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The mechanics of self-similar and self-affine fractal cracks

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Abstract. In this paper we study the mechanical attributes of the fractal nature of fracture surfaces. The structure of stress and strain singularity at the tip of a fractal crack, which can be self-similar or self-affine, is studied. The three classical modes of fracture and the fourth mode of fracture are discussed for fractal cracks in two-dimensional and three- dimensional solid bodies. It is discovered that there are six modes of fracture in fractal fracture mechanics. The J-integral is shown to be path-dependent. It is explained that the proposed modified J-integrals in the literature that are argued to be path-independent are only locally path-independent and have no physical meaning. It is conjectured that a fractal J-integral should be the rate of potential energy release per unit of a fractal measure of crack growth. The powers of stress and strain singularities at the tip of a fractal crack in a strain-hardening material are calculated. It is shown that stresses and strains have weaker singularities at the tip of a fractal crack than they do at the tip of a smooth crack.

Key words: Fractal crack, fractal geometry, fractal fracture.

1. Introduction

Many of nature's irregular and fragmented patterns exhibit a much greater level of complexity than can be described with standard Euclidean geometry. Such features have eluded the application of classical mathematics for a long time. The nature of fractals is reflected in the word itself, coined by Mandelbrot (1983) from the Latin verb frangere, 'to break,' and the related adjective fractus, 'irregular and fragmented.' Before Mandelbrot, mathematicians believed that most of the patterns of nature were far too complex, irregular, fragmented, and amorphous to be described mathematically. Mandelbrot found order in places where others saw only chaos. The fractal universe has infinite precision and is infinitely complex. Usually noninteger, a fractal dimension indicates the extent to which the fractal object fills the embedding Euclidean space.

As in many other fields of science and engineering, fractal geometry has found applications in various branches of solid mechanics, and mainly in fracture mechanics and contact mechanics. Studies in fractal fracture mechanics started in the mid 1980s after the experimental study of Mandelbrot and his coworkers (Mandelbrot et al., 1984). They did the first experimental study in fractal fracture mechanics and found that the fracture surfaces of steel are fractals. Since then many experimental investigations have been done. For example, Saouma et al. (1990) and Saouma and Barton (1994) showed experimentally that fracture surfaces of concrete are fractals.

Experimentalists have observed fractality in the fracture surfaces of many engineering materials. Like all other natural objects, cracks can be described by fractals in a finite range of length scales, $\ell_0 \leq r \leq \ell_1$ (Cherepanov et al., 1995; Balankin, 1997). Here the lower cutoff ℓ_0 is a function of the micromechanical characteristics of the material and the upper cutoff ℓ_1 is related to the geometric size of the cracked specimen, the crack size, and so forth. To date, most theoretical work in fracture mechanics is based on the fundamental assumption that cracks have smooth surfaces. This assumption makes fracture mechanics problems mathematically tractable. But if experiments show that cracks are fractals within a wide range of length scales, what are the consequences of this fractality on the behavior of cracks?

In most papers, the only difference between a fractal crack and a corresponding smooth crack is assumed to be the difference between the actual lengths of the two cracks; it is usually assumed that the specific surface energy is the same for fractal and smooth cracks. Borodich (1992, 1994, 1997, 1999) noticed that this assumption leads to the contradiction that fractal cracking is impossible because the actual length of a fractal curve is infinity. He introduced the concept of a specific surface energy for a unit measure of fractal crack.

An interesting theoretical problem in fractal fracture mechanics is the change of the order of stress singularity at the crack tip. Mosolov (1991a) studied this problem for the first time. Utilizing Griffith's criterion for a mode I self-similar fractal crack, he found the following asymptotic stress distribution at the tip of the crack:

$$\sigma(r, \theta) \sim r^{\frac{D-2}{2}},$$

where $1 \leq D \leq 2$ is the fractal dimension of the crack trajectory. Gol'dshteĭn and Mosolov (1991, 1992) obtained the same order of stress singularity using a cascade energy transfer method. Balankin (1997) found a similar result for self-affine cracks using a dimensional analysis method. Self-affine fractal cracks were studied by Balankin and his coworkers (Balankin, 1996a, b, 1997; Balankin et al., 1996, 1997; Balankin and Susarey, 1996, 1999). It is noteworthy that the upper cutoff of fractality ℓ_1 for a self-affine crack is not necessarily equal to the crossover length ε_x (Balankin and Susarey, 1999). Mode II and mode III self-similar and self-affine fractal cracks were studied by Yavari et al. (2000) and Yavari (2000), who pointed out that stresses have the same order of singularity for all classical modes of fracture.

Mosolov and Borodich (1992) and Mosolov (1993) proposed an explanation for crack growth in compression and found that stresses are singular at the tip of a fractal crack in an infinite medium under a uniform state of compressive stress parallel to the axis of the crack. While they introduced an interesting idea, unfortunately the order of stress singularity they found was incorrect, as shown by Balankin (1997). Balankin (1997) investigated crack growth in compression for both two-dimensional and three-dimensional cracked bodies. Unfortunately, his results for the power of stress singularities were not correct, as was noted by Yavari et al. (2000). Inspired by these works (Mosolov and Borodich, 1992; Mosolov, 1993; Balankin, 1997), Yavari et al. (2000) and Yavari (2000) introduced a fourth mode of fracture in fractal fracture mechanics. They realized that the stress distribution around the tip of a fractal crack in an infinite medium under a uniform (either compressive or tensile) state of stress along the axis of the crack cannot be expressed in terms of the three classical modes of fracture. In other words, they showed that the fractality of fracture surfaces dictates the existence of a new mode of fractal fracture for fractal cracks. In this paper, we will show that there are two other modes of fractal fracture.

Recently, Yavari et al. (2002) studied fractal cracks in micropolar elastic solids. After generalizing Griffith's fracture theory for smooth and fractal cracks in micropolar solids, they showed that the orders of stress and couple-stress singularity at the tip of a fractal crack are equal. Yavari (2002) generalized Barenblatt's cohesive fracture theory for fractal cracks. He defined a fractal cohesive stress and showed that the fractal cohesive fracture theory is equivalent to Borodich's generalization of Griffith's theory.

In this paper we study some of the ramifications of the fractality of fracture surfaces in fracture mechanics. The order of stress singularity at the tip of a fractal crack is obtained. Modes of fractal fracture are discussed in detail. The J-integral for fractal cracks is shown to be path-dependent in general. It is pointed out that the proposed modified J-integrals in the literature are only locally path-independent and have no physical meaning. Finding a fractal J-integral remains to be an open problem.

The paper is organized as follows. Section 2 discusses traction on fractal surfaces. In Section 3 the structure of stress distribution around the tip of a fractal crack is investigated. Section 4 discusses modes of fractal fracture. In Section 5, three-dimensional solid bodies with fractal cracks are studied. In Section 6, the path dependence of J-integral for fractal cracks is studied. HRR singularity for fractal cracks in a strain-hardening material is investigated in Section 7. Conclusions are given in Section 8. For the sake of self-containedness, in the Appendix some basic concepts of fractal geometry are reviewed.

2. Traction on a fractal surface

Consider a two-dimensional solid body with a fractal crack. Stress vector (traction) is defined for all points except points that lie on the crack surfaces, which are of measure (2-measure) zero. Therefore, still we can talk about traction and stress tensor for points that do not lie on crack surfaces. For a fractal crack, classical zero traction boundary condition on crack surfaces is not valid because traction, in its classical sense, does not exist on a fractal surface. As was mentioned in Yavari (2002), on a fractal crack with fractal dimension D, D-fractal traction is zero. Here D-fractal pseudo-traction $\mathbf{t}^D$ at a material point $\mathbf{x}$ is defined as

$$\mathbf{t}^D(\mathbf{x}) = \lim_{\Delta m_D \to 0} \frac{\Delta \mathbf{F}}{\Delta m_D}, \tag{1}$$

where m_D is a D-fractal measure. A solid body partitioned into two sub-bodies by a fractal surface is shown in Figure 1a and the system of internal forces is shown in Figure 1b. As was explained in Yavari (2002), Eq. (1) is a naive generalization of traction and this is why we call it 'D-fractal pseudo-traction'. For more details see Yavari (2002). We should mention that there are generalizations of integral theorems for domains with fractal boundaries (see Harrison and Norton, 1991; Harrison, 1994; Borodich and Volovikov, 2000, and references therein). In all these works, it is assumed that the integrand is a well-defined vector field. But here we have a more fundamental problem: there is no generalization of the concept of stress tensor for a material point on a fractal surface. In what follows, we study the mechanics of fractal cracks by some engineering arguments. A rigorous mathematical theory of fractal fracture mechanics remains to be developed in future.

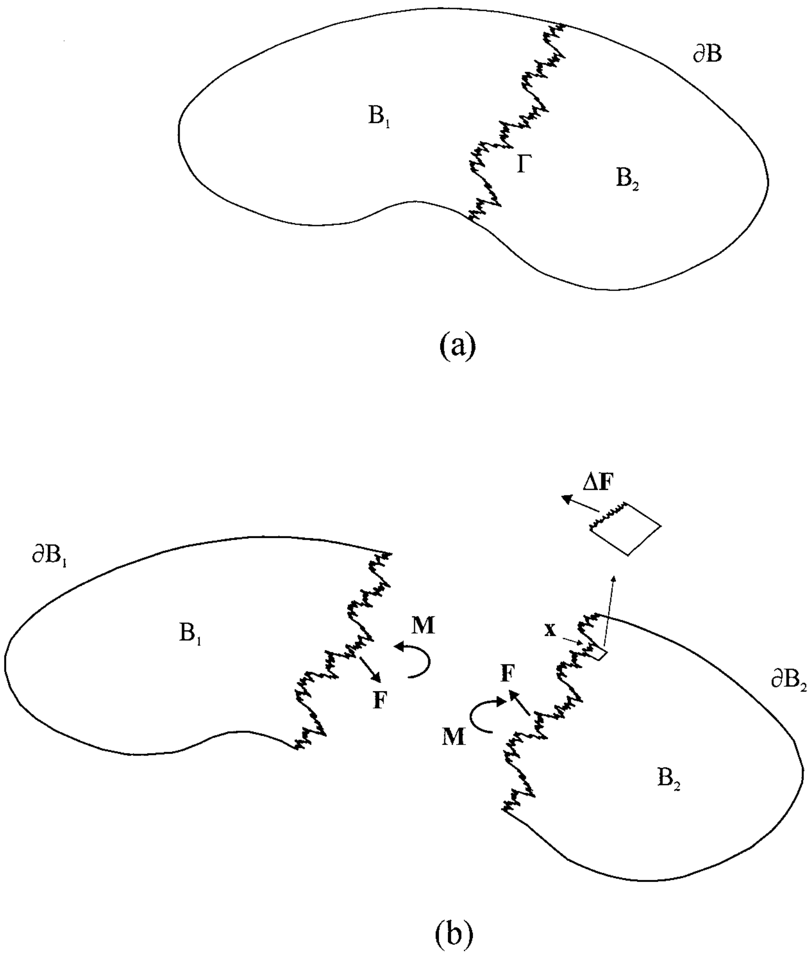

Figure 1. (a) A fractal curve Γ that partitions a solid body B into two bodies B_1 and B_2, (b) the internal force system.

