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## A Geometric Theory of Growth Mechanics

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**Abstract** In this paper we formulate a geometric theory of the mechanics of growing solids. Bulk growth is modeled by a material manifold with an evolving metric. The time dependence of the metric represents the evolution of the stress-free (natural) configuration of the body in response to changes in mass density and “shape”. We show that the time dependency of the material metric will affect the energy balance and the entropy production inequality; both the energy balance and the entropy production inequality have to be modified. We then obtain the governing equations covariantly by postulating invariance of energy balance under time-dependent spatial diffeomorphisms. We use the principle of maximum entropy production in deriving an evolution equation for the material metric. In the case of isotropic growth, we find those growth distributions that do not result in residual stresses. We then look at Lagrangian field theory of growing elastic solids. We will use the Lagrange–d’Alembert principle with Rayleigh’s dissipation functions to derive the governing equations. We make an explicit connection between our geometric theory and the conventional multiplicative decomposition of the deformation gradient,  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$ , into growth and elastic parts. We linearize the nonlinear theory and derive a linearized theory of growth mechanics. Finally, we obtain the stress-free growth distributions in the linearized theory.

**Keywords** Growth mechanics · Geometric elasticity · Material manifold

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Dedicated to the memory of Professor James K. Knowles (1931–2009).

Communicated by A. Mielke.

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**Mathematics Subject Classification (2000)** 74Axx · 74Fxx · 53Zxx**1 Introduction**

In classical continuum mechanics, one usually models mass-conserving bodies. The traditional framework of continuum mechanics is suitable for many practical applications. However, in some natural phenomena mass is added or lost in a deformation process. This is particularly important in biological systems where growth and remodeling are closely linked to mechanical loads. In the case of soft tissues, elastic deformations are large and the theory of linear elasticity is not adequate. This has been the main motivation for extensive studies of large deformations in biomechanics in recent years (see Cowin and Hegedus 1976; Skalak et al. 1982; Fung 1983; Naumov 1994; Hoger 1997; Humphrey 2003; Klarbring et al. 2007 and references therein).

It has been realized for a long time that mechanical forces directly affect the growth and remodeling in biological systems (Hsu 2003). A continuum theory capable of modeling biological tissues must take into account changes of mass and the coupling between growth/remodeling and mechanical stresses. In continuum mechanics, one starts by postulating that a body is made of a large number of “material points” that can be treated as mathematical points. A material point consists of “enough” number of particles (atoms, molecules, cells, etc.), so that it can represent the mechanical properties of the body, e.g. the density. Material points are then identified with their positions in the so-called reference configuration. This is called the material manifold. It is always assumed that the body is macroscopically stress free in the material manifold. The material manifold is not necessarily Euclidean and even not Riemannian; in general, the material manifold is a Riemann–Cartan manifold in the case of solid bodies with distributed dislocations, for example. It is relevant to mention that in most of the existing formulations of finite-strain plasticity, instead of working with a Riemann–Cartan material manifold, one assumes a multiplicative decomposition of the deformation gradient into elastic and plastic parts, i.e. (Eckart 1948; Kröner 1960; Lee and Liu 1967; Lee 1967)

$$\mathbf{F} = T\varphi = \mathbf{F}_e\mathbf{F}_p, \quad (1.1)$$

where  $\varphi$  is the deformation mapping. This means that locally the material deforms plastically followed by elastic deformations to ensure compatibility. In other words, one assumes that both the material and the ambient space manifolds are Euclidean and one locally decomposes the total deformation mapping into incompatible elastic and plastic parts. Motivated by plasticity, in the case of growth several researchers (Kondaurov and Nikitin 1987; Takamizawa and Matsuda 1990; Takamizawa 1991; Rodriguez et al. 1994) postulated a similar decomposition of  $\mathbf{F}$  into elastic and growth parts, i.e.

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_g. \quad (1.2)$$

This has been the fundamental kinematical idea of all the existing models of growth mechanics to this date (see Ben Amar and Goriely 2005; Hoger 1997; Lubarda and Hoger 2002 and references therein).

Recently, Ozakin and Yavari (2010) introduced a geometric theory of thermoelasticity in which thermal strains are buried in a temperature-dependent Riemannian material manifold. In that theory a change of temperature leads to a rescaling of the material metric with a clear physical meaning. In this paper we introduce a geometric theory for growing bodies using similar ideas. We should mention that the analogy between growth and thermal distortions was first realized by Skalak and his coworkers (Skalak et al. 1996).

There are two possibilities in a growth process: (i) the number of material points is fixed, and (ii) material points are removed or are added. Note that in a continuum model the material points are assumed to contain several (or a large number of) particles, cells, etc. Erosion or accretion of cells corresponds to changes in volume (and the corresponding mass) and shape of the material body. In our continuum model, similar to many of the earlier models of bulk growth, we assume that the number of material points is fixed. This means that we work with a fixed set  $\mathcal{B}$  as the material manifold and model growth by allowing  $\mathcal{B}$  to have an evolving geometry. Consider a two-dimensional problem, where the relaxed state of the material is described by a surface. If the bulk of the material grows, as, for example, in a thin shell of biological material undergoing cell division, the shape of the surface describing the relaxed state will change. The stresses for a given configuration should be calculated in terms of the map from the surface describing the relaxed state, to the current configuration.

While the multiplicative decomposition of the deformation gradient has been a source of useful approaches to nonlinear problems, we believe that, in many cases, such an approach obfuscates the underlying natural geometry. A multiplicative decomposition seems natural if one starts with a stress-free material body and considers processes such as plasticity which, in general, induce stresses. However, an initial stress-free Euclidean configuration may not even exist in certain problems. Mathematically, one can still consider an incompatible local deformation that brings the material to a relaxed, Euclidean state, and measure deformations from this state, as in the multiplicative decomposition described above. However, we believe that a more natural way of looking at the problem involves treating the material manifold as a non-Euclidean manifold, and giving its geometry explicitly in terms of the physics of this problem. In passing, we should mention Miehe's (1998) work, in which instead of  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p$  he introduces a "plastic metric", although with no clear physical meaning/interpretation for this metric. We will come back to a geometric interpretation of  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$  in Sect. 3.

In this paper, we model growth by introducing a Riemannian material manifold with an evolving metric. As will be seen, this formalism is very similar to the approach of Ozakin and Yavari (2010) to thermoelasticity; however, there are a few important modifications. First, although we had a version of mass conservation in thermoelasticity in terms of the changing material manifold, for the case of growth, mass will in general be added to (or removed from) the material body; we will have a mass balance. Thus, one can represent the amount of mass being added (or removed) in terms of the changes of the differential form describing the material mass density.

Secondly, for the case of thermal stresses, the material metric was explicitly given in terms of the temperature, but no such simple dependence exists for the material metric in biological growth. We will begin by exploring the consequences of various simple modes of growth, such as a cylindrically symmetric growth represented by a radius-dependent conformal scaling of the metric. Assuming simple constitutive relations, we will write the equations for the equilibrium configurations in terms of the time-dependent metric, much like the case in thermoelasticity. We will also establish the connection to the formulations involving a multiplicative decomposition of the deformation gradient.

One should note that there is in fact no guarantee that a time-dependent Riemannian metric and its Levi-Civita connection is capable of modeling all kinds of growth. We believe that at the very least one needs to consider time-dependent connections with torsion; however, “Riemannian growth” is a good starting point. We aim to investigate the case of growth with torsion, as well as non-metricity in future communications.

Efrati et al. (2009) have recently studied similar problems in the framework of linearized elasticity by modifying the definition of linearized strain. Here, we start with nonlinear elasticity, and instead of modifying any definition of strain we will work with an evolving material manifold. We should mention that the idea of using differential geometry in elasticity goes back to more than fifty years ago in the work of Eckart (1948) who realized that the stress-free configuration of a material body evolves in time and an Euclidean stress-free configuration is not always possible. Later developments are due to Kondo (1955a, 1955b) and Bilby et al. (1955, 1957).

There have been growth models in the literature using mixture theories. For growth mechanics purposes, a mixture theory is certainly more realistic than a mono-phasic continuum theory. However, in this paper for the sake of simplicity and clarity of presentation, we restrict ourselves to mono-phasic continua. We should mention that our ideas are similar, in spirit, to those of Rajagopal and Srinivasa (2004b) who have been advocating the idea of material bodies with evolving natural configurations. Here, we work in a fully geometric framework and model a growing body by a continuum that has an evolving Riemannian material metric. One should note that this is a very special case of a possible evolving material manifold that we believe is sufficient for bulk growth purposes. In particular, we work with Levi-Civita connections that are by construction torsion-free. We should also emphasize that we are not, by any means, questioning the usefulness of the traditionally used  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$  decomposition of the deformation gradient. However, we believe that although the existing models based on this multiplicative decomposition have been very useful in growth mechanics (see Ben Amar and Gorieli 2005; Garikipati et al. 2004 for some concrete examples.) they all lack a rigorous mathematical foundation.

This paper is organized as follows. In Sect. 2 we modify the existing geometric theory of elasticity for growing solids. We show that energy balance has to be modified and then study its covariance. We study the entropy production inequality and the restrictions it imposes on constitutive equations. We also show how the Principle of Maximum Entropy Production can be used to obtain thermodynamically-consistent evolution equations for the material metric. We then look at isotropic growth and

model it by a time-dependent rescaling of an initial material metric. We solve three examples of isotropic and non-isotropic growth analytically. We then discuss how an evolving material manifold can be visualized using embeddings. In the last part of this section we obtain stress-free isotropic growth distributions. We then study growth in the Lagrangian field theory of elasticity and show how the governing equations can be obtained using the Lagrange–d’Alembert principle and using Rayleigh’s dissipation functions. In Sect. 3, we make a connection between the exiting theories of growth based on the decomposition  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$  and the present geometric theory. The nonlinear geometric theory is linearized in Sect. 4. In particular, we obtain those isotropic growth distributions that are stress free in the linearized setting. Conclusions are given in Sect. 5.

## 2 Evolving Material Metrics and Bulk Growth

There have been previous works on the continuum mechanics formulation of bodies with variable mass (see Lubarda and Hoger 2002; Ben Amar and Gorieli 2005; DiCarlo and Quiligotti 2002; Epstein and Maugin 2000, and references therein). In these works it is assumed that the growth part of the deformation gradient is an unknown tensor field and its evolution is governed by a kinetic equation. In writing the energy balance, the corresponding thermodynamic forces show up. In the present geometric theory, we work with an evolving material manifold instead of introducing new fields other than material mass density. Before going into the details of the proposed theory, let us first briefly review the geometric theory of classical nonlinear elasticity.

*Geometric Elasticity* A body  $\mathcal{B}$  is identified with a Riemannian manifold  $\mathcal{B}$  and a configuration of  $\mathcal{B}$  is a mapping  $\varphi : \mathcal{B} \rightarrow \mathcal{S}$ , where  $\mathcal{S}$  is another Riemannian manifold. The set of all configurations of  $\mathcal{B}$  is denoted by  $\mathcal{C}$ . A motion is a curve  $c : \mathbb{R} \rightarrow \mathcal{C}; t \mapsto \varphi_t$  in  $\mathcal{C}$ . It is assumed that the body is stress free in the material manifold. For a fixed  $t$ ,  $\varphi_t(\mathbf{X}) = \varphi(\mathbf{X}, t)$  and for a fixed  $\mathbf{X}$ ,  $\varphi_{\mathbf{X}}(t) = \varphi(\mathbf{X}, t)$ , where  $\mathbf{X}$  is the position of material points in the undeformed configuration  $\mathcal{B}$ . Let  $\varphi : \mathcal{B} \rightarrow \mathcal{S}$  be a  $C^1$  configuration of  $\mathcal{B}$  in  $\mathcal{S}$ , where  $\mathcal{B}$  and  $\mathcal{S}$  are manifolds. The deformation gradient is the tangent map of  $\varphi$  and is denoted by  $\mathbf{F} = T\varphi$ . Thus, at each point  $\mathbf{X} \in \mathcal{B}$ , it is a linear map

$$\mathbf{F}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \rightarrow T_{\varphi(\mathbf{X})}\mathcal{S}. \tag{2.1}$$

If  $\{x^a\}$  and  $\{X^A\}$  are local coordinate charts on  $\mathcal{S}$  and  $\mathcal{B}$ , respectively, the components of  $\mathbf{F}$  are

$$F^a{}_A(\mathbf{X}) = \frac{\partial \varphi^a}{\partial X^A}(\mathbf{X}). \tag{2.2}$$

The material velocity is the map  $\mathbf{V}_t : \mathcal{B} \rightarrow T_{\varphi_t(\mathbf{X})}\mathcal{S}$  given by

$$\mathbf{V}_t(\mathbf{X}) = \mathbf{V}(\mathbf{X}, t) = \frac{\partial \varphi(\mathbf{X}, t)}{\partial t} = \frac{d}{dt} \varphi_{\mathbf{X}}(t). \tag{2.3}$$

The material acceleration is defined by

$$\mathbf{A}_t(\mathbf{X}) = \mathbf{A}(\mathbf{X}, t) = \frac{\partial \mathbf{V}(\mathbf{X}, t)}{\partial t} = \frac{d}{dt} \mathbf{V}_{\mathbf{X}}(t). \quad (2.4)$$

In components

$$A^a = \frac{\partial V^a}{\partial t} + \gamma_{bc}^a V^b V^c, \quad (2.5)$$

where  $\gamma_{bc}^a$  is the Christoffel symbol of the local coordinate chart  $\{x^a\}$ . Note that  $\mathbf{A}$  does not depend on the connection coefficients of the material manifold.  $\varphi_t$  is assumed to be invertible and regular. The spatial velocity of a regular motion  $\varphi_t$  is defined as

$$\mathbf{v}_t : \varphi_t(\mathcal{B}) \rightarrow T_{\varphi_t(\mathbf{X})}\mathcal{S}, \quad \mathbf{v}_t = \mathbf{V}_t \circ \varphi_t^{-1}, \quad (2.6)$$

and the spatial acceleration  $\mathbf{a}_t$  is defined as

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \nabla_{\mathbf{v}} \mathbf{v}. \quad (2.7)$$

In components

$$a^a = \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^b} v^b + \gamma_{bc}^a v^b v^c. \quad (2.8)$$

Suppose  $\mathcal{B}$  and  $\mathcal{S}$  are Riemannian manifolds with inner products  $\langle \cdot, \cdot \rangle_{\mathbf{G}}$  and  $\langle \cdot, \cdot \rangle_{\mathbf{g}}$  based at  $\mathbf{X} \in \mathcal{B}$  and  $\mathbf{x} \in \mathcal{S}$ , respectively. The transpose of  $\mathbf{F}$  is defined by

$$\mathbf{F}^T : T_{\mathbf{X}}\mathcal{S} \rightarrow T_{\mathbf{X}}\mathcal{B}, \quad \langle \langle \mathbf{F}\mathbf{v}, \mathbf{v} \rangle \rangle_{\mathbf{g}} = \langle \langle \mathbf{V}, \mathbf{F}^T \mathbf{v} \rangle \rangle_{\mathbf{G}} \quad \forall \mathbf{V} \in T_{\mathbf{X}}\mathcal{B}, \mathbf{v} \in T_{\mathbf{X}}\mathcal{S}. \quad (2.9)$$

In components

$$(\mathbf{F}^T(\mathbf{X}))^A{}_a = g_{ab}(\mathbf{x}) F^b{}_B(\mathbf{X}) G^{AB}(\mathbf{X}). \quad (2.10)$$

The right Cauchy–Green deformation tensor is defined by

$$\mathbf{C}(X) : T_{\mathbf{X}}\mathcal{B} \rightarrow T_{\mathbf{X}}\mathcal{B}, \quad \mathbf{C}(\mathbf{X}) = \mathbf{F}(\mathbf{X})^T \mathbf{F}(\mathbf{X}), \quad (2.11)$$

where  $\mathbf{g}$  and  $\mathbf{G}$  are metric tensors on  $\mathcal{S}$  and  $\mathcal{B}$ , respectively. In components

$$C^A{}_B = (\mathbf{F}^T)^A{}_a F^a{}_B. \quad (2.12)$$

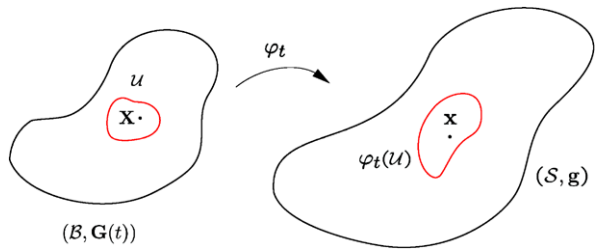
One can show that

$$\mathbf{C}^b = \varphi^*(\mathbf{g}), \quad \text{i.e.} \quad C_{AB} = (g_{ab} \circ \varphi) F^a{}_A F^b{}_B. \quad (2.13)$$

For bulk growth, we assume that the material manifold  $\mathcal{B}$  remains unchanged but the metric evolves, i.e.  $\mathbf{G} = \mathbf{G}(\mathbf{X}, t)$ .<sup>1</sup> When mass is added or removed, the stress-free

<sup>1</sup>In mathematics, evolving metrics have been studied extensively. The most celebrated example is Ricci flow (Hamilton 1982; Topping 2006), which was used in proving the Poincaré Conjecture by Perelman (2002). Interestingly, for a seemingly very different application, i.e. growth mechanics, an evolving geometry plays a key role.

**Fig. 1** Deformation of a growing body as a map between a Riemannian material manifold with a dynamic metric and an ambient space with a fixed background metric



state of the body changes. Local changes in mass change the stress-free configuration of the body. This is modeled by a time-dependent material metric that represents local changes in volume and “shape” in the relaxed configuration (see Fig. 1). In Sect. 3, we will make a connection between this approach and the conventional  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$  decomposition of deformation gradient.

