Study of Edge-Zone Equation of Mindlin-Reissner Plate Theory

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Abstract: Analytical solutions are obtained for the interior and edge-zone equations of Mindlin-Reissner plate theory in bending of composite circular sector plates laminated of transversely isotropic layers. Circular sector laminates, under various boundary conditions, are considered. It is shown that, depending on the boundary conditions of the laminate, the boundary-layer effect on the response quantities of the laminate will be strong, weak, or nonexistent.

INTRODUCTION

The classical fourth-order plate theory of Kirchhoff (1850), although it has been a very useful engineering approximation, has some drawbacks. This theory neglects shear deformations, and as a result it underestimates deflections and overestimates stresses. Though in some cases sufficient, the neglect of shear deformations does not always lead to an acceptable approximation. For example, laminated composites usually undergo considerable shear deformations, and so the effects of shear deformations should be taken into account when these composites are analyzed. The other drawback of classical plate theory is the inconsistency between the order of the governing equilibrium equations and the number of boundary conditions. For example, at a free edge one physically expects three boundary conditions, whereas the order of the governing differential equation dictates only two edge conditions. The reduction of boundary conditions from three to two is due to Kelvin and Tait (Timoshenko and Woinowsky-Krieger 1959). It is known that due to this inconsistency, polynomial plates loaded laterally produce concentrated reactions at corner points, in addition to the distributed reactions along the boundaries. This phenomenon is called the ''corner condition.'' Recently, Yavari et al. (2000) presented a mathematical explanation for this phenomenon in classical plate theory using theory of distributions.

Over the years, researchers have tried to modify the classical plate theory to relax its restrictions. Several alternative plate theories have appeared in the literature, among which those of Mindlin (1951) and Reissner (1945) are the most well known. Both Mindlin's and Reissner's theories are sixth order. Mindlin's theory is displacement based, whereas Reissner's is stress based.

Reissner (1944, 1945, 1947, 1985) found that for a homogeneous isotropic plate his sixth-order theory can be uncoupled into two equations: edge-zone and interior equations. Nosier and Reddy (1991, 1992a,b,c) studied edge-zone and interior equations of several shear deformation plate theories for composite plates laminated of transversely isotropic layers. Nosier et al. (2000) investigated the edge-zone equation of Mindlin-Reissner plate theory in bending of symmetric laminated rectangular plates. They showed that the edge-zone equation contributes to deflection and stresses only in a very localized region near the boundaries of the plate. They numerically showed that the width of this boundary layer is approximately equal to the plate's thickness. They also demonstrated that there is no boundary layer for simply supported edges, and there are weak and strong boundary layers for clamped and free edges, respectively. Finally, they showed that there is no boundary-layer phenomenon for a simply supported Timoshenko beam.

This article considers the bending of laminated circular sector plates under various boundary conditions and demonstrates analytically the contribution to the solution made by the edge-zone, or boundary-layer, equation. The bending equations of laminated circular sector plates laminated of transversely isotropic layers are uncoupled into two equations. The analytical solutions are obtained for both edge-zone (boundary-layer) and interior equations for circular sector plates with two edges simply supported under uniform loads. The effects of the boundary-layer function are studied numerically, and the dependence of the width of the boundary layer on the plate's thickness and on the boundary conditions is studied.

GOVERNING EQUATIONS

According to the first-order shear deformation theory, the bending equations of a plate are obtained in polar coordinates from the following displacement fields (Mindlin and Deresiewicz 1954):

$$u_r(r, \theta, z) = u(r, \theta) + z\psi_r(r, \theta) \quad (1a)$$

$$u_\theta(r, \theta, z) = v(r, \theta) + z\psi_\theta(r, \theta); \quad u_z(r, \theta, z) = w(r, \theta) \quad (1b,c)$$

where z = thickness coordinate; u and v = displacements of the middle surface of the plate in the r- and θ-directions, respectively; and ψ_r and ψ_θ are known as the rotation functions (Reddy 1984, 1997, 1999). Here it is assumed that the plate is symmetrically laminated with respect to its middle surface. Hence the stretching and bending equations are uncoupled. Furthermore, if each layer (or lamina) is made of transversely isotropic material, with the plane of isotropy parallel to the middle surface, then by introducing a new function Φ, which will be referred to as the boundary-layer function, such that