3. Stress distribution around the tip of a fractal crack

In this section, we investigate the radial variation of stresses at the tip of a fractal crack. The power of stress singularity at the tip of a fractal crack is obtained using a modified Griffith's criterion and dimensional analysis considerations.

For obtaining stress and displacement fields around the tip of a fractal crack an elasticity problem with fractal boundaries should be solved. One difficulty in solving this problem is applying stress boundary conditions on crack faces because stress vector is not defined on a fractal surface. Because of the difficulties encountered in solving differential equations with nonsmooth boundaries, this problem is best attacked by an indirect method. First a singular stress field is assumed and then the order of singularity is found using an asymptotic analysis. However, the angular variation of stresses cannot be found using this method. It should be noted that what follows is applicable for only some very special problems; for most problems in fractal fracture mechanics this method is not applicable and the corresponding boundary-value problem has to be solved. However, what we present here for some special cases gives us an understanding of the effects of the fractality of cracks.

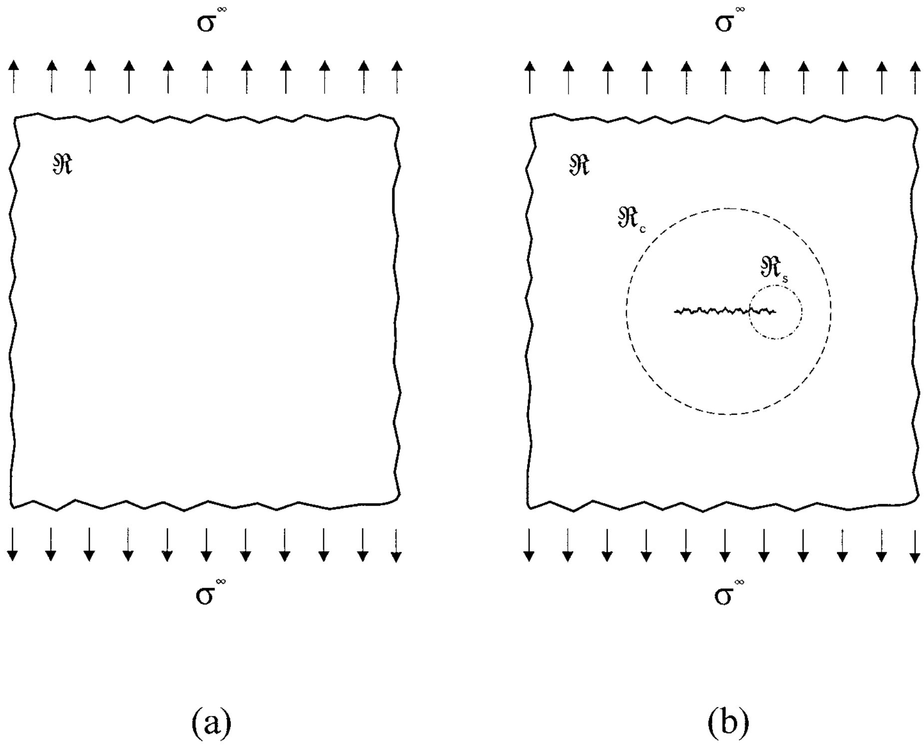

Figure 2. (a) An infinite uncracked solid under uniaxial tension at infinity, (b) an infinite solid with a finite fractal crack perpendicular to the applied stresses and the crack-effect zone.

Here we assume that a fractal crack is a deterministic mathematical fractal curve (or surface). It is also assumed that a single fractal with fractal dimension D (or Hurst exponent H) models the whole crack. In a two-dimensional cracked solid, the two end points of the fractal curve are the crack tips. The crack axis is defined to be line that connects these two points. In the case of a three-dimensional cracked solid, the crack edge is a fractal curve with fractal dimension between one and two.

Consider an infinite solid body $\Re$ under a uniform state of stress σ^{∞} at infinity (Figure 2a). Stress has a uniform distribution at all points. Now suppose that a fractal crack with apparent length 2a and the divider (latent) fractal dimension D_D is formed (a brief description of fractal geometry and the concepts we use in this paper are given in the Appendix). For the system shown in Figure 2b, stress distribution is almost uniform at all points except for points in a finite region Ω_c (crack-effect zone) around the crack. There exists a disk $\Re_c$ such that $\Re_c \subseteq \Omega_c$.

For a smooth crack, surface energy required for crack propagation is proportional to the length (area) of the newly created free surfaces. In the case of a fractal crack the true length (area) of new free surfaces should be considered. Because the true length (area) of a fractal curve (surface) is infinity, a fractal measure should be utilized. The surface energy required to create the fractal crack is:

$$U_s = 2t\gamma_f(D_D)m_{D_D}, \tag{2}$$

where t is the plate thickness, $\gamma_f = \gamma_f\ (D_D)$ is the specific surface energy per unit (divider) fractal measure and m_{D_D} is the latent fractal measure and is proportional to a^{D_D} (see the

Appendix). The specific surface energy per unit fractal measure was defined by Borodich (1992, 1994, 1997, 1999) and has the dimension $[\gamma_f] = FL^{-D_D}$, where F and L are dimensions of force and length, respectively. There are two important problems that should be carefully explained in Borodich's generalization of Griffith's criterion: (1) it should be noted that γ_f is not a material property. In general, it is possible to have cracks with different fractal dimensions in the same material. Therefore, in Equation (2) γ_f cannot be a material property; it depends on both the material and the fractal dimension of the fractal crack. (2) 'fractal measure' is an ambiguous term. There are different definitions of dimension and consequently these different dimensions have different corresponding measures. For self-similar fractals all different dimensions have the same value and hence the corresponding measures they define are identical. Therefore, for self-similar fractals 'fractal measure' is not an ambiguous term. However, this is not the case for self-affine fractals; different definitions of dimension give completely different values. Obviously, the relevant fractal dimension for calculating the surface energy of a fractal crack is the divider (latent) fractal dimension. Therefore, specific surface energy should be defined per unit divider fractal measure, although it can be defined for other fractal measures as well.

It is known that the true length of a fractal crack $\tilde{a}$ is larger than its nominal length and is proportional to a^{D_D}, i.e., $\tilde{a} \propto a^{D_D}$.[1] Hence,

$$U_s \propto a^{D_D} = \begin{cases} a^{1/H} & \frac{1}{2} \leq H < 1, \\ a^2 & 0 < H \leq \frac{1}{2}, \end{cases} \tag{3}$$

where H is the Hurst (roughness) exponent of the self-affine crack. Now suppose that one of the crack tips (for example, the right one) moves by an infinitesimal (apparent) amount δa. The required surface energy for this infinitesimal crack growth is:

$$\delta U_e \propto \delta \left(a^{D_D}\right) \tag{4}$$

The strain energy of the system shown in Figure 2b can be written as:

$$U_e = \int_{\Re} \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\ \mathrm{d}A = \int_{\Re-\Re_c} \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\ \mathrm{d}A + \int_{\Re_c} \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\ \mathrm{d}A. \tag{5}$$

The change in strain energy due to this virtual crack growth may be written as:

$$\delta U_e = \delta \int_{\Re} \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\ \mathrm{d}A \cong \delta \int_{\Re_c} \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\ \mathrm{d}A, \tag{6}$$

because strain energy change in $\Re_c$ is dominant. Suppose that $\Re$ is made of an isotropic, linear elastic material. At the crack tip the following asymptotic stresses and strains are assumed:

$$\sigma_{ij}(r,\theta) = K_I^f r^{-\alpha} f_{ij}(\theta), \tag{7a}$$

$$\varepsilon_{ij}(r,\theta) = K_I^f r^{-\alpha} C_{ijkl} f_{kl}(\theta), \tag{7b}$$

where K_f^I is the mode I fractal stress intensity factor (with dimension $[K_I^f] = FL^{\alpha-2}$), C_{ijkl} is the fourth-order elasticity tensor, and $\alpha = \alpha\ (D_D)$. The above asymptotic expressions are valid only in a disk $\Re_s$ with radius r_s (or more precisely, in a region with area proportional to r_s^2). Obviously, $r_s = r_s\ (a, D_D)$, and according to dimensional analysis we must have $r_s = a\Phi(D_D)$, i.e., $r_s \propto a$ ($r_s = ka$). The strain energy change in $\Re_s$ is much greater than that of $\Re_c - \Re_s$ and hence $\Re_c$ can be replaced by $\Re_s$ in Equation (6), i.e.:

[1] Actually the true length of a fractal is infinity. By $\tilde{a} \propto a^{D_D}$ we mean that the lengths of all the prefractal cracks are proportional to a^{D_D}.

$$\delta U_e = \delta \left(\int_{\Re_c - \Re_s} \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} \, \mathrm{d}A + \int_{\Re_s} \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} \, \mathrm{d}A \right) \cong \delta \int_{\Re_s} \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} \, \mathrm{d}A. \tag{8}$$

The strain energy release due to an infinitesimal crack growth of apparent length δa may be expressed as:

$$\delta U_e \propto \delta(r_s^{-\alpha} r_s^{-\alpha} r_s^2) \propto \delta(a^{2-2\alpha}). \tag{9}$$

According to Griffith's (1920, 1924) criterion $\delta U_e = \delta U_s$; and hence from (4) and (9) we obtain:

$$\alpha = \frac{2 - D_D}{2} \quad 1 \le D_D \le 2, \tag{10}$$

in terms of D_D and

$$\alpha = \begin{cases} \dfrac{2H - 1}{2H} & \tfrac{1}{2} \le H \le 1, \\ 0 & 0 < H \le \tfrac{1}{2} \end{cases} \tag{11}$$

in terms of the Hurst exponent. As can be seen for a self-affine crack, stresses and strains are singular at the crack tip only if $\frac{1}{2} \le H \le 1$. This method is applicable to mode II and mode III fractal cracks as well. It can easily be shown that stresses have the same order of singularity for mode II and mode III as they do for mode I fractal cracks.

