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Generalized solutions of beams with jump discontinuities on elastic foundations

A. Yavari, S. Sarkani, J. N. Reddy

Summary The bending solutions of the Euler–Bernoulli and the Timoshenko beams with material and geometric discontinuities are developed in the space of generalized functions. Unlike the classical solutions of discontinuous beams, which are expressed in terms of multiple expressions that are valid in different regions of the beam, the generalized solutions are expressed in terms of a single expression on the entire domain. It is shown that the boundary-value problems describing the bending of beams with jump discontinuities on discontinuous elastic foundations have more compact forms in the space of generalized functions than they do in the space of classical functions. Also, fewer continuity conditions need to be satisfied if the problem is formulated in the space of generalized functions. It is demonstrated that using the theory of distributions (i.e. generalized functions) makes finding analytical solutions for this class of problems more efficient compared to the traditional methods, and, in some cases, the theory of distributions can lead to interesting qualitative results. Examples are presented to show the efficiency of using the theory of generalized functions.

Keywords Beam Theory, Elastic Foundation, Jump Discontinuities, Distributions Theory

1 Introduction

In beam bending problems, situations arise in practice for which the governing equilibrium equations of a beam cannot be written in the space of classical functions (with the classical definition of a derivative). There are two possible reasons for this: (1) the loading condition may be singular (which results in an external discontinuity), or (2) there may be jump discontinuities in the deflection, its derivative, or mechanical properties of the beam (leading to an internal discontinuity). A point load and an internal hinge are examples that lead to discontinuous solutions. The traditional method of solving these problems is to partition the domain of the beam into segments such that the solution on each beam segment is continuous, and then solve the problem by applying continuity conditions at the interface of the segments. One drawback of this method is that a large number of differential equations must be solved, so many continuity conditions must be applied, which makes the method cumbersome.

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A. Yavari
Department of Mechanical and Aerospace Engineering,
The George Washington University,
Washington, DC 20052, USA
Present address:
Graduate Aeronautical Laboratories,
California Institute of Technology, Pasadena, CA 91125, USA

S. Sarkani (✉)
Department of Engineering Management and Systems
Engineering, The George Washington University,
Washington, DC 20052, USA
Fax: (202) 994-0092

J. N. Reddy
Department of Mechanical Engineering,
Texas A & M University, College Station,
Texas 77483, USA

The problem of a beam with external discontinuity (e.g. discontinuous loading) was studied in [1] and the solution was simplified by writing a single expression for the bending moment. The so-called Macauly bracket, which is also known in the literature as the singularity function method, was utilized in [2]. Macauly's bracket is indeed the same as the Heaviside unit step function. This method is discussed in almost every elementary study of mechanics of materials. The advantage of the method is to reduce a system of uncoupled second-order differential equations to a single second-order differential equation.

The singularity function method was used in [3] for rectangular plates with opposite edges simply supported and the other two edges having arbitrary boundary conditions, and subjected to a point load. The method was used in [4] to study bending of beams and axisymmetric circular plates. The generalization of the method to two-dimensional problems governed by partial differential equations can be found in [5] and [6].

All the above-mentioned works deal with beams or plates with only external discontinuities. Beams with internal hinges and jump discontinuities in flexural stiffness were analyzed in [7] using the singularity function method. However, the author did not analyze the beams by solving the boundary-value problem of beams with jump discontinuities; instead, he started from the bending moment expression.

The concept of distributions or generalized functions (see [8] and [9]; also see the Appendix) can be used to develop solutions to boundary-value problems with discontinuities. Certain types of generalized functions were used in engineering problems years before the development of distribution theory; the "delta function" $\delta(x)$ and its derivatives are good examples of these functions. The delta function dates back to the nineteenth century and the works of Hermite, Cauchy, Poisson, Kirchhoff, Helmholtz, Lord Kelvin, and Heaviside, [10]. Dirac [11] introduced this function in quantum mechanics, and since then it has been known as the Dirac delta function. Distribution theory interprets the Dirac delta function as a linear functional instead of as a function.

Recently, the theory of generalized functions was utilized for analyzing beams with internal and external discontinuities, [12]. In [12], it was shown that the equivalent distributed force for a point moment of order n can be expressed by the nth distributional derivative of the Dirac delta function. The governing differential equations of Euler–Bernoulli and Timoshenko beams with jump discontinuities were found in the space of generalized functions. For Euler–Bernoulli beams, only the forcing term is different from that of the classical equation. But for Timoshenko beams with jump discontinuities, the forcing terms and one of the operators of the governing differential equations change. It was demonstrated that utilizing distribution theory can make the bending analysis of beams with jump discontinuities more efficient. The same problem was investigated in [13] for Euler–Bernoulli beam-columns with jump discontinuities. It was found that unlike beams, the governing differential equation of an Euler–Bernoulli beam-column can not, generally, be written in terms of a single displacement function; it is possible to express the governing equilibrium equation in terms of a single displacement function only under some conditions. It was also demonstrated that calculation of buckling loads for Euler–Bernoulli columns with internal discontinuities is easier, in some cases, using the theory of distributions than using conventional methods.

In the present study, the governing differential equations of the Euler–Bernoulli and Timoshenko beams with jump discontinuities on discontinuous elastic foundations are obtained in the space of generalized functions. It is shown that the governing differential equations of beams can always be expressed in terms of the displacement functions in the space of generalized functions.

2
Euler–Bernoulli beams on elastic foundations

The Euler–Bernoulli beam theory is the simplest beam theory, and it is based on the following displacement field:

$$u_1(x,y,z) = u(x) - \frac{\mathrm{d}w(x)}{\mathrm{d}x} z, \quad u_2(x,y,z) = 0, \quad u_3(x,y,z) = w(x) \quad . \tag{1}$$

Here, u_1, u_2, and u_3 denote the displacement components along the longitudinal axis, width, and height of the beam, respectively. In this theory, the transverse shear deformation is neglected. The governing bending equation of an Euler–Bernoulli beam on an elastic foundation may be written as [14–16],

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w(x)}{dx^2}\right)+K_w w(x)=q(x) \quad , \tag{2}$$

where EI, K_w, and q are the flexural stiffness, foundation modulus and the transverse load density, respectively. For a beam with a constant flexural stiffness EI, Eq. (2) may be written as

$$\frac{d^4w(x)}{dx^4}+4\beta^4w(x)=\frac{q(x)}{EI} \quad , \tag{3}$$

where

$$\beta=\sqrt[4]{\frac{K_w}{4EI}} \quad . \tag{4}$$

The model used in Eq. (3), the simplest model for an elastic foundation, is called the Winkler model. For a Winkler foundation, it is assumed that at any point the pressure the foundation exerts on the beam is proportional to the deflection of the beam at that point and is independent of the deflection of other parts of the foundation. This approximate model is acceptable for many engineering applications.

