively. Figures 7(d)–7(f) show the same things for a deformed subcomplex that intersects ∂K.

B. Invariance of energy balance

It is known that in continuum mechanics, one can obtain all the balance laws by postulating balance of energy and its invariance under rigid translations and rotations of the ambient space. This is the statement of the Green-Naghdi-Rivilin theorem.[17] This theorem is useful as in some cases one may be able to write an energy balance unambiguously and then the nontrivial form of other balance laws can be obtained using invariance arguments. This theorem was the starting point of the covariant elasticity theory introduced by Marsden and Hughes,[27] where ambient space is no longer Euclidean. For more discussions and details, see Refs. 40 and 41. In our discrete theory, we started by postulating the existence of some discrete scalar and vector-valued differential forms defined on a primal complex and its dual complex. Having a balance of energy for a dual subcomplex, in the sequel, we postulate its invariance under time-dependent rigid translations and rotations of the Euclidean ambient space.

Rigid translations. Let us first consider a rigid translation of the form

$$\mathbf{x}' = \xi_t(\mathbf{x}) = \mathbf{x} + (t - t_0)\mathbf{c}, \tag{4.36}$$

for some constant vector $\mathbf{c}$. We assume that energy balance is invariant under this (time-dependent) rigid translation, i.e.,

$$\begin{aligned}
&\frac{d}{dt}\sum_{\star\sigma^0\in\star\mathcal{U}}\sum_{\substack{\star\sigma^1\prec\star\sigma^0\\ \star\sigma^1\notin\partial\star\mathcal{U}}}\langle\rho',\star\sigma'^0(t)\rangle\frac{|\overline{\star\sigma'^1(t)}\cap\star\sigma'^0(t)|}{|\star\sigma'^0(t)|}\langle e',\overline{\star\sigma'^1(t)}\rangle\\
&\quad+\frac{d}{dt}\sum_{\star\sigma^0\in\star\mathcal{U}}\frac{1}{2}\langle\rho',\star\sigma'^0(t)\rangle\langle\langle\,\langle\mathbf{v}',\sigma'^0(t)\rangle,\langle\mathbf{v}',\sigma'^0(t)\rangle\,\rangle\rangle\\
&\quad=\sum_{\star\sigma^0\in\star\mathcal{U}}\langle\rho',\star\sigma'^0(t)\rangle[\langle\langle\,\langle\mathbb{b}',\star\sigma'^0(t)\rangle,\langle\mathbf{v}',\sigma'^0(t)\rangle\,\rangle\rangle+\langle r',\star\sigma'^0(t)\rangle]\\
&\quad+\sum_{\sigma^1\in\star\mathcal{U}}[\epsilon(\star\sigma^1,\star\sigma^0)\langle\langle\,\langle\mathbb{t}',\star\sigma'^1(t)\rangle,\langle\mathbf{v}',\sigma'^0(t)\rangle\,\rangle\rangle+\langle h',\star\sigma'^1(t)\rangle].
\end{aligned}\tag{4.37}$$

For the new deformation mapping $\varphi_t' = \xi_t \circ \varphi_t$, we have

$$\mathbf{v}'(\sigma'^0(t)) = \mathbf{v}(\sigma^0(t)) + \mathbf{c}, \qquad \forall\ \sigma^0 \in K. \tag{4.38}$$

At time $t = t_0$, energy balance reads

$$\begin{aligned}
&\sum_{\star\sigma^0\in\star\mathcal{U}}\sum_{\substack{\star\sigma^1\prec\star\sigma^0\\ \star\sigma^1\notin\partial\star\mathcal{U}}}[\dot{\rho}(\star\sigma^0(t))\bar{e}(\overline{\star\sigma^1(t)})+\rho(\star\sigma^0(t))\dot{\bar{e}}(\overline{\star\sigma^1(t)})]\\
&\quad+\sum_{\star\sigma^0\in\star\mathcal{U}}\frac{1}{2}\dot{\rho}(\star\sigma^0(t))[\mathbf{v}(\sigma^0(t))+\mathbf{c}]\cdot[\mathbf{v}(\sigma^0(t))+\mathbf{c}]+\sum_{\star\sigma^0\in\star\mathcal{U}}\rho(\star\sigma^0(t))\mathbb{a}(\sigma^0(t))\cdot[\mathbf{v}(\sigma^0(t))+\mathbf{c}]\\
&\quad=\sum_{\star\sigma^0\in\star\mathcal{U}}\rho(\star\sigma^0(t))\{\mathbb{b}(\star\sigma^0(t))\cdot[\mathbf{v}(\sigma^0(t))+\mathbf{c}]+r(\star\sigma^0(t))\}\\
&\quad+\sum_{\sigma^1\in\star\mathcal{U}}\{\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t))\cdot[\mathbf{v}(\sigma^0(t))+\mathbf{c}]+\langle h,\star\sigma^1(t)\rangle\}.
\end{aligned}\tag{4.39}$$

