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On the wedge dispiration in an inhomogeneous isotropic nonlinear elastic solid

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ABSTRACT

In this short note we find the stress field of a wedge dispiration (a combination of a screw dislocation and a wedge dislocation along the same line) in an inhomogeneous incompressible isotropic nonlinear solid. We discuss the effect of the radial inhomogeneity of energy function on both the stress field and the energy per unit length of the dispiration and compare with those in an isotropic linear elastic solid.

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1. Introduction

Vito Volterra pioneered the mathematical study of defects in solids in his landmark work [25] more than a century ago. His work appeared many years before the experimental observation of defects in solids. Volterra classified the line defects into six groups, three translational and three rotational. Love [16] and Frank [5] referred to the translational and rotational (line) defects, dislocations and disclinations, respectively. There are only a handful of exact solutions for defects in nonlinear elastic solids in the literature and they are all restricted to homogeneous solids. We should mention Gairola [6], Rosakis and Rosakis [21], Zubov [30], Acharya [1] and Yavari and Goriely [26] for dislocations, Zubov [30], Derezin and Zubov [4] and Yavari and Goriely [28] for disclinations, and Yavari and Goriely [27,29] for point defects. Very little is known about the effects of material inhomogeneities on the stress fields and energetics of defects in solids. In the setting of linearized elasticity we are aware of the works of Barnett [2], Kroupa [14] and Lazar [15]. Both Barnett [2] and Lazar [15] assume that shear modulus in a linear elastic solid with a single screw dislocation depends only on a scalar variable. In the cylindrical polar coordinates (r, φ, z), Kroupa [14] considers a hollow circular cylinder with a screw dislocation and assumes that shear modulus depends on (r, φ) but the dependence on these two variables is separable.

In the 1950s Kondo [12,13] and Bilby et al. [3] independently investigated the deep connections between non-Riemannian geometries and the mechanics of defects. In particular, Kondo [12] observed that in the presence of defects, the reference configuration, which describes the stress-free state of a solid, is not necessarily Euclidean and referred to the affine connection of this manifold as the material connection. He also realized that the curvature of this connection is a measure of incompatibility, and that the Bianchi identities are conservation equations for incompatibility. Kondo [13] showed that torsion tensor is a measure of the density of dislocations. Recently, the geometric theory of solids with distributed defects was revisited and it was demonstrated how to calculate the stress fields of nonlinear solids with distributed defects [26–29].

To the best of our knowledge there are no exact solutions for the stress fields of defects in inhomogeneous nonlinear solids in the literature. A combination of a single screw dislocation and a single wedge disclination along the same line was called a wedge *dispiration* by Harris [7]. Ichikawa et al. [10] showed that, in the setting of linearized elasticity, a wedge dispiration is a stable defect. Yavari and Goriely [29] defined a *discombination* to be any combination of line and point defects in nonlinear solids. In this note we find the stress field of a wedge dispiration in an incompressible isotropic solid with an energy function that explicitly depends on the distance from the dispiration axis.

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2. Geometric elasticity and anelasticity

In classical nonlinear elasticity one starts with a stress-free reference configuration $\mathcal{B}$ and deformation is a mapping $\varphi : \mathcal{B} \to \mathcal{S}$, where $\mathcal{S}$ is a Riemannian manifold that here we can assume is the Euclidean space. In the case of defective solids the initial configuration is residually stressed; the classical techniques of nonlinear elasticity cannot be directly used. One approach for analyzing isolated defects is to use Volterra's cut-and-weld approach [21]. This method, however, cannot be used in the case of distributed defects [26]. In the geometric formulation of the nonlinear mechanics of defects the stress-free configuration is a Riemannian manifold $(\mathcal{B}, \mathbf{G})$, where $\mathbf{G}$ is the material metric that explicitly depends on the density of defects. One should note that the metric may need to be calculated indirectly, see Yavari and Goriely [29] for a detailed discussion. In the case of a wedge dispiration the material manifold is calculated directly following Volterra's construction [23,29].