*Incompressibility* In growth mechanics it is usually assumed that elastic deformations are incompressible. In the classical theory in which  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$  is assumed, incompressibility implies  $\det \mathbf{F}_e = 1$ , i.e. all the volume changes are due to growth. In the geometric theory the following relation holds between volume elements of  $(\mathcal{B}, \mathbf{G})$  and  $(\mathcal{S}, \mathbf{g})$ :

$$dv = J dV, \tag{2.14}$$

where

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}. \tag{2.15}$$

Incompressibility of elastic deformations means that  $J = 1$ . Note that even when  $J = 1$ , still  $dv$  is time dependent as a result of the time evolution of the material metric that makes  $dV$  time dependent. In other words, an observer in the ambient space sees changes in volume that are only due to volume changes in the material manifold. We will show the equivalence of  $J_e = 1$  in the classical theory with  $J = 1$  in the geometric theory in both some simple examples in Sect. 2.9 and in the general case in Sect. 3.

### 2.1 Energy Balance

Let us look at energy balance for a growing body. The standard material balance of energy for a subset  $\mathcal{U} \subset \mathcal{B}$  reads (Yavari et al. 2006)

$$\frac{d}{dt} \int_{\mathcal{U}} \rho_0 \left( E + \frac{1}{2} \langle \mathbf{V}, \mathbf{V} \rangle \right) dV = \int_{\mathcal{U}} \rho_0 (\langle \mathbf{B}, \mathbf{V} \rangle + R) dV + \int_{\partial \mathcal{U}} (\langle \mathbf{T}, \mathbf{V} \rangle + H) dA, \tag{2.16}$$

where  $E = E(\mathbf{X}, \mathbf{N}, \mathbf{G}, \mathbf{F}, \mathbf{g} \circ \varphi)$  is the material internal energy density,  $\mathbf{N}$ ,  $\rho_0$ ,  $\mathbf{B}$ ,  $\mathbf{T}$ ,  $R$ , and  $H$  are specific entropy, material mass density, body force per unit undeformed mass, traction vector, heat supply, and heat flux, respectively.

We first note that the energy balance should be modified in the case of growing bodies with time-dependent material metrics. Note that when the metric is time

dependent, the material density mass form  $m(\mathbf{X}, t) = \rho_0(\mathbf{X}, t) dV(\mathbf{X}, t)$  is time dependent even if  $\rho_0$  is not time dependent. For a subbody  $\mathcal{U} \subset \mathcal{B}$ , the rate of change of mass reads

$$\frac{d}{dt} \int_{\mathcal{U}} \rho_0(\mathbf{X}, t) dV(\mathbf{X}, t) = \int_{\mathcal{U}} \left[ \frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) \right] dV. \quad (2.17)$$

Note that if  $\rho_0$  is time independent, then the term  $\frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right)$  represents the change in mass due to growth. Here, we assume that a scalar field of mass source/sink  $S_m(\mathbf{X}, t)$  is given.<sup>2</sup> This mass source will change the stress-free configuration of the body and  $(\mathcal{B}, \mathbf{G}(\mathbf{X}, t))$  represents the time-dependent stress-free configuration of the body.

The rate of change of the material metric is a kinematical variable that contributes to power. Therefore, the energy balance for a growing body with a time-dependent material metric can be written as<sup>3</sup>

$$\begin{aligned} & \frac{d}{dt} \int_{\mathcal{U}} \rho_0 \left( E + \frac{1}{2} \langle \langle \mathbf{V}, \mathbf{V} \rangle \rangle \right) dV \\ &= \int_{\mathcal{U}} \left\{ \rho_0 (\langle \langle \mathbf{B}, \mathbf{V} \rangle \rangle + R) + \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} \right. \\ & \quad \left. + S_m \left( E + \frac{1}{2} \langle \langle \mathbf{V}, \mathbf{V} \rangle \rangle \right) \right\} dV + \int_{\partial \mathcal{U}} (\langle \langle \mathbf{T}, \mathbf{V} \rangle \rangle + H) dA. \end{aligned} \quad (2.18)$$

## 2.2 Covariance of Energy Balance

It turns out that in continuum mechanics (and even discrete systems) one can obtain all the balance laws using energy balance and postulating its invariance under some groups of transformations. This idea was introduced by Green and Rivlin (1964) in the case of Euclidean ambient spaces and was extended to manifolds by Marsden and Hughes (1983). See also Simo and Marsden (1984), Yavari et al. (2006), Yavari and Ozakin (2008), Yavari (2008), Yavari and Marsden (2009a, 2009b) for applications of covariance ideas in different continuous and discrete systems.

In order to covariantly obtain all the balance laws, we postulate that energy balance is form invariant under an arbitrary time-dependent spatial diffeomorphism

<sup>2</sup>Note that by definition

$$\frac{d}{dt} \int_{\mathcal{U}} \rho_0(X, t) dV = \int_{\mathcal{U}} S_m(X, t) dV = \int_{\mathcal{U}} \overset{\circ}{S}_m(X, t) d\overset{\circ}{V},$$

where  $\overset{\circ}{S}_m(X, t)$  is mass source in the initial material manifold with volume element  $d\overset{\circ}{V}$ . Note also that physically  $\overset{\circ}{S}$  is given.

<sup>3</sup>Note that in Lubarda and Hoger (2002) the term analogous to  $\frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right)$  is denoted by  $r_g$ . There, instead of the term  $\rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t}$  they introduce a term  $\rho \mathcal{R}_g r_g$ . One should note that even if mass is conserved at a point, still a change in shape can contribute to energy balance and is captured in our formulation. See also Epstein and Maugin (2000) and Lubarda and Hoger (2002).



$\xi_t : \mathcal{S} \rightarrow \mathcal{S}$ , i.e.

$$\begin{aligned} & \frac{d}{dt} \int_{\mathcal{U}} \rho'_0 \left( E' + \frac{1}{2} \langle \langle \mathbf{V}', \mathbf{V}' \rangle \rangle \right) dV \\ &= \int_{\mathcal{U}} \left\{ \rho'_0 \left( \langle \langle \mathbf{B}', \mathbf{V}' \rangle \rangle + R' \right) + \rho'_0 \frac{\partial E'}{\partial \mathbf{G}'} : \frac{\partial \mathbf{G}'}{\partial t} \right. \\ & \quad \left. + S'_m \left( E' + \frac{1}{2} \langle \langle \mathbf{V}', \mathbf{V}' \rangle \rangle \right) \right\} dV + \int_{\partial \mathcal{U}} \left( \langle \langle \mathbf{T}', \mathbf{V}' \rangle \rangle + H' \right) dA. \end{aligned} \tag{2.19}$$

Note that (Yavari et al. 2006)

$$R' = R, \quad H' = H, \quad \rho'_0 = \rho_0, \quad \mathbf{T}' = \xi_{t*} \mathbf{T}, \quad \mathbf{V}' = \xi_{t*} \mathbf{V} + \mathbf{W}, \tag{2.20}$$

where  $\mathbf{W} = \frac{\partial}{\partial t} \xi_t \circ \varphi$ . Note also that

$$\mathbf{G}' = \mathbf{G}, \quad \frac{\partial \mathbf{G}'}{\partial t} = \frac{\partial \mathbf{G}}{\partial t} \quad \text{and} \quad E'(\mathbf{X}, \mathbf{N}', \mathbf{G}, \mathbf{F}', \mathbf{g} \circ \varphi') = E(\mathbf{X}, \mathbf{N}, \mathbf{G}, \mathbf{F}, \xi_t^* \mathbf{g} \circ \varphi). \tag{2.21}$$

Thus, at  $t = t_0$

$$\frac{d}{dt} E' = \frac{\partial E}{\partial \mathbf{N}} : \frac{d\mathbf{N}}{dt} + \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial E}{\partial \mathbf{g} \circ \varphi} : \mathcal{L}_W \mathbf{g} \circ \varphi. \tag{2.22}$$

We also assume that body forces are transformed such that (Marsden and Hughes 1983)  $\mathbf{B}' - \mathbf{A}' = \xi_{t*}(\mathbf{B} - \mathbf{A})$ . Therefore, (2.19) at  $t = t_0$  reads

$$\begin{aligned} & \int_{\mathcal{U}} \left[ \frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) \right] \left( E + \frac{1}{2} \langle \langle \mathbf{V} + \mathbf{W}, \mathbf{V} + \mathbf{W} \rangle \rangle \right) dV \\ & \quad + \int_{\mathcal{U}} \rho_0 \left( \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial E}{\partial \mathbf{g} \circ \varphi} : \mathcal{L}_W \mathbf{g} \circ \varphi + \langle \langle \mathbf{V} + \mathbf{W}, \mathbf{A} \rangle \rangle \right) dV \\ &= \int_{\mathcal{U}} \left\{ \rho_0 \left( \langle \langle \mathbf{B}, \mathbf{V} + \mathbf{W} \rangle \rangle + R \right) + \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} \right. \\ & \quad \left. + S_m \left( E + \frac{1}{2} \langle \langle \mathbf{V} + \mathbf{W}, \mathbf{V} + \mathbf{W} \rangle \rangle \right) \right\} dV + \int_{\partial \mathcal{U}} \left( \langle \langle \mathbf{T}, \mathbf{V} + \mathbf{W} \rangle \rangle + H \right) dA. \end{aligned} \tag{2.23}$$

Subtracting (2.18) from (2.23), one obtains

$$\begin{aligned} & \int_{\mathcal{U}} \left[ \frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) - S_m \right] \left( \frac{1}{2} \langle \langle \mathbf{W}, \mathbf{W} \rangle \rangle + \langle \langle \mathbf{V}, \mathbf{W} \rangle \rangle \right) dV \\ & \quad + \int_{\mathcal{U}} \rho_0 \left( \frac{\partial E}{\partial \mathbf{g} \circ \varphi} : \mathcal{L}_W \mathbf{g} \circ \varphi + \langle \langle \mathbf{A}, \mathbf{W} \rangle \rangle \right) dV \\ &= \int_{\mathcal{U}} \rho_0 \left( \langle \langle \mathbf{B}, \mathbf{W} \rangle \rangle \right) dV + \int_{\partial \mathcal{U}} \langle \langle \mathbf{T}, \mathbf{W} \rangle \rangle dA. \end{aligned} \tag{2.24}$$

From this and the arbitrariness of  $\mathcal{U}$  and  $\mathbf{W}$  we conclude that (Yavari et al. 2006)

$$\frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) = S_m, \tag{2.25}$$

$$\operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}, \tag{2.26}$$

$$2 \rho_0 \frac{\partial E}{\partial \mathbf{g} \circ \varphi} = \boldsymbol{\tau}, \tag{2.27}$$

$$\boldsymbol{\tau}^\top = \boldsymbol{\tau}, \tag{2.28}$$

where  $\mathbf{P}$  is the first Piola–Kirchhoff stress and  $\boldsymbol{\tau} = J \boldsymbol{\sigma}$  is the Kirchhoff stress. It is seen that instead of conservation of mass we have a balance of mass and the remaining balance laws are unchanged. Note, however, that divergence and acceleration both explicitly depend on  $\mathbf{G}$ , i.e. the time dependency of material metric affects the governing balance equations. We will see examples in Sect. 2.9.

### 2.3 Local Form of Energy Balance

Let us now localize the energy balance. First note that

$$\frac{d}{dt} E = \mathbf{L}_V E = \frac{\partial E}{\partial \mathbf{N}} \frac{d\mathbf{N}}{dt} + \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial E}{\partial \mathbf{F}} : \mathbf{L}_V \mathbf{F} + \frac{\partial E}{\partial \mathbf{g}} : \mathbf{L}_V \mathbf{g} \circ \varphi. \tag{2.29}$$

Note that  $\mathbf{L}_V \mathbf{F} = \mathbf{0}$  because for an arbitrary  $\mathbf{Z} \in T_X \mathcal{B}$

$$\mathbf{L}_V \mathbf{F} = \frac{\partial}{\partial t} \varphi^* (\mathbf{F} \cdot \mathbf{Z}) = \frac{\partial}{\partial t} \varphi^* (\varphi_* \mathbf{Z}) = \frac{\partial}{\partial t} \mathbf{Z} = \mathbf{0}. \tag{2.30}$$

Using this and also noting that because the background metric is time independent, we have

$$\frac{d}{dt} E = \frac{\partial E}{\partial \mathbf{N}} \frac{d\mathbf{N}}{dt} + \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial E}{\partial \mathbf{g}} : \mathbf{d}, \tag{2.31}$$

where  $\mathbf{d} = \frac{1}{2} \mathcal{L}_V \mathbf{g} \circ \varphi$  is the rate of deformation tensor.<sup>4</sup> We know that  $H = -\langle\langle \mathbf{Q}, \hat{\mathbf{N}} \rangle\rangle$  and (Yavari et al. 2006)

$$\int_{\partial \mathcal{U}} \langle\langle \mathbf{T}, \mathbf{V} \rangle\rangle dA = \int_{\mathcal{U}} (\langle\langle \operatorname{Div} \mathbf{P}, \mathbf{V} \rangle\rangle + \boldsymbol{\tau} : \boldsymbol{\Omega} + \boldsymbol{\tau} : \mathbf{d}) dV, \tag{2.32}$$

where  $\Omega_{ab} = \frac{1}{2} (V_{a|b} - V_{b|a})$ , and  $\boldsymbol{\tau}$  is Kirchhoff stress. Thus, from (2.18) and using the balances of linear and angular momenta we obtain the local form of energy balance:

$$\rho_0 \frac{dE}{dt} + \operatorname{Div} \mathbf{Q} = \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\tau} : \mathbf{d} + \rho_0 R. \tag{2.33}$$

<sup>4</sup>This is the symmetric part of  $\nabla \mathbf{v}$ , i.e. the symmetric part of the so-called “velocity gradient”.

In terms of the first Piola–Kirchhoff stress, this can be written as

$$\rho_0 \frac{dE}{dt} + \text{Div } \mathbf{Q} = \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \mathbf{P} : \nabla_0 \mathbf{V} + \rho_0 R, \tag{2.34}$$

where  $\mathbf{P} : \nabla_0 \mathbf{V} = P^{aA} V^a|_A$ .

*Material Metric Evolution* The evolution of the material metric is assumed to be given through a kinetic equation of the form<sup>5</sup>

$$\frac{\partial \mathbf{G}}{\partial t} = \Upsilon(\mathbf{X}, \mathbf{G}, \mathbf{F}, \mathbf{g}) = \Phi(\mathbf{X}, \mathbf{G}, \mathbf{C}). \tag{2.35}$$

See Ambrosi and Mollica (2004), Lorete and Simoes (2005), Fusi et al. (2006), Ambrosi and Guana (2007) for some examples written in terms of the evolution of  $\mathbf{F}_g$ . We will come back to this problem after first discussing the Second Law of Thermodynamics for a growing body.

### 2.4 The Second Law of Thermodynamics and Restrictions on Constitutive Equations

In the absence of growth, the entropy production inequality in material coordinates has the following form (Coleman and Noll 1963):

$$\frac{d}{dt} \int_{\mathcal{U}} \rho_0 \mathbf{N} dV \geq \int_{\mathcal{U}} \frac{\rho_0 R}{\Theta} dV + \int_{\partial \mathcal{U}} \frac{H}{\Theta} dA, \tag{2.36}$$

where  $\mathbf{N} = \mathbf{N}(\mathbf{X}, t)$  is the material entropy density and  $\Theta = \Theta(\mathbf{X}, t)$  is the absolute temperature. This is called the Clausius–Duhem inequality.

When the material metric is time dependent, using balance of mass, the Clausius–Duhem inequality is modified to read<sup>6</sup>

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{U}} \rho_0 \mathbf{N} dV &\geq \int_{\mathcal{U}} \frac{\rho_0 R}{\Theta} dV + \int_{\partial \mathcal{U}} \frac{H}{\Theta} dA + \int_{\mathcal{U}} \mathbf{N} \left[ \frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \text{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) \right] dV \\ &+ \int_{\mathcal{U}} \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} dV. \end{aligned} \tag{2.37}$$

This inequality can be localized to read

$$\rho_0 \frac{d\mathbf{N}}{dt} \geq \frac{\rho_0 R}{\Theta} - \text{Div} \left( \frac{\mathbf{Q}}{\Theta} \right) + \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t}. \tag{2.38}$$

<sup>5</sup>A simple example that has been extensively studied is the Ricci flow (Hamilton 1982), which reads

$$\frac{\partial \mathbf{G}}{\partial t} = -2\mathbf{R},$$

where  $\mathbf{R}$  is the Ricci curvature of the metric  $\mathbf{G}$ . Note that this flow smooths out a distorted 3-sphere (Topping 2006).

<sup>6</sup>See also Epstein and Maugin (2000).

Note that  $E = \Psi + N\Theta$  and hence<sup>7</sup>

$$\frac{d}{dt}N = \frac{1}{\Theta} \left( \frac{dE}{dt} - \frac{d\Psi}{dt} \right) - \frac{\dot{\Theta}}{\Theta^2} (E - \Psi). \tag{2.39}$$

Substituting this into (2.38) yields

$$\frac{\rho_0}{\Theta} \frac{dE}{dt} - \frac{\rho_0}{\Theta} \frac{d\Psi}{dt} - \rho_0 \frac{\dot{\Theta}}{\Theta^2} (E - \Psi) \geq \frac{\rho_0 R}{\Theta} - \text{Div} \left( \frac{\mathbf{Q}}{\Theta} \right) + \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t}. \tag{2.40}$$

Now substituting the local energy balance (2.34) into the above inequality we obtain

$$\mathbf{P} : \nabla_0 \mathbf{V} - \rho_0 \frac{d\Psi}{dt} - \rho_0 N \dot{\Theta} \geq \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q}. \tag{2.41}$$

We know that  $\Psi = \Psi(\mathbf{X}, \Theta, \mathbf{G}, \mathbf{F}, \mathbf{g} \circ \varphi)$  and thus<sup>8</sup>

$$\frac{d\Psi}{dt} = \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_0 \mathbf{V}. \tag{2.42}$$

Therefore, (2.41) is simplified to read

$$\rho_0 \left( \frac{\partial \Psi}{\partial \Theta} + N \right) \dot{\Theta} + \left( \rho_0 \frac{\partial \Psi}{\partial \mathbf{F}} - \mathbf{P} \right) : \nabla_0 \mathbf{V} + \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q} + \rho_0 \frac{\partial \Psi}{\partial \mathbf{G}} : \dot{\mathbf{G}} \leq 0. \tag{2.43}$$

Following Coleman and Noll (1963) and Marsden and Hughes (1983) we conclude that

$$\frac{\partial \Psi}{\partial \Theta} = -N \quad \text{and} \quad \rho_0 \frac{\partial \Psi}{\partial \mathbf{F}} = \mathbf{P}, \tag{2.44}$$

and the entropy production inequality reduces to  $\rho_0 \frac{\partial \Psi}{\partial \mathbf{G}} : \dot{\mathbf{G}} + \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q} \leq 0$ .