Subtracting (4.34) from (4.39) yields

$$\begin{aligned}
&\sum_{\star\sigma^0\in\star\mathcal{U}}\dot{\rho}(\star\sigma^0(t))\left[\tfrac{1}{2}\mathbf{c}\cdot\mathbf{c}+\mathbf{v}(\sigma^0(t))\cdot\mathbf{c}\right]+\sum_{\star\sigma^0\in\star\mathcal{U}}\rho(\star\sigma^0(t))\mathbb{a}(\sigma^0(t))\cdot\mathbf{c}\\
&\quad=\sum_{\star\sigma^0\in\star\mathcal{U}}\rho(\star\sigma^0(t))\mathbb{b}(\star\sigma^0(t))\cdot\mathbf{c}+\sum_{\sigma^1\in\star\mathcal{U}}\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t))\cdot\mathbf{c}.
\end{aligned}\tag{4.40}$$

Because $\mathbf{c}$ and $\star\mathcal{U}$ are arbitrary, we conclude that

$$\dot{\rho}(\star\sigma^0(t)) = \frac{d}{dt}\langle\rho,\varphi_t(\star\sigma^0)\rangle = 0, \qquad \forall\ \sigma^0 \in K \tag{4.41}$$

and

$$\sum_{\star\sigma^1\in\star\mathcal{U}}\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t))+\sum_{\star\sigma^0\in\star\mathcal{U}}\rho(\star\sigma^0(t))\mathbb{b}(\star\sigma^0(t))=\sum_{\star\sigma^0\in\star\mathcal{U}}\rho(\star\sigma^0(t))\mathbb{a}(\sigma^0(t)). \tag{4.42}$$

Now, assuming that $\star\mathcal{U}=\star\sigma^0$ for some internal dual p-cell, we obtain (discrete localization)

$$\sum_{\sigma^1\succ\sigma^0}\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t))+\rho(\star\sigma^0(t))\mathbb{b}(\star\sigma^0(t))=\rho(\star\sigma^0(t))\mathbb{a}(\sigma^0(t)),\quad\forall\star\sigma^0\in\star K. \tag{4.43}$$

Therefore

$$\langle\mathbb{d}\mathbb{t},\star\sigma^0(t)\rangle=\sum_{\sigma^1\succ\sigma^0}\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t)),\quad\forall\star\sigma^0\in\star K. \tag{4.44}$$

Rigid rotations. Now, let us consider rigid rotations of the deformed configuration. A time-dependent rigid rotation can be represented as

$$\mathbf{x}'=e^{\mathbf{\Omega}(t-t_0)}\mathbf{x}, \tag{4.45}$$

where Ω is a skew-symmetric matrix. Therefore, the 0-cell σ_t^0 is mapped to $e^{\mathbf{\Omega}(t-t_0)}\sigma_t^0$ and hence at $t=t_0$

$$\langle\mathbf{v}',\sigma'^0(t)\rangle=\langle\mathbf{v},\sigma^0(t)\rangle+\mathbf{\Omega}\sigma^0(t). \tag{4.46}$$

Postulating invariance of energy balance under arbitrary rigid rotations, one obtains

$$\sum_{\star\sigma^1\in\star\mathcal{U}}\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t))\cdot\mathbf{\Omega}\sigma^0(t)+\sum_{\star\sigma^0\in\star\mathcal{U}}\rho(\star\sigma^0(t))\mathbb{b}(\star\sigma^0(t))\cdot\mathbf{\Omega}\sigma^0(t)$$
$$=\sum_{\star\sigma^0\in\star\mathcal{U}}\rho(\star\sigma^0(t))\mathbb{a}(\sigma^0(t))\cdot\mathbf{\Omega}\sigma^0(t). \tag{4.47}$$

Considering a single internal dual p-cell ($p=2$ or 3) at time $t=t_0$ and subtracting balance of energy for the original deformed dual cell from this yield

$$\rho(\star\sigma^0(t))\mathbb{a}(\sigma^0(t))\cdot\mathbf{\Omega}\sigma^0(t)=\rho(\star\sigma^0(t))\mathbb{b}(\star\sigma^0(t))\cdot\mathbf{\Omega}\sigma^0(t)+\sum_{\sigma^1\succ\sigma^0}\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t))\cdot\mathbf{\Omega}\sigma^0(t), \tag{4.48}$$

which is trivially satisfied as a consequence of balance of linear momentum (4.43).

Now, let us consider two neighboring dual p-cells sharing a dual 1-cell $\star\Sigma_t$. Applying (4.47) to $\star\mathcal{U}_t=\star\sigma_a^0(t)\cup\star\sigma_b^0(t)$ and using balance of linear momentum yield

$$\mathbb{t}(\star\Sigma_t)\cdot\mathbf{\Omega}(\sigma_a^0(t)-\sigma_b^0(t))=0. \tag{4.49}$$

This implies that

$$\mathbb{t}(\star\Sigma_t)\otimes(\sigma_a^0(t)-\sigma_b^0(t))=(\sigma_a^0(t)-\sigma_b^0(t))\otimes\mathbb{t}(\star\Sigma_t). \tag{4.50}$$