It is worth mentioning that Equation (11) has an interesting conceptual implication. There is a stress singularity at the tip of a smooth crack; stresses are unbounded near the crack tip. This stress singularity is pathological because in reality stresses are finite at the crack tip. At first glance it might appear that this singularity is due to the (unrealistic) assumption that fracture surfaces are smooth. If this is true, we must have nonsingular stresses for fractal cracks. However, self-similar fractal cracks, although they have weaker stress singularities than smooth cracks do, introduce the same pathological problem: stresses are again unbounded at the tip of self-similar fractal cracks.

Sternberg and Muki (1967), with the hope of finding nonsingular stresses, analyzed the problem of a finite crack in a linear couple-stress medium. They showed that even in the presence of couple-stresses, stresses have an $r^{-1/2}$ singularity. Similar results were obtained by Atkinson and Leppington (1977) for smooth cracks in a micropolar medium. Yavari et al. (2002) showed that even for self-similar fractal cracks in a micropolar solid, stresses are singular at the crack tip, although they have a weaker singularity than stresses have at the tip of a smooth crack.

From the results in the literature for smooth cracks, and from what others and we have found for self-similar fractal cracks, we conclude that the singularity of stresses and strains at the tip of a smooth or self-similar fractal crack must be caused by something else. Eringen et al. (1977), showed that a smooth crack in a nonlocal elastic medium has no stress singularity and the maximum stress criterion is applicable. Therefore, stress singularity at the tip of a smooth crack appears because the constitutive equations are local. However, this does not mean that the local theory should be ruled out. As a matter of fact, nonlocal theories are very complicated and most problems in nonlocal elasticity can only be solved numerically.

For self-affine fractal cracks, it has been shown that for $H \le \frac{1}{2}$, stresses are not singular at the crack tip. This interesting result implies that for very rough self-affine cracks ($H \le \frac{1}{2}$), even when a local theory is utilized, stresses are finite at the crack tip and hence the maximum stress criterion may be applied. It is worth mentioning that for self-affine cracks in most engineering materials, H has been found to be close to 0.7. Therefore, the main cause of stress singularity is still that the constitutive equations are local. In the following we discuss fractal stress intensity factor and fractal driving force.

3.1. Fractal stress intensity factor

For a mode I fractal crack stresses have the following asymptotic distribution:

$$\sigma_{ij}^{f}(r,\theta) = K_I^f r^{\frac{D-2}{2}} f_{ij}(\theta, D) + \text{higher-order terms}, \tag{12a}$$

for a self-similar fractal crack and

$$\sigma_{ij}^{f}(r,\theta) = K_I^f r^{\frac{1-2H}{2H}} f_{ij}(\theta, H) + \text{higher-order terms}, \tag{12b}$$

for a self-affine fractal crack. This is a linear elastic problem and hence K_I^f must be proportional to σ^∞, i.e.,

$$K_I^f = \sigma^\infty \Phi(a, D) \tag{13}$$

A simple dimensional analysis dictates the following dependence of the fractal stress intensity factor on other parameters of the problem:

$$K_I^f = \chi(D)\sigma^\infty \sqrt{\pi a^{2-D}}, \tag{14}$$

where $\chi(D)$ is an unknown function of $D(\chi(D = 1) = 1)$. It is seen that when D tends to zero we recover the classical relation for the stress intensity factor. Wnuk and Yavari (2002) found an estimation for $\chi(D)$.

3.2. Driving force for a fractal crack

For a fractal crack with dimension D we define the fractal driving force G_f as the generalized force corresponding to the generalized displacement Δm_D, i.e.,

$$\Delta U_e = G_f \Delta m_D \quad \text{or} \quad G_f = \frac{\Delta U_e}{\Delta m_D}. \tag{15}$$

Consider the mode I fractal crack shown in Figure 2b. The fractal driving force G_f is a function of K_I^f, D, E, and ν. Here the effects of 'a' and σ^∞ are hidden in K_I^f. Thus

$$G_f = \Phi(K_I^f, E, D, \nu). \tag{16}$$

In Equation (16), independent variables are K_I^f and E. According to Buckingham's Π-theorem, we must have

$$\frac{G_f}{(K_I^f)^2 E^{-1}} = \psi(D, \nu) \quad \text{or} \quad G_f = \psi(D,\nu)\frac{\left(K_I^f\right)^2}{E}, \tag{17}$$

which is similar to the classical equation.

Here the following question might arise: how can a crack propagate when the orders of stress and strain singularity are not one-half? When a smooth crack propagates by an amount δa the strain energy release may be calculated as follows (Sih and Liebowitz, 1968):

$$\delta U_e = \int_0^{\delta a} \sigma_{yy}(\delta a - \xi, 0) u_y(\xi, \pi)\, \mathrm{d}\xi. \tag{18}$$

The strain energy release rate may be written as:

$$G = \lim_{\delta a \to 0} \frac{1}{\delta a} \int_0^{\delta a} \sigma_{yy}(\delta a - \xi, 0) u_y(\xi, \pi)\, \mathrm{d}\xi. \tag{19}$$

If an asymptotic stress distribution of the form $\sigma_{yy}(r, \theta) = K_I r^{-\alpha} f(\theta)$ is assumed, it can easily be shown that δU_e is nonzero for the crack growth if and only if $\alpha = \frac{1}{2}$. For fractal cracks, Equations (18) and (19) cannot be used to calculate the strain energy release. Suppose the fractal dimension of the crack is D and that the crack propagates by an amount δm_D. The fractal strain energy release rate (per unit of D-fractal measure) is:

$$G_f = \lim_{\delta m_D \to 0} \frac{1}{\delta m_D} \int_0^{\delta m_D} \sigma_y^D \, \mathrm{d}m_D, \tag{20}$$

where σ_y^D is the y-component of the D-fractal pseudo stress vector. This quantity can be nonzero even if $\alpha \neq \frac{1}{2}$.

4. Modes of fractal fracture

In classical linear elastic fracture mechanics there are three modes of fracture (Irwin, 1958). These are known as mode I (opening mode), mode II (shearing mode), and mode III (tearing mode). In the framework of linear elastic fracture mechanics, stress distribution around the tip of any crack can be written as the superposition of mode I, II, and III stresses, i.e.,

$$\sigma_{ij}(r, \theta) = K_I \bar{r}^{1/2} f_{ij}^{I}(\theta) + K_{II} r^{-1/2} f_{ij}^{II}(\theta) + K_{III} f^{-1/2} f_{ij}^{III}(\theta). \tag{21}$$

Consider the cracked plate shown in Figure 3a. The crack is in pure mode I, and a combination of mode I and mode II for $\alpha = \pi/2$ and $0 < \alpha < \pi/2$, respectively. There is a discontinuity at $\alpha = 0$ because for this value of α there is no interaction between the crack and the applied stresses. This is physically unacceptable. We know that when the applied far field stress is compressive we could have crack propagation. The problem of fracture in compression has been investigated by many researchers (see Bhattacharya et al., 1998, and references therein). This pathology in linear elastic fracture mechanics arises from the assumption that cracks are smooth surfaces. Mosolov and Borodich (1992) and Mosolov (1993) tried to explain this phenomenon considering fractal cracks. Now consider the same plate with a fractal crack (Figure 3b). We can define the axis of the fractal crack as the line segment that connects the two crack tips. Here again for $\alpha = \pi/2$ we have a pure mode I fractal crack and for $0 < \alpha < \pi/2$ a combination of mode I and mode II. Because of its irregularity, even for $\alpha = 0$ the crack interacts with the applied stresses and there is a stress singularity at the crack tip. However, this loading condition cannot be expresses in terms of classical modes of fracture. Therefore fractality of fracture surfaces dictates the existence of at least one more fracture mode (Yavari et al., 2000). Here we show that there are two other fractal modes of fracture, i.e., there are six modes of fracture in fractal fracture mechanics.

Consider an infinite medium under a uniform tensile stress σ^∞ at infinity. A smooth crack parallel to the tensile stresses does not interact with applied loads and does not change the stress and displacement of this system. In other words, the crack is not active and there is no

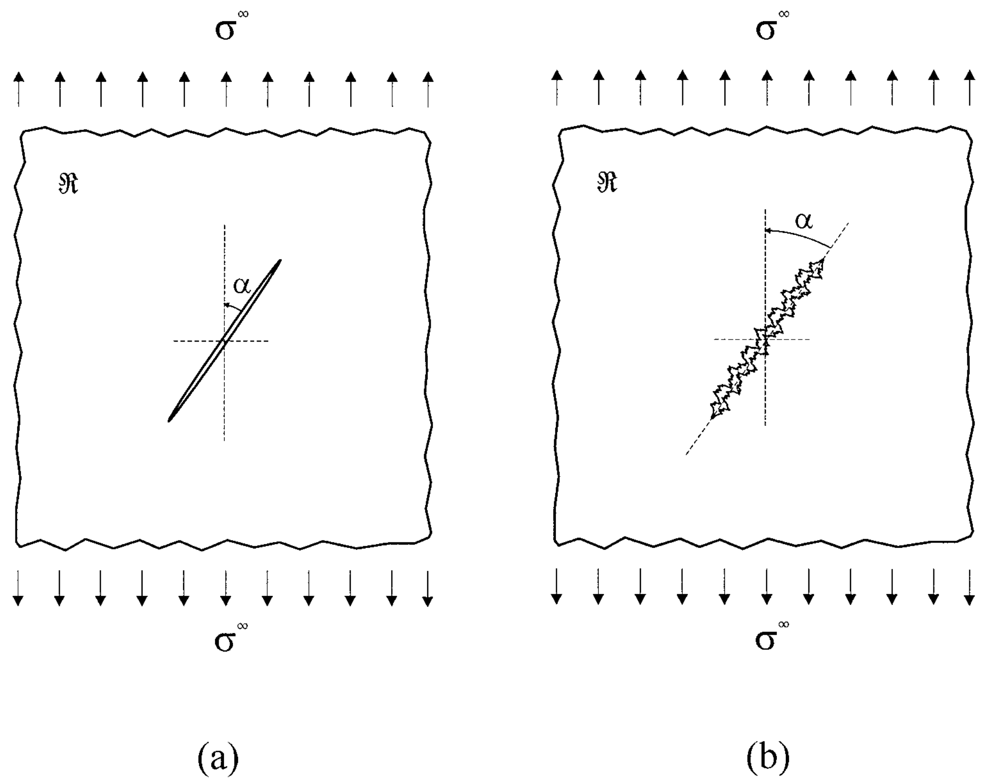

Figure 3. Mixed-mode smooth and fractal cracks.