*Remark* There have been objections in the literature on using the Clausius–Duhem inequality in continuum mechanics (Green and Naghdi 1977; Marsden and Hughes 1983). Next, we show that in growth mechanics energy balance and a more general

<sup>7</sup>Note that  $\Psi = \Psi(\mathbf{X}, \Theta, \mathbf{G}, \mathbf{F}, \mathbf{g} \circ \varphi)$  and  $E = E(\mathbf{X}, N, \mathbf{G}, \mathbf{F}, \mathbf{g} \circ \varphi)$ .

<sup>8</sup>Nota that because  $\Psi$  is a scalar its time derivative is equal to its Lie derivative along the velocity vector field, and thus when the material metric is time independent we have

$$\frac{d}{dt} \Psi = \mathbf{L}_V \Psi = \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \Psi}{\partial \mathbf{F}} : \mathbf{L}_V \mathbf{F} + \frac{\partial \Psi}{\partial \mathbf{g}} : \mathbf{L}_V \mathbf{g} = \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \Psi}{\partial \mathbf{g}} : \mathbf{L}_V \mathbf{g}.$$

We also know that the same time derivative is equal to the covariant derivative of N with respect to velocity vector, i.e.

$$\begin{aligned} \frac{d}{dt} \Psi &= \frac{\partial \Psi}{\partial t} \Psi + \nabla_V \Psi = \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_V \mathbf{F} + \frac{\partial \Psi}{\partial \mathbf{g}} : \nabla_V \mathbf{g} \\ &= \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_{\frac{\partial}{\partial t}} \mathbf{F} = \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_0 \mathbf{V}, \end{aligned}$$

where we used the fact that  $\frac{\partial}{\partial t} \Psi = 0$  and  $\nabla_{\frac{\partial}{\partial t}} \mathbf{F} = \nabla_0 \mathbf{V}$ . See Nishikawa (2002) for a proof.

notion of covariance are enough to obtain the restrictions (2.44) on the constitutive equations. In passing we should mention that Green and Naghdi (1991) were able to obtain their entropy balance using energy balance and invariance arguments in the case of Euclidean ambient space. What we will show next is consistent with their results.

### 2.5 Restrictions on Constitutive Equations Using a Thermomechanical Covariance of Energy Balance

In this subsection, we follow Marsden and Hughes (1983) and obtain the restrictions (2.44) on the constitutive equations using the covariance of the energy balance with no reference to the entropy production inequality. In the case of classical elasticity, Marsden and Hughes (1983) started with the local form of energy balance and postulated its covariance under the simultaneous action of time-dependent spatial diffeomorphisms and time-dependent monotonically increasing temperature rescalings. Here we start with the integral form of the energy balance.

Let us consider spatial diffeomorphisms  $\xi_t : \mathcal{S} \rightarrow \mathcal{S}$  and monotonically increasing temperature rescalings  $\zeta_t : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ . We assume that at  $t = t_0$ ,  $\zeta = 1$  and  $\frac{d}{dt}\zeta_t = z$ . We also assume that under these transformations, the energy balance is invariant, i.e.

$$\begin{aligned} & \frac{d}{dt} \int_{\mathcal{U}} \rho'_0 \left( E' + \frac{1}{2} \langle\langle \mathbf{V}', \mathbf{V}' \rangle\rangle \right) dV \\ &= \int_{\mathcal{U}} \left\{ \rho'_0 \left( \langle\langle \mathbf{B}', \mathbf{V}' \rangle\rangle + R' \right) + \rho'_0 \frac{\partial E'}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + S'_m \left( E' + \frac{1}{2} \langle\langle \mathbf{V}', \mathbf{V}' \rangle\rangle \right) \right\} dV \\ &+ \int_{\partial \mathcal{U}} \left( \langle\langle \mathbf{T}', \mathbf{V}' \rangle\rangle + H' \right) dA. \end{aligned} \tag{2.45}$$

Note that  $E = \Psi + \Theta N$ . In the new frame  $\varphi' = \xi \circ \varphi$  and  $\Theta' = \zeta \Theta$ . We assume that  $E$  transforms tensorially, i.e.

$$E'(\mathbf{X}, N', \mathbf{G}, \mathbf{F}', \mathbf{g}') = E(\mathbf{X}, N, \mathbf{G}, \xi_* \mathbf{F}, \xi_* \mathbf{g}). \tag{2.46}$$

The same transformation is assumed for the free energy density, and hence

$$\frac{d}{dt} \Psi' = \frac{d}{dt} \Psi(\mathbf{X}, \zeta \Theta, \mathbf{G}, \xi_* \mathbf{F}, \xi_* \mathbf{g}) = \frac{\partial \Psi}{\partial \Theta'} \dot{\Theta}' + \frac{\partial \Psi}{\partial \mathbf{G}} : \dot{\mathbf{G}} + \frac{\partial \Psi}{\partial \xi_* \mathbf{F}} : \nabla_0 \mathbf{V}', \tag{2.47}$$

where  $(\nabla_0 \mathbf{V})^a_A = V^a_{|A}$ . Therefore, at  $t = t_0$

$$\frac{d}{dt} \Psi' = \frac{\partial \Psi}{\partial \Theta} (\dot{\Theta} + z\Theta) + \frac{\partial \Psi}{\partial \mathbf{G}} : \dot{\mathbf{G}} + \frac{\partial \Psi}{\partial \mathbf{F}} : (\nabla_0 \mathbf{V} + \nabla_0 \mathbf{W}). \tag{2.48}$$

Thus, at  $t = t_0$ , we can write

$$\frac{d}{dt} E' = \frac{d}{dt} E + \frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_0 \mathbf{W} + z \left( \frac{\partial \Psi}{\partial \Theta} + N \right) \Theta + \left( \frac{dN'}{dt} \Theta' - \frac{dN}{dt} \Theta \right)_{t=t_0}. \tag{2.49}$$

Energy balance in the new frame at  $t = t_0$  is simplified to read

$$\begin{aligned}
 & \int_{\mathcal{U}} \left[ \frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) \right] \left( E + \frac{1}{2} \langle \mathbf{V}, \mathbf{V} \rangle + \langle \mathbf{V}, \mathbf{W} \rangle + \frac{1}{2} \langle \mathbf{W}, \mathbf{W} \rangle \right) dV \\
 & + \int_{\mathcal{U}} \rho_0 \left[ \frac{d}{dt} E + \frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_0 \mathbf{W} + z \left( \frac{\partial \Psi}{\partial \Theta} + \mathbf{N} \right) \Theta + \left( \frac{d\mathbf{N}'}{dt} \Theta' - \frac{d\mathbf{N}}{dt} \Theta \right) \right]_{t=t_0} \\
 & + \langle \mathbf{V} + \mathbf{W}, \mathbf{A}'|_{t=t_0} \rangle dV \\
 & = \int_{\mathcal{U}} \left[ \rho_0 \left( \langle \mathbf{B}'|_{t=t_0}, \mathbf{V} + \mathbf{W} \rangle + R'|_{t=t_0} \right) + \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} \right. \\
 & + S_m \left( E + \frac{1}{2} \langle \mathbf{V}, \mathbf{V} \rangle + \langle \mathbf{V}, \mathbf{W} \rangle + \frac{1}{2} \langle \mathbf{W}, \mathbf{W} \rangle \right) \left. \right] dV \\
 & + \int_{\partial \mathcal{U}} \left( \langle \mathbf{T}, \mathbf{V} + \mathbf{W} \rangle + H'|_{t=t_0} \right) dA. \tag{2.50}
 \end{aligned}$$

Assuming that  $\mathbf{B}' - \mathbf{A}' = \xi_* (\mathbf{B} - \mathbf{A})$  and subtracting the balance of energy (2.18) from (2.45), we obtain

$$\begin{aligned}
 & \int_{\mathcal{U}} \left[ \frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) - S_m \right] \left( \langle \mathbf{V}, \mathbf{W} \rangle + \frac{1}{2} \langle \mathbf{W}, \mathbf{W} \rangle \right) dV \\
 & + \int_{\mathcal{U}} \rho_0 \left[ \frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_0 \mathbf{W} + z \left( \frac{\partial \Psi}{\partial \Theta} + \mathbf{N} \right) \Theta + \left( \frac{d\mathbf{N}'}{dt} \Theta' - \frac{d\mathbf{N}}{dt} \Theta \right) \right]_{t=t_0} \\
 & + \langle \mathbf{W}, \mathbf{A} \rangle dV \\
 & = \int_{\mathcal{U}} \rho_0 \left[ \langle \mathbf{B}, \mathbf{W} \rangle + (R' - R)|_{t=t_0} \right] dV \\
 & + \int_{\mathcal{U}} \left[ \langle \operatorname{Div} \mathbf{P}, \mathbf{W} \rangle + \boldsymbol{\tau} : (\nabla \mathbf{W})^b - (\operatorname{Div} \mathbf{Q}' - \operatorname{Div} \mathbf{Q})|_{t=t_0} \right] dV, \tag{2.51}
 \end{aligned}$$

where  $\boldsymbol{\tau} : (\nabla \mathbf{W})^b = \tau^{ab} W_{a|b}$ . Assuming that  $R$  and  $\mathbf{Q}$  are transformed such that (Marsden and Hughes 1983)

$$\frac{d\mathbf{N}'}{dt} \Theta' - R' = \zeta \left( \frac{d\mathbf{N}}{dt} \Theta - R \right) \quad \text{and} \quad \mathbf{Q}' = \zeta \xi_* \mathbf{Q}, \tag{2.52}$$

we obtain

$$\begin{aligned}
 & \int_{\mathcal{U}} \left[ \frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) - S_m \right] \left( \langle \mathbf{V}, \mathbf{W} \rangle + \frac{1}{2} \langle \mathbf{W}, \mathbf{W} \rangle \right) dV \\
 & - \int_{\mathcal{U}} \langle \operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} - \rho_0 \mathbf{A}, \mathbf{W} \rangle dV + \int_{\mathcal{U}} \left[ \left( \rho_0 \frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_0 \mathbf{W} - \boldsymbol{\tau} : (\nabla \mathbf{W})^b \right) \right. \\
 & \left. + z \rho_0 \left( \frac{\partial \Psi}{\partial \Theta} + \mathbf{N} \right) \Theta \right] dV = 0. \tag{2.53}
 \end{aligned}$$

Note that

$$(\nabla_0 \mathbf{W})^a{}_A = g^{ab} [(\nabla \mathbf{W})^b]_{bc} F^c{}_A. \tag{2.54}$$

Therefore, the arbitrariness of  $\mathcal{U}$ ,  $\mathbf{W}$ , and  $z$  implies that<sup>9</sup>

$$\frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) = S_m, \tag{2.55}$$

$$\operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}, \tag{2.56}$$

$$\boldsymbol{\tau}^\top = \boldsymbol{\tau}, \tag{2.57}$$

$$\rho_0 \frac{\partial \Psi}{\partial \mathbf{F}} = \mathbf{P}, \tag{2.58}$$

$$\frac{\partial \Psi}{\partial \Theta} = -\mathbf{N}. \tag{2.59}$$

Thus, we have proven the following proposition.

**Proposition** *Covariance of energy balance under spatial diffeomorphisms and temperature rescalings gives all the balance laws and the constitutive restrictions imposed by the Clausius–Duhem inequality.*

### 2.6 Covariance of the Entropy Production Inequality

In this subsection we study the consequences of covariance of the Clausius–Duhem inequality. Again, let us consider the diffeomorphisms  $\xi_t : \mathcal{S} \rightarrow \mathcal{S}$  and monotonically increasing temperature rescalings  $\zeta_t : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ . Let us postulate that the entropy production inequality is invariant under the simultaneous action of these two trans-

<sup>9</sup>In a previous footnote it was shown that

$$\frac{d}{dt} \Psi = \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_0 \mathbf{V} = \frac{\partial \Psi}{\partial \Theta} \dot{\Theta} + \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \Psi}{\partial \mathbf{g}} : \mathcal{L}_{\mathbf{V}} \mathbf{g}.$$

Thus

$$\frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_0 \mathbf{V} = \frac{\partial \Psi}{\partial \mathbf{g}} : \mathcal{L}_{\mathbf{V}} \mathbf{g}.$$

This holds for an arbitrary change of frame  $\xi_t : \mathcal{S} \rightarrow \mathcal{S}$  as well, i.e. at  $t = t_0$ :

$$\frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_0 (\mathbf{V} + \mathbf{W}) = \frac{\partial \Psi}{\partial \mathbf{g}} : \mathcal{L}_{(\mathbf{V} + \mathbf{W})} \mathbf{g}.$$

Hence, for arbitrary  $\mathbf{W}$

$$\frac{\partial \Psi}{\partial \mathbf{F}} : \nabla_0 \mathbf{W} = \frac{\partial \Psi}{\partial \mathbf{g}} : \mathcal{L}_{\mathbf{W}} \mathbf{g}.$$

Noting that (Marsden and Hughes 1983)  $\frac{\partial E}{\partial \mathbf{g} \circ \varphi} = \frac{\partial \Psi}{\partial \mathbf{g} \circ \varphi}$ , this means that  $\rho_0 \frac{\partial \Psi}{\partial \mathbf{F}} = \mathbf{P}$  is equivalent to  $2\rho_0 \frac{\partial E}{\partial \mathbf{g} \circ \varphi} = \boldsymbol{\tau}$ , i.e. we have the Doyle–Ericksen formula.

formations, i.e.

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{U}} \rho'_0 \mathbf{N}' dV &\geq \int_{\mathcal{U}} \frac{\rho'_0 R'}{\Theta'} dV + \int_{\partial \mathcal{U}} \frac{H'}{\Theta'} dA + \int_{\mathcal{U}} \mathbf{N}' \left[ \frac{\partial \rho'_0}{\partial t} + \frac{1}{2} \rho'_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) \right] dV \\ &+ \int_{\mathcal{U}} \rho'_0 \frac{\partial E'}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} dV. \end{aligned} \quad (2.60)$$

Note that (2.52) implies that

$$\frac{d\mathbf{N}'}{dt} - \frac{R'}{\Theta'} = \frac{d\mathbf{N}}{dt} - R \quad \text{and} \quad \frac{H'}{\Theta'} = \frac{H}{\Theta}. \quad (2.61)$$

It can easily be shown that the inequality (2.60) is identical to (2.37), i.e. assuming the transformations (2.52), entropy production inequality is trivially covariant.

## 2.7 Principle of Maximum Entropy Production

In this subsection we use the so-called maximum entropy production principle to obtain a kinetic equation for  $\dot{\mathbf{G}}$ . This principle states that a non-equilibrium system with some possible constraints evolves in such a way as to maximize its entropy production (Ziegler 1983; Rajagopal and Srinivasa 2004a). This principle has found applications in many different fields of science. For a recent review see Martyushev and Seleznev (2006). This principle has recently been used in growth mechanics for obtaining kinetic equations for the “growth velocity gradient” (Loret and Simoes 2005; Ambrosi and Guana 2007; Fusi et al. 2006). Here, we use it in our geometric framework.