Hence

$$\mathbb{t}(\star\Sigma_t)\parallel(\sigma_a^0(t)-\sigma_b^0(t)) \tag{4.51}$$

or

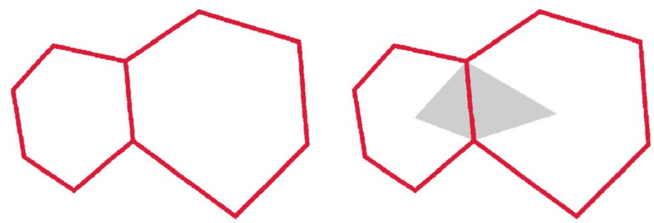

FIG. 8. (Color online) (a) Two neighboring dual 2-cells. Writing energy balance for the two cells separately, there is no internal energy contribution. (b) In writing balance of energy for the union of the two neighboring dual 2-cells, internal energy has a contribution in the shaded region (a support volume).

$$\mathbb{t}(\star\Sigma_t) \parallel \mathbb{F}(\Sigma). \tag{4.52}$$

Therefore, traction on each dual $(p-1)$-cell has to be along the corresponding primal 1-cell. It can be shown that if discrete stress field satisfies (4.51), Eq. (4.47) would be satisfied for any $\star\mathcal{U}\subset\star K$.

Balance of linear and angular momenta on boundary dual cells. Let us now consider boundary dual cells. In the continuous case, on a boundary point $\mathbf{x}\in\partial\varphi_t(\mathcal{U})$, one may have the boundary condition $\boldsymbol{\sigma}\hat{\mathbf{n}}=\overline{\mathbf{t}}$, where $\overline{\mathbf{t}}$ is a known traction. In the discrete case, the interior dual complex is connected to the boundary of K by some dual 1-cells (see Fig. 6). As boundary conditions, the primal 0-cell can be given a position vector (displacement boundary condition) or if not the body force covector corresponding to the dual of the 0-cell would be an unknown. In either case, balance of linear momentum for boundary dual cells has the same form as that for interior dual cells. Balance of angular momentum has the same form for all dual p-cells, i.e., it is independent of boundary conditions. In other words, in this discrete theory, one does not need to look at balances of linear and angular momenta for boundary dual cells separately.

Localization of energy balance. In the continuous case and in the absence of heat sources and heat fluxes, material energy balance has the following localized form:[27]

$$\rho_0\frac{\partial E}{\partial t}=\mathbf{P}:\frac{\partial \mathbf{F}}{\partial t}=\mathbf{S}:\mathbf{D}, \tag{4.53}$$

where E is the material energy density, $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor, and $\mathbf{D}$ is the material rate of deformation tensor, with components $D_{AB}=\frac{1}{2}(F^a{}_A V_{a|B}+F^a{}_B V_{a|A})$. In the following, we obtain a discrete analogue of (4.53).

Let us consider a dual subcomplex $\star\mathcal{U}$ consisting of two dual p-cells, i.e., $\star\mathcal{U}=\star\sigma_a^0\cup\star\sigma_b^0$ (see Fig. 8). Energy balances for $\star\sigma_a^0$ and $\star\sigma_b^0$ separately read (note that there are no internal support volumes for these subcomplexes)

$$\rho(\star\sigma_a^0(t))\mathbb{a}(\sigma_a^0(t))\cdot\mathbf{v}(\sigma_a^0(t))=\rho(\star\sigma_a^0(t))\mathbb{b}(\sigma_a^0(t))\cdot\mathbf{v}(\sigma_a^0(t))+\sum_{\sigma^1>\sigma_a^0}\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t))\cdot\mathbf{v}(\sigma_a^0(t)), \tag{4.54}$$

$$\rho(\star\sigma_b^0(t))\mathbb{a}(\sigma_b^0(t))\cdot\mathbf{v}(\sigma_b^0(t))=\rho(\star\sigma_b^0(t))\mathbb{b}(\sigma_b^0(t))\cdot\mathbf{v}(\sigma_b^0(t))+\sum_{\sigma^1>\sigma_b^0}\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t))\cdot\mathbf{v}(\sigma_b^0(t)). \tag{4.55}$$

Balance of energy for $\star\sigma_a^0\cup\star\sigma_b^0$ reads

$$\frac{d}{dt}e(\overline{\star\sigma^1(t)}) + \rho(\star\sigma_a^0(t))\mathbb{a}(\sigma_a^0(t)) \cdot \mathbf{v}(\sigma_a^0(t)) + \rho(\star\sigma_b^0(t))\mathbb{a}(\sigma_b^0(t)) \cdot \mathbf{v}(\sigma_b^0(t))$$
$$= \rho(\star\sigma_a^0(t))\mathbb{b}(\sigma_a^0(t)) \cdot \mathbf{v}(\sigma_a^0(t)) + \rho(\star\sigma_b^0(t))\mathbb{b}(\sigma_b^0(t)) \cdot \mathbf{v}(\sigma_b^0(t))$$
$$+ \sum_{\sigma^1 > \sigma_a^0(t)}{}' \epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t)) \cdot \mathbf{v}(\sigma_a^0(t)) + \sum_{\sigma^1 > \sigma_b^0}{}' \epsilon(\star\sigma^1,\star\sigma^0)\mathbb{t}(\star\sigma^1(t)) \cdot \mathbf{v}(\sigma_b^0(t)), \tag{4.56}$$