Now consider a beam with jump discontinuities on a Winkler foundation, whose constant undergoes abrupt changes. Obviously, for this discontinuous beam Eq. (3) is not the governing equilibrium equation. For the sake of simplicity, consider a beam with one point of jump discontinuity, as shown in Fig. 1. Generalization to the case of n jump discontinuity points is straightforward. The beam shown in Fig. 1 has arbitrary boundary conditions at $x=0, L$, and jump discontinuities at $x=x_0$ in slope, deflection, and flexural stiffness. At this point, we have a combination of an internal hinge with a rotational spring and a shear-free connection with a translational spring. The spring stiffnesses are K_r and K_t, respectively. It is also assumed that the foundation constant has an abrupt change at $x=x_0$. Now let

$$w(x_0^+)-w(x_0^-)=\Delta, \quad \frac{dw(x_0^+)}{dx}-\frac{dw(x_0^-)}{dx}=\Theta \quad . \tag{5}$$

The beam is composed of two beam segments in the intervals $[0,x_0]$ and $[x_0,L]$ with respective displacements w_1 and w_2. Using Heaviside's function, the deflection of the beam can be written as

$$w(x)=w_1(x)+[w_2(x)-w_1(x)]H(x-x_0) \quad . \tag{6}$$

For each beam segment, Eq. (3) is the governing equilibrium equation. Hence,

$$\begin{aligned}\frac{d^4w_1}{dx^4}+\frac{K_1}{EI_1}w_1&=\frac{q(x)}{EI_1}; \quad 0\le x\le x_0,\\ \frac{d^4w_2}{dx^4}+\frac{K_2}{EI_2}w_2&=\frac{q(x)}{EI_2}; \quad x_0\le x\le L \quad .\end{aligned} \tag{7}$$

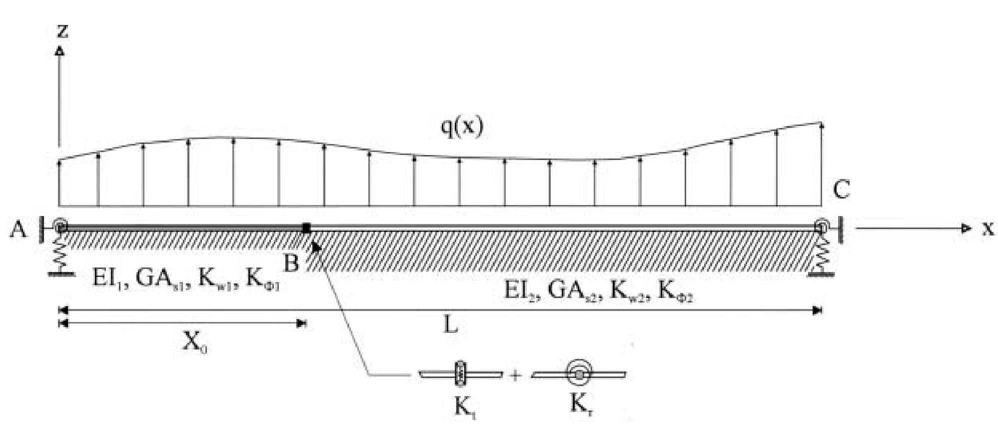

Fig. 1. A beam with jump discontinuities on a discontinuous elastic foundation

Note that $q(x)$ could be a discontinuous function. Now let

$$EI_1 = EI, \quad EI_2 = \lambda\, EI, \quad K_{w1} = K_w, \quad K_{w2} = \mu\, K_w \ . \tag{8}$$

Substituting (8) into (7) yields

$$\frac{\mathrm{d}^4 w_1}{\mathrm{d}x^4} + \frac{K_w}{EI} w_1 = \frac{q(x)}{EI}; \quad 0 \le x \le x_0 \ , \tag{9}$$

$$\frac{\mathrm{d}^4 w_2}{\mathrm{d}x^4} + \frac{\mu\, K_w}{\lambda\, EI} w_2 = \frac{q(x)}{\lambda\, EI}; \quad x_0 \le x \le L \ . \tag{10}$$

Differentiating both sides of (6), we obtain

$$\frac{\bar{\mathrm{d}}w}{\mathrm{d}x} = \frac{\mathrm{d}w_1}{\mathrm{d}x} + \left[\frac{\mathrm{d}w_2}{\mathrm{d}x} - \frac{\mathrm{d}w_1}{\mathrm{d}x}\right] H(x - x_0) + \Delta\delta(x - x_0) \ , \tag{11}$$

$$\frac{\bar{\mathrm{d}}^2 w}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} + \left[\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2}\right] H(x - x_0) + \Theta\delta(x - x_0) + \Delta\delta^{(1)}(x - x_0) \ . \tag{12}$$

Here, a bar over the differentiation symbol means distributional differentiation. For this Euler–Bernoulli beam, we know that moment and force equilibrium at $x = x_0$ yield

$$\begin{aligned} M_1(x_0) &= EI \frac{\mathrm{d}^2 w_1(x_0)}{\mathrm{d}x^2} = M_2(x_0) = \lambda\, EI \frac{\mathrm{d}^2 w_2(x_0)}{\mathrm{d}x^2} = K_r \Theta, \\ V_1(x_0) &= EI \frac{\mathrm{d}^3 w_1(x_0)}{\mathrm{d}x^3} = V_2(x_0) = \lambda\, EI \frac{\mathrm{d}^3 w_2(x_0)}{\mathrm{d}x^3} = K_t \Delta \ . \end{aligned} \tag{13}$$

Therefore,

$$\left[\frac{\mathrm{d}^2 w_2(x)}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1(x)}{\mathrm{d}x^2}\right]_{x=x_0} = \frac{K_r \Theta}{EI}\left(\frac{1}{\lambda} - 1\right) \ , \tag{14}$$

$$\left[\frac{\mathrm{d}^3 w_2(x)}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 w_1(x)}{\mathrm{d}x^3}\right]_{x=x_0} = \frac{K_t \Delta}{EI}\left(\frac{1}{\lambda} - 1\right) \ . \tag{15}$$

Differentiating (12) with respect to x and considering (14), we obtain

$$\begin{aligned} \frac{\bar{\mathrm{d}}^3 w}{\mathrm{d}x^3} = \frac{\mathrm{d}^3 w_1}{\mathrm{d}x^3} &+ \left[\frac{\mathrm{d}^3 w_2}{\mathrm{d}x^3} - \frac{\mathrm{d}^3 w_1}{\mathrm{d}x^3}\right] H(x - x_0) \\ &+ \frac{K_r \Theta}{EI}\left(\frac{1}{\lambda} - 1\right)\delta(x - x_0) + \Theta\delta^{(1)}(x - x_0) + \Delta\delta^{(2)}(x - x_0) \ . \end{aligned} \tag{16}$$

Similarly, differentiating (16) and considering (15) yields

$$\begin{aligned} \frac{\bar{\mathrm{d}}^4 w}{\mathrm{d}x^4} = \frac{\mathrm{d}^4 w_1}{\mathrm{d}x^4} &+ \left[\frac{\mathrm{d}^4 w_2}{\mathrm{d}x^4} - \frac{\mathrm{d}^4 w_1}{\mathrm{d}x^4}\right] H(x - x_0) + \frac{K_t \Delta}{EI}\left(\frac{1}{\lambda} - 1\right)\delta(x - x_0) \\ &+ \frac{K_r \Theta}{EI}\left(\frac{1}{\lambda} - 1\right)\delta^{(1)}(x - x_0) + \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0) \ . \end{aligned} \tag{17}$$