For a growing body, entropy production in a subbody  $\mathcal{U} \subset \mathcal{B}$  is defined as

$$\begin{aligned} \Gamma(\mathcal{U}, t) &= \frac{d}{dt} \int_{\mathcal{U}} \rho_0 \mathbf{N} dV - \int_{\mathcal{U}} \frac{\rho_0 R}{\Theta} dV - \int_{\partial \mathcal{U}} \frac{H}{\Theta} dA \\ &- \int_{\mathcal{U}} \mathbf{N} \left[ \frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \left( \frac{\partial \mathbf{G}}{\partial t} \right) \right] dV - \int_{\mathcal{U}} \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} dV \\ &= \int_{\mathcal{U}} \left[ \rho_0 \frac{d\mathbf{N}}{dt} - \rho_0 \frac{R}{\Theta} + \operatorname{Div} \left( \frac{\mathbf{Q}}{\Theta} \right) - \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} \right] dV \\ &= \int_{\mathcal{U}} \frac{\Lambda}{\Theta} dV, \end{aligned} \quad (2.62)$$

where  $\Lambda = \Theta \left[ \rho_0 \frac{d\mathbf{N}}{dt} - \rho_0 \frac{R}{\Theta} + \operatorname{Div} \left( \frac{\mathbf{Q}}{\Theta} \right) - \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} \right]$  is the rate of entropy production. Using energy balance, we have

$$\Lambda = -\rho_0 \frac{d\Psi}{dt} + \mathbf{P} : \nabla_0 \mathbf{V} - \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q} - \dot{\Theta} \mathbf{N}. \quad (2.63)$$

Note that

$$-\rho_0 \frac{d\Psi}{dt} = \dot{\Theta} \mathbf{N} - \rho_0 \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} - \mathbf{P} : \nabla_0 \mathbf{V}. \quad (2.64)$$



Thus, we can write

$$\Lambda = -\rho_0 \frac{\partial \Psi}{\partial \mathbf{G}} : \dot{\mathbf{G}} - \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q}. \tag{2.65}$$

We now maximize  $\Lambda$  with respect to  $\dot{\mathbf{G}}$  under the constraint (2.65). Let us define

$$\Phi = \Lambda + \lambda \left( \Lambda + \rho_0 \frac{\partial \Psi}{\partial \mathbf{G}} : \dot{\mathbf{G}} + \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q} \right), \tag{2.66}$$

where  $\lambda$  is a Lagrange multiplier. Maximizing  $\Phi$  with respect to  $\dot{\mathbf{G}}$  gives

$$\frac{\partial \Lambda}{\partial \dot{\mathbf{G}}} = -\frac{\lambda}{\lambda + 1} \rho_0 \frac{\partial \Psi}{\partial \mathbf{G}}. \tag{2.67}$$

Note that part of the entropy production rate is constitutively given, i.e.  $\Lambda = \bar{\Lambda}(\Theta, \mathbf{G}, \dot{\mathbf{G}}, \mathbf{F}, \mathbf{g}) - \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q}$ . As the simplest example let us assume that

$$\Lambda = \beta \operatorname{tr} \dot{\mathbf{G}}^2 - \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q} = \beta \dot{G}_{AB} \dot{G}_{CD} G^{AC} G^{BD} - \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q}. \tag{2.68}$$

Thus

$$\frac{\partial \Lambda}{\partial \dot{G}_{AB}} = 2\beta G^{AC} G^{BD} \dot{G}_{CD}. \tag{2.69}$$

Or

$$\dot{\mathbf{G}}^\sharp = \frac{1}{2\beta} \frac{\partial \Lambda}{\partial \dot{\mathbf{G}}} = -\frac{\lambda}{2\beta(\lambda + 1)} \rho_0 \frac{\partial \Psi}{\partial \mathbf{G}}. \tag{2.70}$$

Using (2.70) and (2.68) we can write

$$\Lambda = \frac{\lambda^2}{4\beta(\lambda + 1)^2} \rho_0^2 \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \Psi}{\partial \mathbf{G}} - \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q}. \tag{2.71}$$

At the same time using (2.70) and (2.65) we have

$$\Lambda = \frac{\lambda}{2\beta(\lambda + 1)} \rho_0^2 \frac{\partial \Psi}{\partial \mathbf{G}} : \frac{\partial \Psi}{\partial \mathbf{G}} - \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q}. \tag{2.72}$$

Looking at (2.71) and (2.72) we see that  $\lambda = -2$  and hence

$$\dot{\mathbf{G}}^\sharp = -\frac{1}{\beta} \rho_0 \frac{\partial \Psi}{\partial \mathbf{G}}. \tag{2.73}$$

### 2.8 Isotropic Growth

For isotropic growth, the material metric has the following time dependent form:

$$\mathbf{G}(\mathbf{X}, t) = e^{2\Omega(\mathbf{X}, t)} \mathbf{G}_0(\mathbf{X}), \tag{2.74}$$

i.e. a family of conformal material metrics model the growth. Thus

$$\frac{\partial \mathbf{G}(\mathbf{X}, t)}{\partial t} = 2 \frac{\partial \Omega}{\partial t} \mathbf{G}(\mathbf{X}, t). \quad (2.75)$$

Therefore, balance of mass is simplified to read

$$\frac{\partial \rho_0(\mathbf{X}, t)}{\partial t} + \rho_0(\mathbf{X}, t) \frac{\partial \Omega(\mathbf{X}, t)}{\partial t} e^{2\Omega(\mathbf{X}, t)} \operatorname{tr} \mathbf{G}_0(\mathbf{X}) = S_m(\mathbf{X}, t). \quad (2.76)$$

Given  $\mathbf{G} = \mathbf{G}(\mathbf{X}, t)$ , one has the following relation between volume elements at  $t_0$  and  $t$ :

$$dV(\mathbf{X}, t) = \sqrt{\frac{\det \mathbf{G}(\mathbf{X}, t)}{\det \mathbf{G}(\mathbf{X}, t_0)}} dV(\mathbf{X}, t_0). \quad (2.77)$$

Or

$$dV(\mathbf{X}, t) = e^{N\Omega(\mathbf{X}, t)} dV_0(\mathbf{X}), \quad (2.78)$$

where  $N = \dim \mathcal{B}$ . The mass form has the following representation:

$$m(\mathbf{X}, t) = \rho_0(\mathbf{X}, t) dV(\mathbf{X}, t) = e^{N\Omega(\mathbf{X}, t)} m_0(\mathbf{X}). \quad (2.79)$$

Note that  $\dot{m}(\mathbf{X}, t) = N \frac{\partial \Omega}{\partial t} m(\mathbf{X}, t)$ . The mass of a subbody  $\mathcal{U} \subset \mathcal{B}$  will have the following time-dependent form:

$$M_t(\mathcal{U}) = \int_{\mathcal{U}} m(\mathbf{X}, t). \quad (2.80)$$

Hence

$$\frac{d}{dt} M_t(\mathcal{U}) = \int_{\mathcal{U}} N \frac{\partial \Omega}{\partial t} m(\mathbf{X}, t). \quad (2.81)$$

In the decomposition of the deformation gradient one has

$$\mathbf{F} = \mathbf{F}^c \mathbf{F}^g \quad \text{and} \quad \mathbf{F}^g = g(\mathbf{X}, t) \mathbf{I}, \quad (2.82)$$

where  $\mathbf{I}$  is the identity map and  $g$  is a scalar field. We will discuss this decomposition in more detail in Sect. 3. In the sequel we will obtain those isotropic growth distributions that are stress free. But let us first look at some simple examples of the evolution of the material metric.

## 2.9 Examples of Bulk Growth

In this subsection we look at three examples of bulk growth and show how analytical solutions for residual stresses can be generated for both isotropic and non-isotropic growth.

*Example 1* (Isotropic Growth of a Neo-Hookean Annulus) Let us consider a two-dimensional, incompressible neo-Hookean material in a flat two-dimensional spatial

manifold. The free energy density of a neo-Hookean material in two dimensions has the form

$$\Psi = \Psi(\mathbf{X}, \mathbf{C}) = \mu(\text{tr} \mathbf{C} - 2), \tag{2.83}$$

where  $\mathbf{C}$  is the Cauchy–Green tensor, or equivalently, the pull-back of the spatial metric,  $C_{AB} = F^a{}_A F^b{}_B g_{ab}$ , and  $\mu$  is a material constant. In components

$$\Psi = \mu(F^a{}_A F^b{}_B g_{ab} G^{AB} - 2). \tag{2.84}$$

The “2” is of no particular significance: when the material metric is fixed, it simply shifts the free energy by a constant. When the material metric changes, its contribution to the free energy is proportional to the time-dependent material volume, which, for a given growth distribution, is independent of the spatial configuration. We ignore this term, and use  $\Psi = \Psi(\mathbf{X}, \mathbf{C}) = \mu \text{tr} \mathbf{C}$  as our definition of the free energy.

Let us assume that initially the material has a flat annular shape  $R_1 \leq R \leq R_2$  without any stresses. We would like to calculate the stresses that occur in the new equilibrium configuration after a rotationally symmetric growth,  $\Omega = \Omega(R, t)$ . In polar coordinates, the spatial metric and its inverse read

$$\mathbf{g} = \begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad \mathbf{g}^{-1} = \begin{pmatrix} g^{rr} & g^{r\theta} \\ g^{\theta r} & g^{\theta\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}, \tag{2.85}$$

and thus  $\det \mathbf{g} = r^2$ . The only nonzero connection coefficients are  $\gamma_{\theta\theta}^r = -r$ ,  $\gamma_{r\theta}^\theta = \gamma_{\theta r}^\theta = 1/r$ . For the rotationally symmetric time-dependent material metric we have

$$\begin{aligned} \mathbf{G} &= \begin{pmatrix} G_{RR} & G_{R\Theta} \\ G_{\Theta R} & G_{\Theta\Theta} \end{pmatrix} = e^{2\Omega(R,t)} \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}, \\ \mathbf{G}^{-1} &= \begin{pmatrix} G^{RR} & G^{R\Theta} \\ G^{\Theta R} & G^{\Theta\Theta} \end{pmatrix} = e^{-2\Omega(R,t)} \begin{pmatrix} 1 & 0 \\ 0 & 1/R^2 \end{pmatrix}, \end{aligned} \tag{2.86}$$

and thus,  $\det \mathbf{G} = R^2 e^{4\Omega(R,t)}$ . The following nonzero connection coefficients are needed in the balance of linear momentum:

$$\begin{aligned} \Gamma_{RR}^R &= \Omega'(R, t), & \Gamma_{\Theta\Theta}^R &= -R - R^2 \Omega'(R, t), \\ \Gamma_{R\Theta}^\Theta &= \Gamma_{\Theta R}^\Theta = 1/R + \Omega'(R, t), \end{aligned} \tag{2.87}$$

where  $\Omega'(R, t) = \frac{\partial \Omega}{\partial R}$ . Given  $\Omega = \Omega(R, t)$ , we are looking for solutions of the form

$$\varphi(R, \Theta) = (r, \theta) = (r(R, t), \Theta). \tag{2.88}$$

Thus<sup>10</sup>

$$\mathbf{F} = \begin{pmatrix} r'(R, t) & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{F}^{-1} = \begin{pmatrix} 1/r'(R, t) & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.89}$$

This gives the Jacobian as

$$J = \frac{r r'}{R e^{2\Omega(R,t)}}. \tag{2.90}$$

Incompressibility dictates that

$$r r' = R e^{2\Omega(R,t)}. \tag{2.91}$$

This differential equation has the following solution:

$$r^2(R, t) = r_1^2(R, t) + \int_{R_1}^R 2\xi e^{2\Omega(\xi,t)} d\xi. \tag{2.92}$$

Note that  $r_1(R)$  is not known a priori and will be obtained after imposing the traction boundary conditions at  $r_1$  and  $r_2$ . In incompressible elasticity,  $P^{aA}$  is replaced by  $P^{aA} - J p (F^{-1})^{-A}{}_b g^{ab}$ , where  $p$  is an unknown scalar field (pressure) that will be determined using the constraint  $J = 1$  (Marsden and Hughes 1983), i.e.

$$P^{aA} = 2\mu F^a{}_B G^{AB} - p(R) (F^{-1})^A{}_b g^{ab}. \tag{2.93}$$

Therefore, using (2.91), we obtain the nonzero-stress components

$$P^{rR} = \frac{2\mu R}{r} - p(R) \frac{r}{R} e^{-2\Omega(R,t)} \quad \text{and} \quad P^{\theta\Theta} = \frac{2\mu}{R^2} e^{-2\Omega(R,t)} - \frac{p(R)}{r^2}, \tag{2.94}$$

where  $p(R)$  is an unknown pressure.

Balance of linear momentum in components reads

$$P^{aA}{}_{|A} = \frac{\partial P^{aA}}{\partial X^A} + \Gamma_{AB}^A P^{aB} + P^{bA} \gamma_{bc}^a F^c{}_A = 0. \tag{2.95}$$

For the radial direction,  $a = r$ , we have

$$\begin{aligned} P^{rA}{}_{|A} &= \frac{\partial P^{rA}}{\partial X^A} + \Gamma_{AB}^A P^{rB} + P^{bA} \gamma_{bc}^r F^c{}_A \\ &= \frac{\partial P^{rR}}{\partial R} + (\Gamma_{RR}^R + \Gamma_{\Theta R}^\Theta) P^{rR} + P^{\theta\Theta} \gamma_{\theta\theta}^r F^{\theta\Theta} \end{aligned}$$

<sup>10</sup>If one does not consider the intrinsic metric and instead uses the standard metric of the Euclidean space,  $\mathbf{F}$  has the following representation:

$$\mathbf{F} = \begin{pmatrix} r'(R, t) & 0 \\ 0 & \frac{r(R)}{R} \end{pmatrix} \quad \text{and} \quad \mathbf{F}_g = \begin{pmatrix} e^{\Omega(R,t)} & 0 \\ 0 & e^{\Omega(R,t)} \end{pmatrix}.$$

Thus,  $\det \mathbf{F}_e = \frac{r r'}{R} e^{-2\Omega}$  and hence  $J_e = 1$  is equivalent to (2.91), as expected.

$$= \frac{\partial P^{rR}}{\partial R} + \left( \frac{1}{R} + 2\Omega'(R, t) \right) P^{rR} - r P^{\theta\theta} = 0. \tag{2.96}$$

This gives

$$p'(R) = \frac{2\mu R}{r^2} e^{2\Omega(R,t)} \left[ 2(1 + R\Omega') - \frac{R^2}{r^2} e^{2\Omega(R,t)} - \frac{r^2}{R^2} e^{-2\Omega(R,t)} \right]. \tag{2.97}$$

Assuming that  $p(R_i) = 0$ , we obtain

$$p(R) = \int_{R_i}^R \frac{2\mu\xi}{r^2(\xi)} e^{2\Omega(\xi,t)} \left[ 2(1 + \xi\Omega'(\xi)) - \frac{\xi^2}{r^2(\xi)} e^{2\Omega(\xi,t)} - \frac{r^2(\xi)}{\xi^2} e^{-2\Omega(\xi,t)} \right] d\xi. \tag{2.98}$$

For  $a = \theta$ , balance of momentum, (2.95), gives

$$\begin{aligned} P^{\theta A} |_{A} &= \frac{\partial P^{\theta\theta}}{\partial \Theta} + \Gamma_{A\Theta}^A P^{\theta\theta} + P^{\theta R} \gamma_{rr}^{\theta} F^r_R + P^{\theta\theta} \gamma_{\theta\theta}^{\theta} F^{\theta}_{\Theta} \\ &= (\Gamma_{R\Theta}^R + \Gamma_{\Theta\Theta}^{\Theta}) P^{\theta\theta} = 0, \end{aligned} \tag{2.99}$$

i.e., this equilibrium equation is trivially satisfied. Note that  $\text{tr}(\frac{\partial \mathbf{G}}{\partial t}) = 4\Omega'$  and hence balance of mass reads

$$\frac{\partial \rho_0(R, t)}{\partial t} + 2\Omega'(R, t)\rho_0(R, t) = S_m(R, t). \tag{2.100}$$

This differential equation can be easily solved for mass density.

Note that if one considers a cylinder with the kinematics assumptions  $r = r(R, t), \theta = \Theta, z = kZ$  for a constant  $k$ , the residual stresses will be very similar to what was just calculated.

*Example 2 (Anisotropic Growth of a Neo-Hookean Annulus)* Let us consider an anisotropic growth represented by the following material metric:

$$\mathbf{G} = \begin{pmatrix} e^{2\Omega(R,t)} & 0 \\ 0 & R^2 e^{-2\Omega(R,t)} \end{pmatrix}, \quad \mathbf{G}^{-1} = \begin{pmatrix} e^{-2\Omega(R,t)} & 0 \\ 0 & 1/R^2 e^{2\Omega(R,t)} \end{pmatrix}, \tag{2.101}$$

and thus,  $\det \mathbf{G} = R^2$ . The following nonzero connection coefficients are needed in the balance of linear momentum:

$$\begin{aligned} \Gamma_{RR}^R &= \Omega'(R, t), & \Gamma_{\Theta\Theta}^R &= R e^{-4\Omega(R,t)} [\Omega'(R, t) - 1], \\ \Gamma_{R\Theta}^{\Theta} &= \Gamma_{\Theta R}^{\Theta} = 1/R - \Omega'(R, t). \end{aligned} \tag{2.102}$$

Given  $\Omega = \Omega(R, t)$ , we are looking for solutions of the form  $\varphi(R, \Theta) = (r, \theta) = (r(R, t), \Theta)$ . Thus,  $\mathbf{F}$  and  $\mathbf{F}^{-1}$  have the forms given in (2.89) and this gives the Jacobian as  $J = \frac{r r'}{R}$ . Incompressibility dictates that  $r r' = R$ .<sup>11</sup> This simple differential equation has the following solution:

$$r(R, t) = \sqrt{R^2 + C(t)} = \sqrt{R^2 - R_1^2 + r_1^2}. \tag{2.103}$$

<sup>11</sup>In the classical formulation

Note that  $r_1(R, t)$  is not known a priori and will be obtained after imposing the traction boundary conditions at  $r_1$  and  $r_2$ . Now, we get the nonzero-stress components, thus:

$$\begin{aligned}
 P^{rR} &= 2\mu e^{-2\Omega(R,t)} \frac{R}{r(R,t)} - p(R,t) \frac{r(R,t)}{R} \quad \text{and} \\
 P^{\theta\Theta} &= 2\mu \frac{e^{2\Omega(R,t)}}{R^2} - \frac{p(R,t)}{r^2(R,t)},
 \end{aligned}
 \tag{2.104}$$

where  $p(R, t)$  is an unknown pressure.

Balance of linear momentum in components reads

$$P^{aA}{}_{|A} = \frac{\partial P^{aA}}{\partial X^A} + \Gamma_{AB}^A P^{aB} + P^{bA} \gamma_{bc}^a F^c{}_A = 0.
 \tag{2.105}$$

For the radial direction,  $a = r$ , we have

$$\begin{aligned}
 P^{rA}{}_{|A} &= \frac{\partial P^{rA}}{\partial X^A} + \Gamma_{AB}^A P^{rB} + P^{bA} \gamma_{bc}^r F^c{}_A \\
 &= \frac{\partial P^{rR}}{\partial R} + (\Gamma_{RR}^R + \Gamma_{\Theta R}^\Theta) P^{rR} + P^{\theta\Theta} \gamma_{\theta\theta}^r F^\theta{}_\Theta \\
 &= \frac{\partial P^{rR}}{\partial R} + \frac{1}{R} P^{rR} - r P^{\theta\Theta} = 0.
 \end{aligned}
 \tag{2.106}$$

This gives

$$p'(R, t) = \frac{2\mu R}{r^2} e^{-2\Omega(R,t)} \left[ 2 - 2R\Omega'(R, t) - \frac{r^2}{R^2} e^{4\Omega(R,t)} - \frac{R^2}{r^2} \right].
 \tag{2.107}$$

Assuming that  $p(R_1, t) = 0$ , we obtain

$$p(R, t) = \int_{R_1}^R \frac{2\mu\xi}{r^2(\xi)} e^{-2\Omega(\xi,t)} \left[ 2 - 2\xi\Omega'(\xi, t) - \frac{r^2(\xi)}{\xi^2} e^{4\Omega(\xi,t)} - \frac{\xi^2}{r^2(\xi)} \right] d\xi.
 \tag{2.108}$$

Note that  $r^2 = R^2 + C$  and thus

$$p(R, t) = \int_{R_1}^R \frac{2\mu\xi}{\xi^2 + C} e^{-2\Omega(\xi,t)} \left[ 2 - 2\xi\Omega'(\xi, t) - \frac{\xi^2 + C}{\xi^2} e^{4\Omega(\xi,t)} - \frac{\xi^2}{\xi^2 + C} \right] d\xi.
 \tag{2.109}$$

Assuming that  $p(R_2, t) = 0$ ,  $C(t)$  can be calculated using the above equation.