stress singularity (Figure 4a). Now suppose that this crack is a self-affine fractal with Hurst (roughness) exponent H ($0 < H < 1$). In this case the crack affects the stress distribution in a strip of width proportional to a^H (see Figure 4b). Again the following singular stress and strain fields are assumed:

$$\sigma_{ij}(r,\theta) = K_{IV}^f r^{-\alpha} f_{ij}(\theta), \tag{22a}$$

$$\varepsilon_{ij}(r,\theta) = K_{IV}^f r^{-\alpha} C_{ijkl} f_{kl}(\theta), \tag{22b}$$

where K_{IV}^f is the fractal mode IV stress intensity factor and α is the order of stress singularity to be determined. The above asymptotic expressions are valid in a disk $\Re_s$ with radius proportional to a_H, i.e., $r_s \propto a^H$. When the crack propagates by an infinitesimal (apparent) amount δa, the strain energy is released in a zone with area proportional to $a^H \delta a$. Hence the strain energy release rate may be written as:

$$\delta U_s \cong \delta \int_{\Re_c} \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\, \mathrm{d}A \cong \delta \int_{\Re_s} \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}\, \mathrm{d}A \propto r_s^{-\alpha} r_s^{-\alpha} a^H \delta a. \tag{23}$$

We know that $r_s \propto a^H$, hence,

$$\delta U_e \propto a^{-2\alpha H + H}\delta \propto \delta(a^{1+H-2\alpha H}). \tag{24}$$

The surface energy required for this crack growth has the same form as the surface energy required for growth of a mode I fractal crack does. Thus:

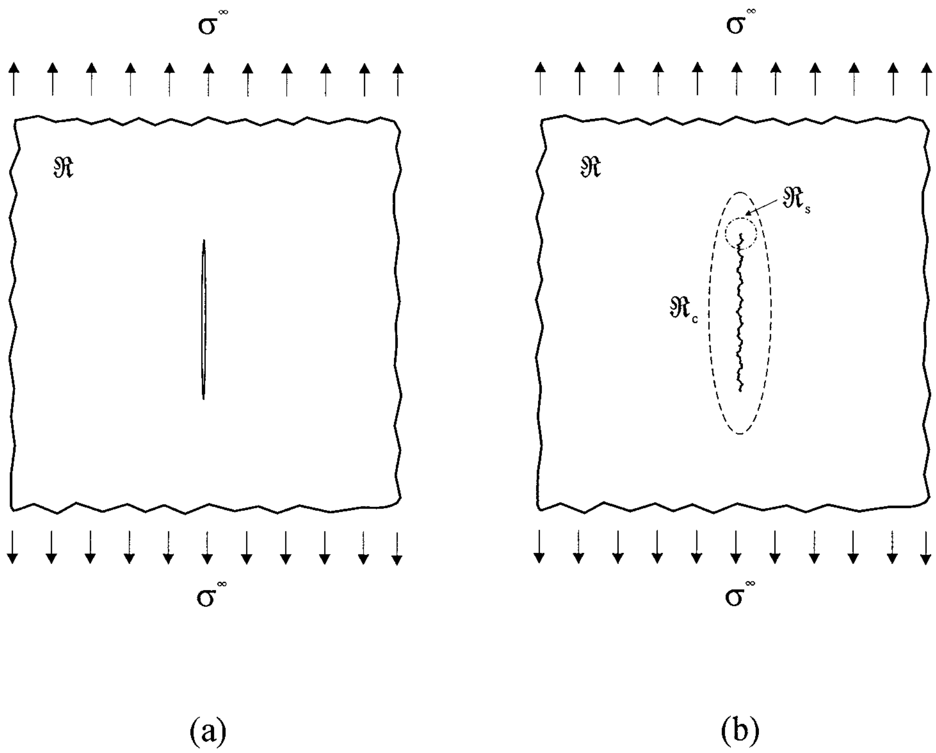

Figure 4. (a) An infinite solid with a finite smooth crack parallel to the applied stresses, (b) an infinite solid with a finite fractal crack parallel to the applied stresses and the crack-effect zone.

$$\delta U_s \propto \delta(a^{D_D}) = \begin{cases} \delta(a^{1/H}) & \frac{1}{2} \leq H < 1, \\ \delta(a^2) & 0 < H \leq \frac{1}{2}. \end{cases} \tag{25}$$

Applying Griffith's criterion yields:

$$\alpha = \begin{cases} \dfrac{H^2 + H - 1}{2H^2} & \dfrac{1}{g} = \dfrac{\sqrt{5}-1}{2} \leq H < 1, \\ 0 & 0 < H \leq \dfrac{1}{g} = \dfrac{\sqrt{5}-1}{2}, \end{cases} \tag{26}$$

where g is the Golden mean. As was shown by Yavari et al. (2000) and Yavari (2000), stresses have a weaker singularity in this mode than they do in the classical modes. However, for self-similar fractal cracks this mode introduces the same order of stress singularity as other modes do. As was noticed by Yavari et al. (2000) and Yavari (2000), existence of mode IV could make some of the single-mode problems of classical fracture mechanics mixed-mode problems in fractal fracture mechanics.

Now consider the smooth and fractal cracks shown in Figure 5. The state of stress at a material point on the crack edges can be completely specified by six independent stress components. For the case of a fractal crack, each of these independent stress components defines a mode of fracture. For a crack in xy-plane (for the fractal crack xy-plane is the nominal plane of the crack), σ_{zz}, σ_{yz}, and σ_{xz} correspond to modes I, II, and III, respectively. For the fractal crack, σ_{yy} corresponds to mode IV or in-plane axial mode. The stress component σ_{xx} defines

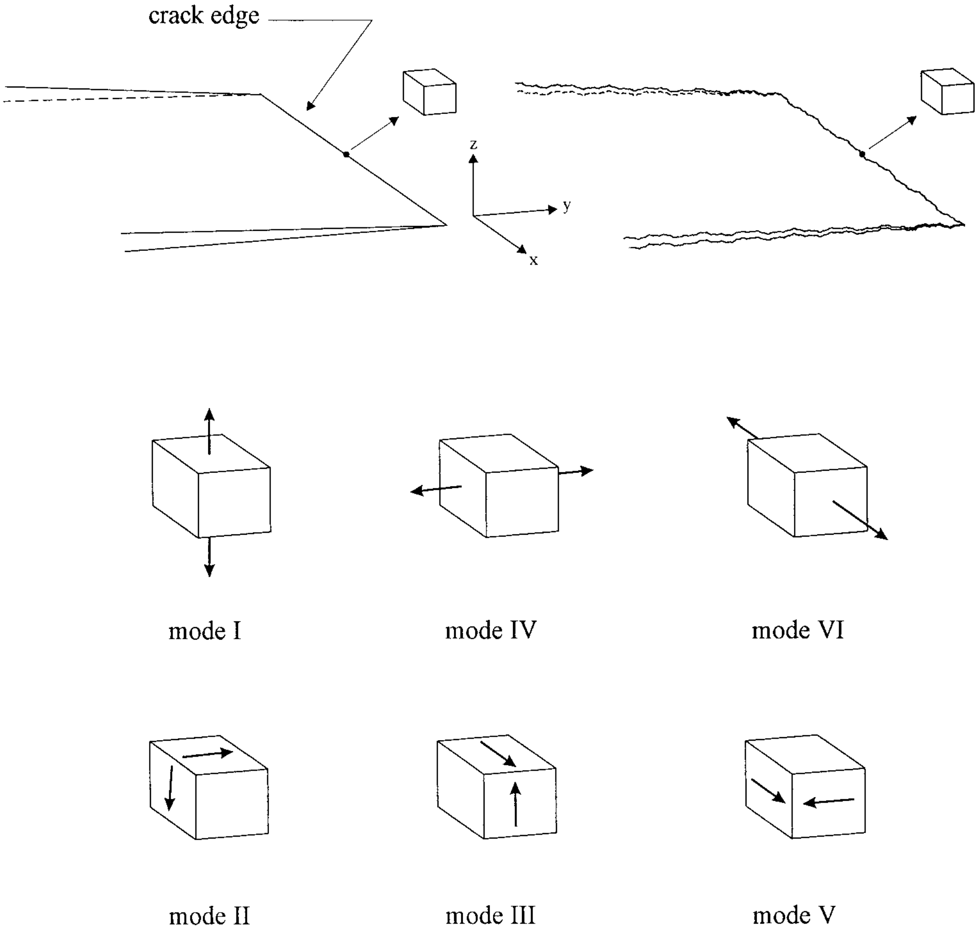

Figure 5. A crack in a three-dimensional solid and six modes of loading. Modes IV, V, and VI do not exist for smooth cracks.

a similar mode that we call mode VI or out-of-plane axial mode. Finally, the stress component σ_{xy} defines a new shearing mode of fractal fracture that we call mode V or distorting mode. It can be easily shown that mode IV, V, and VI fractal cracks have the same order of stress singularity. Figure 6a shows a circular shaft with a penny-shaped crack in uniform torsion. This crack is in mode III for both smooth and fractal cases. The cracked shaft of Figure 6b is in mode II for both fractal and smooth cases. The stress distribution in the cracked shaft shown in Figure 6c is identical to that of an uncracked shaft in the case of a smooth crack. In other words the smooth crack does not interact with the applied stresses. But if the crack is a fractal surface it does interact with applied shearing stresses and is in mode V. All six modes of fractal fracture are shown in Figure 7.