The so-called deformation gradient is the tangent map of $\varphi : \mathcal{B} \to \mathcal{S}$. It is denoted by $\mathbf{F} = T\varphi$ and at each point $\mathbf{X} \in \mathcal{B}$, it is a linear map $\mathbf{F}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \to T_{\varphi(\mathbf{X})}\mathcal{S}$. Choosing local coordinate charts $\{x^a\}$ and $\{X^A\}$ for $\mathcal{S}$ and $\mathcal{B}$, respectively, $\mathbf{F}$ in components reads

$$F^a{}_A(\mathbf{X}) = \frac{\partial \varphi^a}{\partial X^A}(\mathbf{X}). \tag{2.1}$$

The adjoint (transpose) of deformation gradient $\mathbf{F}^\mathsf{T} : T_{\mathbf{x}}\mathcal{S} \to T_{\mathbf{X}}\mathcal{B}$ is defined through the relation $\langle\!\langle \mathbf{FV}, \mathbf{v} \rangle\!\rangle_{\mathbf{g}} = \langle\!\langle \mathbf{V}, \mathbf{F}^\mathsf{T}\mathbf{v} \rangle\!\rangle_{\mathbf{G}}$, for all $\mathbf{V} \in T_{\mathbf{X}}\mathcal{B}$, $\mathbf{v} \in T_{\mathbf{x}}\mathcal{S}$ and has components $(F^\mathsf{T}(\mathbf{X}))^A{}_a = g_{ab}(\mathbf{x})F^b{}_B(\mathbf{X})G^{AB}(\mathbf{X})$. The right Cauchy–Green deformation tensor $\mathbf{C}(\mathbf{X}) = \mathbf{F}(\mathbf{X})^\mathsf{T}\mathbf{F}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \to T_{\mathbf{X}}\mathcal{B}$ has components, $C^A_B = (F^\mathsf{T})^A{}_a F^a{}_B$. One can show that $\mathbf{C}^\flat = \varphi^*(\mathbf{g})$, i.e. $C_{AB} = (g_{ab} \circ \varphi)F^a{}_A F^b{}_B$. The left Cauchy–Green deformation tensor $\mathbf{B}^\sharp = \varphi^*(\mathbf{g}^\sharp)$ has components $B^{AB} = (F^{-1})^A{}_a (F^{-1})^B{}_b\, g^{ab}$. The spatial analogues of $\mathbf{C}^\flat$ and $\mathbf{B}^\sharp$ are

$$\begin{aligned} \mathbf{c}^\flat &= \varphi_*(\mathbf{G}), \quad c_{ab} = (F^{-1})^A{}_a (F^{-1})^B{}_b\, G_{AB}, \\ \mathbf{b}^\sharp &= \varphi_*(\mathbf{G}^\sharp), \quad b^{ab} = F^a{}_A F^b{}_B G^{AB}. \end{aligned} \tag{2.2}$$

$\mathbf{b}^\sharp$ is called the Finger deformation tensor. It is straightforward to show that $\mathbf{C}$ and $\mathbf{b}$ have the same principal invariants that are denoted by I_1, I_2, and I_3 [17]. The Jacobian of deformation gives the Riemannian volume element of the deformed configuration dv in terms of that of the reference configuration dV through the relation $dv = JdV$. One can show that $J = \sqrt{\det \mathbf{g} / \det \mathbf{G}} \det \mathbf{F}$. For an isotropic solid the strain energy function W only depends on the principal invariants of $\mathbf{b}$, i.e. $W = W(I_1, I_2, I_3)$ [20]. For an incompressible solid $I_3 = J^2 = 1$. It is known that for an incompressible and isotropic solid the Cauchy stress has the classical representation [24,22]

$$\boldsymbol{\sigma} = \left(-p + 2I_2 \frac{\partial W}{\partial I_2}\right)\mathbf{g}^\sharp + 2\frac{\partial W}{\partial I_1}\mathbf{b}^\sharp - 2\frac{\partial W}{\partial I_2}\mathbf{b}^{-1}, \tag{2.3}$$

where p is the Lagrange multiplier corresponding to the incompressibility constraint $J = 1$.

3. Dispiration: combination of a single screw dislocation and a single wedge disclination along the same line

In a previous work [29], we constructed the material manifold of a combination of a screw dislocation and a wedge disclination lying on the same line (dispiration). Note that the same problem was analyzed by Zubov [30] but using a different approach. In these works specific energy functions were used and the body was assumed to be homogeneous. Here, we consider an arbitrary incompressible isotropic solid with an energy function that explicitly depends on the distance from the dispiration axis. This models an axisymmetric inhomogeneous solid with a single wedge dispiration.