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$$\mathbf{F} = \begin{pmatrix} r'(R,t) & 0 \\ 0 & \frac{r(R,t)}{R} \end{pmatrix} \quad \text{and} \quad \mathbf{F}_g = \begin{pmatrix} e^{\Omega(R,t)} & 0 \\ 0 & e^{-\Omega(R,t)} \end{pmatrix}.$$

Thus,  $J_e = \det \mathbf{F}_e = 1$  would lead to the same incompressibility constraint  $rr' = R$ .

For  $a = \theta$ , balance of momentum (2.95) gives

$$P^{\theta A} |_{A} = \frac{\partial P^{\theta\Theta}}{\partial \Theta} + \Gamma_{A\Theta}^A P^{\theta\Theta} + P^{\theta R} \gamma_{rr}^{\theta} F^r{}_R + P^{\theta\Theta} \gamma_{\theta\theta}^{\theta} F^{\theta}{}_{\Theta} = (\Gamma_{R\Theta}^R + \Gamma_{\Theta\Theta}^{\Theta}) P^{\theta\Theta} = 0, \tag{2.110}$$

i.e., this equilibrium equation is trivially satisfied. It is seen that a growth that results in only a change in shape and no change in volume can still result in residual stresses. Note that  $\text{tr}(\frac{\partial \mathbf{G}}{\partial t}) = 0$  and hence the balance of mass reads

$$\frac{\partial \rho_0(R, t)}{\partial t} = S_m(R, t). \tag{2.111}$$

*Example 3 (Spherical Growth of a Neo-Hookean Hollow Sphere)* Let us consider a hollow sphere with inner and outer radii  $R_i$  and  $R_o$ , initially in a coordinate system  $(R, \Theta, \Phi)$ . Let us denote the spatial coordinates by  $(r, \theta, \phi)$ . The spatial metric has the following form:

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \phi \end{pmatrix}, \tag{2.112}$$

with the nonzero connection coefficients  $\gamma_{\theta\theta}^r = -r, \gamma_{\phi\phi}^r = -r \sin^2 \phi, \gamma_{r\theta}^{\theta} = \gamma_{\theta}^{\theta} = 1/r, \gamma_{r\phi}^{\phi} = \gamma_{\phi}^{\phi} = 1/r$ . For isotropic growth of the hollow sphere we consider the following material metric:

$$\mathbf{G} = e^{2\Omega(R,t)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & R^2 \sin^2 \Phi \end{pmatrix}. \tag{2.113}$$

The nonzero connection coefficients are

$$\begin{aligned} \Gamma_{RR}^R &= \Omega', & \Gamma_{\Theta\Theta}^R &= -R - R^2 \Omega', & \Gamma_{\Phi\Phi}^R &= -(R + R^2 \Omega') \sin^2 \Phi, \\ \Gamma_{R\Theta}^{\Theta} &= \Gamma_{\Theta R}^{\Theta} = \Gamma_{R\Phi}^{\Phi} = \Gamma_{\Phi R}^{\Phi} &= \frac{1}{R} + \Omega'. \end{aligned} \tag{2.114}$$

Under this symmetric change of material metric (growth) we look for solutions of the form  $r = r(R, t), \theta = \Theta, \phi = \Phi$ . Thus<sup>12</sup>

$$\mathbf{F} = \begin{pmatrix} r'(R) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.115}$$

<sup>12</sup>In the classical formulation

$$\mathbf{F} = \begin{pmatrix} r'(R, t) & 0 & 0 \\ 0 & \frac{r(R, t)}{R} & 0 \\ 0 & 0 & \frac{r(R, t)}{R} \end{pmatrix} \text{ and } \mathbf{F}_g = \begin{pmatrix} e^{\Omega(R, t)} & 0 & 0 \\ 0 & e^{\Omega(R, t)} & 0 \\ 0 & 0 & e^{\Omega(R, t)} \end{pmatrix}.$$

Hence,  $J_e = \det \mathbf{F}_e = 1$  would lead to the same incompressibility constraint. See Chen and Hoger (2000) for more details.

and hence

$$J = \frac{r^2}{R^2} e^{-3\Omega} r'. \tag{2.116}$$

Incompressibility gives us

$$r^3(R) = r_1^3(R) + \int_{R_1}^R 3\xi^2 e^{3\Omega(\xi,t)} d\xi. \tag{2.117}$$

The only nonzero stresses are

$$\begin{aligned} P^{rR} &= 2\mu \frac{R^2}{r^2} e^{\Omega} - p \frac{r^2}{R^2} e^{-3\Omega}, & P^{\theta\Theta} &= \frac{2\mu}{R^2} e^{-2\Omega} - \frac{p}{r^2}, \\ P^{\phi\Phi} &= \frac{2\mu}{R^2 \sin^2 \Phi} e^{-2\Omega} - \frac{p}{r^2 \sin^2 \Phi}. \end{aligned} \tag{2.118}$$

Again, the equilibrium equations for  $P^{\theta\Theta}$  and  $P^{\phi\Phi}$  are trivially satisfied. The only nontrivial equilibrium equation reads

$$\begin{aligned} P^{rA}{}_{|A} &= \frac{\partial P^{rA}}{\partial X^A} + \Gamma_{AB}^A P^{rB} + P^{bA} \gamma_{bc}^r F^c{}_A \\ &= \frac{\partial P^{rR}}{\partial R} + (\Gamma_{RR}^R + \Gamma_{\Theta R}^\Theta + \Gamma_{\Phi R}^\Phi) P^{rR} + P^{\theta\Theta} \gamma_{\theta\theta}^r F^\theta{}_\Theta + P^{\phi\Phi} \gamma_{\phi\phi}^r F^\phi{}_\Phi \\ &= \frac{\partial P^{rR}}{\partial R} + \left( \frac{2}{R} + 3\Omega' \right) P^{rR} - r P^{\theta\Theta} - r \sin^2 \Phi P^{\phi\Phi} = 0. \end{aligned} \tag{2.119}$$

This gives

$$p'(R, t) = \frac{4\mu R^4}{r^4} e^{4\Omega(R,t)} \left[ \frac{2}{R} + \Omega'(R, t) - \frac{R^2}{r^3} e^{3\Omega(R,t)} - \frac{r^3}{R^4} e^{-3\Omega(R,t)} \right]. \tag{2.120}$$

Assuming that  $p(R_1) = 0$ , we obtain

$$p(R, t) = \int_{R_1}^R \frac{4\mu \xi^4}{r^4(\xi)} e^{4\Omega(\xi,t)} \left[ \frac{2}{\xi} + \Omega'(\xi, t) - \frac{\xi^2}{r^3(\xi)} e^{3\Omega(\xi,t)} - \frac{r^3(\xi)}{\xi^4} e^{-3\Omega(\xi,t)} \right] d\xi. \tag{2.121}$$

Note that  $\text{tr}(\frac{\partial \mathbf{G}}{\partial t}) = 6\Omega'$ , and hence balance of mass reads

$$\frac{\partial \rho_0(R, t)}{\partial t} + 3\Omega'(R, t) \rho_0(R, t) = S_m(R, t). \tag{2.122}$$

This differential equation can be easily solved for mass density.

### 2.10 Visualizing Material Manifolds with Evolving Metrics

In our geometric theory, we model growth in a fixed material manifold  $\mathcal{B}$ . We can visualize the evolution of  $\mathbf{G}(t)$  by embedding  $\mathcal{B}$  in some material ambient space



$\mathcal{X}$  with a fixed metric  $\mathbf{H}$ . For us this larger space would be the Euclidean space with its standard metric. Consider a one-parameter family of isometric embeddings  $\iota_t : \mathcal{B} \hookrightarrow \mathcal{X}$ , i.e.  $\iota_t^* \mathbf{H} = \mathbf{G}(t)$ . For the sake of simplicity, let us restrict ourselves to rotationally symmetric metrics, i.e. we look at metrics of the form

$$\mathbf{G} = \begin{pmatrix} M^2(R, t) & 0 \\ 0 & N^2(R, t) \end{pmatrix}, \tag{2.123}$$

in some coordinate patch  $(R, \Theta)$ , i.e. the metric has the form  $M^2(R, t) dR^2 + N^2(R, t) d\Theta^2$ , where  $t$  is the time. Note that  $M$  and  $N$  are independent of  $\Theta$ . We now look for solutions in the set of surfaces of revolution. Let us consider a time-dependent curve  $\gamma(s, t) = (\rho(s, t), \xi(s, t))$  in the plane. The surface obtained from this curve by revolution about the  $z$ -axis has the following parametric representation:

$$\Phi(s, \Theta, t) = (\rho(s, t) \cos \Theta, \rho(s, t) \sin \Theta, \xi(s, t)). \tag{2.124}$$

The induced Euclidean metric is (Peterson 1997):

$$\Phi^*(dX^2 + dY^2 + dZ^2) = (\dot{\rho}^2(s, t) + \dot{\xi}^2(s, t)) ds^2 + \rho(s, t)^2 d\Theta^2, \tag{2.125}$$

where a superimposed dot means differentiation with respect to  $s$ . Given  $M(R, t)^2 dR^2 + N(R, t)^2 d\Theta^2$ , let us assume that  $\rho(s, t) = N(s, t)$  and hence

$$\dot{\xi}(s, t) = \sqrt{M^2(s, t) - \dot{N}^2(s, t)}. \tag{2.126}$$

Therefore

$$\xi(s, t) = \int_{s_0}^s \sqrt{M^2(\ell, t) - \dot{N}^2(\ell, t)} d\ell. \tag{2.127}$$

Of course, a solution may not exist. This happens when  $M^2 < \dot{N}^2$ . This is not surprising, as not every rotationally symmetric metric arises from a surface of revolution. In the following we consider an initially stress-free annulus under different rotationally symmetric growth distributions.

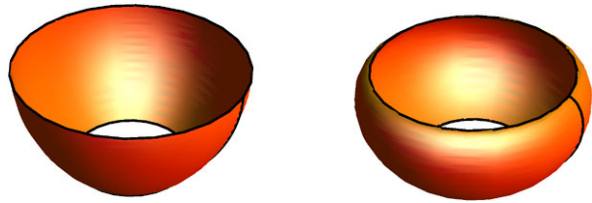
*Example 1* Consider isotropic growth, i.e.

$$\mathbf{G} = \begin{pmatrix} e^{2\Omega(R)} & 0 \\ 0 & R^2 e^{2\Omega(R)} \end{pmatrix}, \quad M = e^\Omega, \quad N = Re^\Omega. \tag{2.128}$$

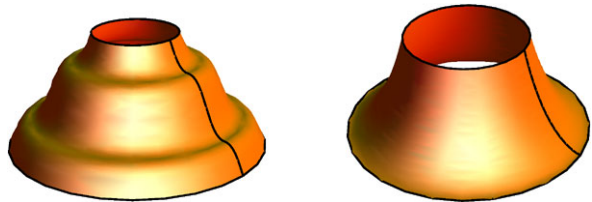
Hence  $M^2 - \dot{N}^2 = -R\Omega' e^{2\Omega} (R\Omega' + 2)$ . Let us look at two cases:

- (i)  $\Omega(R) = -R$ : We have  $M^2 - \dot{N}^2 = e^{-2R} (2R - R^2)$ , which for  $0 < R < 2$  gives the material manifold shown in Fig. 2(left).
- (ii)  $\Omega(R) = -R^2$ : We have  $M^2 - \dot{N}^2 = 4R^2 e^{-2R^2} (1 - R^2)$ , which for  $0 < R < 1$  gives the material manifold shown in Fig. 2(right).

**Fig. 2** Visualization of the material manifolds of two isotropic growth distribution of an annulus as embeddings in  $\mathbb{R}^3$ . *Left:*  $\Omega(R) = -R$ . *Right:*  $\Omega(R) = -R^2$



**Fig. 3** Visualization of the material manifolds of two anisotropic growth distribution of an annulus as embeddings in  $\mathbb{R}^3$ . *Left:*  $\Omega(R) = \cos^2 R$ ,  $\Pi(R) = 0$ . *Right:*  $\Omega(R) = 0$ ,  $\Pi(R) = -\ln R^2$



*Example 2* We look at anisotropic metric evolutions represented by

$$\mathbf{G} = \begin{pmatrix} e^{2\Omega(R)} & 0 \\ 0 & R^2 e^{2\Pi(R)} \end{pmatrix}, \quad M = e^\Omega, \quad N = Re^\Pi. \tag{2.129}$$

We look at two cases:

- (i)  $\Omega(R) = \cos^2 R$  and  $\Pi(R) = 0$ : We have  $M^2 - \dot{N}^2 = e^{2\cos^2 R} - 1 > 0$ . The material manifold shown in Fig. 3(left).
- (ii)  $\Omega(R) = 0$  and  $\Pi(R) = -\ln R^2$ : We have  $M^2 - \dot{N}^2 = 1 - \frac{1}{R^4}$ , which for  $R > 1$  gives the material manifold shown in Fig. 3(right).

### 2.11 Stress-Free Isotropic Growth

In the context of growth mechanics, Takamizawa and Matsuda (1990) realized that having a stress-free configuration is equivalent to vanishing of Riemann’s curvature tensor, although they did not present any detailed calculations. In this subsection we study this problem in detail and obtain stress-free isotropic growth distributions in both two and three dimensions.

Let us first review some basic concepts in Riemannian geometry. For  $\pi : E \rightarrow \mathcal{S}$  a vector bundle over a manifold  $\mathcal{S}$ ,  $\mathcal{E}(\mathcal{S})$  the space of smooth sections of  $E$ , and  $\mathcal{X}(\mathcal{S})$  the space of vector fields on  $\mathcal{S}$ , a connection on  $E$  is a map  $\nabla : \mathcal{X}(\mathcal{S}) \times \mathcal{E}(\mathcal{S}) \rightarrow \mathcal{E}(\mathcal{S})$  such that  $\forall f, f_1, f_2 \in C^\infty(\mathcal{S}), \forall a_1, a_2 \in \mathbb{R}$

$$(a) \nabla_{f_1 \mathbf{X}_1 + f_2 \mathbf{X}_2} \mathbf{Y} = f_1 \nabla_{\mathbf{X}_1} \mathbf{Y} + f_2 \nabla_{\mathbf{X}_2} \mathbf{Y}, \tag{2.130}$$

$$(b) \nabla_{\mathbf{X}} (a_1 \mathbf{Y}_1 + a_2 \mathbf{Y}_2) = a_1 \nabla_{\mathbf{X}} (\mathbf{Y}_1) + a_2 \nabla_{\mathbf{X}} (\mathbf{Y}_2), \tag{2.131}$$

$$(c) \nabla_{\mathbf{X}} (f \mathbf{Y}) = f \nabla_{\mathbf{X}} \mathbf{Y} + (\mathbf{X}f) \mathbf{Y}. \tag{2.132}$$

A linear connection on  $\mathcal{S}$  is a connection on  $T\mathcal{S}$ , i.e.,  $\nabla : \mathcal{X}(\mathcal{S}) \times \mathcal{X}(\mathcal{S}) \rightarrow \mathcal{X}(\mathcal{S})$ . In a local chart  $\{x^i\}$

$$\nabla_{\partial_i} \partial_j = \gamma_{ij}^k \partial_k, \tag{2.133}$$

where  $\gamma_{ij}^k$  are the Christoffel symbols of the connection and  $\partial_i = \frac{\partial}{\partial x^i}$ . A linear connection is said to be compatible with the metric of the manifold if

$$\nabla_{\mathbf{X}} \langle \mathbf{Y}, \mathbf{Z} \rangle = \langle \nabla_{\mathbf{X}} \mathbf{Y}, \mathbf{Z} \rangle + \langle \mathbf{Y}, \nabla_{\mathbf{X}} \mathbf{Z} \rangle. \tag{2.134}$$

One can show that  $\nabla$  is compatible with  $\mathbf{g}$  if and only if  $\nabla \mathbf{g} = \mathbf{0}$ . Torsion of a connection is defined as

$$\mathcal{T}(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}], \tag{2.135}$$

where

$$[\mathbf{X}, \mathbf{Y}](F) = \mathbf{X}(\mathbf{Y}(F)) - \mathbf{Y}(\mathbf{X}(F)) \quad \forall F \in C^\infty(\mathcal{S}), \tag{2.136}$$

is the commutator of  $\mathbf{X}$  and  $\mathbf{Y}$ .  $\nabla$  is symmetric if it is torsion-free, i.e.  $\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} = [\mathbf{X}, \mathbf{Y}]$ . According to the Fundamental Lemma of Riemannian Geometry (Lee 1997) on any Riemannian manifold  $(\mathcal{S}, \mathbf{g})$  there is a unique linear connection  $\nabla$ , the Levi-Civita connection, that is compatible with  $\mathbf{g}$  and is torsion-free with the following Christoffel symbols:

$$\gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right). \tag{2.137}$$

The curvature tensor,  $\mathcal{R}$ , of a Riemannian manifold  $(\mathcal{S}, \mathbf{g})$  is a  $\binom{1}{3}$ -tensor  $\mathcal{R} : T_{\mathbf{X}}^* \mathcal{S} \times T_{\mathbf{X}} \mathcal{S} \times T_{\mathbf{X}} \mathcal{S} \times T_{\mathbf{X}} \mathcal{S} \rightarrow \mathbb{R}$  defined as

$$\mathcal{R}(\alpha, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \alpha(\nabla_{\mathbf{w}_1} \nabla_{\mathbf{w}_2} \mathbf{w}_3 - \nabla_{\mathbf{w}_2} \nabla_{\mathbf{w}_1} \mathbf{w}_3 - \nabla_{[\mathbf{w}_1, \mathbf{w}_2]} \mathbf{w}_3) \tag{2.138}$$

for  $\alpha \in T_{\mathbf{X}}^* \mathcal{S}$ ,  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in T_{\mathbf{X}} \mathcal{S}$ . In a coordinate chart  $\{x^a\}$

$$\mathcal{R}^a{}_{bcd} = \frac{\partial \gamma_{bd}^a}{\partial x^c} - \frac{\partial \gamma_{bc}^a}{\partial x^d} + \gamma_{ce}^a \gamma_{bd}^e - \gamma_{de}^a \gamma_{bc}^e. \tag{2.139}$$

Note that for an arbitrary vector field  $\mathbf{w}$

$$w^a{}_{|bc} - w^a{}_{|cb} = \mathcal{R}^a{}_{bcd} w^d + \mathcal{T}^d{}_{cb} w^a{}_{|d}. \tag{2.140}$$

An  $n$ -dimensional Riemannian manifold is flat if it is isometric to a Euclidean space. A Riemannian manifold is flat if and only if its curvature tensor vanishes (Lee 1997; Spivak 1999; Berger 2003). The Ricci curvature is defined as

$$R_{ab} = \mathcal{R}^c{}_{acb}. \tag{2.141}$$

The trace of the Ricci curvature is called the scalar curvature:

$$\mathbf{R} = R_{ab} g^{ab}. \tag{2.142}$$

In dimensions two and three, the Ricci curvature algebraically determines the entire curvature tensor. In dimension three (Hamilton 1982):

$$\mathcal{R}_{abcd} = g_{ac} R_{bd} - g_{ad} R_{bc} - g_{bc} R_{ad} + g_{bd} R_{ac} - \frac{1}{2} \mathbf{R} (g_{ac} g_{bd} - g_{ad} g_{bc}). \tag{2.143}$$

In dimension two  $R_{ab} = Rg_{ab}$ , and hence scalar curvature completely characterizes the curvature tensor and is twice the Gauss curvature. Note that any one-dimensional metric is flat. In the following we obtain the stress-free growth distributions in dimensions two and three.