5. Fractal cracks in three-dimensional bodies

In this section three-dimensional solid bodies with fractal cracks are studied. A self-similar fractal crack in a three-dimensional solid body is a fractal surface with fractal dimension D between two and three. In the case of a self-affine crack, the crack is a surface with Hurst

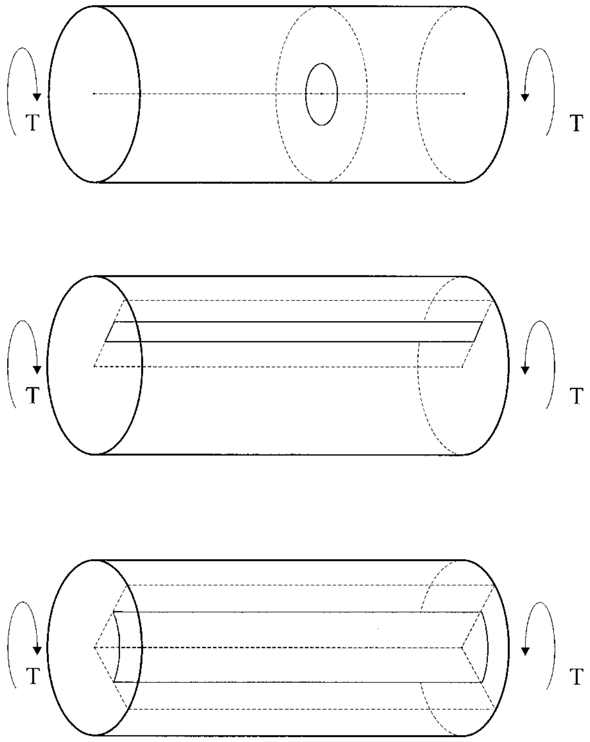

Figure 6. A circular shaft under uniform torsion with (a) a penny-shaped crack in mode III, (b) straight crack in mode II, and (c) cylindrical crack in mode V.

exponent $H (0 < H < 1)$ and latent fractal dimension D_D $(2 < D_D < 3)$. Here we are implicitly assuming that the $x - y$ plane (the plane of the crack) is isotropic and hence a Hurst exponent H can describe the roughness of the self-affine crack. In this section the radial variations of stresses near the crack edges are obtained. Without loss of generality, a mode I fractal crack is considered. For this crack, there is a finite ball $\mathcal{B}_c$ that lies inside the crack-effect zone. The surface energy required for formation of the crack is proportional to a^{D_D}, where 'a' is a characteristic length of the crack. For example, for a disk-shaped crack it is the radius of the disk, and for a rectangular crack it is the diameter of the rectangle. Hence,

$$U_s \propto a^{D_D} = \begin{cases} a^{2/H} & \frac{2}{3} \leq H < 1, \\ a^3 & 0 < h \leq \frac{2}{3}. \end{cases} \tag{27}$$

Now suppose that the crack propagates and the characteristic length of the crack is increased by an infinitesimal amount δa. Note that in this case there are infinitely many possible crack growth forms. The surface energy needed for this infinitesimal crack growth is proportional to:

$$\delta U_s \propto \delta\left(a^{D_D}\right) = \begin{cases} \delta(a^{2/H}) & \frac{2}{3} \leq H < 1, \\ \delta(a^3) & 0 < H \leq \frac{2}{3}. \end{cases} \tag{28}$$

The strain energy release due to the crack growth may be written as:

$$\delta U_e \cong \delta \int_{B_c} \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\, \mathrm{d}V = \delta\left(\int_{B_c - B_s} \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\, \mathrm{d}V + \int_{B_s} \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\, \mathrm{d}V\right) \cong \delta \int_{B_s} \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\, \mathrm{d}V. \tag{29}$$

Around the crack edge stresses and strains have distributions like those shown in Equations (7). As in the two-dimensional problem discussed in Section 3, strain energy release in the ball $\mathcal{B}_s$ of radius r_s (dominant disk of stress singularity) is much greater than it is in $\mathcal{B}_c - \mathcal{B}_s$. Hence in Equation (29) $\mathcal{B}_c$ can be replaced by $\mathcal{B}_s$. Therefore, strain energy release rate due to the crack growth is proportional to:

$$\delta U_e \propto \delta(r_s^{-\alpha} r_s^{-\alpha} r_s^3) = \delta(a^{3-2\alpha}). \tag{30}$$

Griffith's criterion states that $\delta U_e = \delta U_s$; hence,

$$\alpha = \frac{3 - D_D}{2} \quad 2 \le D_D \le 3 \tag{31}$$

in terms of latent fractal dimension (for self-similar cracks $D_D = D$). Similarly, in terms of Hurst exponent we have:

$$\alpha = \begin{cases} \dfrac{3H-2}{2H} & \dfrac{2}{3} < H \le 1, \\ 0 & 0 < H \le \frac{2}{3}, \end{cases} \tag{32}$$

which is valid for mode I, II, and III self-affine fractal cracks. It is seen that for very rough self-affine fractal cracks ($0 < H \le \frac{2}{3}$), stresses are not singular. Now suppose that a self-affine crack (which is a self-affine surface) is under a uniform stress parallel to its plane. The order of stress singularity is (see Yavari et al., 2000):

$$\alpha = \begin{cases} \dfrac{H^{+}2H-1}{2H^2} & \sqrt{3}-1 \le H < 1, \\ 0 & 0 < H \le \sqrt{3}-1, \end{cases} \tag{33}$$

It can be easily shown that this is also the order of stress singularity for mode V and VI. Angular variations of stresses and strains at the tip of a fractal crack cannot be found using this method. If a crack is assumed to be a mathematical fractal for all length scales, the only way to obtain the angular variations of stresses and strains is to solve the corresponding boundary-value problem with fractal boundaries.

6. Path-Dependence of J-integral for fractal cracks

Rice (1968a, b) introduced the following path-independent integral for smooth cracks and called it the J-integral:

$$J = \int_\Gamma (Wn_1 - \boldsymbol{\sigma}.\mathbf{u}_{,1})\, \mathrm{d}s = \int_\Gamma (Wn_1 - \sigma_{ij} n_j u_{i,1})\, \mathrm{d}s, \tag{34}$$

where Γ is any path starting at one face of the crack and ending at the other face, W is the strain energy density, n_1 is the x component of outward unit normal vector to Γ, $\boldsymbol{\sigma}$ is the stress (traction) vector, and $\mathbf{u}$ is the displacement vector. Note that the crack is parallel to the x-axis.

For a smooth crack, the J-integral is path-independent. Using Green's theorem, it can be shown that this integral is zero around any closed path that includes the crack tip. Because the crack faces do not contribute to this integral, the J-integral is path-independent for any path starting from one face of the crack and ending at the other face of the crack. A singularity lies at the crack tip that is dominant inside a disk with radius r_s, which is proportional to the crack half-length 'a.' The paths of integration do not have to lie inside this disk for the J-integral to be path independent. In other words, the J-integral is path-independent regardless of the length of this path (as long as the path lies within the elastic body). Another proof of path-independence of the J-integral involves simply substituting the asymptotic expressions of stresses and strains of Equation (7) into Equation (34). This way, it can easily be shown that the J-integral is path-independent. However, this proof is valid only for paths of integration inside the disk $r \leq r_s$.

Now consider a fractal crack. In this case the crack tip is not the only singularity point; there are infinitely many points on the edges of the crack that all have singularities (but much weaker singularities than those at the crack tip). For a fractal crack J-integral is not path-independent. Rice's argument cannot be applied in this case because, unlike in the classical case, the crack faces are not smooth. Actually, they are nowhere differentiable and the unit normal vector is not defined (the stress vector is not defined on a fractal surface). Path-dependence of J-integral can be seen by simply substituting the asymptotic stresses and strains in (34). Mosolov (1991b) and Balankin (1997) argued that for fractal cracks J-integral scales as:

$$J(\lambda\Gamma) = \lambda^{1-2\alpha} J(\Gamma), \tag{35}$$

where λ is a positive number, α is the power of stress singularity, and $(x, y) \in \lambda\Gamma$ if and only if $(x/\lambda, y/\lambda) \in \Gamma$. Their argument is not correct in general; the asymptotic stress and strain fields (7) are valid only in a disk $\Re_s$ with radius r_s proportional to the apparent half crack length 'a.' Actually, each stress component could have a different $\Re_s$. Here r_s is the minimum of all the radii. Scaling (35) is valid only if both Γ and $\lambda\Gamma$ lie inside $\Re_s$ (see Figure 8). Mosolov (1991b) limited himself to the case $\Gamma \to 0$, i.e., he only considered very local paths of integration. His Equation (1) is not the general definition of J-integral. Actually, one advantage of J-integral techniques is that we can calculate local crack parameters by calculating a path integral over a path that is far from the crack tip. This advantage of J-integral is very useful in computational fracture mechanics; without having a very fine discretization around the crack tip and without utilizing special crack-tip elements very good approximate results can be obtained for J-integral.

Now we define a modified J-integral and call it FJ-integral. Consider the following line integral:

$$FJ = \int_\Gamma r^{1-D}(Wn_1 - \boldsymbol{\sigma}.\mathbf{u}_{,1})\, \mathrm{d}s, \tag{36}$$

for self-similar fractal cracks and

$$FJ = \int_\Gamma r^{(H-1)/H}(Wn_1 - \boldsymbol{\sigma}.\mathbf{u}_{,1})\, \mathrm{d}s, \tag{37}$$

for mode I and mode II self-affine fractal cracks. This path integral is proportional to Mosolov's (1991b) J_f-integral, but here we do not restrict ourselves to very local paths of integration. The FJ-integral can be easily shown to be path independent for all paths lying in the dominant zone of stress singularity around the crack tip (see Figure 9a). As a matter of fact, FJ-integral is locally path-independent for fractal cracks in any linear or nonlinear elastic solid as will be shown in sequel. All these path integrals are, in general, path dependent for paths of integration like the ones shown in Figure 9b.

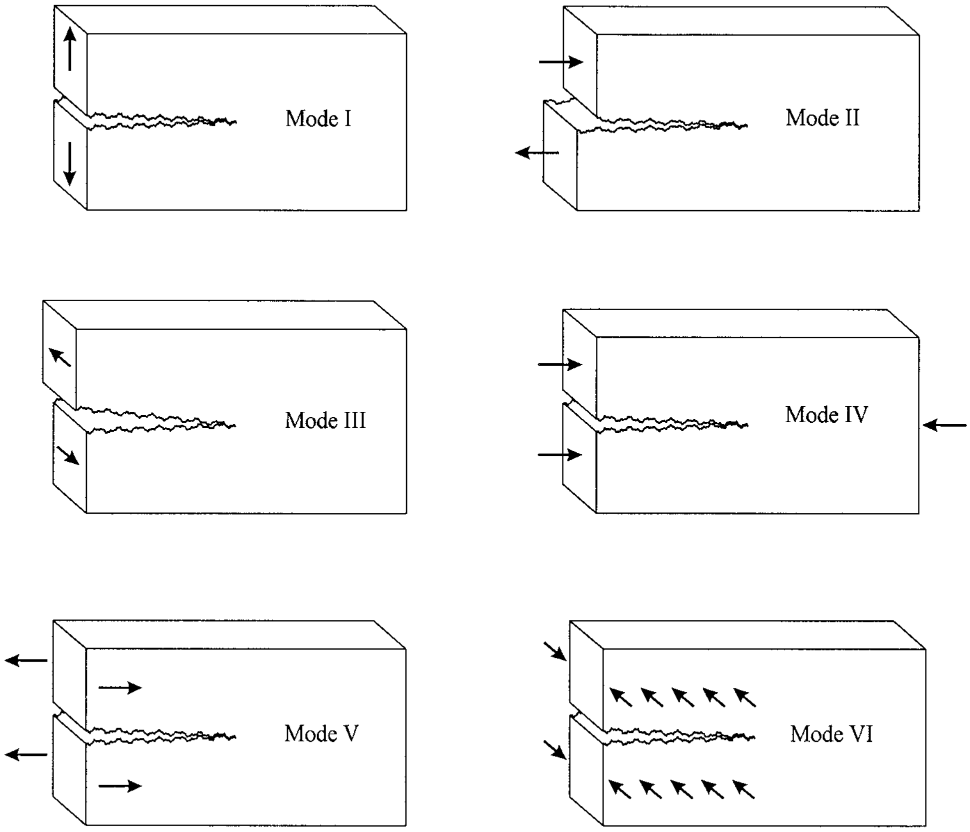

Figure 7. The six modes of fractal fracture: mode I (opening mode), mode II (shearing mode), mode III (tearing mode), mode IV (in-plane axial mode), mode V (distorting mode), and mode VI (out-of-plane axial mode).