where prime on summations means that $\star\sigma^1$ is excluded. Without loss of generality, let us assume that $\epsilon(\star\sigma_t^1, \star\sigma_a^0) = -1$ and $\epsilon(\star\sigma_t^1, \star\sigma_b^0) = 1$. Adding (4.54) and (4.55) and subtracting from (4.56), one obtains

$$\frac{d}{dt}e(\overline{\star\sigma^1(t)}) = \mathbb{t}(\star\sigma^1(t)) \cdot \mathbf{v}(\sigma_b^0(t)) - \mathbb{t}(\star\sigma^1(t)) \cdot \mathbf{v}(\sigma_a^0(t)). \tag{4.57}$$

Therefore, we have proved the following proposition.

Proposition 4.2: In the discrete case, for any $\star\sigma^1(t)$ and its corresponding support volume,

$$\frac{d}{dt}e(\overline{\star\sigma^1(t)}) = \langle \mathbb{t}, \star\sigma^1(t)\rangle \cdot \langle \mathbf{d}\mathbf{v}, \sigma^1(t)\rangle = \langle \mathbb{t} \wedge \mathbf{d}\mathbf{v}, \overline{\star\sigma^1(t)}\rangle. \tag{4.58}$$

Discrete constitutive equations. In the continuous case, the first Piola-Kirchhoff stress is conjugate to the deformation gradient and

$$\mathbf{P} = \frac{\partial\Psi}{\partial\mathbf{F}}, \tag{4.59}$$

where Ψ is the free energy density. Discrete specific entropy N is defined on support volumes and in the absence of heat sources and fluxes, the entropy production inequality on an internal support volume reads

$$\frac{d}{dt}\langle \mathsf{N}, \overline{\star\sigma^1(t)}\rangle \geqslant 0. \tag{4.60}$$

Thus

$$\frac{d}{dt}\langle \psi, \overline{\star\sigma^1(t)}\rangle \leqslant \frac{d}{dt}\langle e, \overline{\star\sigma^1(t)}\rangle = \langle \mathbb{t} \wedge \mathbf{d}\mathbf{v}, \overline{\star\sigma^1(t)}\rangle. \tag{4.61}$$

Assuming locality, i.e., assuming that $\langle \psi, \overline{\star\sigma^1(t)}\rangle = \psi((\langle \mathbb{F}, \sigma^1\rangle)$ and using a Coleman-Noll argument, we obtain

$$\langle \mathbb{t}, \star\sigma^1(t)\rangle = \frac{\partial}{\partial\mathbb{F}(\sigma^1)}\langle \psi, \overline{\star\sigma^1(t)}\rangle. \tag{4.62}$$

Note that, in general, constitutive equations may be nonlocal, i.e., traction may depend on discrete strains on several neighboring 1-cells.

C. Discrete compatibility equations

The Laplace–de Rham operator is defined by $\Delta = \mathbf{d}\delta + \delta\mathbf{d} : \Omega_d^p(K) \to \Omega_d^p(K)$. For a harmonic form α, $\Delta\alpha = 0$. Space of harmonic p-forms is denoted by $K^p = \{\alpha \in \Omega_d^p | \Delta\alpha = 0\}$. Compatibility means that given $\mathbb{F}$, would it be possible to find φ such that $\mathbb{F} = \mathbf{d}\varphi$? Because strain is $\mathbb{R}^2$ (or $\mathbb{R}^3$) valued, we can use the standard discrete Hodge decomposition theorem, which says that[1]

$$\Omega_d^p = \mathbf{d}\Omega_d^{p-1} \oplus \delta\Omega_d^{p+1} \oplus K^k. \tag{4.63}$$

The pth cohomology group of K is defined as

$$H^p(K) = \ker(\mathbf{d}^p)/\mathrm{range}(\mathbf{d}^{p-1}). \tag{4.64}$$

It is known that K^p and H^p are isomorphic.[1] Note also that a form β is harmonic if and only if $\mathbf{d}\beta=0$ and $\delta\beta=0$. If $\mathbb{F}=\mathbb{d}\varphi$, then obviously $\mathbb{dF}=\mathbf{0}$. Now, suppose that $\mathbb{dF}=\mathbf{0}$, then because

$$\mathbb{F} = \mathbb{d}\varphi + \delta\alpha + \beta, \tag{4.65}$$

we have

$$\mathbb{dF} = \mathbb{d}\,\delta\alpha = \mathbf{0}. \tag{4.66}$$

However, this implies

$$\langle \delta\alpha, \delta\alpha \rangle = \langle \alpha, \mathbb{d}\,\delta\alpha \rangle = \mathbf{0} \tag{4.67}$$

and hence $\delta\alpha=\mathbf{0}$, i.e., $\mathbb{F}=\mathbb{d}\varphi+\beta$. Therefore, $\mathbb{dF}=\mathbf{0}$ would guarantee the existence of a deformation mapping up to the form β. These compatibility equations can be written as

$$\langle \mathbb{dF}, \sigma^2 \rangle = \langle \mathbb{F}, \partial\sigma^2 \rangle = \mathbf{0}, \qquad \forall\ \sigma^2 \in K \tag{4.68}$$

or

$$\sum_{\sigma^1 < \sigma^2} \langle \mathbb{F}, \sigma^1 \rangle = \mathbf{0}, \qquad \forall\ \sigma^2 \in K \tag{4.69}$$