Denoting the left-hand sides of (9) and (10) by $\Im$ and $\Re$, respectively, we can write

$$\Im + (\Re - \Im) H(x - x_0) = 0 \ . \tag{18}$$

Substituting (9), (10) into (18), we obtain

$$\left(\frac{d^4 w_1}{dx^4} + \frac{K_w}{EI} w_1\right) + \left[\left(\frac{d^4 w_2}{dx^4} + \frac{\mu}{\lambda}\frac{K_w}{EI} w_2\right) - \left(\frac{d^4 w_1}{dx^4} + \frac{K_w}{EI} w_1\right)\right] H(x - x_0)$$
$$= \frac{q(x)}{EI}\left[1 + \left(\frac{1}{\lambda} - 1\right) H(x - x_0)\right] \ . \qquad (19)$$

Multiplying both sides of (6) by $H(x - x_0)$, we obtain

$$w_2(x) H(x - x_0) = w(x) H(x - x_0) \ . \qquad (20)$$

From (12), (16), and (19), we obtain

$$\frac{\bar{d}^4 w(x)}{dx^4} + \frac{K_w}{EI}\left[1 + \left(\frac{\mu}{\lambda} - 1\right) H(x - x_0)\right] w(x)$$
$$= \frac{q(x)}{EI}\left[1 + \left(\frac{1}{\lambda} - 1\right) H(x - x_0)\right] + \left(\frac{1}{\lambda} - 1\right)\frac{K_t \Delta}{EI}\delta(x - x_0)$$
$$+ \left(\frac{1}{\lambda} - 1\right)\frac{K_r \Theta}{EI}\delta^{(1)}(x - x_0) + \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0) \ . \qquad (21)$$

Equation (21) is the governing equilibrium equation of an Euler–Bernoulli beam with jump discontinuities on a discontinuous Winkler foundation in the space of generalized functions. As may be seen, both the operator and the forcing term of the differential equation are different from those of the classical equation (3).

If $\mu \neq \lambda$, when solving Eq. (21) we have to assume that $w(x) = f(x) + g(x)H(x - x_0)$. This procedure yields two uncoupled differential equations for f and g. Obviously, in this case, it is better to use the classical method than to analyze the beam in the space of generalized functions. When $\mu = \lambda$, Eq. (21) is simplified to

$$\frac{\bar{d}^4 w(x)}{dx^4} + \frac{K_w}{EI} w(x) = \frac{q(x)}{EI}\left[1 + \left(\frac{1}{\lambda} - 1\right) H(x - x_0)\right] + \left(\frac{1}{\lambda} - 1\right)\frac{K_t \Delta}{EI}\delta(x - x_0)$$
$$+ \left(\frac{1}{\lambda} - 1\right)\frac{K_r \Theta}{EI}\delta^{(1)}(x - x_0) + \Theta\delta^{(2)}(x - x_0) + \Delta\delta^{(3)}(x - x_0) \ . \qquad (22)$$

It is observed that when $\mu = \lambda$, the operator of the governing equilibrium equation is the same as that of the classical equation (3).

Now consider an Euler–Bernoulli beam with n discontinuity points at $x = x_1, x_2, x_3, \ldots, x_n$, and let

$$EI_i = \lambda_i EI, \ \ K_{wi} = \mu_i K_w, \ \ w(x_i^+) - w(x_i^-) = \Delta_i, \ \ \frac{dw(x_i^+)}{dx} - \frac{dw(x_i^-)}{dx} = \Theta_i \ . \qquad (23)$$

Also assume that at $x = x_i, K_{ti}$ and K_{ri} are the respective stiffnesses of translational and rotational springs. It can be easily shown that the governing differential equilibrium equation of this beam may be written as

$$\frac{\bar{d}^4 w(x)}{dx^4} + \left[\frac{\mu_1}{\lambda_1} + \sum_{i=1}^{n}\left(\frac{\mu_{i+1}}{\lambda_{i+1}} - \frac{\mu_i}{\lambda_i}\right) H(x - x_i)\right] w(x)$$
$$= \frac{q(x)}{EI}\left[\frac{1}{\lambda_1} + \sum_{i=1}^{n}\left(\frac{1}{\lambda_{i+1}} - \frac{1}{\lambda_i}\right) H(x - x_i)\right] + \sum_{i=1}^{n}\left[\frac{K_{ti}\Delta_i}{EI}\left(\frac{1}{\lambda_{i+1}} - \frac{1}{\lambda_i}\right)\delta(x - x_i)\right.$$
$$\left. + \frac{K_{ri}\Theta_i}{EI}\left(\frac{1}{\lambda_{i+1}} - \frac{1}{\lambda_i}\right)\delta^{(1)}(x - x_i) + \Theta_i\delta^{(2)}(x - x_i) + \Delta_i\delta^{(3)}(x - x_i)\right] \ . \qquad (24)$$

When Euler–Bernoulli and Timoshenko beams with jump discontinuities are supported only at boundary points, auxiliary beams were introduced in [12] as a means of simplifying their analysis. An auxiliary beam has no jump discontinuity, and its response quantities can very easily be obtained in the space of classical functions. However, for the case of a beam resting on an elastic foundation, the governing equilibrium equation that appears as Eq. (21) contains, in addition to the fourth derivative of the deflection function, the function itself. Therefore, similar to beam-columns with jump discontinuities, [13], it can easily be shown that using an auxiliary beam does not make the analysis more efficient. Therefore, the governing equilibrium equation is directly solved in the space of generalized functions using the Laplace transform method.

3
Timoshenko beams on elastic foundations

In this section, the governing differential equations of a Timoshenko beam, with jump discontinuities on a two-parameter elastic foundation whose constants change abruptly, are obtained in the space of generalized functions. The Timoshenko beam theory is the simplest shear deformation beam theory; it has the following displacement field assumptions, [16], [17]:

$$u_1(x,y,z) = z\Phi(x), \quad u_2(x,y,z) = 0, \quad u_3(x,y,z) = w^T(x) \ , \tag{25}$$

where u_1, u_2, and u_3 are displacement components along the x, y and z axes, respectively; Φ is the rotation about the y axis; and w^T is the transverse deflection. This theory assumes that any planar cross section, perpendicular to the beam axis before deformation, remains planar after deformation. For this beam, the shear stress is constant, which is in contradiction with elasticity boundary conditions at free surfaces of the beam. However, using a shear correction factor this theory gives good results. The governing equilibrium equations of a Timoshenko beam on a two-parameter elastic foundation may be written as, [18]

$$\begin{aligned} &GA_s\left(\Phi + \frac{\mathrm{d}w}{\mathrm{d}x}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right) + K_\Phi\Phi = 0, \\ &GA_s\left(\frac{\mathrm{d}\Phi}{\mathrm{d}x} + \frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right) - K_w w + q = 0 \ , \end{aligned} \tag{26}$$

where GA_s and EI are, respectively, the shear and flexural stiffnesses, and K_Φ and K_w are the rotational and extensional moduli of the foundation. It should be noted that $q(x)$ can have discontinuities. When all the mechanical properties of the beam are constant, the governing equilibrium equations may be simplified as

$$GA_s\left(\Phi + \frac{\mathrm{d}w}{\mathrm{d}x}\right) - EI\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} + K_\Phi\Phi = 0, \quad GA_s\left(\frac{\mathrm{d}\Phi}{\mathrm{d}x} + \frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right) - K_w w + q = 0 \ . \tag{27}$$