FJ-integral has an interesting dimension; its dimension is $[FJ] = L^{1-D}FL^{-1} = FL^{-D}$, which is the dimension of energy per unit latent fractal measure. But this does not necessarily mean that FJ-integral has a physical meaning. Rice (1968a, b) demonstrated that J-integral is the rate of potential energy release rate per unit crack length (per unit thickness), i.e.,

$$J = -\lim_{\Delta L \to 0} \frac{\Pi(L + \Delta L) - \Pi(\Delta L)}{\Delta L} = -\frac{\partial \Pi}{\partial L}, \tag{38}$$

where Π is the potential energy of the system and the crack tip is located at $x = L$. As was noticed earlier, J-integral is not path independent for a fractal crack and hence it cannot have a physical meaning. In other words, potential energy release per unit crack length is not defined for a fractal crack. Physical dimension of FJ-integral motivates us to define a fractal J-integral as:

$$J^{\text{fractal}} = -\lim_{\Delta m_D \to 0} \frac{\Pi(m_D + \Delta m_D) - \Pi(m_D)}{\Delta m_D} = -\frac{\partial \Pi}{\partial m_D}, \tag{39}$$

where m_D is a fractal measure and D is the fractal dimension. A generalization of Equation (34) that leads to Equation (39) remains to be done in future.

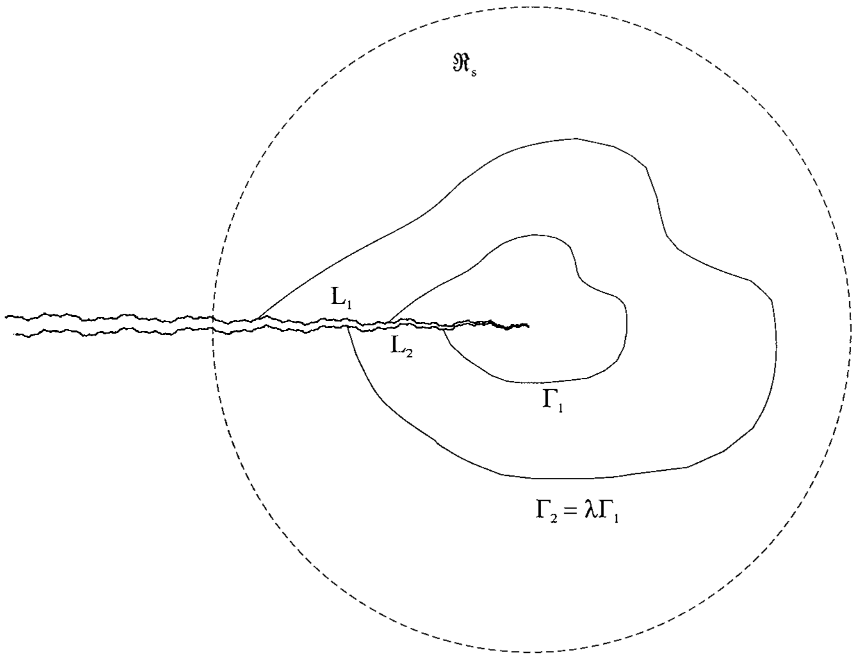

Figure 8. A fractal crack and two similar paths of integration inside the dominant zone of fractal stress singularity ($\Re_s$).

7. Fractal cracks in classical strain-hardening and nonlinear elastic solids

In this section structure of stress and strain distributions around the tip of a fractal crack in a strain-hardening material is studied. Hutchinson (1968) and Rice and Rosengren (1968), independently, found the asymptotic forms of stresses and strains at the tip of a stationary crack in a Ramberg–Osgood-type strain hardening material. They found that stresses and strains are singular at the crack tip and this singularity is now known as HRR singularity. Stress-strain relations in a Ramberg–Osgood material may be written as:

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left(\frac{\sigma}{\sigma_0}\right)^n \tag{40}$$

where ε_0 and σ_0 are material properties and n is the hardening exponent. Using a J_2 deformation plasticity theory, Hutchinson and Rice and Rosengren found the following asymptotic stresses, strains, and displacements:

$$\sigma_{ij}(r,\theta) = \sigma_0 \left(\frac{J}{\alpha\varepsilon_0\sigma_0 I_n r}\right)^{1/(n+1)} \tilde{\sigma}_{ij}(\theta, n) \tag{41a}$$

$$\varepsilon_{ij}(r,\theta) = \alpha\varepsilon_0 \left(\frac{J}{\alpha\varepsilon_0\sigma_0 I_n r}\right)^{1/(n+1)} \tilde{\varepsilon}_{ij}(\theta, n) \tag{41b}$$

$$u_i(r,\theta) = \alpha\varepsilon_0 r \left(\frac{J}{\alpha\varepsilon_0\sigma_0 I_n r}\right)^{1/(n+1)} \tilde{u}_i(\theta, n) \tag{41c}$$

where I_n, $\tilde{\sigma}_{ij}(r,\theta)$, $\tilde{\varepsilon}_{ij}(r,\theta)$, and $\tilde{u}_i(r,\theta)$ are tabulated functions. It is seen that stresses have an $r^{-1/(n+1)}$ singularity and strains have an $r^{-n/(n+1)}$ singularity at the crack tip. Several caveats should be remembered about the HRR solutions:

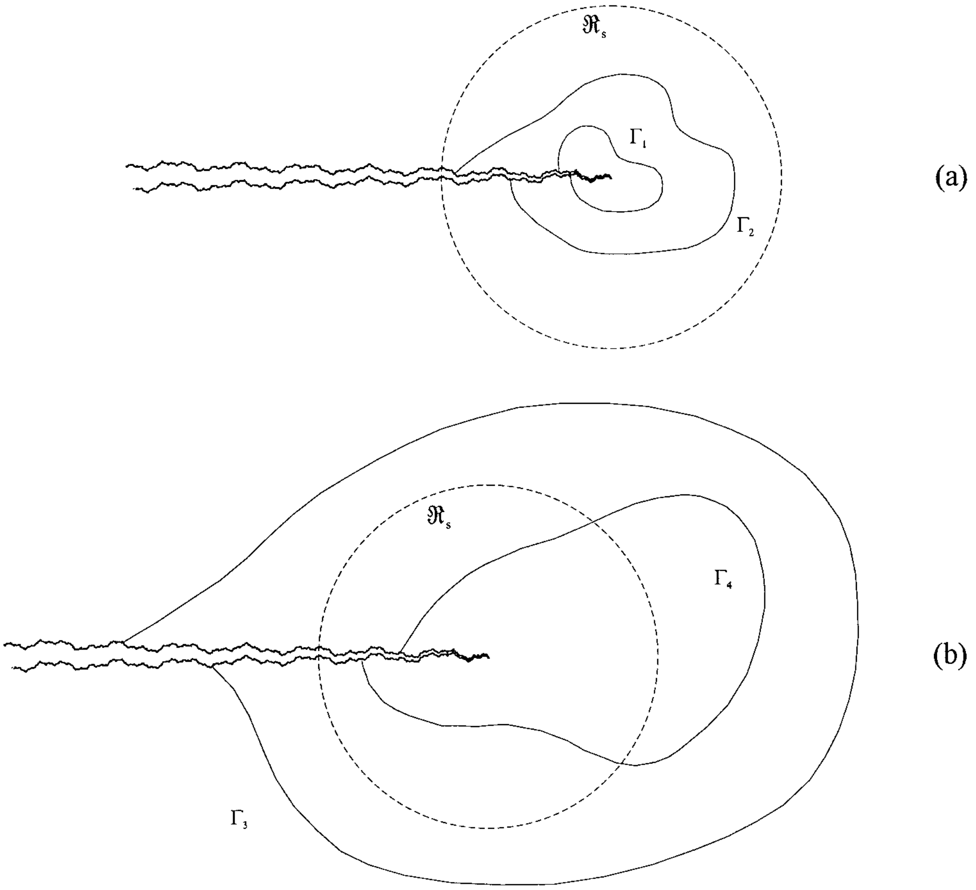

Figure 9. (a) A fractal crack and two paths of integration inside the dominant zone of fractal stress singularity; (b) a fractal crack and two paths of integration, one crossing the dominant zone of fractal stress singularity and one outside it.

(1) Small scale yielding is assumed (plastic zones are concentrated only at the crack tip).
(2) No crack growth occurs.
(3) Infinitesimal strain theory is assumed.
(4) Loading is proportional.

It is worth mentioning that there have been investigations on the structure of stress and strain singularity for propagating cracks. It has been shown that stresses and strains have weaker singularities in the case of propagating cracks. In this work, we consider only stationary cracks.