$$\Phi = \frac{1}{r}\,\psi_{r,\theta} - \psi_{\theta,r} - \frac{1}{r}\,\psi_\theta \quad (2)$$

and following a procedure as in Nosier and Reddy (1992a), the bending equations of the plate may be recast to yield two uncoupled equations as follows:

$$\bar{C}\nabla^2\Phi - K^2\bar{A}\Phi = 0; \quad \bar{D}\nabla^2\nabla^2 w = P_z - \frac{\bar{D}}{K^2\bar{A}}\nabla^2 P_z \quad (3a,b)$$

where ∇^2 = 2D Laplace operator in the polar coordinates; K^2 = shear correction factor; and P_z = transverse loading. Eqs. (3a) and (3b) are known as the edge-zone (or boundary-layer)

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Note. Associate Editor: Roberto Ballarini. Discussion open until November 1, 2000. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on January 27, 1999. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 126, No. 6, June, 2000. ©ASCE, ISSN 0733-9399/00/0006-0647–0651/$8.00 + $.50 per page. Paper No. 20138.

and interior equations of the plate, respectively (Reissner 1985; Nosier and Reddy 1991, 1992a,b,c). It also may be shown that (Nosier and Reddy 1992)

$$\psi_r = -w_{,r} - \frac{\bar{D}}{K^2\bar{A}}\nabla^2 w_{,r} + \frac{\bar{C}}{K^2\bar{A}}\frac{1}{r}\Phi_{,\theta} - \frac{\bar{D}}{(K^2\bar{A})^2}P_{z,r} \quad (4a)$$

$$\psi_\theta = -\frac{1}{r}w_{,\theta} - \frac{\bar{D}}{K^2\bar{A}}\frac{1}{r}\nabla^2 w_{,\theta} - \frac{\bar{C}}{K^2\bar{A}}\Phi_{,r} - \frac{\bar{D}}{(K^2\bar{A})^2}\frac{1}{r}P_{z,\theta} \quad (4b)$$

The rigidity terms $\bar{A}$, $\bar{C}$, and $\bar{D}$ are defined as

$$\bar{A} = \sum_{k=1}^{N}(G_z)_k(z_k - z_{k+1}) \quad (5a)$$

$$\bar{C} = \sum_{k=1}^{N}\frac{1}{6}\left(\frac{E}{1+\nu}\right)_k(z_k^3 - z_{k+1}^3) \quad (5b)$$

$$\bar{D} = \sum_{k=1}^{N}\frac{1}{3}\left(\frac{E}{1-\nu^2}\right)_k(z_k^3 - z_{k+1}^3) \quad (5c)$$

where N = total number of layers; E and ν = Young's modulus and Poisson's ratio in the plane of isotropy (i.e., the r-θ plane), respectively; and G_z = shear modulus in the plane normal to the plane of isotropy.

APPLICATION TO CIRCULAR SECTOR PLATES

Here the bending of a laminated plate in the form of a sector subjected to a uniformly distributed load P_z ($=P_0$) is studied (Fig. 1). To this end, it is assumed that the edges at $\theta = 0$ and $\theta = \theta_0$ have simple supports with the boundary conditions

$$\psi_r = M_{\theta\theta} = w = 0 \quad \text{at } \theta = 0 \text{ and } \theta = \theta_0 \quad (6)$$

where $M_{\theta\theta}$ is given by

$$M_{\theta\theta} = (\bar{D} - 2\bar{C})\psi_{r,r} + \frac{\bar{D}}{r}(\psi_r + \psi_{\theta,\theta}) \quad (7)$$

Because $\psi_r = \psi_{r,r} = 0$ at these two edges, it can be concluded from (7) that $\psi_{\theta,\theta} = 0$. Therefore, the boundary conditions in (6) are equivalent to

$$\psi_r = \psi_{\theta,\theta} = w = 0 \quad \text{at } \theta = 0 \text{ and } \theta = \theta_0 \quad (8)$$