Now consider a Timoshenko beam with jump discontinuities in slope, deflection, shear stiffness, and flexural stiffness at a point $x = x_0$. It is obvious that (27) cannot be the governing equilibrium equations of this discontinuous beam. Suppose that there is only one point of jump discontinuity, and at this point the foundation constants, K_w and K_Φ, have abrupt changes. Generalization to the case of a Timoshenko beam with n discontinuity points is straightforward. Consider the beam shown in Fig. 1. For beam segments AB and BC, the mechanical properties and foundation constants are constant. Therefore, Eqs. (27) are the governing differential equations for each beam segment. Now let

$$\begin{aligned} &EI_1 = EI, \quad EI_2 = \lambda EI, \quad GA_{s1} = GA_s, \quad GA_{s2} = \mu GA_s, \\ &K_{w1} = K_w, \quad K_{w2} = \xi K_w, \quad K_{\Phi1} = K_\Phi, \quad K_{\Phi2} = \eta K_\Phi \ . \end{aligned} \tag{28}$$

Substituting (28) into (27), we obtain the governing differential equations for the beam segments. For $0 \le x \le x_0$

$$\frac{GA_s}{EI}\left(\Phi_1 + \frac{\mathrm{d}w_1^T}{\mathrm{d}x}\right) - \frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x^2} + \frac{K_\Phi}{EI}\Phi_1 = 0, \quad \frac{\mathrm{d}\Phi_1}{\mathrm{d}x} + \frac{\mathrm{d}^2w_1^T}{\mathrm{d}x^2} - \frac{K_w}{GA_s}w_1 + \frac{q}{GA_s} = 0 \ , \tag{29}$$

and for $x_0 \leq x \leq L$

$$\frac{\gamma}{\alpha}\frac{GA_s}{EI}\left(\Phi_2 + \frac{\mathrm{d}w_2^T}{\mathrm{d}x}\right) - \frac{\mathrm{d}\Phi_2}{\mathrm{d}x} + \frac{\eta K_\Phi}{\lambda EI}\Phi_2 = 0,$$
$$\frac{\mathrm{d}\Phi_2}{\mathrm{d}x} + \frac{\mathrm{d}^2 w_2^T}{\mathrm{d}x^2} - \frac{\xi}{\mu GA_s}K_w w_2 + \frac{q}{\mu GA_s} = 0 \ . \tag{30}$$

The deflection and slope of the beam have jump discontinuities; hence

$$w_2^T(x_0) - w_1^T(x_0) = \Delta, \ \ \Phi_2(x_0) - \Phi_1(x_0) = \Theta \ , \tag{31}$$

where Δ and Θ are the respective strengths of the jump discontinuities in deflection and slope. Now let

$$w^T(x) = w_1^T(x) + \left[w_2^T(x) - w_1^T(x)\right]H(x - x_0),$$
$$\Phi(x) = \Phi_1(x) + [\Phi_2(x) - \Phi_1(x)]H(x - x_0) \ , \tag{32}$$

it is known that for this Timoshenko beam

$$M_1(x_0) = EI_1\frac{\mathrm{d}\Phi_1(x_0)}{\mathrm{d}x}, \quad M_2(x_0) = EI_2\frac{\mathrm{d}\Phi_2(x_0)}{\mathrm{d}x},$$
$$V_1(x_0) = GA_{s1}\left[\Phi_1(x_0) + \frac{\mathrm{d}w_1(x_0)}{\mathrm{d}x}\right], \quad V_2(x_0) = GA_{s2}\left[\Phi_2(x_0) + \frac{\mathrm{d}w_2(x_0)}{\mathrm{d}x}\right] \ . \tag{33}$$

The equilibrium of an infinitesimal element including the discontinuity point implies that

$$M_1(x_0) = M_2(x_0) = K_r\Theta^T, \quad V_1(x_0) = V_2(x_0) = K_t\Delta^T \ , \tag{34}$$

From (33) and (34), we obtain

$$\frac{\mathrm{d}\Phi_2(x_0)}{\mathrm{d}x} - \frac{\mathrm{d}\Phi_1(x_0)}{\mathrm{d}x} = \frac{K_r\Theta^T}{EI}\left(\frac{1}{\lambda} - 1\right),$$
$$\frac{\mathrm{d}w_2(x_0)}{\mathrm{d}x} - \frac{\mathrm{d}w_1(x_0)}{\mathrm{d}x} = \frac{K_t\Theta^T}{EI}\left(\frac{1}{\mu} - 1\right) - \Theta^T \ . \tag{35}$$

Differentiating (32) and considering (35), yields

$$\frac{\bar{\mathrm{d}}w^T}{\mathrm{d}x} = \frac{\mathrm{d}w_1^T}{\mathrm{d}x} + \left(\frac{\mathrm{d}w_2^T}{\mathrm{d}x} - \frac{\mathrm{d}w_1^T}{\mathrm{d}x}\right)H(x - x_0) + \Delta^T\delta(x - x_0),$$
$$\frac{\bar{\mathrm{d}}^2 w^T}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 w_1^T}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}^2 w_2^T}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 w_1^T}{\mathrm{d}x^2}\right)H(x - x_0) + \left[\frac{K_t\Delta^T}{GA_s}\left(\frac{1}{\mu} - 1\right) - \Theta^T\right]\delta(x - x_0)$$
$$+ \Delta^T\delta^{(1)}(x - x_0) \ , \tag{36}$$

and

$$\frac{\bar{\mathrm{d}}\Phi}{\mathrm{d}x} = \frac{\mathrm{d}\Phi_1}{\mathrm{d}x} + \left(\frac{\mathrm{d}\Phi_2}{\mathrm{d}x} - \frac{\mathrm{d}\Phi_1}{\mathrm{d}x}\right)H(x - x_0) + \Theta^T\delta(x - x_0),$$
$$\frac{\bar{\mathrm{d}}^2\Phi}{\mathrm{d}x^2} = \frac{\mathrm{d}^2\Phi_2}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}^2\Phi_2}{\mathrm{d}x^2} - \frac{\mathrm{d}^2\Phi_1}{\mathrm{d}x^2}\right)H(x - x_0) + \frac{K_r\Theta^T}{EI}\left(\frac{1}{\lambda} - 1\right)\delta(x - x_0) + \Theta^T\delta^{(1)}(x - x_0) \ . \tag{37}$$