(i) *The two-dimensional case.* Consider a two-dimensional shell restricted to live on a flat planar surface between two rigid planes. We assume that with no external or body forces, initially the shell is stress free. Can one find the growth distributions that will result in equilibrium configurations with zero stress? Uniform growth will obviously result in uniform expansion/contraction, and hence no stress. Are there other isotropic growth distributions with this property?

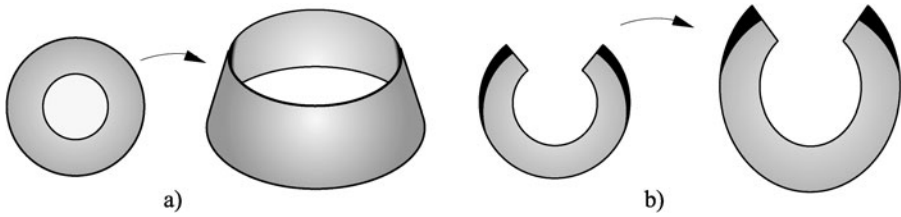
The spatial distances between material points are measured by the ambient space metric (the “spatial metric”), which is Euclidean. A given growth distribution will result in a change in the material metric. A configuration will be stress free if there is no “stretch” in the material, i.e., if the material distance between two points is the same as the spatial distance. This can happen only if the two metric tensors (spatial and material) give the same distance measurements between nearby material points, i.e. if they are isometric. As the spatial metric is assumed to be Euclidean, this means that the material metric, after the change due to a given growth distribution, must be Euclidean.

Riemann defined the curvature tensor of the metric and proved that a metric is flat, i.e., it can be brought into the Euclidean form  $\delta_{IJ}$  locally by a coordinate transformation, if and only if its curvature tensor is zero (Lee 1997; Spivak 1999; Berger 2003). It turns out that in dimension two, a weaker requirement is sufficient (Berger 2003): a metric is flat if and only if its scalar curvature (the Ricci scalar) is zero. Let us now apply this condition to a two-dimensional metric that is obtained from a non-uniform growth distribution on an initially stress-free, planar shell, i.e.,  $G_{IJ} = e^{2\Omega} \delta_{IJ}$ . The Ricci scalar for a metric of this form is given by (Wald 1984)

$$R = -2e^{-2\Omega} \nabla^2 \Omega. \quad (2.144)$$

Thus,  $R = 0$  requires  $\nabla^2 \Omega = 0$ , i.e.,  $\Omega$  has to be a harmonic function. Note that here  $\nabla^2$  is the spatial Laplacian. Growth is a slow process compared to elastic deformations and therefore time can be treated as a parameter and hence inertial effects can be ignored. Hence, time in  $\Omega$  is treated as a parameter.

It is worth emphasizing the distinction between local and global flatness, and the implications for stress-free growth distributions. Although the surface of a right circular cylinder in three dimensions looks curved, it is locally, intrinsically flat. For any given point on the cylinder, one can find a finite-sized region containing the point, and a single-valued coordinate patch on this region, for which the metric has the Euclidean form. Physically, this means that for any given point, we can cut some finite-sized piece containing the point, and can lay the piece on a flat plane, without stretching it. The surface of a sphere in three dimensions, on the other hand, is intrinsically curved; it is impossible to make any finite-sized piece of the sphere, no matter how small, to lie on a flat plane without stretching it. The curvature condition  $R = 0$  (or  $\nabla^2 \Omega = 0$ ) is local. That making a full cylinder lie in a plane nicely (i.e., without tearing, folding, or stretching it) is impossible is due to the global topology of the cylinder; local restrictions on curvature cannot constrain the global properties sufficiently.



**Fig. 4** (a) Zero-stress growth of an annulus to a cone. (b) Zero-stress growth of a simply-connected piece of an annulus

Let us specialize to the case where  $\Omega$  depends only on the radial coordinate  $R$  of an initially flat annular piece of a material,  $R_0 \leq R \leq R_1$ . The flatness condition gives

$$\nabla^2 \Omega = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Omega(R, t)}{\partial R} \right) = 0. \tag{2.145}$$

Solving this gives

$$e^{2\Omega} = \xi(t) R^{2\eta(t)}, \tag{2.146}$$

where  $\xi > 0$  and  $\eta$  are time-dependent constants. The metric rescaling (2.146), with the proper identifications, is describing an annular piece from a conical surface, with deficit angle  $\xi = 2\pi(1 - 1/|c|)$ , where  $c = \frac{1}{1+\eta}$  (Ozakin and Yavari 2010). Now, one can show that it is not possible to make such a conical surface lie on the plane without tearing, stretching, or folding it. Thus, starting with an annular shell between two rigid planes, a growth distribution of the form (2.146) will indeed result in stresses, although the related material metric is intrinsically flat (see Fig. 4a). However, if the material consists only of a simply-connected piece of the annulus (say,  $R_1 < R < R_2$ ,  $0 < \Theta_1 < \Theta < \Theta_2 < 2\pi$ ), the growth distribution (2.146) will just cause a stress-free expansion of the material, between the two rigid planes (see Fig. 4b).

*A remark on conformally flat manifolds and growth mechanics.* A Riemannian manifold  $(\mathcal{B}, \mathbf{G})$  is conformally flat if there exists a smooth map  $f : \mathcal{B} \rightarrow \mathbb{R}$  such that  $\mathbf{G} = f\delta$ , where  $\delta$  is the Euclidean metric. In *isothermal coordinates* the conformally flat Riemannian metric has the following local form:

$$\mathbf{G} = f(\mathbf{X})(dX_1^2 + \dots + dX_n^2). \tag{2.147}$$

It is known that (Berger 2003) any two-dimensional Riemannian manifold is conformally flat and the map  $f$  is unique. A corollary of this theorem in our theory of growth mechanics is that given any smooth curved 2D stress-free solid, there exists a unique growth distribution such that in the new (grown) configuration, the 2D solid is flat and still stress free. Equivalently, starting from a stress-free flat sheet, it is always possible to deform it to any smooth curved shape by growth without imposing any residual stresses.

(ii) *The three-dimensional case.* Let us next consider the three-dimensional case. In three dimensions, a vanishing Ricci scalar is not sufficient to guarantee local flatness.

However, a three-dimensional metric is flat if and only if its Ricci tensor vanishes (Berger 2003). The Ricci tensor  $R_{IJ}$  of the metric  $G_{IJ} = e^{2\Omega} \overset{\circ}{G}_{IJ}$  is given in terms of the Ricci tensor  $\overset{\circ}{R}_{IJ}$  of  $\overset{\circ}{G}_{IJ}$  by the following relation (Wald 1984):

$$R_{IJ} = \overset{\circ}{R}_{IJ} - (n - 2)\nabla_I\nabla_J\Omega - \overset{\circ}{G}_{IJ}\overset{\circ}{G}^{KL}\nabla_K\nabla_L\Omega + (n - 2)\nabla_I\Omega\nabla_J\Omega - (n - 2)\overset{\circ}{G}_{IJ}\overset{\circ}{G}^{KL}\nabla_K\Omega\nabla_L\Omega, \tag{2.148}$$

where  $n = \dim \mathcal{B}$ . Now, once again, assume that the initial metric  $\overset{\circ}{G}_{IJ} = \delta_{IJ}$ ,  $\overset{\circ}{R}_{IJ} = 0$ , and  $n = 3$ , and replace the covariant derivatives with partial derivatives. This gives

$$R_{IJ} = -\partial_I\partial_J\Omega - \delta_{IJ}\delta^{KL}\partial_K\partial_L\Omega + \partial_I\Omega\partial_J\Omega - \delta_{IJ}\delta^{KL}\partial_K\Omega\partial_L\Omega = 0. \tag{2.149}$$

This gives the following system of nonlinear partial differential equations in terms of  $\Omega$ :

$$\Omega_{,12} = \Omega_{,1}\Omega_{,2}, \tag{2.150}$$

$$\Omega_{,13} = \Omega_{,1}\Omega_{,3}, \tag{2.151}$$

$$\Omega_{,23} = \Omega_{,2}\Omega_{,3}, \tag{2.152}$$

$$\Omega_{,11} + \nabla^2\Omega + \Omega_{,2}^2 + \Omega_{,3}^2 = 0, \tag{2.153}$$

$$\Omega_{,22} + \nabla^2\Omega + \Omega_{,1}^2 + \Omega_{,3}^2 = 0, \tag{2.154}$$

$$\Omega_{,33} + \nabla^2\Omega + \Omega_{,1}^2 + \Omega_{,2}^2 = 0. \tag{2.155}$$

This system of nonlinear equations were solved in Ozakin and Yavari (2010). The general solution is

$$\Omega(X^1, X^2, X^3, t) = -\ln\{c_0(t)[(X^1)^2 + (X^2)^2 + (X^3)^2] + c_1(t)X^1 + c_2(t)X^2 + c_3(t)X^3 + c_4(t)\}. \tag{2.156}$$

In a special case if  $c_1 = c_2 = c_3 = c_4 = 0$ , we have

$$\Omega(X^1, X^2, X^3) = -\ln(c_0R^2), \tag{2.157}$$

where  $R = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}$ . In order to understand what this solution represents physically, let us write the metric in polar coordinates.

$$dS^2 = e^{2\Omega} [dR^2 + R^2(d\Theta^2 + \sin^2\Theta d\Phi^2)] = \frac{1}{c^2R^4} [dR^2 + R^2(d\Theta^2 + \sin^2\Theta d\Phi^2)]. \tag{2.158}$$

Now let us define

$$\tilde{R} = \frac{1}{cR}. \tag{2.159}$$

In terms of  $\tilde{R}$ , the metric becomes

$$dS^2 = d\tilde{R}^2 + \tilde{R}^2(d\Theta^2 + \sin^2 \Theta d\phi^2), \tag{2.160}$$

which is precisely the flat Euclidean metric in three dimensions. Thus, after the growth, the metric is still flat, but the radial coordinate in which it is manifestly so is related to the old radial coordinate by (2.159) (up to a simple shift of origin). This means that particles at the two radii  $R_1 < R_2$  move to the new radii  $\tilde{R}_1 > \tilde{R}_2$ , after growth, i.e., the material gets “inverted”. This may not be possible for a solid ball without tearing it apart, but it is perfectly possible for a piece from such a ball.

If only  $c_4$  is nonzero, we recover the trivial uniform growth. If only  $c_1$  is nonzero and assuming that the initial material metric is Euclidean for the half space  $X^1 > 0$ , we have

$$G_{IJ} = \frac{\lambda(t)}{(X^1)^2} \delta_{IJ}, \tag{2.161}$$

where  $\lambda = 1/(c_1)^2$ . This shows that the material manifold is conformal to the Poincaré half space.

### 2.12 Lagrangian Field Theory of Growing Bodies

In the Lagrangian formulation of nonlinear elasticity, one assumes the existence of a Lagrangian density

$$\mathcal{L} = \mathcal{L}(\mathbf{X}, t, \mathbf{G}, \varphi, \dot{\varphi}, \mathbf{F}, \mathbf{g}). \tag{2.162}$$

The Lagrangian is defined in the reference configuration by

$$L = \int_{\mathcal{B}} \mathcal{L}(\mathbf{X}, t, \mathbf{G}(\mathbf{X}), \varphi(\mathbf{X}), \dot{\varphi}(\mathbf{X}), \mathbf{F}(\mathbf{X}), \mathbf{g}(\varphi(\mathbf{X}))) dV(\mathbf{X}). \tag{2.163}$$

In the case of a growing continuum, the material metric will be a dynamical variable too. Thus, for growth of an elastic body we assume the existence of a Lagrangian density  $\mathcal{L} = \mathcal{L}(\mathbf{X}, t, \mathbf{G}, \varphi, \dot{\varphi}, \mathbf{F}, \mathbf{g})$  and write the Lagrangian as

$$L = \int_{\mathcal{B}} \mathcal{L}(\mathbf{X}, t, \mathbf{G}(\mathbf{X}, t), \varphi(\mathbf{X}, t), \dot{\varphi}(\mathbf{X}, t), \mathbf{F}(\mathbf{X}, t), \mathbf{g} \circ \varphi(\mathbf{X}, t)) dV(\mathbf{X}), \tag{2.164}$$

where  $dV(\mathbf{X}) = \sqrt{\det \mathbf{G}} dX^1 \wedge \dots \wedge dX^n = \sqrt{\det \mathbf{G}} d\mathbf{X}$ . Having the Lagrangian, the action is defined as

$$S = \int_{t_0}^{t_1} L dt \tag{2.165}$$

and *Hamilton’s Principle of Least Action* states that

$$\delta S = \mathbf{d}S \cdot (\delta\varphi, \delta\mathbf{G}) = 0. \tag{2.166}$$

The problem with this formulation is that it assumes that the solid is a conservative system. This is obviously not correct here as growth is a dissipative process, in

general. There have been recent works on the Lagrangian formulation of dissipative systems. One idea is to use fractional derivatives and assume that the Lagrangian is a function of some non-integer time derivatives of generalized coordinates (Riewe 1997). It is not clear how one can use this idea for a general field theory and even if successful how useful that theory will be. Another way of considering dissipation in Lagrangian mechanics is to use a Rayleigh dissipation function (Marsden and Ratiu 2003).

Assume that there exists a Rayleigh dissipation function  $\mathcal{R} = \mathcal{R}(\dot{\varphi}, \dot{\mathbf{G}})$ . For a continuum with dissipative forces  $\mathbf{F}$ , the Lagrange–d’Alembert Principle states that (Marsden and Ratiu 2003)

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L} \, dV \, dt + \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathbf{F} \cdot \delta\varphi \, dt = 0. \tag{2.167}$$

Assuming the existence of a dissipation potential  $\mathcal{R}$  for a growing body, the two dissipative forces are represented as

$$\mathbf{F} = -\frac{\partial \mathcal{R}}{\partial \dot{\varphi}} \quad \text{and} \quad \mathbf{F}_G = -\frac{\partial \mathcal{R}}{\partial \dot{\mathbf{G}}}. \tag{2.168}$$

In this case, the Lagrange–d’Alembert Principle states that

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L}(\mathbf{X}, t, \mathbf{G}, \varphi, \dot{\varphi}, \mathbf{F}, \mathbf{g} \circ \varphi) \, dV \, dt + \int_{t_0}^{t_1} \int_{\mathcal{B}} (\mathbf{F} \cdot \delta\varphi + \mathbf{F}_G \cdot \delta\mathbf{G}) \, dV \, dt = 0. \tag{2.169}$$

For the sake of simplicity, let us consider the two variations separately.