Let us denote the Euclidean 3-space by $\mathcal{B}_0$ with the flat metric

$$dS^2 = dR_0^2 + R_0^2 d\Theta_0^2 + dZ_0^2, \tag{3.1}$$

in the cylindrical coordinates (R_0, Θ_0, Z_0). Volterra's cut-and-weld construction of the dispiration is as follows: (i) Cut $\mathcal{B}_0$ along the closed half planes $\Theta_0 = 0$ and $\Theta_0 = \omega_0$ $(0 < \omega_0 < 2\pi)$, (ii) remove the line $R = 0$ and the region $0 < \Theta_0 < \omega_0$, (iii) translate the two closed half planes by b_0 in the Z_0-direction, and (iv) identify the two closed half planes. The material metric is written as [29]

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{R^2}{\Omega_0^2} + \dfrac{b_0^2}{4\pi^2\Omega_0^2} & \dfrac{b_0}{2\pi\Omega_0} \\ 0 & \dfrac{b_0}{2\pi\Omega_0} & 1 \end{pmatrix}, \tag{3.2}$$

where

$$\Omega_0 = \frac{2\pi}{2\pi - \omega_0}. \tag{3.3}$$

Changing the sign of ω_0 one obtains a negative disclination, i.e. instead of removing a wedge one inserts in a wedge.

Note that $\det \mathbf{G} = \frac{R^2}{\Omega_0^2}$ and hence the material volume form is: $dV = \sqrt{\det \mathbf{G}}\, dR d\Theta dZ = \frac{R}{\Omega_0} dR d\Theta dZ$. We assume that there are no body forces. We embed the defective body in the Euclidean ambient space $(\mathcal{S}, \mathbf{g})$ and look for solutions of the following form: $(r, \theta, z) = (r(R), \Theta, Z)$. The deformation gradient is written as $\mathbf{F} = \mathrm{diag}(r'(R), 1, 1)$ and hence from the incompressibility condition we have

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{r'(R) r(R)}{R/\Omega_0} = 1. \tag{3.4}$$

Assuming that $r(0) = 0$ we obtain $r = \frac{1}{\sqrt{\Omega_0}} R$ and thus $\mathbf{F} = \mathrm{diag}(\frac{1}{\sqrt{\Omega_0}}, 1, 1)$. The Finger tensor has components $b^{ab} = F^a{}_A F^b{}_B G^{AB}$. For this problem it reads

$$\mathbf{b}^\sharp = \begin{pmatrix} \dfrac{1}{\Omega_0} & 0 & 0 \\ 0 & \dfrac{\Omega_0^2}{R^2} & -\dfrac{b_0\Omega_0}{2\pi R^2} \\ 0 & -\dfrac{b_0\Omega_0}{2\pi R^2} & 1 + \dfrac{b_0^2}{4\pi^2 R^2} \end{pmatrix}. \tag{3.5}$$

The principal invariants of $\mathbf{b}$ ($\mathbf{b}$ and $\mathbf{C}$ have the same principal invariants) are:

$$\begin{aligned} I_1 &= 1 + \Omega_0 + \frac{1}{\Omega_0} + \frac{b_0^2}{4\pi^2 R^2}, \\ I_2 &= 1 + \Omega_0 + \frac{1}{\Omega_0} + \frac{b_0^2}{4\pi^2 R^2 \Omega_0}. \end{aligned} \tag{3.6}$$

Note that $(b^{-1})^{ab} = c^{ab} = g^{am} g^{bm} c_{mn}$ and hence we have

$$\mathbf{b}^{-1} = \begin{pmatrix} \dfrac{1}{\Omega_0^2} & 0 & 0 \\ 0 & \dfrac{\Omega_0^2}{R^2}\left(\Omega_0 + \dfrac{b_0^2}{4\pi^2 R^2}\right) & -\dfrac{\Omega_0 b_0}{2\pi R^2}\left(1 + \Omega_0 + \dfrac{b_0^2}{4\pi^2 R^2}\right) \\ 0 & -\dfrac{\Omega_0 b_0}{2\pi R^2}\left(1 + \Omega_0 + \dfrac{b_0^2}{4\pi^2 R^2}\right) & 1 + \dfrac{b_0^2}{2\pi^2 R^2}\left(1 + \dfrac{\Omega_0}{2} + \dfrac{b_0^2}{8\pi^2 R^2}\right) \end{pmatrix}. \tag{3.7}$$