From (29) and (30), and after some manipulations similar to what was done for Euler–Bernoulli beams, the governing equilibrium equations of the beam can be written in the space of generalized functions as

$$\frac{GA_s}{EI}\left[1+\left(\frac{\mu}{\lambda}-1\right)H(x-x_0)\right]\frac{\bar{\mathrm{d}}w^T(x)}{\mathrm{d}x}-\frac{\bar{\mathrm{d}}^2\Phi(x)}{\mathrm{d}x^2}$$
$$+\left\{\frac{GA_s}{EI}\left[1+\left(\frac{\mu}{\lambda}-1\right)H(x-x_0)\right]+\frac{K_\Phi}{EI}\left[1+\left(\frac{\eta}{\lambda}-1\right)H(x-x_0)\right]\right\}\Phi(x)$$
$$=\left\{\frac{GA_s}{EI}\left[1+\frac{1}{2}\left(\frac{1}{\lambda}-1\right)\right]\Delta^T-\frac{K_r\Theta^T}{EI}\left(\frac{1}{\lambda}-1\right)\right\}\delta(x-x_0)-\Theta^T\delta^{(1)}(x-x_0),$$
$$\frac{\bar{\mathrm{d}}^2w^T(x)}{\mathrm{d}x^2}+\frac{\bar{\mathrm{d}}\Phi(x)}{\mathrm{d}x}-\frac{K_w}{GA_s}\left[1+\left(\frac{\xi}{\mu}-1\right)H(x-x_0)\right]w(x)+\left[1+\left(\frac{1}{\mu}-1\right)H(x-x_0)\right]\frac{q(x)}{GA_s}$$
$$=\frac{K_t\Delta^T}{EI}\left(\frac{1}{\mu}-1\right)\delta(x-x_0)+\Delta^T\delta^{(1)}(x-x_0)\ . \tag{38}$$

When there are no abrupt changes in mechanical properties of the beam and foundation constants ($\lambda=\mu=\xi=\eta=1$), Eqs. (38) may be simplified as

$$\frac{GA_s}{EI}\frac{\bar{\mathrm{d}}w^T(x)}{\mathrm{d}x}-\frac{\bar{\mathrm{d}}^2\Phi(x)}{\mathrm{d}x^2}+\frac{GA_s+K_\Phi}{EI}\Phi(x)=\frac{GA_s}{EI}\Delta^T\delta(x-x_0)-\Theta^T\delta^{(1)}(x-x_0),$$
$$\frac{\bar{\mathrm{d}}^2w^T(x)}{\mathrm{d}x^2}+\frac{\bar{\mathrm{d}}\Phi(x)}{\mathrm{d}x}-\frac{K_w}{GA_s}w(x)+\frac{q(x)}{GA_s}=\Delta^T\delta^{(1)}(x-x_0)\ . \tag{39}$$

In the next section, three examples are solved. It is shown that finding analytical solutions for discontinuous beams on elastic foundations is easier using generalized functions than it is using traditional methods. Also having the boundary-value problems describing the bending of discontinuous beams on elastic foundations in the space of generalized functions can give us a deeper insight into the behavior of these problems than we have otherwise.

4 Examples

Three examples are solved here. These examples demonstrate the usefulness of distribution theory to the study of discontinuous problems.

Example 1

In this example, an Euler–Bernoulli beam with an internal hinge on a uniform elastic foundation is considered (Fig. 2). The characteristics of this beam are

$$\mu=\lambda=1,\quad \Delta=0,\quad K_r=0\ . \tag{40}$$

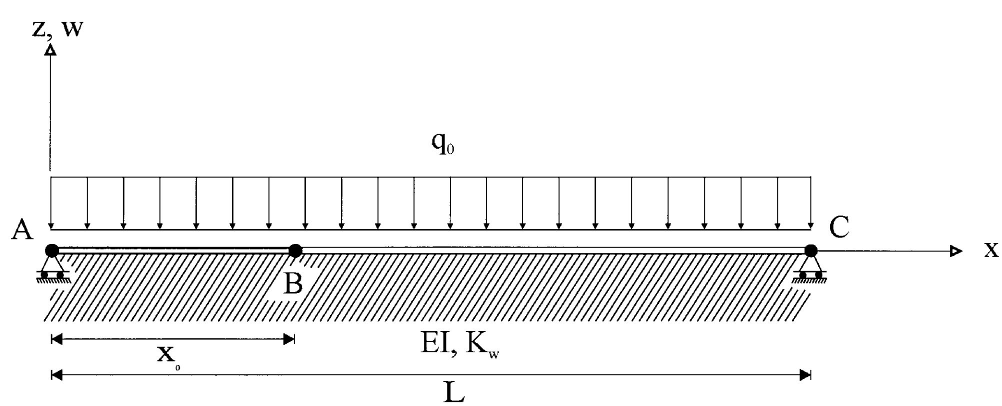

Fig. 2. A simply-supported beam with an internal hinge on a uniform elastic foundation under a uniform distributed force

Substituting (40) into (21) yields

$$\frac{\bar{d}^4 w(x)}{dx^4} + \frac{K_w}{EI} w(x) = -\frac{q_0}{EI} + \Theta \delta^{(2)}(x - x_0) \quad . \tag{41}$$

Therefore, the boundary-value problem in the space of generalized functions can be written as

$$\frac{\bar{d}^4 w(x)}{dx^4} + 4\beta^4 w(x) = -\frac{q_0}{EI} + \Theta \delta^{(2)}(x - x_0) \quad , \tag{42}$$

$$w(0) = w(L) = 0 \tag{43}$$

$$\frac{\bar{d}^2 w(0)}{dx^2} = \frac{\bar{d}^2 w(L)}{dx^2} = \frac{\bar{d}^2 w(x_0)}{dx^2} = 0 \quad . \tag{44}$$

Now let the Laplace transform of $w(x)$ be $W(s)$. Taking the Laplace transform of both sides of (42), we obtain

$$W(s) = A \frac{s^2}{4\beta^4 + s^4} + B \frac{1}{4\beta^4 + s^4} - \frac{q_0}{EI} \frac{1}{s(4\beta^4 + s^4)} + \Theta \frac{s^2 e^{-x_0 s}}{4\beta^4 + s^4} \quad , \tag{45}$$

where

$$A = \frac{\bar{d} w(0)}{dx}, \quad B = \frac{\bar{d}^3 w(0)}{dx^3} \quad .$$

We know that (see [19])

$$\begin{aligned}
L^{-1}\left\{\frac{1}{4\beta^4 + s^4}\right\} &= \frac{1}{4\beta^3}(\sin\beta x \cosh\beta x - \cos\beta x \sinh\beta x), \\
L^{-1}\left\{\frac{s}{4\beta^4 + s^4}\right\} &= \frac{1}{2\beta^2} \sin\beta x \sinh\beta x, \\
L^{-1}\left\{\frac{s^2}{4\beta^4 + s^4}\right\} &= \frac{1}{2\beta}(\sin\beta x \cosh\beta x + \cos\beta x \sinh\beta x), \\
L^{-1}\left\{\frac{s^3}{4\beta^4 + s^4}\right\} &= \cos\beta x \cosh\beta x \quad ,
\end{aligned} \tag{46}$$

where L^{-1} is the inverse Laplace transformation operator. From (45) and (46), we obtain