We know that as a result of de Rham's theorem,[29] H^p and H_p are dual of each other and hence $\dim H_p = \dim H^p$. Note that $\dim H_p$ is the number of $(p-1)$-dimensional holes in K. For 2D problems, this is equal to $h=\dim H^1=\dim K^1$. One can guarantee that $\beta=0$ if the following $2h$ extra conditions are enforced. Denoting the ith hole by $\mathcal{H}_i$ ($\mathcal{H}_i \subset \partial K$), the extra equations are

$$\sum_{\sigma^1 < \mathcal{H}_i} \langle \mathbb{F}, \sigma^1 \rangle = \mathbf{0}, \quad i = 1, \ldots, h. \tag{4.70}$$

We call (4.69) the *first compatibility equations* and (4.70) the *second compatibility equations*.

D. Number of equations and unknowns

Let us see if in this discrete theory the number of unknowns and the number of equations are equal. In a 2D simplicial complex K, let us denote the number of zero and one boundary cells by $\#\sigma_{bd}^0$ and $\#\sigma_{bd}^1$, respectively. Note that $\#\sigma_{bd}^0 = \#\sigma_{bd}^1$. Assume that $m \leqslant \#\sigma_{bd}^0$ boundary 0-cells are fixed. As was mentioned earlier, fixing a boundary 0-cell σ^0, the corresponding body force covector on $\star\sigma^0$ becomes an unknown.

In 2D, unknowns are displacements (velocities), stresses, and body forces. Thus

$$\#\text{velocities} = 2\#\sigma^0 - 2m, \tag{4.71}$$

$$\#\text{stresses} = 2\#\star\sigma^1, \tag{4.72}$$

$$\#\text{unknown body forces} = 2m. \tag{4.73}$$

Governing equations are balances of linear and angular momenta, constitutive equations, and boundary equations (conditions). Thus[11]

$$\#\text{balance of linear momentum equations} = 2\#\star\sigma^0, \tag{4.74}$$

$$\#\text{balance of angular momentum equations} = \#\sigma^1, \tag{4.75}$$

$$\#\text{constitutive equations} = \#\sigma^1. \tag{4.76}$$

Noting that $\#\star\sigma^0 = \#\sigma^0$ and $\#\star\sigma^1 = \#\sigma^1$, we see that the number of unknowns and equations are equal. □

If we formulate the problem in terms of strains, then the number of unknowns is $2\#\sigma^1 + 2\#\star\sigma^1$, i.e., there are $2(\#\sigma^1 - \#\sigma^0)$ extra unknowns. The number of compatibility equations is $2(\#\sigma^2 - 1) + 2h$. Euler's equation for planar graphs with h holes reads

$$\#\sigma^2 - 1 + h = \#\sigma^1 - \#\sigma^0. \tag{4.77}$$

Therefore, it is seen that the number of compatibility equations is exactly equal to the number of extra unknowns. □

Summary of discrete quantities. The following table summarizes the discrete fields of our theory and their types.

Quantity	Symbol	Type
Velocity	$\mathbf{v}$	Vector-valued 0-form
Displacement	$\mathbf{u}$	Vector-valued 0-form
Strain	$\mathbb{F}$	Vector-valued 1-form
Mass density	ρ	Dual p-form
Internal energy density	e	Support volume-form
Specific entropy	N	Support volume-form
Heat flux	h	Dual $(p-1)$-form
Heat supply	r	Dual p-form
Stress	$\mathbb{t}$	Covector-valued $(p-1)$-form
Body force	$\mathbb{b}$	Covector-valued dual p-form
Kinetic energy density	κ	Dual p-form

E. A discrete Cosserat elasticity

In the case of a discrete Cosserat solid, in addition to a discrete deformation mapping, a (time-dependent) rotation is associated with each primal 0-cell, i.e., kinematics is defined by the pair $(\varphi_t(\sigma^0), \vartheta_t(\sigma^0))$. In addition to discrete stress, we postulate the existence of a discrete couple stress $\mathbb{m}$ that associates a couple to each dual $(p-1)$-cell. Rotation velocity is defined as

$$\langle \tilde{\mathbf{v}}, \sigma^0(t)\rangle = \frac{d}{dt}\vartheta(\sigma^0) = \dot{\vartheta}_t(\sigma^0). \tag{4.78}$$

For the sake of simplicity, let us ignore the rotational inertia. We also assume that there is a discrete field of body couples $\mathbb{c}$. The new terms in balance of energy are the power of discrete couple stresses and body couples, which read

[11]Note that we do not need to worry about the number of boundary equations as they are already included in (4.74) and (4.75).