We assume that the body is inhomogeneous, incompressible, and isotropic with an energy function $W = W(R, I_1, I_2)$. Note that

$$\boldsymbol{\sigma} = (-p + I_2\beta)\mathbf{g}^\sharp + \alpha\mathbf{b}^\sharp - \beta\mathbf{b}^{-1}, \tag{3.8}$$

where

$$\alpha(R)=2\frac{\partial W(R,I_1,I_2)}{\partial I_1},\qquad \beta(R)=2\frac{\partial W(R,I_1,I_2)}{\partial I_2}. \tag{3.9}$$

Therefore, the Cauchy stress has the following representation

$$\boldsymbol{\sigma}=\begin{pmatrix} -p+\frac{1}{\Omega_0}\alpha+\left(1+\frac{1}{\Omega_0}+\frac{b_0^2}{4\pi^2R^2\Omega_0}\right)\beta & 0 & 0\\ 0 & -\frac{\Omega_0}{R^2}p+\frac{\Omega_0^2}{R^2}\alpha+\frac{\Omega_0}{R^2}(1+\Omega_0)\beta & -\frac{b_0\Omega_0}{2\pi R^2}\alpha-\frac{b_0}{2\pi R^2}\beta\\ 0 & -\frac{b_0\Omega_0}{2\pi R^2}\alpha-\frac{b_0}{2\pi R^2}\beta & -p+\left(1+\frac{b_0^2}{4\pi^2R^2}\right)\alpha+\left(\Omega_0+\frac{1}{\Omega_0}+\frac{b_0^2}{4\pi^2\Omega_0R^2}\right)\beta \end{pmatrix}. \tag{3.10}$$

The non-trivial radial equilibrium equation reads $\sigma^{ra}{}_{|a}=\sigma^{rr}{}_{,r}+\frac{1}{r}\sigma^{rr}-r\sigma^{\theta\theta}=0$ (the other two equilibrium equations give us $p=p(R)$). Therefore

$$\sigma^{rr}{}_{,r}+\frac{1}{r}\sigma^{rr}-r\sigma^{\theta\theta}=0. \tag{3.11}$$

Or

$$\sigma^{rr}{}_{,R}+\frac{\alpha(R)}{R}\left(\frac{1}{\Omega_0}-\Omega_0\right)+\frac{\beta(R)}{R}\left(\frac{1}{\Omega_0}-\Omega_0+\frac{b_0^2}{4\pi R^2\Omega_0}\right)=0. \tag{3.12}$$

This gives a simple ODE for the unknown function $p=p(R)$. Imposing traction-free boundary conditions at the boundary $R=R_o$, one obtains the Cauchy stress. The non-zero physical components of the Cauchy stress read:

$$\begin{aligned}
\overline{\sigma}^{rr}(R)&=\left(\frac{1}{\Omega_0}-\Omega_0\right)\int_R^{R_o}\frac{\alpha(x)+\beta(x)}{x}dx+\frac{b_0^2}{4\pi\Omega_0}\int_R^{R_o}\frac{\beta(x)}{x^3}dx,\\
\overline{\sigma}^{\theta\theta}(R)&=\left(\Omega_0-\frac{1}{\Omega_0}\right)(\alpha(R)+\beta(R))-\frac{b_0^2}{4\pi^2\Omega_0}\frac{\beta(R)}{R^2}\\
&\quad+\left(\frac{1}{\Omega_0}-\Omega_0\right)\int_R^{R_o}\frac{\alpha(x)+\beta(x)}{x}dx+\frac{b_0^2}{4\pi\Omega_0}\int_R^{R_o}\frac{\beta(x)}{x^3}dx,\\
\overline{\sigma}^{zz}(R)&=\left(1-\frac{1}{\Omega_0}+\frac{b_0^2}{4\pi^2R^2}\right)\alpha(R)+(\Omega_0-1)\beta(R)\\
&\quad+\left(\frac{1}{\Omega_0}-\Omega_0\right)\int_R^{R_o}\frac{\alpha(x)+\beta(x)}{x}dx+\frac{b_0^2}{4\pi\Omega_0}\int_R^{R_o}\frac{\beta(x)}{x^3}dx,\\
\overline{\sigma}^{\theta z}(R)&=-\frac{b_0\sqrt{\Omega_0}}{2\pi R}\left(\alpha(R)+\frac{1}{\Omega_0}\beta(R)\right).
\end{aligned} \tag{3.13}$$