$$\begin{aligned}
w(x) = {} & \frac{A}{2\beta}(\sin\beta x \cosh\beta x + \cos\beta x \sinh\beta x) \\
& + \frac{B}{4\beta^3}(\sin\beta x \ \cosh\beta x - \cos\beta x \ \sinh\beta x) + \frac{q_0}{EI}(\cos\beta x \ \cosh\beta x - 1) \\
& + \frac{\Theta}{2\beta}[\sin\beta(x - x_0)\cosh\beta(x - x_0) + \cos\beta(x - x_0)\sinh\beta(x - x_0)]H(x - x_0) \quad ,
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
A &= \frac{m_2(m_6 m_{11} - m_7 m_{10}) + m_3(m_5 m_{10} - m_6 m_9)}{m_1(m_5 m_{10} - m_6 m_9) + m_2(m_6 m_8 - m_4 m_{10})}, \\
B &= \frac{m_1(m_7 m_{10} - m_6 m_{11}) + m_3(m_6 m_8 - m_4 m_{10})}{m_1(m_5 m_{10} - m_6 m_9) + m_2(m_6 m_8 - m_4 m_{10})}, \\
\Theta &= \frac{m_2^2(m_7 m_8 - m_4 m_{11}) + m_1 m_2(m_5 m_{11} - m_7 m_9) + m_2 m_3(m_4 m_9 - m_5 m_8)}{m_1(m_5 m_{10} - m_6 m_9) + m_2(m_6 m_8 - m_4 m_{10})} \quad ,
\end{aligned} \tag{48}$$

where

$$
\begin{aligned}
m_1 &= \beta(\cos\beta x_0 \ \sinh\beta x_0 - \sin\beta x_0 \ \cosh\beta x_0),\\
m_2 &= \frac{1}{2\beta}(\cos\beta x_0 \ \sinh\beta x_0 + \sin\beta x_0 \ \cosh\beta x_0),\\
m_3 &= \frac{2q_0\beta}{EI}\sin\beta x_0 \ \sinh\beta x_0,\\
m_4 &= \frac{1}{2\beta}(\sin\beta L \ \cosh\beta L + \cos\beta L \ \sinh\beta L),\\
m_5 &= \frac{1}{4\beta^3}(\sin\beta L \ \cosh\beta L - \cos\beta L \ \sinh\beta L),\\
m_6 &= \frac{1}{2\beta}[\sin\beta(L-x_0)\cosh\beta(L-x_0) + \cos\beta(L-x_0)\sinh\beta(L-x_0)],\\
m_7 &= \frac{q_0}{EI}(1-\cos\beta L \ \cosh\beta L),\\
m_8 &= \beta(\cos\beta L \ \sinh\beta L - \sin\beta L \ \cosh\beta L),\\
m_9 &= \frac{1}{2\beta}(\cos\beta L \ \sinh\beta L + \sin\beta L \ \cosh\beta L),\\
m_{10} &= \beta[\cos\beta(L-x_0)\sinh\beta(L-x_0) - \sin\beta(L-x_0)\cosh\beta(L-x_0)],\\
m_{11} &= \frac{2q_0\beta^2}{EI}\sin\beta L \ \sinh\beta L \ .
\end{aligned}
\tag{49}
$$

Example 2
Consider the beams shown in Fig. 3. These beams are of equal length; they have the same boundary conditions at $x = 0$, L and the same jump discontinuities at $x = x_0$. The mechanical properties of the beam I of Fig. 3a change abruptly at $x = x_0$, where the elastic foundation

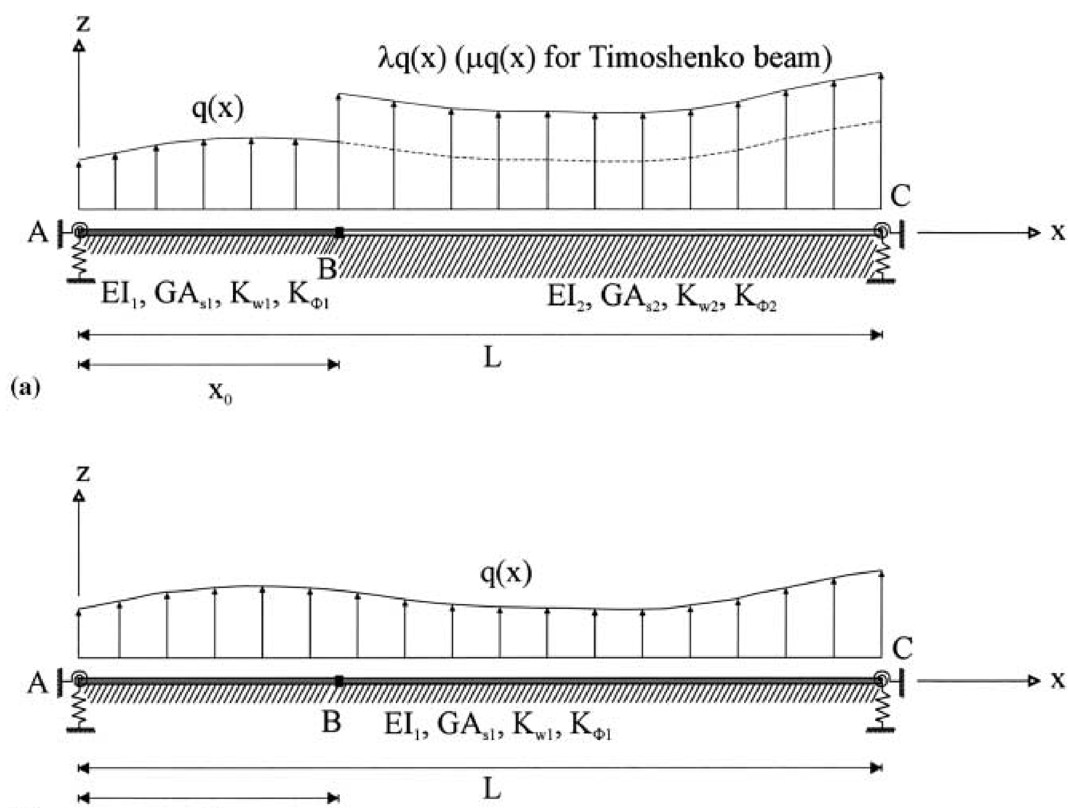

Fig. 3a, b. Comparison between deflections of two beams on elastic foundation constant changes abruptly as well. However, the beam II of Fig. 3b has uniform mechanical properties, and its elastic foundation is uniform. Beam II bears a distributed force $q(x)$. For beam I, the distributed force is $\lambda q(x)$ for $x_0 \leq x \leq L$. The governing differential equation of beam I after replacing $q(x)$ in (21) by $q(x)[1-(\lambda-1)H(x-x_0)]$ is

$$\begin{aligned}&\frac{\bar{d}^4 w_I(x)}{dx^4}+\frac{K_w}{EI}\left[1+\left(\frac{\mu}{\lambda}-1\right)H(x-x_0)\right]w_I(x)\\&\quad=\frac{q(x)}{EI}+\left(\frac{1}{\lambda}-1\right)\frac{K_t\Delta}{EI}\delta(x-x_0)+\left(\frac{1}{\lambda}-1\right)\frac{K_r\Theta}{EI}\delta^{(1)}(x-x_0)\\&\qquad+\Theta\delta^{(2)}(x-x_0)+\Delta\delta^{(3)}(x-x_0)\ .\end{aligned} \tag{50}$$

Assuming $\lambda=\mu$ in (50) yields

$$\begin{aligned}\frac{\bar{d}^4 w_I(x)}{dx^4}+\frac{K_w}{EI}w_I(x)=\frac{q(x)}{EI}+\left(\frac{1}{\lambda}-1\right)\frac{K_t\Delta}{EI}\delta(x-x_0)+\left(\frac{1}{\lambda}-1\right)\frac{K_r\Theta}{EI}\delta^{(1)}(x-x_0)\\+\Theta\delta^{(2)}(x-x_0)+\Delta\delta^{(3)}(x-x_0)\ .\end{aligned} \tag{51}$$

The governing differential equation of beam II is obtained by substituting $\lambda=\mu=1$ in (21)

$$\frac{\bar{d}^4 w_{II}(x)}{dx^4}+\frac{K_w}{EI}w_{II}(x)=\frac{q(x)}{EI}+\Theta\delta^{(2)}(x-x_0)+\Delta\delta^{(3)}(x-x_0)\ . \tag{52}$$

It is seen that Eqs. (51) and (52) have different forcing terms. Therefore, in general, the deflections of beams I and II are not equal. However, because both beams have similar boundary conditions and jump discontinuities, it can be concluded that

$$w_I(x)=w_{II}(x)\quad \forall x\in[0,L]\ , \tag{53}$$

provided that

$$\mu=\lambda,\quad \Delta=\Theta=0\ . \tag{54}$$

This result is not intuitively obvious; it is very difficult to reach this conclusion unless we have the governing differential equation of beams I and II in the space of generalized functions.