*Case 1:* If only the deformation mapping is varied, one has

$$\delta S = \mathbf{d}S \cdot (\delta\varphi, \mathbf{0}) = 0. \tag{2.170}$$

This can be simplified to read (Yavari et al. 2006)

$$\frac{\partial \mathcal{L}}{\partial \varphi^a} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^a} - \left( \frac{\partial \mathcal{L}}{\partial F^a_A} \right)_{|A} - \frac{\partial \mathcal{L}}{\partial F^b_A} F^c_A \gamma_{ac}^b + 2 \frac{\partial \mathcal{L}}{\partial g_{cd}} g_{bd} \gamma_{ac}^b = \frac{\partial \mathcal{R}}{\partial \dot{\varphi}^a}. \tag{2.171}$$

Or

$$P_a^A|_A + \frac{\partial \mathcal{L}}{\partial \varphi^a} + (F^c_A P_b^A - J \sigma^{cd} g_{bd}) \gamma_{ac}^b = \rho_0 g_{ab} A^b + \frac{\partial \mathcal{R}}{\partial \dot{\varphi}^a}. \tag{2.172}$$

*Case 2:* In a material representation of classical nonlinear elasticity, the density is not a dynamical variable but rather it is a parameter appearing in the Lagrangian. It is through a reduction process (material to spatial) that the density ends up satisfying the continuity or advection equation (see Holm et al. 1998 for more details). Here, we should note that unlike classical nonlinear elasticity, mass density varies, in general, when the material metric changes. In other words,  $\delta\rho_0$  and  $\delta\mathbf{G}$  are related



through the nonholonomic constraint of the mass balance. For a similar discussion of the Lagrangian formulation of fluid mechanics in Eulerian (spatial) coordinates, see Brethert (1970). We know that by definition of  $S_m$

$$\frac{d}{dt} \int_{\mathcal{U}} \rho_0(X, t) dV = \int_{\mathcal{U}} S_m(X, t) dV = \int_{\mathcal{U}} \overset{\circ}{S}_m(X, t) d\overset{\circ}{V}, \tag{2.173}$$

where  $\overset{\circ}{S}_m(X, t)$  is mass source in the initial material manifold with volume element  $d\overset{\circ}{V}$ . Note that  $\overset{\circ}{S}$  is the quantity that can be given physically. Now balance of mass can be rewritten as

$$\int_{\mathcal{U}} \rho_0(X, t) dV = \int_{\mathcal{U}} \rho_0(X, t_0) dV + \int_{t_0}^t \int_{\mathcal{U}} \overset{\circ}{S}_m(X, \tau) d\overset{\circ}{V} d\tau. \tag{2.174}$$

For a fixed  $\overset{\circ}{S}_m$ , let us consider mass density and material metric variation fields  $\rho_0(X, t; \epsilon)$  and  $\mathbf{G}(X, t; \epsilon)$ . For an arbitrary  $\epsilon$  the above integral mass balance reads

$$\int_{\mathcal{U}} \rho_0(X, t; \epsilon) dV_{\epsilon} = \int_{\mathcal{U}} \rho_0(X, t_0; \epsilon) dV_{\epsilon} + \int_{t_0}^t \int_{\mathcal{U}} \overset{\circ}{S}_m(X, \tau) d\overset{\circ}{V} d\tau. \tag{2.175}$$

Let us take derivatives with respect to  $\epsilon$  of both sides, evaluate them at  $\epsilon = 0$  and note that all variations vanish at  $t = t_0$ . This gives

$$\int_{\mathcal{U}} \left( \delta\rho_0 + \frac{1}{2} \rho_0 \operatorname{tr}(\delta\mathbf{G}) \right) dV = 0. \tag{2.176}$$

As  $\mathcal{U}$  is arbitrary, we obtain

$$\delta\rho_0 + \frac{1}{2} \rho_0 \operatorname{tr}(\delta\mathbf{G}) = 0. \tag{2.177}$$

We can write  $\mathcal{L} = \rho_0 \bar{\mathcal{L}}$ , where  $\bar{\mathcal{L}}$  is Lagrangian density per unit mass. Hence

$$\begin{aligned} \delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L} dV dt &= \delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \rho_0 \bar{\mathcal{L}} dV dt \\ &= \int_{t_0}^{t_1} \int_{\mathcal{B}} \left[ \rho_0 \delta \bar{\mathcal{L}} + \bar{\mathcal{L}} \left( \delta\rho_0 + \frac{1}{2} \rho_0 \operatorname{tr}(\delta\mathbf{G}) \right) \right] dV dt. \end{aligned} \tag{2.178}$$

Thus, using (2.177)

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L} dV dt = \int_{t_0}^{t_1} \int_{\mathcal{B}} \delta \mathcal{L} dV dt, \tag{2.179}$$

where in  $\delta \mathcal{L}$  the mass density is assumed to be fixed. Now substituting this in the Lagrange–d’Alembert Principle and assuming that  $\mathbf{G}$  varies while  $\varphi$  is fixed, we obtain

$$\int_{t_0}^{t_1} \int_{\mathcal{B}} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{G}} - \frac{\partial \mathcal{R}}{\partial \dot{\mathbf{G}}} \right) : \delta \mathbf{G} dV dt = 0. \tag{2.180}$$

Therefore, the corresponding Euler–Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \mathbf{G}} = \frac{\partial \mathcal{R}}{\partial \dot{\mathbf{G}}}. \tag{2.181}$$

Note that this is very similar to what we obtained using the principle of maximum entropy production in Sect. 2.7.

*Example* Assuming that  $\mathcal{R}(\mathbf{G}, \dot{\mathbf{G}})$  is a quadratic function, i.e.  $\mathcal{R}(\mathbf{G}, \dot{\mathbf{G}}) = \omega \operatorname{tr} \dot{\mathbf{G}}^2 = \omega \dot{G}_{AM} \dot{G}_{BN} G^{AB} G^{MN}$ , we have

$$\frac{\partial \mathcal{R}}{\partial \dot{G}_{AB}} = 2\omega \dot{G}_{MN} G^{AM} G^{BN}. \tag{2.182}$$

Thus

$$\dot{\mathbf{G}}^\sharp = \frac{1}{2\omega} \frac{\partial \mathcal{L}}{\partial \mathbf{G}}. \tag{2.183}$$

Note that  $\mathcal{L} = \mathcal{T} - \mathcal{V}$ , where

$$\mathcal{T} = \frac{1}{2} \rho_0 \langle \langle \mathbf{V}, \mathbf{V} \rangle \rangle \quad \text{and} \quad \mathcal{V} = \rho_0 E + \mathcal{V}_B, \tag{2.184}$$

where  $\mathcal{V}_B$  is the potential of body forces. Therefore, the evolution equation (2.183) reads

$$\dot{\mathbf{G}}^\sharp = -\frac{1}{2\omega} \rho_0 \frac{\partial E}{\partial \mathbf{G}}. \tag{2.185}$$

Note that  $E = \Psi + \mathbf{N}\Theta$  and hence

$$\frac{\partial E}{\partial \mathbf{G}} = \left( \frac{\partial \Psi}{\partial \mathbf{G}} + \frac{\partial \Psi}{\partial \Theta} \frac{\partial \Theta}{\partial \mathbf{G}} \right) + \mathbf{N} \frac{\partial \Theta}{\partial \mathbf{G}} = \frac{\partial \Psi}{\partial \mathbf{G}}. \tag{2.186}$$

Thus

$$\dot{\mathbf{G}}^\sharp = -\frac{1}{2\omega} \rho_0 \frac{\partial \Psi}{\partial \mathbf{G}}, \tag{2.187}$$

which is identical to (2.73) if we choose  $\omega = \frac{1}{2}\beta$ .

### 3 Connection Between $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$ and the Geometric Theory

In the literature of growth mechanics the idea of multiplicative decomposition of the deformation gradient into elastic and growth parts is usually attributed to Rodriguez et al. (1994), although it can be seen in several earlier works, like Kondaurov and Nikitin (1987), Takamizawa and Matsuda (1990), Takamizawa (1991). Takamizawa and Matsuda (1990) and Takamizawa (1991) considered a local stress-free configuration by using a multiplicative decomposition of the deformation gradient, although this decomposition is implicit in their presentation. They realized that the local stress-free configurations are not unique but a corresponding metric is unique and defined

a global stress-free configuration by equipping the original reference configuration with this metric, giving it a Riemannian structure. Here we look at this metric and its rigorous connection with  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$ .

It should be mentioned that similar ideas were used in plasticity and thermoelasticity before the growth mechanics applications. For the less familiar application in thermal stresses, the idea of decomposition of the deformation gradient goes back to the works of Stojanović and his coworkers (Stojanović et al. 1964; Stojanović 1969). See Vujošević and Lubarda (2002), Lubrada (2004) and Ozakin and Yavari (2010) for a detailed discussion. These researchers extended Kondo’s (1955a, 1955b, 1963, 1964) and Bilby’s et al. (1955, 1957) idea of local elastic relaxation in the continuum theory of distributed defects to the case of thermal stresses.<sup>13</sup>

We have posed the following question in this paper: which space, as opposed to the Euclidean space, would be compatible with a relaxed state of the body? We claim that the answer to this question is: a Riemannian manifold whose metric is related to the nonuniform growth. This metric describes the relaxed state of the material with respect to which the strains in a given configuration should be measured. In this framework, the constitutive relations are given in terms of the material metric, the (Euclidean) spatial metric, and the deformation gradient  $\mathbf{F}$ .

Let us consider one of the above-mentioned imaginary relaxed pieces. Relaxation of this piece corresponds to a linear deformation (linear, because the piece is small) denoted by  $\mathbf{F}_g$ . If this piece is deformed in some arbitrary way after the relaxation, one can calculate the induced stresses by using the tangent map of this deformation in the constitutive relations. In order to calculate the stresses induced for a given deformation of the full body, we focus attention to one such particular piece. The deformation gradient of the full body at this piece  $\mathbf{F}$  can be decomposed as  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$ , where, by definition,  $\mathbf{F}_e = \mathbf{F} \mathbf{F}_g^{-1}$ . Thus, as far as this piece is concerned, the deformation of the body consists of a relaxation, followed by a linear deformation given by  $\mathbf{F}_e$ . The stresses induced on this piece, for an arbitrary deformation of the body, can be calculated by substituting  $\mathbf{F}_e$  in the constitutive relations. One should note that  $\mathbf{F}_e$  and  $\mathbf{F}_g$  are not necessarily compatible. However, as long as we have a prescription for obtaining  $\mathbf{F}_e$  and  $\mathbf{F}_g$  directly for a given deformation map  $\varphi$  for the body and a growth distribution, we can calculate the stresses by the following procedure. Note also that if the material manifold is one-dimensional the decomposition of the deformation mapping into elastic and growth parts is always possible. This is implicitly assumed, for example, in Senan et al. (2008).

For isotropic growth, one has the following expression for  $\mathbf{F}_g$ :

$$(F_g)^A{}_B = g \delta^A_B. \tag{3.1}$$

Given this formula for  $\mathbf{F}_g$ , we can calculate  $\mathbf{F}_e = \mathbf{F} \mathbf{F}_g^{-1}$  for a given deformation and use a constitutive relation that gives the stresses in terms of  $\mathbf{F}_e$ . At first glance these two approaches seem very different; however, they are related, as we demonstrate next. We should emphasize that the following discussion is not restricted to isotropic growth; given any  $\mathbf{F}_g$  our arguments can be repeated.

<sup>13</sup>Note that the idea of local elastic relaxation was first proposed in the work of Eckart (1948).

The constitutive relations of the two approaches are formulated in terms of different quantities:  $\mathbf{G}(\mathbf{X}, t)$  and  $\mathbf{F}$  on one side, and  $\mathbf{F}_e = \mathbf{F}\mathbf{F}_g^{-1}$  on the other. Let us start with our approach, namely, assume that a constitutive relation is given in terms of  $\mathbf{G}(\mathbf{X}, t)$  and  $\mathbf{F}$ . This takes the form of a scalar free energy density function that depends on  $\mathbf{G}(\mathbf{X}, t)$ ,  $\mathbf{F}$ , as well as on the spatial metric tensor  $\mathbf{g}$ , and possibly  $\mathbf{X}$  explicitly:

$$\Psi = \Psi(\mathbf{X}, \Theta, \mathbf{G}(\mathbf{X}, t), \mathbf{F}, \mathbf{g} \circ \varphi). \tag{3.2}$$

$\mathbf{G}$ ,  $\mathbf{F}$ , and  $\mathbf{g}$  are tensors, expressed in terms of specific bases for the material and the ambient spaces. A change of basis changes the components of these tensors, but  $\Psi$  does not change, as it is a scalar. Let us consider a change of basis from the original coordinate basis  $\mathbf{E}_A$  of the material space, with the following property:

$$\langle\langle \mathbf{E}_A, \mathbf{E}_B \rangle\rangle_{\mathbf{G}} = G_{AB}, \tag{3.3}$$

to an orthonormal basis  $\hat{\mathbf{E}}_{\hat{A}}$  that satisfies

$$\langle\langle \hat{\mathbf{E}}_{\hat{A}}, \hat{\mathbf{E}}_{\hat{B}} \rangle\rangle_{\mathbf{G}} = \delta_{\hat{A}\hat{B}}. \tag{3.4}$$

A matrix  $F_{\hat{A}}^B$  represents the transformation between the two bases:

$$\hat{\mathbf{E}}_{\hat{A}} = F_{\hat{A}}^B \mathbf{E}_B. \tag{3.5}$$

The orthonormality condition gives

$$F_{\hat{A}}^C F_{\hat{B}}^D G_{CD} = \delta_{\hat{A}\hat{B}}. \tag{3.6}$$

Any  $F_{\hat{A}}^C$  that satisfies this equation gives an orthonormal basis. Given such an  $F_{\hat{A}}^C$ , we can also obtain an orthonormal basis for the dual space by using its inverse. Defining  $F^{\hat{C}}_D$  as the inverse of the matrix  $F_{\hat{A}}^B$ , i.e.,  $F_{\hat{A}}^B F^{\hat{A}}_C = \delta^B_C$  and  $F_{\hat{A}}^B F^{\hat{C}}_B = \delta^{\hat{C}}_{\hat{A}}$ , we obtain the dual orthonormal basis  $\{\hat{\mathbf{E}}^{\hat{A}}\}$  in terms of the original dual basis  $\{\mathbf{E}^A\}$ , thus:

$$\hat{\mathbf{E}}^{\hat{A}} = F^{\hat{A}}_B \mathbf{E}^B. \tag{3.7}$$

For isotropic growth,  $G_{CD} = e^{2\Omega(\mathbf{X},t)} \delta_{CD} = g(\mathbf{X}, t)^2 \delta_{CD}$  gives<sup>14</sup>

$$F_{\hat{A}}^C = \delta^C_{\hat{A}} e^{-\Omega(\mathbf{X},t)} = \delta^C_{\hat{A}} g^{-1}(\mathbf{X}, t), \tag{3.9}$$

as a solution to (3.6). Here,  $\delta^B_{\hat{A}}$  is 1 for  $A = B$ , and 0 otherwise, i.e.,  $\delta^1_{\hat{1}} = \delta^2_{\hat{2}} = \delta^3_{\hat{3}} = 1$ , etc. One should note that (3.6) has other solutions, as well, which we will comment

<sup>14</sup>This means that

$$e^{2\Omega} = \mathbf{g}^2 \quad \text{or} \quad e^{\Omega} = \mathbf{g}, \tag{3.8}$$

as  $\mathbf{g}$  is always positive. If growth is anisotropic, having an expression for  $G_{CD}$  all these arguments can be repeated.

on in the sequel. Now let us write the components of the total deformation gradient  $\mathbf{F}$  in the orthonormal basis  $\{\hat{\mathbf{E}}_{\hat{A}}\}$ . The components are transformed by using  $\mathbf{F}$  as

$$F^a_{\hat{A}} = F_{\hat{A}}^B F^a_B. \tag{3.10}$$

Using (3.9), (3.8), and (3.1), we can clearly see that the components  $F^a_{\hat{A}}$  are given precisely by those of  $\mathbf{F}_e$ , the “elastic part” of the deformation gradient in the  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$  approach:

$$F^a_{\hat{A}} = F_{\hat{A}}^B F^a_B = \delta_{\hat{A}}^B e^{-\Omega(\mathbf{X},t)} F^a_B = (\mathbf{g}(\mathbf{X}, t))^{-1} \delta_{\hat{A}}^B F^a_B = (F_g^{-1})_A^B F^a_B = (F_e)^a_A. \tag{3.11}$$

Thus,  $\mathbf{F}_e$  is the original deformation gradient, written in terms of an orthonormal basis in the material space.<sup>15</sup> We have also shown that there is no need for a mysterious “intermediate configuration” as the target space of  $\mathbf{F}_g$ ; the latter simply gives an orthonormal frame in the material manifold, and as such can be treated as a linear map from the tangent space of the material manifold to itself.

Although a coordinate basis  $\{\mathbf{E}_A = \partial/\partial X^A\}$  is not necessarily orthonormal, one can always obtain an orthonormal basis by applying a pointwise change of basis  $F_{\hat{A}}^B$ . Moreover, giving an orthonormal basis in this way is equivalent to giving a metric tensor at each point; the inner product of any two vectors can be calculated by using their components in the orthonormal basis. We have seen above that in the context of growth mechanics, this means that a change in the material metric due to a growth distribution can be given in terms of the “growth deformation gradient” of the local relaxation approach. Given an orthonormal basis  $\{\hat{\mathbf{E}}_{\hat{A}}\}$ , it is possible to obtain another one,  $\{\hat{\mathbf{E}}'_{\hat{A}}\}$ , by using an orthogonal transformation  $\Lambda_{\hat{A}}^{\hat{B}}$ :

$$\hat{\mathbf{E}}'_{\hat{A}} = \Lambda_{\hat{A}}^{\hat{B}} \hat{\mathbf{E}}_{\hat{B}}, \tag{3.12}$$

where  $\Lambda_{\hat{A}}^{\hat{B}}$  satisfies  $\Lambda_{\hat{A}}^{\hat{C}} \Lambda_{\hat{B}}^{\hat{D}} \delta_{\hat{C}\hat{D}} = \delta_{\hat{A}\hat{B}}$ . Let the relation between the original coordinate basis  $\{\mathbf{E}_A\}$  and the new orthonormal basis be given by the matrix  $F'_{\hat{A}}^B$ , as follows:

$$\hat{\mathbf{E}}'_{\hat{A}} = F'_{\hat{A}}^B \mathbf{E}_B. \tag{3.13}$$

The relation between  $\mathbf{F}$  and  $\mathbf{F}'$  is given by

$$F'_{\hat{A}}^B = \Lambda_{\hat{A}}^{\hat{C}} F_{\hat{C}}^B. \tag{3.14}$$

Going in the opposite direction, one can see that  $\mathbf{F}$  and  $\mathbf{F}'$  represent the same material metric  $\mathbf{G}$ , if and only if they are related through (3.14) for some orthogonal matrix

<sup>15</sup>Note that given  $F_{\hat{A}}^B$ , the material metric can be recovered as

$$G_{AB} = F_{\hat{A}}^{\hat{C}} F_{\hat{B}}^{\hat{D}} \delta_{\hat{C}\hat{D}}.$$

$\Lambda_{\hat{A}}^{\hat{B}}$ . This means that there is an  $SO(3)$  ambiguity in the choice of  $\mathbf{F}$ , and hence, in that of  $\mathbf{F}_g$ .