As far as (3) is concerned, the boundary conditions in (8) must be restated in terms of Φ and w. With the help of (8), (4*a*), and the governing equations of motion, it can be shown that these conditions are

$$\Phi_{,\theta} = \frac{1}{r^2}w_{,\theta\theta} + \frac{P_z}{K^2\bar{A}} = w = 0 \quad \text{at } \theta = 0 \text{ and } \theta = \theta_0 \quad (9)$$

Next with $\beta_n = n\pi/\theta_0$, the uniformly distributed load may be represented as

$$P_z(r, \theta) = \sum_{n=1,3,\ldots}^{\infty}\frac{4P_0}{n\pi}\sin\beta_n\theta \quad (10)$$

It is also seen that the solution representations

$$\Phi(r, \theta) = \sum_{n=1,3,\ldots}^{\infty}\Phi_n(r)\cos\beta_n\theta \quad (11a)$$

$$w(r, \theta) = \sum_{n=1,3,\ldots}^{\infty}w_n(r)\sin\beta_n\theta \quad (11b)$$

satisfy identically the boundary conditions at $\theta = 0$ and $\theta = \theta_0$. Substitution of (11*a*) into (3*a*) yields

$$r^2\frac{d^2\Phi_n(r)}{dr^2} + r\frac{d\Phi_n(r)}{dr} - \left(\beta_n^2 + \frac{K^2\bar{A}}{\bar{C}}r^2\right)\Phi_n(r) = 0 \quad (12)$$

which is the modified Bessel equation with the general solution

$$\Phi_n(r) = C_{n1}I_{\beta_n}(\mu r) + C_{n2}K_{\beta_n}(\mu r) \quad (13)$$

where I_{β_n} and K_{β_n} = modified Bessel functions of the first and second kinds, respectively; and

$$\mu^2 = \frac{K^2\bar{A}}{\bar{C}} \quad (14)$$

Because Φ must be finite at $r = 0$, it should be concluded that $C_{n2} = 0$. Thus

$$\Phi_n(r) = C_{n1}I_{\beta_n}(\mu r) \quad (15)$$

Also, substitution of (10) and (11*b*) into (3*b*) yields

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\beta_n^2}{r^2}\right)\left(\frac{d^2w_n}{dr^2} + \frac{1}{r}\frac{dw_n}{dr} - \frac{\beta_n^2}{r^2}w_n\right) = \frac{4P_0}{n\pi\bar{D}},$$
$$n = 1, 3, \ldots \quad (16)$$

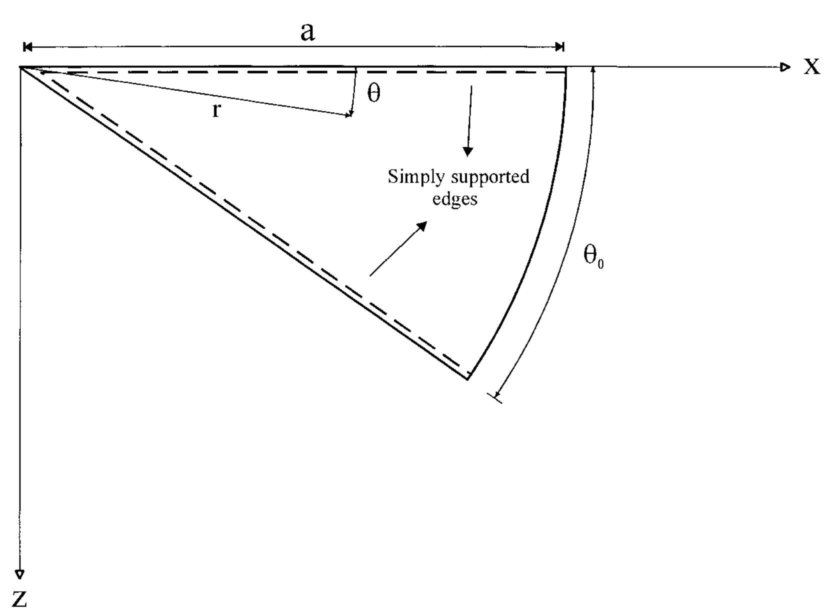

FIG. 1. Circular Sector Plate with Simple Supports at $\theta = 0$ and $\theta = \theta_0$