Example 3
In this example, the two beams of Fig. 3 are reconsidered. Assume that both beams are Timoshenko beams. The governing equilibrium equations of beam I may be written as (replace $q(x)$ in (38) by $q(x)[1-(\mu-1)H(x-x_0)]$)

$$\begin{aligned}&\frac{GA_s}{EI}\left[1+\left(\frac{\mu}{\lambda}-1\right)H(x-x_0)\right]\frac{\bar{d}w_I^T(x)}{dx}-\frac{\bar{d}^2\Phi_I(x)}{dx^2}\\&\quad+\left\{\frac{GA_s}{EI}\left[1+\left(\frac{\mu}{\lambda}-1\right)H(x-x_0)\right]+\frac{K_\Phi}{EI}\left[1+\left(\frac{\eta}{\lambda}-1\right)H(x-x_0)\right]\right\}\Phi_I(x)\\&\quad=\left\{\frac{GA_s}{EI}\left[1+\frac{1}{2}\left(\frac{\mu}{\lambda}-1\right)\right]\Delta^T-\frac{K_r\Theta^T}{EI}\left(\frac{1}{\lambda}-1\right)\right\}\delta(x-x_0)-\Theta^T\delta^{(1)}(x-x_0),\\&\frac{\bar{d}^2 w_I^T(x)}{dx^2}+\frac{\bar{d}\Phi_I(x)}{dx}-\frac{K_w}{GA_s}\left[1+\left(\frac{\xi}{\mu}-1\right)H(x-x_0)\right]w_I(x)+\frac{q(x)}{GA_s}\\&\quad=\frac{K_t\Delta^T}{EI}\left(\frac{1}{\mu}-1\right)\delta(x-x_0)+\Delta^T\delta^{(1)}(x-x_0)\ .\end{aligned} \tag{55}$$

Similarly, the governing equilibrium equations of beam II are obtained by substituting $\lambda=\mu=\xi=\eta=1$

$$\frac{GA_s}{EI}\frac{\bar{\mathrm{d}}w_{\mathrm{II}}^T(x)}{\mathrm{d}x} - \frac{\bar{\mathrm{d}}^2\Phi_{\mathrm{II}}(x)}{\mathrm{d}x^2} + \frac{GA_s + K_\Phi}{EI}\Phi_{\mathrm{II}}(x) = \frac{GA_s}{EI}\Delta^T\delta(x-x_0) - \Theta^T\delta^{(1)}(x-x_0),$$
$$\frac{\bar{\mathrm{d}}^2 w_{\mathrm{II}}^T(x)}{\mathrm{d}x^2} + \frac{\bar{\mathrm{d}}\Phi_{\mathrm{II}}(x)}{\mathrm{d}x} - \frac{K_w}{GA_s}w_{\mathrm{II}}(x) + \frac{q(x)}{GA_s} = \Delta^T\delta^{(1)}(x-x_0) \ . \tag{56}$$

Now, suppose that for beam I, $\lambda = \mu = \xi = \eta$. Hence, from (55) we have

$$\frac{GA_s}{EI}\frac{\bar{\mathrm{d}}w_{\mathrm{I}}^T(x)}{\mathrm{d}x} - \frac{\bar{\mathrm{d}}^2\Phi_{\mathrm{I}}(x)}{\mathrm{d}x^2} + \frac{GA_s + K_\Phi}{EI}\Phi_{\mathrm{I}}(x)$$
$$= \left\{\frac{GA_s}{EI}\Delta^T - \frac{K_r\Theta^T}{EI}\left(\frac{1}{\alpha} - 1\right)\right\}\delta(x-x_0) - \Theta^T\delta^{(1)}(x-x_0),$$
$$\frac{\bar{\mathrm{d}}^2 w_{\mathrm{I}}^T(x)}{\mathrm{d}x^2} + \frac{\bar{\mathrm{d}}\Phi_{\mathrm{I}}(x)}{\mathrm{d}x} - \frac{K_w}{GA_s}w_{\mathrm{I}}(x) + \frac{q(x)}{GA_s} = \frac{K_t\Delta^T}{EI}\left(\frac{1}{\mu} - 1\right)\delta(x-x_0) + \Delta^T\delta^{(1)}(x-x_0) \ . \tag{57}$$

Because beams I and II have the same boundary conditions, from (56) and (57) it can be concluded that $w_{\mathrm{I}}(x) = w_{\mathrm{II}}(x)\ \forall x \in [0, L]$, provided that $\Theta^T = \Delta^T = 0$. This is an interesting result which is not intuitively obvious.

5 Conclusions

In this article, the governing equilibrium equations of an Euler–Bernoulli beam with jump discontinuities on a Winkler elastic foundation, whose constant changes abruptly, is obtained in the space of generalized functions. It is shown that the governing equilibrium equation of the Euler–Bernoulli beam can always be expressed in terms of a single displacement function $w = w(x)$. It is observed that the boundary-value problem describing the bending of the beam has a more compact form in the space of generalized functions than it does in the space of classical functions.

The governing equilibrium equations of a Timoshenko beam with jump discontinuities on a two-parameter elastic foundation with abrupt changes in foundation constants are found in the space of generalized functions. It is demonstrated that the governing equilibrium equations of the Timoshenko beam can always be written in terms of a single displacement function $w = w(x)$ and a single rotation function $\Phi = \Phi(x)$. As is true for Euler–Bernoulli beams, the boundary-value problem describing the bending of the Timoshenko beam has a more compact form in the space of generalized functions than in the space of classical functions. Three examples are solved to show the usefulness of distribution theory in finding analytical solutions for beams with jump discontinuities on elastic foundations. In Example 2 and 3, some interesting qualitative results are found using the governing equilibrium equations of discontinuous beams in the space of generalized functions.

Appendix
The Schwartz–Sobolev theory of distributions

For the sake of self-containedness, in this appendix some definitions and theorems from the distribution theory of Schwartz–Sobolev are given. For more details, the reader may refer to [20–25].