In this section we calculate the corresponding orders of singularity for stresses and strains at the tip of a fractal crack. Obviously, the orders of stress and strain singularity may be functions of both hardening exponent and the fractal dimension of the crack. Now consider a fractal crack in a power strain-hardening material with the following strain- stress relations:

$$\varepsilon_{ij} = A_{ijkl}\sigma_{kl} + B_{ijkl}\sigma_{ij}^{n}; \quad n \geq 1, \tag{42}$$

where A_{ijkl} and B_{ijkl} are fourth-order tensors (and are mechanical properties) and n is the hardening exponent. Therefore, if $\sigma_{ij}(r) \propto r^{-\alpha}$ as $r \to 0$, we will have $\varepsilon_{ij}(r) \propto r^{-n\alpha}$ as $r \to 0$. For checking the applicability of the method proposed in Section 3, first consider a mode I smooth crack. Suppose that the following asymptotic stresses and strains are dominant around the crack tip:

$$\sigma_{ij}(r,\theta) = K_1 r^{-\alpha} f_{ij}(\theta, n), \tag{43a}$$

$$\varepsilon_{ij}(r,\theta) = K_2 r^{-n\alpha} g_{ij}(\theta, n), \tag{43b}$$

where K_1 and K_2 are independent of r and θ. A disk of radius r_s lies inside the dominant zone of the above asymptotic expressions. It should be noted that the dominant zone of the singularity is not necessarily a disk and it may be different for different stress and strain components. Deng and Rosakis (1992a, b) numerically studied this interesting problem for both linear hardening and power hardening materials. Here r_s is the radius of the smallest disk and is in general a function of crack length and strain-hardening exponent, i.e.,

$$r_s = \psi(a, n). \tag{44}$$

Both 'a' and r_s have dimensions of length and n and H are dimensionless, hence according to Buckingham's Π theorem, we have:

$$\frac{r_s}{a} = \Phi(n) \quad \text{or} \quad r_s = a\Phi(n). \tag{45}$$

The strain energy release due to an infinitesimal crack growth of length δa may be written as:

$$\delta U_e \cong \delta \int_{\Re_c} \frac{1}{2}\sigma_{ij}\varepsilon_{ij}\, \mathrm{d}A \cong \delta \int_{\Re_s} \frac{1}{2}\sigma_j \varepsilon_{ij}\, \mathrm{d}A \propto \delta\left(a^{-\alpha}a^{-n\alpha}a^2\right) = \delta\left(a^{2-(n+1)\alpha}\right). \tag{46}$$

The surface energy required for the crack growth is proportional to δa, thus applying Griffith's criterion yields:

$$2 - (n+1)\alpha = 1 \quad \text{or} \quad \alpha = \frac{1}{n+1}, \tag{47}$$

which is the correct order of stress singularity. Therefore, the method is applicable. It should be noted that here we have utilized Orowan's (1952) generalization of Griffith's theory for ductile fracture.

In this section, the radial variation of stresses and strains are found for self-affine fractal cracks. The case of self-similar cracks can be treated similarly. First, orders of stress and strain singularity are obtained for three classical modes of fracture. Without loss of generality, consider an infinite medium made of the above-mentioned strain-hardening material with a mode I finite self-affine fractal crack with apparent length 2a. The crack is assumed to be a self-affine fractal with Hurst exponent $H(0 < H < 1)$. Again, the method of crack-effect zone is utilized. The following asymptotic stress and strain forms are assumed:

$$\sigma_{ij}(r,\theta) = K_1 r^{-\alpha} f_{ij}(\theta, n, H), \tag{48a}$$

$$\varepsilon_{ij}(r,\theta) = K_2 r^{-n\alpha} g_{ij}(\theta, n, H). \tag{48b}$$

The above asymptotic stresses and strains are valid in a region that covers by a disk $\Re_s$ of radius r_s (more precisely, the area of the dominant zone of singularity is proportional to r_s^2). r_s is, in general, a function of the apparent crack length, strain-hardening exponent, and the Hurst exponent, i.e.,

$$r_s = \psi(a, n, H). \tag{49}$$

Here both 'a' and r_s have dimensions of length and n and H are dimensionless. According to Buckingham's Π theorem, we must have:

$$\frac{r_s}{a} = \Phi(n, H) \tag{50a}$$

or

$$r_s = a\Phi(n, H). \tag{50b}$$

Therefore, r_s is again proportional to 'a' and this is all we need to find the asymptotic form of the released strain energy. The strain energy release due to an infinitesimal crack growth of apparent length δa may be written as:

$$\delta U_e \cong \delta \int_{\Re_c} \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \, \mathrm{d}A \cong \delta \int_{\Re_s} \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \, \mathrm{d}A \propto \delta \left(a^{-\alpha} a^{-n\alpha} a^2\right) = \delta \left(a^{2-(n+1)\alpha}\right). \tag{51}$$

The surface energy required for the crack growth is independent of the strain-stress relations and hence:

$$\delta U_s \sim \left(a^{D_D}\right) = \begin{cases} \delta(a^{1/H}) & \frac{1}{2} \le H < 1, \\ \delta(a^2) & 0 < H \le \frac{1}{2}. \end{cases} \tag{52}$$

Making use of Griffith's criterion, from (51) and (52) we must have:

$$\alpha = \begin{cases} \left(\dfrac{2H-1}{H}\right)\dfrac{1}{n+1} & \dfrac{1}{2} \le H < 1 \\ 0 & 0 < H \le \frac{1}{2} \end{cases} ; \quad n \ge 1. \tag{53}$$

It is seen that the fractality of the crack makes the stress singularity weaker compared to the stress singularity at the tip of a smooth crack. From (48) and (53), the stresses and strains have the following asymptotic behavior near the crack tip:

$$\sigma_{ij} \propto r^{-\left(\frac{2H-1}{h}\right)\frac{1}{n+1}}, \quad \varepsilon_{ij} \propto r^{-\left(\frac{2H-1}{H}\right)\frac{n}{n+1}} \quad \text{as} \quad r \to 0. \tag{54}$$

Balankin (1997) found the following asymptotic stresses and strains using a dimensional analysis technique:

$$\sigma_{ij} \propto r^{-\frac{1}{n+1}+\frac{1-H}{2H}}, \quad \varepsilon_{ij} \propto r^{-\frac{n}{n+1}+\frac{n(1-H)}{2H}} \quad \text{as} \quad r \to 0. \tag{55}$$

Balankin's asymptotic stresses and strains are incorrect because when n tends to infinity ($n \to \infty$), we must have $\sigma_{ij} \propto 1$ even if the crack is a fractal curve. In our stress fields, for $n \to \infty$ we obtain:

$$\sigma_{ij} \propto r^0 = 1. \tag{56}$$

But for Balankin's stress field we have:

$$\sigma_{ij} \propto r^{H/(1-H)} \ne 1. \tag{57}$$

Balankin (2000) corrected his dimensional analysis and found the asymptotic form (54)[2].

Now consider a mode IV self-affine fractal crack (our conclusions are also valid for modes V and VI). We again assume the asymptotic forms (48a) and (48b) for stresses and strains. Similar to the argument presented for the case of a mode IV fractal crack in a linear elastic medium, the radius of the dominant zone of stress and strain singularities is proportional to a^H, i.e., $r_s \sim a^H$ (more precisely, the area of the dominant zone of stress and strain singularity is proportional to a^{H+1}). Therefore, the strain energy release due to an infinitesimal crack growth of apparent length δa has the following asymptotic form:

$$\delta U_e \cong \delta \int_{\Re_c} \sigma_{ij}\varepsilon_{ij}\, \mathrm{d}A \cong \delta \int_{\Re_s} \sigma_{ij}\varepsilon_{ij}\, \mathrm{d}A \propto \delta\left(a^{-\alpha H} a^{-n\alpha H} a a^H\right) = \delta\left(a^{-(n+1)\alpha H+1+H}\right). \quad (58)$$

The asymptotic form of the surface energy is the same as that of mode I, mode II, and mode III fractal cracks Equation (52). Now applying Griffith's criterion yields:

$$\alpha = \begin{cases} \left(\dfrac{H^2+H-1}{H^2}\right)\dfrac{1}{n+1} & \dfrac{1}{g} \le H < 1 \\ 0 & 0 < H \le \dfrac{1}{g} \end{cases} ; \quad n \ge 1, \quad (59)$$

where g is the Golden ratio. Therefore, stresses and strains have the following asymptotic forms near the crack tip:

$$\sigma_{ij} \propto r^{-\left(\frac{H^2+H-1}{H^2}\right)\frac{1}{n+1}}, \quad \varepsilon_{ij} \propto r^{-\left(\frac{H^2+H-1}{H^2}\right)\frac{1}{n+1}}, \quad \text{as} \quad r \to 0 \quad \frac{1}{g} \le H < 1, \quad (60a)$$

$$\sigma_{ij} \propto r^0, \quad \varepsilon_{ij} \propto r^0 \quad \text{as} \quad r \to 0 \quad 0 < H \le \frac{1}{g}. \quad (60b)$$

Similarly, for a self-similar fractal crack in a strain-hardening material asymptotic stresses and strains are:

$$\sigma_{ij} \propto r^{-\left(\frac{2-D}{2}\right)\frac{1}{n+1}}, \quad \varepsilon_{ij} \propto r^{-\left(\frac{2-D}{2}\right)\frac{n}{n+1}} \quad \text{as} \quad r \to 0 \quad 1 \le D \le 2, \quad (61)$$

for all modes of fractal fracture. From (36) and (59), it can be seen that FJ-integral is locally path-independent.

7.1. Fractal cracks in a nonlinear elastic solid

Consider a nonlinear elastic material with the following stress-strain relations:

$$\sigma_{ij} = f(\varepsilon_{ij}). \quad (62)$$

Because the material is elastic, f has to be a single-valued and one-to-one function. Now suppose that there is a self-affine fractal crack with Hurst exponent H in a solid with constitutive equations (62). The following asymptotic stresses and strains are assumed:

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[2]The dimensional analysis presented in Balankin (1997) gives the correct result if Balankin's (1997) Equation (78) is modified to read:

$$\beta = 0, \quad \text{if} \quad \varepsilon = 0, \quad \text{and} \quad \beta = \frac{n}{n+1}, \quad \text{if} \quad \varepsilon = 1.$$

It should be noted that Balankin's n is 1/n in our formulation.

$$\sigma_{ij} \propto r^{-\alpha}, \quad \varepsilon_{ij} \propto r^{-\beta} \quad \text{as} \quad r \to 0, \tag{63}$$

where $\beta = \beta(\alpha)$. Because the constitutive equations are single-valued and one-to-one, β is a single-valued and one-to-one function of α. Without loss of generality, consider a mode I self-affine crack. We have the following expressions for the surface energy and strain energy release:

$$\delta U_s \propto \begin{cases} \delta(a^{1/H}) & \frac{1}{2} \leq H < 1, \\ \delta(a^2) & 0 < H \leq \frac{1}{2}, \end{cases} \tag{64a}$$

$$\delta U_e \propto \delta(a^{-\alpha} a^{-\beta} a^2). \tag{64b}$$

Applying Griffith's criterion yields:

$$\alpha + \beta = \begin{cases} \dfrac{2H - 1}{H} & \frac{1}{2} \leq H < 1, \\ 0 & 0 < H \leq \frac{1}{2}. \end{cases} \tag{65}$$

Similarly for a mode IV self-affine fractal crack we have:

$$\delta U_s \propto \begin{cases} \delta(a^{1/H}) & \frac{1}{2} \leq H < 1, \\ \delta(a^2) & 0 < H \leq \frac{1}{2}, \end{cases} \tag{66a}$$

$$\delta U_e \propto \delta(a^{-\alpha} a^{-\beta} a^{1+H}). \tag{66b}$$

Thus:

$$\alpha + \beta = \begin{cases} \dfrac{H^2 + H - 1}{H} & \dfrac{1}{g} \leq H < 1, \\ 0 & 0 < H \leq \dfrac{1}{g}. \end{cases} \tag{67}$$

From (36), (37), and (67), it is seen that FJ-integral is locally path-independent even for a fractal crack in any elastic solid.