whose general solution may be represented as

$$w_n(r) = w_c(r) + w_p(r) \tag{17}$$

where w_c and w_p represent, respectively, the complementary and particular solutions of (16). When $\beta_n^2 \neq 4$, $\beta_n^2 \neq 16$, and $\theta_0 < 2\pi$, these solutions are given by

$$w_c(r) = C_{n3}r^{\beta_n} + C_{n4}r^{-\beta_n} + C_{n5}r^{\beta_n+2} + C_{n6}r^{-\beta_n+2} \tag{18a}$$

$$w_p(r) = Cr^4 \tag{18b}$$

where

$$C = \frac{\lambda^2}{(\beta_n^2 - 4)(\beta_n^2 - 16)}; \quad \lambda^2 = \frac{4P_0}{n\pi\bar{D}} \tag{19a,b}$$

When $\beta_n^2 = 4$ or $\beta_n^2 = 16$, (18a) still remains valid, but the particular solution will be given by

$$w_p = -\frac{\lambda^2}{96} r^4 \ln r \quad \text{when } \theta_0 = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \tag{20a}$$

$$w_p = \frac{\lambda^2}{48} r^4 \ln r \quad \text{when } \theta_0 = \frac{2\pi}{4}, \frac{6\pi}{4} \tag{20b}$$

For $\theta_0 = \pi$, $w_{c1} = w_c(n = 1)$ is given by

$$w_{c1} = C_{13}r + C_{14}r^{-1} + C_{15}r^3 + C_{16}r \ln r \tag{21}$$

For $n \geq 2$, (18a) still holds. Also, since w and Q_r must be finite at $r = 0$, it is concluded that $C_{n4} = C_{n6} = 0$. The remaining unknown constants of integration C_{n1}, C_{n3}, and C_{n5} must be determined by imposing three boundary conditions at $r = a$. For example, when the edge of the plate at $r = a$ is simply supported, the following conditions are imposed:

$$w = M_{rr} = \psi_\theta = 0 \quad \text{at } r = a \tag{22}$$

Alternatively, in terms of w and Φ, these conditions may be shown to be

$$w = w_{,rr} + \frac{P_z}{K^2\bar{A}} = \Phi_{,r} = 0 \quad \text{at } r = a \tag{23}$$

To impose the boundary conditions in (22), the general solutions for ψ_r and ψ_θ must be known. These solutions are readily obtained by substituting the general solution of w and Φ into (4). In either case, whether the boundary conditions of (22) or (23) are imposed, a set of three nonhomogeneous algebraic equations will be obtained whose solution yields the integration constants C_{n1}, C_{n3}, and C_{n5}. For the sake of brevity, we do not include the coefficients of these systems of linear equations. Here, however, it should be noted that the imposition of the last condition in (23) yields

$$\Phi(r, \theta) = 0 \tag{24}$$

That is, there is no boundary-layer effect when the sector plate is completely simply supported. When the edge at $r = a$ is clamped, the following boundary conditions will be imposed:

$$w = \psi_r = \psi_\theta = 0 \quad \text{at } r = a \tag{25}$$

where, again, the expressions for rotation functions ψ_r and ψ_θ are obtained from (4). When the boundary conditions in (25) are imposed, again three linear algebraic equations will be obtained whose solution will yield the integration constants. Finally, when the sector plate has a free edge at $r = a$, the boundary conditions will be

$$M_{rr} = \hat{Q}_r = M_{r\theta} = 0 \quad \text{at } r = a \tag{26}$$

Once the expressions for w, ψ_r, and ψ_θ have been obtained, the stress components in any lamina of the plate at (r, θ, z) are obtained from the following relations

$$\sigma_{rr} = \frac{E}{1 - \nu^2}\left[\psi_{r,r} + \frac{\nu}{r}(\psi_r + \psi_{\theta,\theta})\right] z \tag{27a}$$

$$\sigma_{\theta\theta} = \frac{E}{1 - \nu^2}\left[\nu\psi_{r,r} + \frac{1}{r}(\psi_r + \psi_{\theta,\theta})\right] z \tag{27b}$$

$$\sigma_{r\theta} = \frac{E}{2(1 + \nu)}\left(\frac{1}{r}\psi_{r,\theta} + \psi_{\theta,r} - \frac{1}{r}\psi_\theta\right) z \tag{27c}$$

$$\sigma_{rz} = G_z(w_{,r} + \psi_r); \quad \sigma_{\theta z} = G_z\left(\frac{1}{r}w_{,\theta} + \psi_\theta\right) \tag{27d,e}$$

where E and ν = Young's modulus and Poisson's ratio in the r-θ plane, respectively; and G_z = shear modulus in the plane normal to the r-θ plane.