In the case of a neo-Hookean solid $W=\frac{\mu(R)}{2}(I_1-3)$ and hence $\alpha(R)=\mu(R)$ and $\beta(R)=0$, where μ is the shear modulus for infinitesimal strains. In this case the dispiration stress field has the following non-zero components:

$$\begin{aligned}
\overline{\sigma}^{rr}(R)&=\left(\frac{1}{\Omega_0}-\Omega_0\right)\int_R^{R_o}\frac{\mu(x)}{x}dx\\
\overline{\sigma}^{\theta\theta}(R)&=\left(\Omega_0-\frac{1}{\Omega_0}\right)\mu(R)+\left(\frac{1}{\Omega_0}-\Omega_0\right)\int_R^{R_o}\frac{\mu(x)}{x}dx,\\
\overline{\sigma}^{zz}(R)&=\left(1-\frac{1}{\Omega_0}+\frac{b_0^2}{4\pi^2R^2}\right)\mu(R)+\left(\frac{1}{\Omega_0}-\Omega_0\right)\int_R^{R_o}\frac{\mu(x)}{x}dx,\\
\overline{\sigma}^{\theta z}(R)&=-\frac{b_0\sqrt{\Omega_0}}{2\pi}\frac{\mu(R)}{R}.
\end{aligned} \tag{3.14}$$

3.1. The resultant axial force and energy per unit length of a wedge dispiration

The resultant force parallel to the dispiration axis is written as

$$F_Z=\int_0^{2\pi}\int_0^{R_o}\overline{\sigma}^{zz}(R)\frac{R}{\Omega_0}dRd\Theta=\frac{2\pi}{\Omega_0}\int_0^{R_o}\overline{\sigma}^{zz}(R)RdR. \tag{3.15}$$

The energy per unit length of the dispiration reads

$$\mathcal{W}=\frac{2\pi}{\Omega_0}\int_0^{R_o}W(R,I_1,I_2)RdR. \tag{3.16}$$

In the case of a wedge disclination both F_Z and $\mathcal{W}$ are finite. However, this is not the case, in general, when $b_0\neq0$. For a screw dislocation in the linear approximation $\overline{\sigma}^{zz}(R)=0$ and hence $F_Z=0$. There are some well-known subtle issues regarding the energy per unit length in the case of a screw dislocation that we discuss next.

3.2. Stress field of a wedge disclination

For a single wedge disclination ($b_0=0$) we have the following nonzero Cauchy stress components

$$\begin{aligned}
\overline{\sigma}^{rr}(R)&=\left(\frac{1}{\Omega_0}-\Omega_0\right)\int_R^{R_o}\frac{\alpha(x)+\beta(x)}{x}dx,\\
\overline{\sigma}^{\theta\theta}(R)&=\left(\Omega_0-\frac{1}{\Omega_0}\right)(\alpha(R)+\beta(R))\\
&\quad+\left(\frac{1}{\Omega_0}-\Omega_0\right)\int_R^{R_o}\frac{\alpha(x)+\beta(x)}{x}dx,\\
\overline{\sigma}^{zz}(R)&=\left(1-\frac{1}{\Omega_0}\right)\alpha(R)+(\Omega_0-1)\beta(R)\\
&\quad+\left(\frac{1}{\Omega_0}-\Omega_0\right)\int_R^{R_o}\frac{\alpha(x)+\beta(x)}{x}dx.
\end{aligned} \tag{3.17}$$

Note that in this case $I_1=I_2=1+\Omega_0+\frac{1}{\Omega_0}$.

3.3. Stress field and energy per unit length of a screw dislocation

For a single screw dislocation ($\Omega_0 = 1$) the non-zero Cauchy stress components read

$$\begin{aligned}
\overline{\sigma}^{rr}(R) &= \frac{b_0^2}{4\pi}\int_R^{R_o}\frac{\beta(x)}{x^3}dx,\\
\overline{\sigma}^{\theta\theta}(R) &= -\frac{b_0^2}{4\pi^2}\frac{\beta(R)}{R^2} + \frac{b_0^2}{4\pi}\int_R^{R_o}\frac{\beta(x)}{x^3}dx,\\
\overline{\sigma}^{zz}(R) &= \frac{b_0^2}{4\pi^2}\frac{\alpha(R)}{R^2} + \frac{b_0^2}{4\pi}\int_R^{R_o}\frac{\beta(x)}{x^3}dx,\\
\overline{\sigma}^{\theta z}(R) &= -\frac{b_0}{2\pi}\frac{\alpha(R)+\beta(R)}{R}.
\end{aligned} \tag{3.18}$$

Note that $I_1 = I_2 = 3 + \frac{b_0^2}{4\pi^2 R^2} \geq 3$. It is seen that in the limit of small b_0 (linear dislocation mechanics) only $\overline{\sigma}^{\theta z}(R)$ is nonzero.