8. Conclusions

In this paper we study some consequences of fractality of fracture surfaces. Radial variations of stresses and strains around the tip of a fractal crack are obtained using a modified Griffith's criterion and dimensional analysis considerations. Both self-similar and self-affine fractals are considered as crack trajectory models. It is observed that for some range of roughness exponent ($0 < H \leq \frac{1}{2}$) stresses are not singular at the tip of self-affine cracks and hence the maximum stress criterion may be utilized for these cracks. It is shown that stresses are always less singular at the tip of a fractal crack than stresses at the tip of a smooth (or rectilinear crack) and consequently stress intensity factors for smooth and fractal cracks have different physical dimensions.

Three new modes of fractal fracture exist which make some single-mode problems of classical fracture mechanics mixed-mode problems in fractal fracture mechanics. The stress singularity power is found for these modes using the method of crack-effect zone. All modes of fractal fracture introduce the same order of stress singularity for self-similar fractal cracks. However, in the case of self-affine fractal cracks modes IV, V, and VI stress singularity is always weaker than that of classical modes.

The J-integral is shown to be path-dependent for fractal cracks. This path-dependence implies that the potential energy release rate per unit of crack length (area) is not defined because J-integral has different values for different paths of integration. It is conjectured that a fractal J-integral should be equal to the rate of energy release per unit of fractal crack growth measure.

Self-similar and self-affine fractal cracks in a power-law strain-hardening medium are studied. The orders of stress and strain singularities are found. It is shown that stresses and strains have weaker singularities for fractal cracks than they do for smooth cracks.

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Appendix. Fractal geometry

This appendix presents some concepts and definitions of fractal geometry. Here we discuss only those aspects of fractal geometry that are directly relevant to our investigation.

Consider a set $S \subset \mathbb{R}^n$. An affine transformation of real scaling ratios $r_1, r_2, \ldots, r_n$ $(0 < r_i < 1)$ transforms each $x = (x_1, x_2, \ldots, x_n) \in S$ into $r(x) = (r_1x_1, r_2x_2, \ldots, r_nx_n) \in r(S)$. The set S is self-affine if it is composed of N nonoverlapping subsets congruent to $r(S)$. If the above property holds for S when $r_1 = r_2 = \ldots = r_n = r$, it is called a self-similar set. A self-similar fractal is invariant under an isotropic length-scale transformation, while a self-affine fractal is invariant under a transformation with different length scales in different directions.

Roughly speaking, the measure of a set $S \subset \mathbb{R}^n$ tells us about the size of the set and is denoted by $\mu(S)$. In other words, measure is a generalized size. Here, μ is a measure on $\mathbb{R}^n$ if it assigns a nonnegative number (possibly $+\infty$) to each subset of $\mathbb{R}^n$ and satisfies the following requirements:

1. $\mu(\phi) = 0$,
2. $\mu(A) \leq \mu(B) \quad \text{if} \quad A \subset B$,
3. If $A_1, A_2, \ldots$ is a finite or countable sequence of subsets of $\mathbb{R}^n$ then:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i), \tag{A1}$$

with equality if the A_i's are disjoint subsets of $\mathbb{R}^n$.

Suppose that $U \neq \phi$ is a subset of $\mathbb{R}^n$. The diameter of U is defined as:

$$\operatorname{diam}(U) = \sup\{|x - y| : x, y \in U\}. \tag{A2}$$

An ε-cover of S is a countable or finite collection of sets $\{U_i\}$ such that:

1. $0 < \mathrm{diam}(U_i) \leq \varepsilon$,
2. $S \subset \bigcup_{i=1}^{\infty} U_i$.

Now suppose that $S \subset \mathbb{R}^n$ and $D \in \mathbb{R}^+ \cup \{0\}$. The D-dimensional Hausdorff measure of S is denoted by $\mathcal{H}^D(S)$ and is defined as:

$$\mathcal{H}^D(S) = \lim_{\varepsilon \to 0} \mathcal{H}_\varepsilon^D(S), \tag{A3}$$

where

$$\mathcal{H}_\varepsilon^D(S) = \inf \left\{ \sum_{i=1}^{\infty} \mathrm{diam}(U_i)^D : \{U_i\} \text{is and } \varepsilon - c \text{ over of } S \right\}. \tag{A4}$$

It can be shown that $\mathcal{H}^D$ has all the properties of a measure. It can be proven that for any set S, $\mathcal{H}^D(S)$ has a jump from $+\infty$ to 0 for one and only one value of D; this is called the Hausdorff dimension of S, i.e.:

$$D_H = \inf\{D : \mathcal{H}^D(S) = 0\} = \sup\{D : \mathcal{H}^D(S) = +\infty\}. \tag{A5}$$

Therefore:

$$\mathcal{H}^D(S) = \begin{cases} +\infty & D < D_H \\ 0 & D > D_H \end{cases} \tag{A6}$$

Note that when $D = D_H$, $0 \leq \mathcal{H}^D(S) \leq +\infty$. If, when $D = D_H$, $\mathcal{H}^D(S)$ is nonzero and finite, the set S is called a D-set. A finite (or countable) collection of isolated points is a 0-set, a line segment is a 1-set, a disk is a 2-set, and a cube is a 3-set.

There are many other definitions of dimension. One disadvantage of Hausdorff dimension is the difficulty of calculating it, which makes it impractical. Here we discuss two other important dimensions, namely the box dimension and the divider dimension. All dimensions somehow measure the complexity of irregularity of a set. In most definitions there is a measurement at scale ε. For each ε irregularities below this scale are ignored and the behavior of measurements as $\varepsilon \to 0$ is studied.

Box dimension: Let $S \neq \phi$ be a subset of $\mathbb{R}^n$ and let $N_\varepsilon^B(S)$ be the smallest number of sets of diameter at most ε which can cover S. The box dimension of S is D_B if

$$N_\varepsilon^B(S) = O(\varepsilon^{D_B}) \quad \text{as} \quad \varepsilon \to 0 \quad \text{or} \quad D_B = \lim_{\varepsilon \to 0} \frac{\log N_\varepsilon^B(S)}{-\log \varepsilon}, \tag{A7}$$

where O is Landau's order symbol. It can be shown that always $D_H \leq D_B$. For self-similar fractals the equality holds. The Box measure m_{D_B} is defined as:

$$m_{D_B}^\varepsilon = N_\varepsilon^B(S)\varepsilon^{D_B} = \inf \left\{ \sum_i \varepsilon^{D_B} : \{U_i\} \text{is a finite } \varepsilon - c \text{ over of } S \right\}, \quad m_{D_B} = \lim_{\varepsilon \to 0} m_{D_B}^\varepsilon. \tag{A8}$$

In calculating Hausdorff measure, different weights $|U_i|^s$ are assigned to covering sets U_i while, in the box measure the same weight is used for all covering sets. It should be noted that

m_{D_B} is not a mathematical measure on subsets of $\mathbb{R}^n$; it does not have all the properties of a measure.

Divider (latent) dimension: This is the most important dimension in fractal fracture mechanics. Consider a Jordan curve C (a curve that does not intersect itself) $f : [a, b] \rightarrow \mathbb{R}^n$. Here f is a bijection (a one-to-one and onto function). For $\varepsilon > 0$ define $N_\varepsilon^D(C)$ to be the maximum number of points $x_0, x_1, x_2, \ldots, x_m$ on C such that $|x_k - x_{k-1}| = \varepsilon$ for $k = 1, 2, \ldots, m$. Therefore, the approximate length of the curve $L_\varepsilon(C)$ is $L_\varepsilon(C) = O\left[\left(N_\varepsilon^D(C) - 1\right)\varepsilon\right]$. The divider dimension of C is D_D if:

$$N_\varepsilon^D(C) = O\left(\varepsilon^{-D_D}\right) \quad \text{as} \quad \varepsilon \rightarrow 0 \quad \text{or} \quad L_\varepsilon(C) = O\left(\varepsilon^{-D_D+1}\right) \quad \text{as} \quad \varepsilon \rightarrow 0. \tag{A9}$$

We know that $N_\varepsilon^D(C)$ is dimensionless, while ε has the dimension of length. Therefore, from (A.9) we conclude that:

$$N_\varepsilon(C) \propto \left(\frac{\varepsilon}{L_0}\right)^{-D_D} \quad \text{or} \quad N_\varepsilon(C) \propto \varepsilon^{-D_D} L_0^{D_D}, \tag{A10}$$

where L_0 is the apparent length of C. It can be shown that for any Jordan curve C, $D_D \geq D_B$. For self-similar curves the equality holds. The divider measure m_{D_D} is defined as:

$$m_{D_D}^\varepsilon = N_\varepsilon^D(C)\varepsilon^{D_D}, \quad m_{D_D} \lim_{\varepsilon \rightarrow 0} m_{D_D}^\varepsilon. \tag{A11}$$

From (A.10) and (A.11) we can write:

$$m_{D_D} \propto L_0^{D_D}. \tag{A12}$$

Like the box measure, the divider measure is not a mathematical measure because it is not σ-additive.

It can be shown that for a self-affine fractal with Hurst exponent H we locally have:

$$D_D = \begin{cases} \dfrac{1}{H} & \frac{1}{2} \leq H < 1, \\ 2 & 0 < H \leq \frac{1}{2}. \end{cases} \tag{A13}$$

And globally $D_D = 1$. In general, for a self-affine fractal (with Hurst exponent H) embedded in $\mathbb{R}^n$, the divider and box dimensions are locally related to roughness exponent by:

$$D_D = \begin{cases} \dfrac{n-1}{H} & \dfrac{n-1}{n} \leq H < 1, \\ n & 0 < H \leq \dfrac{n-1}{n}, \end{cases} \tag{A14a}$$

$$D_B = n - H. \tag{A14b}$$

And globally, $D_D = D_B = n - 1$.

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