For a complete circular plate (i.e., $\theta_0 = 2\pi$) it could easily be shown that in the axisymmetric bending problem there exists no boundary-layer effect. The details are omitted here for the sake of brevity.

DISCUSSION OF NUMERICAL RESULTS

To study the behavior of the boundary-layer function, we assume the laminated plate to be a single-layer transversely isotropic plate with the following properties:

$$E = 5 \times 10^5; \quad G_z = 0.15E; \quad \nu = 0.25 \tag{28}$$

Also, it is assumed that $K^2 = 5/6$, $a = 50$, and $P_0 = 1$. It should be noted that the units must be consistent. For example, if E is given in pounds per square inch, then a will be in inches.

The variation of the function $\Phi(r, \theta)$ at $\theta = \pi/36$ for a clamped sector plate is shown in Fig. 2(a), for various values of the sector angle θ_0. It is assumed that $h = 5$, where h is the thickness of the plate. It is seen that, except near the edge of the plate (i.e., in the edge zone of the plate), the function Φ is zero everywhere. This is the justification for calling Φ the boundary-layer function. The region where Φ is zero is also called the interior region of the plate (Nosier and Reddy 1992a,c).The width of the edge zone may be considered to be almost equal to the thickness of the plate. This conclusion may be more fully supported by Figs. 2(b and c). In Fig. 2(b) it is assumed that the edge of the sector plate at $r = a$ is clamped, and the variation of Φ is plotted along $\theta = \pi/36$ with the sector angle $\theta_0 = \pi/2$ and various values for the thickness of the plate. Fig. 2(c) shows the variation of the boundary-layer function for a circular sector plate with the sector angle $\theta_0 = \pi/2$ and free edge at $r = a$ for different values of θ. As can be seen, the boundary-layer width is independent of θ; it only depends on the thickness of the plate. The variation of the boundary-layer function Φ for free and clamped sector plates for $\theta_0 = \pi/3$ are compared in Fig. 3(a) along $\theta = \pi/12$. It is observed that Φ has a larger magnitude in the free case than in the clamped case. In other words, it may be said that in the case of a clamped plate, the boundary-layer effect is weak, whereas for a free plate this effect is strong. Also, it should be recalled that when the edge at $r = a$ is simply supported (i.e., in the case of a completely simply supported plate), it can be analytically shown that there exists no boundary-layer effect.

Next, the effect of the boundary-layer function Φ on stress components is studied. To this end, the following quantity is defined:

$$\delta\sigma_{\theta\theta} = \frac{\sigma_{\theta\theta} - \sigma_{\theta\theta}(\Phi = 0)}{\sigma_{\theta\theta}} \times 100 \tag{29}$$

where $\sigma_{\theta\theta}(\Phi = 0)$ means that in calculating $\sigma_{\theta\theta}$ the function Φ is set equal to zero. The variations of stresses in clamped and free sector plates with $\theta_0 = \pi/3$ are shown in Fig. 3(b) along $\theta = \pi/6$. It is seen that the effect of Φ on the stress components is confined to the edge zone of the plate. Again,