3.3.1. A few special classes of incompressible isotropic solids

Next we look at a few special classes of incompressible isotropic solids and the corresponding stress field and energy per unit length of the screw dislocation.

Generalized neo-Hookean solids. For a generalized neo-Hookean solid, $W = W(R, I_1)$ and hence $\beta(R) = 0$. Noting that $I_1 \geq 3$, the Baker–Ericksen inequality reads $\partial W(R, I_1)/\partial I_1 > 0$, and hence $\alpha(R) > 0$ [11]. In this case the non-zero stress components for a screw dislocation read

$$\begin{aligned}
\overline{\sigma}^{zz}(R) &= \frac{b_0^2}{4\pi^2}\frac{\alpha(R)}{R^2},\\
\overline{\sigma}^{\theta z}(R) &= -\frac{b_0}{2\pi}\frac{\alpha(R)}{R}.
\end{aligned} \tag{3.19}$$

For a screw dislocation in a generalized neo-Hookean solid, the longitudinal force is written as

$$F_Z = \frac{b_0^2}{2\pi}\int_0^{R_o}\frac{\alpha(R)}{R}dR. \tag{3.20}$$

Neo-Hookean solids. It is known that for a screw dislocation energy per unit length may not be bounded for certain choices of energy functions. For a neo-Hookean solid

$$W = \frac{\mu(R)}{2}(I_1 - 3) = \frac{b_0^2}{4\pi^2}\frac{\mu(R)}{R^2}. \tag{3.21}$$

Therefore

$$\mathcal{W} = \frac{b_0^2}{2\pi}\int_0^{R_o}\frac{\mu(R)}{R}dR. \tag{3.22}$$

Note that for a neo-Hookean solid from (3.20) $F_Z = \mathcal{W}$. For $\mu(R) = \mu_0$ it is known that $\mathcal{W}$ cannot be defined due to a logarithmic singularity at $R = 0$. The same issue is encountered in linear elasticity [9]. A way out is to remove a core of radius R_i (dislocation core) and assume that it has a finite energy $\mathcal{W}_{\text{core}}$. Energy of the dislocation per unit length in a cylinder of radius R_o is then written as (for a homogeneous solid $\mu(R) = \mu_0$)

$$\mathcal{W} = \mathcal{W}_{\text{core}} + \frac{b_0^2\mu_0}{2\pi}\ln\frac{R_o}{R_i}. \tag{3.23}$$

This is the expression of energy per unit length of dislocation for both isotropic linear elastic and neo-Hookean solids. In the case of homogeneous solids Zubov [30] noticed that for the neo-Hookean energy function F_Z is unbounded, which is not physical. However, this is not surprising as the longitudinal force and the energy per unit length are equal for neo-Hookean solids. For an inhomogeneous neo-Hookean solid with $\mu = \mu(R)$, assuming that $\mu(R) = \mu_0 + O(R)$ as $R \to 0$, we observe that $\mu(R)/R = \mu_0/R + O(1)$ and hence the same singularity is observed in energy per unit length. Note that $\mu(R) > 0$, and hence $\mu_0 > 0$.

Power-law materials. Rosakis and Rosakis [21] considered a screw dislocation in power law materials with the following energy function

$$W = W(I_1) = \frac{\mu}{2c}\left\{\left[1 + \frac{c}{n}(I_1 - 3)\right]^n - 1\right\}, \tag{3.24}$$

where μ, c, and n are material constants. They showed that for the special case of $n = \frac{1}{2}$ energy per unit length of the screw dislocation is finite. This means that one does not need to remove a core to have a finite energy for this material. One can show that for this material F_Z is finite as well.