Definition 1: The *Heaviside function* $H(x - x_0)$ is defined as

$$H(x-x_0) = \begin{cases} 0; & x < x_0 \\ 1; & x > x_0 \ . \end{cases} \tag{A1}$$

This function has a jump discontinuity at $x = x_0$. Its value at this point is usually taken to be one-half. The reason for choosing one-half at the point of discontinuity is explained later in this appendix. The Heaviside function is very useful for problems that involve functions with jump discontinuities. As an example, consider a function $f(x)$ which is continuous everywhere on the real line except at the point $x = x_0$, where it has a jump discontinuity, i.e.

$$f(x) = \begin{cases} f_1(x); & x < x_0 \\ f_2(x); & x > x_0 \end{cases} . \quad \text{(A2)}$$

This function can be written as

$$f(x) = f_1(x) + [f_2(x) - f_1(x)]H(x - x_0) \ . \quad \text{(A3)}$$

Definition 2: *Test functions* are real-valued functions $\varphi(x)$ that satisfy the following two conditions: (1) φ is infinitely smooth, i.e. $\varphi \in C^\infty$; (2) φ has compact support; i.e. φ is zero outside a finite interval. The space of all test functions is denoted by D.

Definition 3: A *distribution* (or *generalized function*) is a continuous linear functional on the space D of test functions. The set of all distributions is denoted by D'; this space is itself linear and is called the dual space of D. The space D' forms a generalization of the class of locally integrable functions because it contains functions that are not locally integrable.

A locally integrable function is integrable in the Lebesgue sense over every finite interval. Every locally integrable function $f(x)$ generates a distribution by means of the following formula:

$$\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(x)\varphi(x)\mathrm{d}x \ . \quad \text{(A4)}$$

This is called a *regular distribution*. All other distributions are called *singular distributions*.

Definition 4: Two generalized functions f and g in D' are said to be equal if

$$\langle f, \varphi \rangle = \langle g, \varphi \rangle \quad \forall \varphi \in D \ . \quad \text{(A5)}$$

Definition 5: The *Dirac delta function* is a singular distribution and is defined as

$$\langle \delta(x - x_0), \varphi(x) \rangle = \varphi(x_0) \ . \quad \text{(A6)}$$

Corollary: Suppose that f is a generalized function and $f(x_0)$ is defined. Then,

$$\langle \delta(x - x_0) f(x), \varphi(x) \rangle = \langle \delta(x - x_0), f(x)\varphi(x) \rangle = f(x_0)\varphi(x_0) \ . \quad \text{(A7)}$$

Hence,

$$\delta(x - x_0) f(x) = f(x_0)\delta(x - x_0) \ . \quad \text{(A8)}$$

Definition 6: The nth *distributional derivative* of a generalized function $f(x)$ is defined as

$$\left\langle f^{(n)}(x), \varphi(x) \right\rangle = \left\langle f(x), (-1)^n \varphi^{(n)}(x) \right\rangle, \quad \forall \varphi \in D \ . \quad \text{(A9)}$$

The nth distributional derivative of the delta function is, therefore, defined as

$$\left\langle \delta^{(n)}(x - x_0), \varphi(x) \right\rangle = \left\langle \delta(x - x_0), (-1)^n \varphi^{(n)}(x) \right\rangle = (-1)^n \varphi(x_0) \ . \quad \text{(A10)}$$

Corollary: The Dirac delta function is the first distributional derivative of Heaviside's unit step function. Thus,

$$\begin{aligned} \left\langle H^{(1)}(x - x_0), \varphi(x) \right\rangle &= \left\langle H(x - x_0), -\varphi^{(1)}(x) \right\rangle = -\int_{x_0}^{+\infty} \varphi^{(1)}(x)\mathrm{d}x = \varphi(x_0) \\ &= \langle \delta(x - x_0), \varphi(x) \rangle \ . \end{aligned} \quad \text{(A11)}$$

Hence,

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}x}H(x-x_0)=\delta(x-x_0)\ , \tag{A12}$$

where a bar over the distributional differentiation symbol distinguishes it from classical differentiation. Now we show why the value of Heaviside's function at the discontinuity point is assumed to be one-half. It is known that the Dirac delta function is an even function. Hence,

$$\int_{x_0}^{+\infty}\delta(x-x_0)\mathrm{d}x=\frac{1}{2}=\int_{-\infty}^{+\infty}\delta(x-x_0)H(x-x_0)\mathrm{d}x=H(x-x_0)\big|_{x=x_0}\ . \tag{A13}$$

Theorem 1: If f is a generalized function and if $f(x_0)$ is defined, then,

$$f(x)\delta^{(1)}(x-x_0)=f(x_0)\delta^{(1)}(x-x_0)-f^{(1)}(x_0)\delta(x-x_0)\ . \tag{A14}$$

Similarly

$$\begin{aligned}f(x)\delta^{(n)}(x-x_0)&=(-1)^nf^{(n)}(x_0)\delta(x-x_0)+(-1)^{n-1}nf^{(n-1)}(x_0)\delta^{(1)}(x-x_0)\\&\quad+(-1)^{n-2}\frac{n(n-1)}{2!}f^{(n-2)}(x_0)\delta^{(2)}(x-x_0)+\cdots+f(x_0)\delta^{(n)}(x-x_0)\ .\end{aligned} \tag{A15}$$

Corollary: The nth distributional of the product of a function $f(x)$ and Heaviside's function may be expressed as

$$\begin{aligned}[f(x)H(x-x_0)]^{(n)}&=f^{(n)}(x_0)H(x-x_0)+f^{(n-1)}(x_0)\delta(x-x_0)\\&\quad+f^{(n-2)}(x_0)\delta^{(1)}(x-x_0)+\cdots+f(x_0)\delta^{(n-1)}(x-x_0)\ .\end{aligned} \tag{A16}$$

Definition 7: The space of distributions D'_R having their supports bounded on the left is called the space of *right-sided distributions*, $D'_R\subset D'$ (proper subspace).

Definition 8: Assume that f is a locally integrable function with the following properties:

(1) $f(x)=0\quad\text{for}\ -\infty<x<T$.
(2) $\exists c\in\mathbb{R}$ such that $f(x)\mathrm{e}^{-cx}$ is absolutely integrable over $-\infty<x<+\infty$.

The Laplace transformation is an operator L that assigns a function $F(s)$ of the complex variable s to each function f that satisfies the above conditions. *The Laplace transform of* $f(x)$ is defined by

$$L\{f(x)\}=F(s)=\int_{-\infty}^{+\infty}f(x)\mathrm{e}^{-sx}\,\mathrm{d}x=\langle f(x),\mathrm{e}^{-sx}\rangle\ . \tag{A17}$$

The Laplace transform of Heaviside's function, the delta function and its distributional derivatives, can be obtained directly from the definition as

$$L\{H(x-x_0)\}=\frac{1}{s}\mathrm{e}^{-sx_0},\quad L\{\delta(x-x_0)\}=\mathrm{e}^{-sx_0},\quad L\left\{\delta^{(k)}(x-x_0)\right\}=s^k\mathrm{e}^{-sx_0}\ . \tag{A18}$$

Theorem 2: If $f^{(k)}(x)$ exists and is continuous for all x, then

$$L\left\{f^{(k)}(x)\right\}=s^kF(s)-f(0^+)s^{k-1}-f^{(1)}(0^+)s^{k-2}-\cdots-f^{(k-1)}(0^+)\ . \tag{A19}$$

6
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