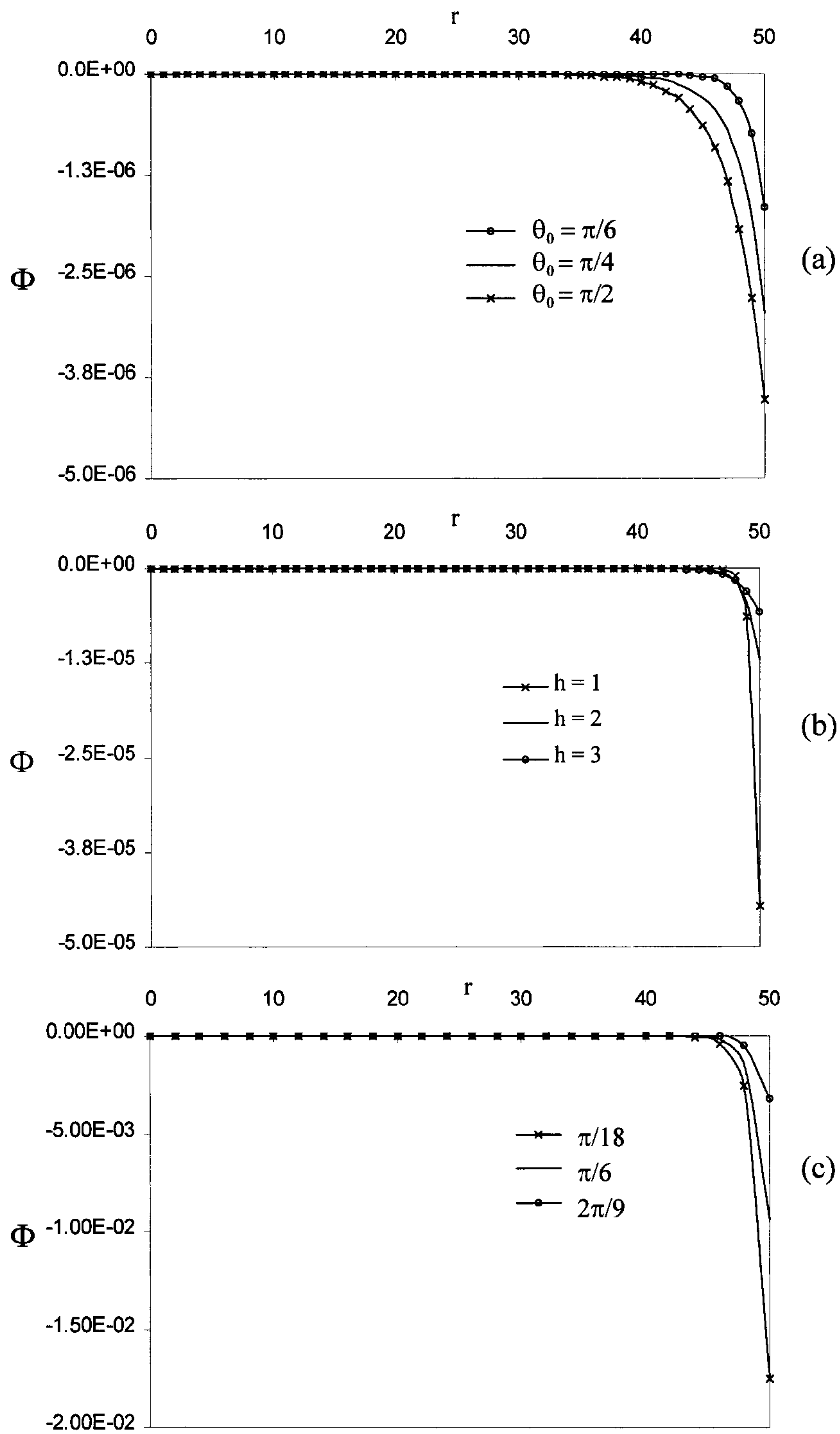

FIG. 2. Variation of Boundary-Layer Function for Circular Sector Plates: (a) $\theta = \pi/36$, $h = 5$, Clamped Edge at $r = a$; (b) $\theta = \pi/36$, $\theta_0 = \pi/2$, Clamped Edge at $r = a$; (c) $h = 2$, $\theta_0 = \pi/2$, Free Edge at $r = a$ ($a = 50$)

it is seen that for a clamped sector plate the boundary-layer effect is weaker than it is when the edge is free. We found that Φ has boundary-layer effects on all of the other stress components, but for the sake of brevity we have not presented the results here.

CONCLUSIONS

The bending problem for laminated circular sector plates is considered in the present work. It is assumed that each layer (lamina) is made of a transversely isotropic material. The theory considered is called the first-order shear deformation plate theory, also known as the Mindlin-Reissner plate theory. Analytical expressions are obtained for primary response quantities of laminated circular sector plates with various boundary conditions under uniform loading. It is analytically shown that when the edges of circular sector plates are simply supported, the boundary-layer effect disappears. Numerical results indicate that the boundary-layer function and its influence on the stress components are confined to the edge zone of the plate. It is also seen that the boundary-layer effect is stronger in the presence of a free edge than it is near a clamped edge.