$$\sum_{\star\sigma^1\in\star\mathcal{U}}\left[\epsilon(\star\sigma^1,\star\sigma^0)\mathbb{m}(\star\sigma^1(t))+\epsilon(\star\sigma^1,\star\sigma^0)\mathbf{r}(\star\sigma^0,\star\sigma^1)\times\mathbb{t}(\star\sigma^1(t))\right]\cdot\tilde{\mathbf{v}}(\sigma^0(t))$$
$$+\sum_{\star\sigma^0\in\star\mathcal{U}}\langle\rho,\star\sigma^0(t)\rangle\langle\langle\,\langle\mathbb{e},\star\sigma^0(t)\rangle,\langle\tilde{\mathbf{v}},\sigma^0(t)\rangle\,\rangle\rangle, \tag{4.79}$$

where $\mathbf{r}(\star\sigma^0,\star\sigma^1)$ is the vector connecting σ_t^0 to the point $\sigma_t^1\cap\star\sigma_t^1$. Under a rigid translation, rotation velocities remain unchanged and hence balance of linear momentum still has the form (4.42) and (4.44). Under a rigid rotation, rotation velocities have the following transformation:

$$\tilde{\mathbf{v}}'(\sigma^0(t))=\tilde{\mathbf{v}}(\sigma^0(t))+\alpha(t). \tag{4.80}$$

At time $t=t_0$

$$\tilde{\mathbf{v}}'(\sigma^0(t))=\tilde{\mathbf{v}}(\sigma^0(t))+\alpha, \tag{4.81}$$

where α is an arbitrary constant. Assuming invariance of energy balance and using balance of linear momentum, for each internal dual cell, we obtain

$$\sum_{\sigma^1>\sigma^0}\epsilon(\star\sigma^1,\star\sigma^0)[\mathbb{m}(\star\sigma^1(t))+\mathbf{r}(\star\sigma^0,\star\sigma^1)\times\mathbb{t}(\star\sigma^1(t))]+\langle\rho,\star\sigma^0(t)\rangle\langle\mathbb{e},\star\sigma^0(t)\rangle=\mathbf{0},\quad\forall\star\sigma^0\in\star K. \tag{4.82}$$

Note that in this case, traction on dual p-cells is not necessarily along the corresponding primal 1-cell. Note also that balances of linear and angular momenta for boundary dual cells have the same forms as those of the internal dual cells.

F. A geometric formulation of linear elasticity

In this section, we study the geometric structure of discrete linearized elasticity. There have been previous efforts in the literature in formulating consistent discrete theories of elasticity. As was mentioned earlier, an example is the so-called cell method, which is a numerical method that aims to formulate discrete problems *ab initio*, i.e., without any reference to the corresponding continuum formulations. Cosmi,[10] Ferretti,[14] and Pani *et al.*[30] extended Tonti's idea[35] for linear elasticity and defined the displacements on primal 0-cells and assumed that deformation is homogeneous within each primal 2-cell (for a 2D elasticity problem). Then, they associated a strain tensor to each primal 2-cell. In other words, they enter a continuous elasticity quantity into the discrete formulation. In this sense, cell method cannot be considered as a geometric discretization of linearized elasticity. With the uniform strain in each primal two-cell, they assumed a uniform stress in each primal 2-cell. This is again a direct use of a continuous concept and makes the method not a geometric discretization. In other words, this immediately contradicts the original idea of the cell method. The only geometric idea in the cell method is in writing the equilibrium equations on dual 2-cells.

Let us consider a discretized solid K and identify it with its representing complex K. Strain $\mathbb{e}$ is an $\mathbb{R}^p$-valued primal 1-form defined as follows. Discrete strain associates to each primal 1-cell the difference between displacements at its boundary points. If $\sigma^1=[\sigma_a^0,\sigma_b^0]$, then[12]

$$\langle\mathbb{e},\sigma^1\rangle=\langle\mathbf{u},\sigma_b^0\rangle-\langle\mathbf{u},\sigma_a^0\rangle. \tag{4.83}$$

Stress $\mathbb{t}$ is an $\mathbb{R}^p$-valued dual $(p-1)$-form. The discretized body is under a field of body forces, which is a discrete $\mathbb{R}^p$-valued dual p-form $\mathbb{b}$ and mass density is a discrete dual p-form ρ. Instead of looking at deformation as a mapping between the undeformed and current complexes, we define a displacement field $\mathbf{u}$ on K (reference and current configurations are not distinguishable in this case). In the discrete case, the main difference between linear and nonlinear elasticities is in the

[12] Note that $\mathbb{e}=\mathbb{F}-\mathbb{I}$, where $\mathbb{I}$ is the identity map.

constitutive equations. A discrete Hodge star operator relates discrete primal forms to discrete dual forms. This should explicitly depend on both metric properties of the ambient space and also material properties of the given medium. Let us denote the discrete material Hodge star operator of linear elasticity by $*_E$. Thus

$$*_E: \mathbb{V} \otimes \Omega^1(K) \to \mathbb{V} \otimes \Omega^{p-1}(\star K), \tag{4.84}$$

where $\mathbb{V}$ is a linear space (in this case $\mathbb{V}=\mathbb{R}^p$). Therefore, given a discrete strain field, we have

$$\mathbb{t} = *_E(\mathbb{e}) \tag{4.85}$$

or

$$\langle \mathbb{t}, \star\sigma^1 \rangle = *_E\left(\sum_{\sigma^1 \in \mathcal{I}(\star\sigma^1)} \langle \mathbb{e}, \sigma^1 \rangle \right), \quad \forall \star\sigma^1 \in \star K, \tag{4.86}$$

where $\mathcal{I}(\star\sigma^1)$ is a subset of K whose 1-cells influence the stress at $\star\sigma^1$. Note that, in general, $*_E$ could be a nonlocal operator, i.e., stress on a dual p-cell may depend on strains in a fairly large domain. Note also that one may have $\mathcal{I}(\star\sigma^1) = \star K^{(1)}$.