Hencky material. For an incompressible Hencky solid [8,18]

$$W = \mu[(\log\lambda_1)^2 + (\log\lambda_2)^2 + (\log\lambda_3)^2], \quad \lambda_1\lambda_2\lambda_3 = 1, \tag{3.25}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches (eigenvalues of $\mathbf{U} = \sqrt{\mathbf{C}}$). For the case of a single screw dislocation we have

$$\begin{aligned}
\lambda_1 &= 1 + \frac{b_0^2}{8\pi^2R^2}\left(1 + \sqrt{1 + \frac{16\pi^2R^2}{b_0^2}}\right),\\
\lambda_2 &= 1 + \frac{b_0^2}{8\pi^2R^2}\left(1 - \sqrt{1 + \frac{16\pi^2R^2}{b_0^2}}\right),\\
\lambda_3 &= 1.
\end{aligned} \tag{3.26}$$

For small R, the energy density has the following asymptotic expansion

$$W = 2\mu\left[\ln\left(\frac{b_0^2}{4\pi^2}\right) - 2\ln R\right]^2 + O(R^2) \quad \text{as} \quad R \to 0, \tag{3.27}$$

and hence energy per unit length of the screw dislocation is finite, i.e. one does not need to remove a core.

For an incompressible exponential Hencky solid [19]

$$W = \mu\; e^{(\log\lambda_1)^2 + (\log\lambda_2)^2 + (\log\lambda_3)^2}, \quad \lambda_1\lambda_2\lambda_3 = 1. \tag{3.28}$$

One can show that in this case energy per unit length of the dislocation is unbounded.

Remark 1. One may conclude that a screw dislocation is a possible defect in incompressible and isotropic solids with the energies (3.24) when $n = \frac{1}{2}$ and (3.25) while it is not a possible defect in the neo-Hookean and exponential Hencky solids.

3.4. A hollow solid cylinder with a wedge dispiration

Instead of a solid cylinder with a wedge dispiration, let us consider a hollow cylinder with inner and outer radii R_i and R_o, respectively. We assume that the interior boundary is traction free, i.e. $\sigma^{rr}(R_i) = 0$. In this case instead of (3.13) we obtain a different Cauchy stress field that has the following nonzero physical components

$$
\begin{aligned}
\overline{\sigma}^{rr}(R) &= -\left(\frac{1}{\Omega_0} - \Omega_0\right)\int_{R_i}^{R}\frac{\alpha(x)+\beta(x)}{x}dx - \frac{b_0^2}{4\pi\Omega_0}\int_{R_i}^{R}\frac{\beta(x)}{x^3}dx,\\
\overline{\sigma}^{\theta\theta}(R) &= \left(\Omega_0 - \frac{1}{\Omega_0}\right)(\alpha(R)+\beta(R)) - \frac{b_0^2}{4\pi^2\Omega_0}\frac{\beta(R)}{R^2}\\
&\quad - \left(\frac{1}{\Omega_0} - \Omega_0\right)\int_{R_i}^{R}\frac{\alpha(x)+\beta(x)}{x}dx + \frac{b_0^2}{4\pi\Omega_0}\int_{R}^{R_o}\frac{\beta(x)}{x^3}dx,\\
\overline{\sigma}^{zz}(R) &= \left(1 - \frac{1}{\Omega_0} + \frac{b_0^2}{4\pi^2R^2}\right)\alpha(R) + (\Omega_0 - 1)\beta(R)\\
&\quad - \left(\frac{1}{\Omega_0} - \Omega_0\right)\int_{R_i}^{R}\frac{\alpha(x)+\beta(x)}{x}dx - \frac{b_0^2}{4\pi\Omega_0}\int_{R_i}^{R}\frac{\beta(x)}{x^3}dx,\\
\overline{\sigma}^{\theta z}(R) &= -\frac{b_0\sqrt{\Omega_0}}{2\pi R}\left(\alpha(R) + \frac{1}{\Omega_0}\beta(R)\right).
\end{aligned}
\tag{3.29}
$$

The traction at the outer boundary $R = R_o$ reads

$$
\begin{aligned}
\sigma^{rr}(R_o) &= -\left(\frac{1}{\Omega_0} - \Omega_0\right)\int_{R_i}^{R_o}\frac{\alpha(x)+\beta(x)}{x}dx\\
&\quad - \frac{b_0^2}{4\pi\Omega_0}\int_{R_i}^{R_o}\frac{\beta(x)}{x^3}dx.
\end{aligned}
\tag{3.30}
$$

It is seen that the outer boundary cannot be traction-free, in general. However, for a screw dislocation in a generalized neo-Hookean solid the outer boundary is traction-free as well. Note that energy per unit length of the dispiration is bounded for any material in this case.

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