The numerical calculations consider a single-layer plate with transversely isotropic material; however, the conclusions drawn here are also valid for isotropic plates and laminated transversely isotropic plates.

In this article we have reached the same conclusions as did

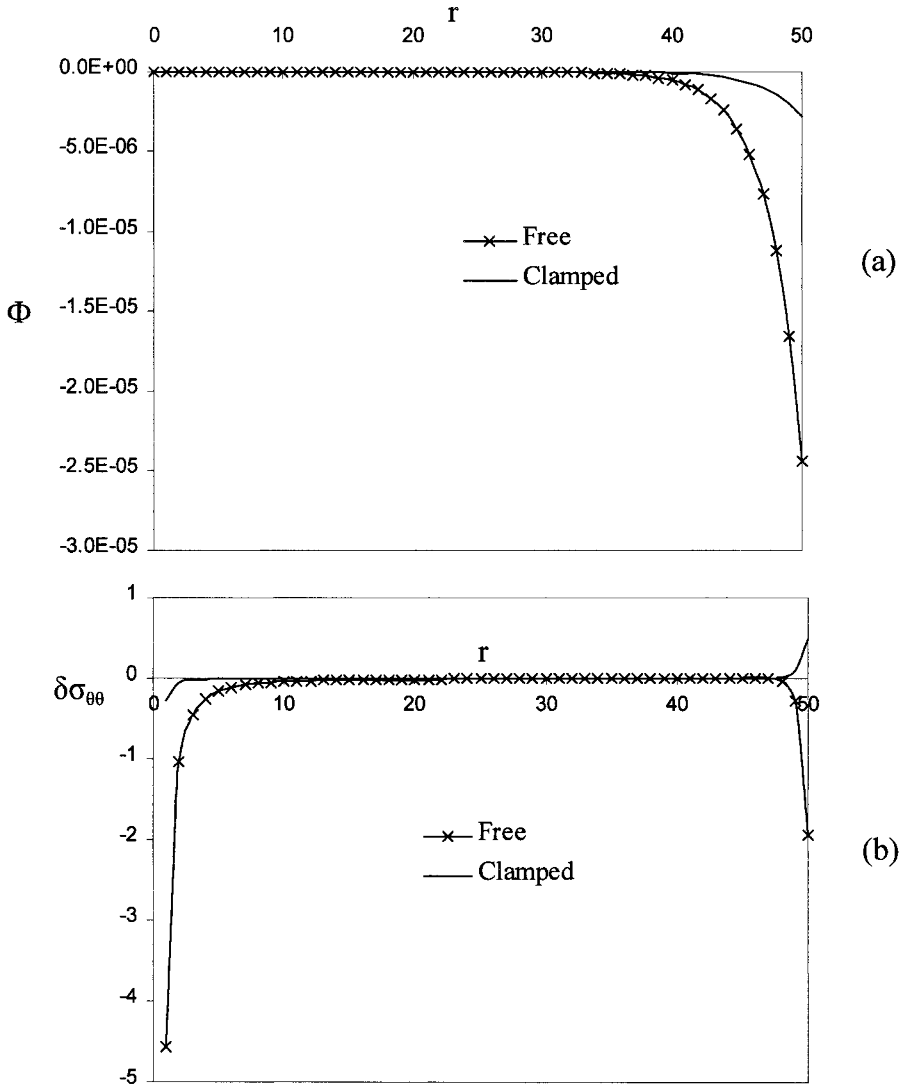

FIG. 3. Effect of Boundary Condition at $r = a$ on: (a) Variation of Boundary-Layer Function in Circular Sector Plate ($h = 5$, $\theta = \pi/12$, $\theta_0 = \pi/3$); (b) Stress Component $\sigma_{\theta\theta}$ in Circular Sector Plate ($h = 1$, $\theta_0 = \pi/3$, $\theta = \pi/6$)

by Nosier et al. (2000). Therefore, it can be concluded that this boundary-layer phenomenon occurs independently of geometry.

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