In a discrete problem with a three-dimensional ambient space, the coboundary operator can be uniquely specified by three incidence matrices that we denote by $\mathbf{M}_0$, $\mathbf{M}_1$, and $\mathbf{M}_2$.[4,3,5] The matrix $\mathbf{M}_0$ is an $\#\sigma^1 \times \#\sigma^0$ matrix with entries 0, 1, or -1, as defined below

$$M_0(\sigma^1,\sigma^0) = \begin{cases} 0 & \text{if } \sigma^0 \nprec \sigma^1 \\ 1 & \text{if } \sigma^0 \prec \sigma^1 \text{ and } \sigma^1 = [\sigma'^0, \sigma^0] \\ -1 & \text{if } \sigma^0 \prec \sigma^1 \text{ and } \sigma^1 = [\sigma^0, \sigma'^0], \end{cases} \tag{4.87}$$

for some 0-simplex σ'^0. The matrix $\mathbf{M}_1$ is an $\#\sigma^2 \times \#\sigma^1$ matrix with entries 0, 1, or -1, as defined below

$$M_1(\sigma^2,\sigma^1) = \begin{cases} 0 & \text{if } \sigma^1 \nprec \sigma^2 \\ 1 & \text{if } \sigma^1 \prec \sigma^2 \text{ and orientation}(\sigma^1) = \text{orientation}(\sigma^1;\sigma^2) \\ -1 & \text{if } \sigma^1 \prec \sigma^2 \text{ and orientation}(\sigma^1) = -\text{orientation}(\sigma^1;\sigma^2), \end{cases} \tag{4.88}$$

where orientation $(\sigma^1;\sigma^2)$ is the orientation of σ^1 induced from σ^2. $\mathbf{M}_2$ is a $\#\sigma^3 \times \#\sigma^2$ matrix with entries 0, 1, or -1 and is defined similarly to $\mathbf{M}_1$. The fact that coboundary of coboundary is null implies that[4,3]

$$\mathbf{M}_1\mathbf{M}_0 = \mathbf{0}, \quad \mathbf{M}_2\mathbf{M}_1 = \mathbf{0}. \tag{4.89}$$

In 2D

$$\mathbf{M}_0 \in \mathbb{R}^{\#\sigma^1 \times \#\sigma^0}, \quad \mathbf{M}_1 \in \mathbb{R}^{\#\sigma^2 \times \#\sigma^1}. \tag{4.90}$$

Similar matrices can be defined for dual cells and are denoted by $\tilde{\mathbf{M}}_0$, $\tilde{\mathbf{M}}_1$, and $\tilde{\mathbf{M}}_2$. Again in 2D

$$\tilde{\mathbf{M}}_0 \in \mathbb{R}^{\#\star\sigma^1 \times \#\star\sigma^2}, \quad \tilde{\mathbf{M}}_1 \in \mathbb{R}^{\#\star\sigma^0 \times \#\star\sigma^1}. \tag{4.91}$$

We can write the discrete governing equations using these matrices. Let us first write balance of linear momentum. We define a discrete stress matrix $\mathbf{T}$, which is a $\#\star\sigma^1 \times p$ ($p=2$ or 3) matrix. Each row is the covector associated with the corresponding dual $(p-1)$-cell. We can similarly define discrete body force and acceleration matrices $\mathbf{B}$ and $\mathbf{A}$. Each row of $\mathbf{B}$ is a covector of body force on a dual p-cell multiplied by the mass density of the same dual p-cell. Balance of linear momentum in 3D in matrix form reads

$$\widetilde{\mathbf{M}}_2\mathbf{T}+\mathbf{B}=\mathbf{A}. \tag{4.92}$$

In 2D

$$\widetilde{\mathbf{M}}_1\mathbf{T}+\mathbf{B}=\mathbf{A}. \tag{4.93}$$

Balance of angular momentum says that each traction is parallel to the corresponding (deformed) primal 1-cell. This means that there is a constraint on each row of $\mathbf{T}$ and these are the $\#\sigma^1$ angular momentum balance equations. Let us denote by $\hat{\mathbf{n}}_i$ the unit vector in the direction of the deformed 1-simplex $\sigma_i^1(t)$. Balance of angular momentum for $\star\sigma_i^1(t)$ is written as $\mathbf{T}_i\cdot\hat{\mathbf{n}}_i=0$. Define a matrix of unit vectors $\mathbf{N}$ such that the ith column of $\mathbf{N}$ is $\hat{\mathbf{n}}_i$. Balance of angular momentum in matrix form reads

$$\mathbf{TN}=\mathbf{0}. \tag{4.94}$$

Note that $\mathbf{N}$ is metric dependent.

If $\mathbf{U}$ is the matrix of displacements, we can define a matrix of discrete strains by $\mathbf{E}=\mathbf{M}_0\mathbf{U}$. Discrete elasticity Hodge operator has the following matrix representation:

$$\mathbf{T}=\mathbf{AE}. \tag{4.95}$$

The ith row of $\mathbf{T}$ has the following form (summation on j):

$$\mathbf{T}_i=\mathbf{A}_{ij}\mathbf{E}_j, \tag{4.96}$$

where $\mathbf{E}_j$ is the jth row of the discrete strain and $\mathbf{A}_{ij}$ are some matrices. Therefore, $\mathbf{A}$ is a $\#\sigma^1\times\#\sigma^1$ matric of $p\times p$ submatrices. This reminds us of the so-called global stiffness matrix in structural mechanics.

Compatibility equations can also be written in matrix form as follows. The first compatibility equations $\mathbb{d}\mathbb{e}=\mathbf{0}$ are written as[13]

$$\mathbf{M}_1\mathbf{E}=\mathbf{0}. \tag{4.97}$$

Note that if discrete strains are compatible, i.e., if $\mathbf{E}=\mathbf{M}_0\mathbf{U}$, then the first compatibility equations are trivially satisfied because $\mathbf{M}_1\mathbf{E}=\mathbf{M}_1\mathbf{M}_0\mathbf{U}=\mathbf{0}$.[14] Second compatibility equations correspond to the h holes $\mathcal{H}_i$, $i=1,\ldots,h$. Orienting the h holes arbitrarily, one can define a hole incidence matrix $\mathbf{H}$ as follows:

$$H(i,\sigma^1)=\begin{cases}0 & \text{if } \sigma^1\nprec\mathcal{H}_i\\ 1 & \text{if } \sigma^1\prec\mathcal{H}_i \text{ and orientation}(\sigma^1)=\text{orientation}(\sigma^1;\mathcal{H}_i)\\ -1 & \text{if } \sigma^1\prec\mathcal{H}_i \text{ and orientation}(\sigma^1)=-\text{orientation}(\sigma^1;\mathcal{H}_i).\end{cases} \tag{4.98}$$

Then, the second compatibility equations are written as[15]

$$\mathbf{HE}=\mathbf{0}. \tag{4.99}$$

Matrix compatibility equations can be compactly written as

$$\mathbf{CE}=\mathbf{0}, \tag{4.100}$$

where

[13]Note that $\mathbb{d}\mathbb{e}=\mathbb{d}(\mathbb{F}+\mathbb{I})=\mathbf{0}$.

[14]This was realized in the literature of structural mechanics in Ref. 31.

[15]Again note that because $\mathbb{e}=\mathbb{F}+\mathbb{I}$, the second compatibility equations of $\mathbb{e}$ are identical to those of $\mathbb{F}$.

$$\mathbf{C} = \begin{pmatrix} \mathbf{M}_1 \\ \mathbf{H} \end{pmatrix} \tag{4.101}$$

is the compatibility matrix. Note that in this theory the only metric-dependent matrices are **A** and **N**; all the other matrices are topological.

Remark: If a boundary point is fixed (essential boundary condition), the corresponding displacement is given but the body force would be an unknown.

V. CONCLUSIONS

In this paper, we presented a geometric discrete elasticity theory for discretized solids embedded in Euclidean space. We built this theory using ideas from algebraic topology, exterior calculus, and the recent developments of discrete exterior calculus. We reviewed geometric developments in the continuous case (and also presented some new results) and compared with previous works on the geometric discretization of Maxwell's equations.

Our discrete elasticity theory does not use any continuum concept. Instead, we start by postulating the existence of some discrete differential forms and discrete vector-valued differential forms as discrete fields defined on a triangulated domain. Similar to discrete electromagnetism, kinematical quantities are defined on a primal complex, while kinetic quantities are defined on a dual complex. The main difference between our discrete elasticity theory and discrete electromagnetism is the appearance of some discrete vector and covector-valued differential forms as discrete fields.

Instead of heuristically discretizing the governing field equations of elasticity written in terms of bundle-valued differential forms, we started from a balance of energy. It is seen that there are subtleties in writing balance of energy compared to the continuous case. For example, in a 2D problem, power of tractions is written on a layer of boundary dual 2-cells with a nonzero 2-volume. Postulating invariance of energy balance under time-dependent rigid translations and rotations of the Euclidean ambient space, we obtained discrete conservation of mass and discrete balance of linear and angular momenta. Finally, we wrote these balance laws on single dual cells (discrete localization). Using the discrete Hodge decomposition theorem, we obtained the discrete compatibility equations for a discrete 2D body with h holes. We studied a discrete Cosserat elasticity and obtained its discrete governing equations. We also formulated a discrete linearized elasticity and wrote its governing equations in a matrix form. Topological and metric-dependent equations are clearly separated in the discrete theory.

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