Using an orthonormal basis for the material manifold, we rewrite the constitutive relation (3.2) as

$$\Psi = \Psi(\mathbf{X}, \Theta, G_{AB} = \delta_{AB}, F^a{}_B = (F_e)^a{}_B, g_{ab}). \tag{3.15}$$

Hence, given a constitutive relation  $\Psi^{\text{Riem}}$  in our (Riemannian) approach, one can obtain a constitutive relation  $\Psi^{\text{LR}}$  in the “local relaxation” approach by simply going to an orthonormal basis by (3.5) and (3.6), and ignoring the constant terms  $G_{AB} = \delta_{AB}$  and  $g_{ab} = \delta_{ab}$  in the functional dependence.

$$\Psi^{\text{LR}}(\mathbf{X}, \Theta, (F_e)^a{}_B) = \Psi^{\text{Riem}}(\mathbf{X}, \Theta, G_{AB} = \delta_{AB}, F^a{}_B = (F_e)^a{}_B, g_{ab} = \delta_{ab}). \tag{3.16}$$

Going in the opposite direction is also possible; starting with a free energy function for the  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$  approach, one can derive an equivalent free energy in the geometric approach.<sup>16</sup>

*Balance of Mass* In the  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$  approach, the mass balance reads  $S_m = \frac{\partial \rho_0}{\partial t} + \rho_0 \text{tr} \mathbf{L}_g$ , where  $\mathbf{L}_g = \dot{\mathbf{F}}_g \mathbf{F}_g^{-1}$ . Usually, it is assumed that growth is density preserving (Lubarda and Hoger 2002). We show that the term  $\text{tr} \mathbf{L}_g$  is equivalent to  $\frac{1}{2} \text{tr}_G(\frac{\partial \mathbf{G}}{\partial t})$ , where by  $\text{tr}_G$  we emphasize the  $\mathbf{G}$ -dependence of the trace operator. Note that

$$\text{tr}\left(\frac{\partial \mathbf{G}}{\partial t}\right) = \frac{\partial G_{AB}}{\partial t} G^{AB} = \frac{\partial}{\partial t} (F^{\hat{A}}{}_A F^{\hat{B}}{}_B \delta_{\hat{A}\hat{B}}) (F_{\hat{C}}{}^A F_{\hat{D}}{}^B \delta^{\hat{C}\hat{D}}) = 2 \dot{F}^{\hat{A}}{}_A F_{\hat{A}}{}^A = 2 \text{tr} \mathbf{L}_g.$$

*Incompressibility* In the  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$  approach, incompressibility is equivalent to  $J_e = 1$ . In the geometric theory, incompressibility means  $J = 1$ . These are equivalent as is shown below:

$$1 = J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{1}{\sqrt{\det(F^{\hat{A}}{}_A F^{\hat{B}}{}_B \delta_{\hat{A}\hat{B}})}} \det(F^a{}_{\hat{C}} F^{\hat{C}}{}_C) = \det F^a{}_{\hat{C}} = J_e. \tag{3.17}$$

*Absolutely Parallelizable Manifolds and Their Connection with Growth Mechanics* Whenever deformation is coupled with other phenomena, e.g. plasticity, growth/remodeling, thermal expansion/contraction, etc., all one can hope for is to locally decouple the elastic deformations from the inelastic deformations. Many related works start from a decomposition of the deformation gradient  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_a$ , where

<sup>16</sup>A simple example can make this clearer. Let us assume that free energy density in the classical approach is  $\mu \text{tr} \mathbf{C}_e$ . In components this reads

$$\begin{aligned} \Psi &= \mu (C_e)_{\hat{A}\hat{B}} \delta^{\hat{A}\hat{B}} = \mu (F_{\hat{A}}{}^A F^a{}_A) (F_{\hat{B}}{}^B F^b{}_B) \delta_{ab} \delta^{\hat{A}\hat{B}} = \mu (F^a{}_A F^b{}_B \delta_{ab}) (F_{\hat{A}}{}^A F_{\hat{B}}{}^B \delta^{\hat{A}\hat{B}}) \\ &= \mu F^a{}_A F^b{}_B \delta_{ab} G^{AB}. \end{aligned}$$

Thus,  $\Psi = \mu \text{tr}_G \mathbf{C}$ .

$\mathbf{F}_e$  is the elastic deformation gradient and  $\mathbf{F}_a$  is the remaining local deformation or anelastic deformation gradient. Given an (inelastic) growth deformation gradient, a vector in the tangent space of  $\mathbf{X} \in \mathcal{B}$ , i.e.  $\mathbf{W} \in T_{\mathbf{X}}\mathcal{B}$  is mapped to another vector,  $\hat{\mathbf{W}} = \mathbf{F}_a\mathbf{W}$ . Traditionally, these vectors are assumed to lie in the tangent bundle of an “intermediate configuration.” In the literature, the intermediate configuration is not clearly defined and at first glance it seems to be more or less mysterious. These vectors are closely related to parallelizable manifolds (or absolutely parallelizable (AP) manifolds) (Eisenhart 1926, 1927; Youssef and Sid-Ahmed 2007; Wanas 2008). In an  $n$ -dimensional AP-manifold  $M$ , one starts with a field of  $n$  linearly independent vectors  $\{\mathbf{E}_{(A)}\}$  that span the tangent space at each point. We denote the components of  $\mathbf{E}_{(A)}$  by  $\mathbf{E}_{(A)}^I$ . The dual vectors, i.e. the corresponding basis vectors for the cotangent space are denoted by  $\{\mathbf{E}^{(A)}\}$  with components  $\{\mathbf{E}_I^{(A)}\}$ . Note that

$$\mathbf{E}_I^{(A)}\mathbf{E}_{(B)}^I = \delta_B^A \quad \text{and} \quad \mathbf{E}_I^{(A)}\mathbf{E}_{(A)}^I = \delta_J^J. \tag{3.18}$$

One can equip  $M$  with a connection  $\Gamma_{JK}^I$  such that the basis vectors  $\{\mathbf{E}_{(A)}\}$  are covariantly constant, i.e.<sup>17</sup>

$$\mathbf{E}_{(A)|J}^I = 0. \tag{3.19}$$

Note that

$$\mathbf{E}_{(A)|JK}^I - \mathbf{E}_{(A)|KJ}^I = \mathcal{R}^I{}_{LJK}\mathbf{E}_{(A)}^L + \mathcal{T}^L{}_{KJ}\mathbf{E}_{(A)|L}^I. \tag{3.20}$$

Therefore, (3.19) implies that

$$\mathcal{R}^I{}_{LJK} = 0, \tag{3.21}$$

i.e.,  $M$  is flat with respect to the connection  $\Gamma_{JK}^I$ . Note that

$$\mathbf{E}_{(A)|J}^I = \frac{\partial \mathbf{E}_{(A)}^I}{\partial X^J} + \Gamma_{JK}^I\mathbf{E}_{(A)}^K. \tag{3.22}$$

Thus

$$\mathbf{E}_L^{(A)}\frac{\partial \mathbf{E}_{(A)}^I}{\partial X^J} + \Gamma_{LK}^I = 0. \tag{3.23}$$

Hence

$$\Gamma_{JK}^I = -\mathbf{E}_J^{(A)}\frac{\partial \mathbf{E}_{(A)}^I}{\partial X^K} = \mathbf{E}_I^{(A)}\frac{\partial \mathbf{E}_J^{(A)}}{\partial X^K}. \tag{3.24}$$

This connection has been used by many authors, e.g. by Bilby et al. (1955) and Kondo (1955a) for dislocations, by Epstein and Elżanowski (2007) for material inhomogeneities, and by Stojanović et al. (1964) for thermal stresses. This connection is curvature free by construction, but it has a non-vanishing torsion.

<sup>17</sup>Equivalently, the tangent bundle is a trivial bundle, so that the associated principal bundle of linear frames has a section on  $M$ .

For a growing body, in the local charts  $\{X^A\}$  and  $\{U^I\}$  for the reference and intermediate configurations, we have

$$dU^I = (F_g)^I_A dX^A. \quad (3.25)$$

$(F_g)^I_A$  can be identified with  $\mathbf{E}_{(A)}^I$ , and hence

$$\Gamma_{JK}^I = (F_g)^I_A \frac{\partial (F_g^{-1})^A_J}{\partial X^K}. \quad (3.26)$$

Note that this (growth) connection is curvature free but has a non-vanishing torsion. In plasticity it is shown that torsion of this connection has a physical meaning; it can be identified with the dislocation density tensor. For a growing body such a quantity does not seem to have a physical interpretation and we prefer to work with a Riemannian material manifold, whose curvature quantifies the tendency of the growth distribution in causing residual stresses.

In summary, our geometric approach has a concrete connection with that of  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$ : in the geometric approach we use a Riemannian manifold with a time-dependent metric as the material manifold, while  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$  implicitly uses the same metric but in an absolutely parallelizable manifold that is not Riemannian. We believe that our approach is more straightforward as we do not introduce an unnecessary torsion in the material manifold but of course the Riemannian material manifold has a non-vanishing curvature tensor, in general.

#### 4 Linearized Theory of Growth Mechanics

The geometric linearization of elasticity was first introduced by Marsden and Hughes (1983) and was further developed by Yavari and Ozakin (2008). See also Mazzucato and Rachele (2006). In this section, we start with a body with a time-dependent material manifold and its motion in an ambient space, which is assumed to be Euclidean. Suppose a given body with a material metric  $\mathbf{G}$  is in a static equilibrium configuration,  $\varphi$ . The balance of linear momentum for this material body reads<sup>18</sup>

$$\text{Div } \mathbf{P} + \rho_0 \mathbf{B} = \mathbf{0}. \quad (4.1)$$

Now suppose the body grows by a small amount represented by a small change in the material metric  $\delta \mathbf{G}$ .  $\varphi$  will no longer describe a static equilibrium configuration. Stress in this new equilibrium configuration  $\varphi' = \varphi + \delta \varphi$  will be  $\mathbf{P}' = \mathbf{P} + \delta \mathbf{P}$ . We are interested in calculating the change in the stress (or the configuration), for a given small amount of growth.

The linearization procedure can be formulated rigorously if instead of thinking about two nearby configurations and the differences between various quantities for these configurations, we describe the situation in terms of a one-parameter family

<sup>18</sup>Growth is a “slow” process compared to elastic deformations and hence inertial effects can be ignored. Throughout this paper, time is treated as a parameter.



of configurations around a reference motion, and calculate the derivatives of various quantities with respect to the parameter. Let  $\mathbf{G}_\epsilon(\mathbf{X})$  be a one-parameter family of material metrics, and let  $\varphi_\epsilon$  be the corresponding equilibrium configurations and  $\mathbf{P}_\epsilon$  the corresponding stresses. Let  $\epsilon = 0$  describe the reference configuration. Now, for a fixed point  $\mathbf{X}$  in the material manifold,  $\varphi_\epsilon(\mathbf{X})$  describes a curve in the spatial manifold, and its derivative at  $\epsilon = 0$  gives a vector  $\mathbf{U}(\mathbf{X})$  at  $\varphi(\mathbf{X})$  (Yavari and Ozakin 2008):

$$\mathbf{U}(\mathbf{X}) = \left. \frac{d\varphi_\epsilon(\mathbf{X})}{d\epsilon} \right|_{\epsilon=0}. \tag{4.2}$$

Considering  $\delta\varphi \approx \epsilon \frac{d\varphi_\epsilon}{d\epsilon}$ , we see that a more rigorous version of  $\delta\varphi$  is the vector field  $\mathbf{U}$ .  $\mathbf{U}$  is the geometric analog of what is called displacement field in classical linear elasticity.

The first variation (or linearization) of the deformation gradient is defined as

$$\mathcal{L}(\mathbf{F}) := \left. \nabla_{\frac{\partial}{\partial \epsilon}} \mathbf{F}_\epsilon \right|_{\epsilon=0} = \left. \nabla_{\frac{\partial}{\partial \epsilon}} \left( \frac{\partial \varphi_{t,\epsilon}}{\partial \mathbf{X}} \right) \right|_{\epsilon=0} = \nabla \mathbf{U}. \tag{4.3}$$

Or, in components

$$\mathcal{L}(\mathbf{F})^a{}_A = U^a|_A = \frac{\partial U^a}{\partial X^A} + \gamma^a_{bc} F^b{}_A U^c, \tag{4.4}$$

where the  $\gamma^a_{bc}$  are the connection coefficients of the Riemannian manifold  $(\mathcal{S}, \mathbf{g})$ . Note that for different values of  $\epsilon$ , the spatial leg of  $\mathbf{F}_\epsilon$  lies in different tangent spaces, and this is why a covariant derivative with respect to  $\frac{\partial}{\partial \epsilon}$  should be used. The right Cauchy–Green strain tensor for the perturbed motion  $\varphi_{t,\epsilon}$  is defined as

$$C_{AB}(\epsilon) = F^a{}_A(\epsilon) F^b{}_B(\epsilon) g_{ab}(\epsilon). \tag{4.5}$$

Note that  $\mathbf{C}_\epsilon$  lies in the same linear space for all  $\epsilon \in I$ , and the first variation of  $\mathbf{C}$  can be calculated as

$$\frac{d}{d\epsilon} C_{AB}(\epsilon) = \nabla_{\frac{\partial}{\partial \epsilon}} F^a{}_A(\epsilon) F^b{}_B(\epsilon) g_{ab}(\epsilon) + F^a{}_A(\epsilon) \nabla_{\frac{\partial}{\partial \epsilon}} F^b{}_B(\epsilon) g_{ab}(\epsilon). \tag{4.6}$$

Therefore,

$$\mathcal{L}(\mathbf{C})_{AB} := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} C_{AB}(\epsilon) = F^b{}_B g_{ab} U^a|_A + F^a{}_A g_{ab} U^b|_B. \tag{4.7}$$

The transpose of the deformation gradient has the following linearization (Yavari and Ozakin 2008):  $\mathcal{L}(\mathbf{F}^T) = (\nabla \mathbf{U})^T$ . Spatial and material strain tensors are defined, respectively, by (Marsden and Hughes 1983)

$$\mathbf{e} = \frac{1}{2}(\mathbf{g} - \varphi_{t*} \mathbf{G}) \quad \text{and} \quad \mathbf{E} = \frac{1}{2}(\varphi_t^* \mathbf{g} - \mathbf{G}). \tag{4.8}$$

Or, in components

$$e_{ab} = \frac{1}{2}(g_{ab} - G_{AB} (F^{-1})^A{}_a (F^{-1})^B{}_b), \quad E_{AB} = \frac{1}{2}(C_{AB} - G_{AB}). \tag{4.9}$$

We now show that linearization of  $\mathbf{E}$  is related to  $\boldsymbol{\epsilon} = \frac{1}{2} \mathcal{L}_{\mathbf{u}} \mathbf{g}$ , where  $\mathbf{u} = \mathbf{U} \circ \varphi^{-1}$ . We know that

$$\begin{aligned} \mathcal{L}(\mathbf{C})_{AB} &= g_{ab} F^a_A F^c_B u^b_{|c} + g_{ab} F^b_B F^c_A u^a_{|c} = F^a_A F^c_B u_{a|c} + F^b_B F^c_A u_{b|c} \\ &= 2 F^a_A F^b_B \epsilon_{ab}, \end{aligned} \tag{4.10}$$

where  $\epsilon_{ab} = \frac{1}{2}(u_{a|b} + u_{b|a})$  is the linearized strain. Therefore

$$\mathcal{L}(\mathbf{C}) = 2\varphi_t^* \boldsymbol{\epsilon}. \tag{4.11}$$

Thus

$$\boldsymbol{\epsilon} = \varphi_{t*} \mathcal{L}(\mathbf{E}). \tag{4.12}$$

In other words, the linearized strain is the push-forward of the linearized Lagrangian strain. Obviously, if the ambient space is Euclidean and the coordinates are Cartesian, then the covariant derivatives reduce to partial derivatives and one recovers the classical definition of linear strain in terms of partial derivatives, i.e.

$$\epsilon_{ab} = \frac{1}{2} \left( \frac{\partial u_a}{\partial x^b} + \frac{\partial u_b}{\partial x^a} \right). \tag{4.13}$$

Note that when the linearized strain is zero the variation field is a Killing vector field for the spatial metric  $\mathbf{g}$ . In other words, this shows that this definition of linearized strain is consistent when the variation field generates an isometry of the ambient space.

For the one-parameter family of material metrics  $\mathbf{G}_\epsilon$ , the variation of the material metric is defined as

$$\delta \mathbf{G} \approx \epsilon \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathbf{G}_\epsilon. \tag{4.14}$$

In the case of isotropic growth

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathbf{G}_\epsilon = \frac{d}{d\epsilon} e^{2\Omega\epsilon} \mathbf{G}_0 = 2 \left. \frac{d\Omega}{d\epsilon} \right|_{\epsilon=0} \mathbf{G} = \beta \mathbf{G}, \tag{4.15}$$

where  $\beta = 2\delta\Omega$ . Now consider, in the absence of body forces, the equilibrium equations  $\text{Div} \mathbf{P} = \mathbf{0}$  for the family of material metrics parametrized by  $\epsilon$ :  $\text{Div}_\epsilon \mathbf{P}_\epsilon = \mathbf{0}$ . Linearization of the equilibrium equations is defined as (Yavari and Ozakin 2008)

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (\text{Div}_\epsilon \mathbf{P}_\epsilon) = \mathbf{0}. \tag{4.16}$$

Once again, one should note that since the equilibrium configuration is different for each  $\epsilon$ ,  $\mathbf{P}_\epsilon$  is based at different points in the ambient space for different values of  $\epsilon$ , and in order to calculate the derivative with respect to  $\epsilon$ , one in general needs to use the connection (parallel transport) in the ambient space. For the case of Euclidean

ambient space that we are considering and for a Cartesian coordinate system  $\{x^a\}$ , (4.16) is simplified and in components reads

$$\frac{\partial P^{aA}(\epsilon)}{\partial X^A} + \Gamma_{AB}^A(\epsilon) P^{aB}(\epsilon) = 0. \tag{4.17}$$

Thus, the linearized balance of linear momentum can be written as

$$\frac{\partial}{\partial X^A} \frac{d}{d\epsilon} \Big|_{\epsilon=0} P^{aA}(\epsilon) + \left[ \frac{d}{d\epsilon} \Big|_{\epsilon=0} \Gamma_{AB}^A(\epsilon) \right] P^{aB} + \Gamma_{AB}^A \frac{d}{d\epsilon} \Big|_{\epsilon=0} P^{aB}(\epsilon) = 0. \tag{4.18}$$

Note that

$$P^{aA} = g^{ac} \frac{\partial \Psi}{\partial F_A^c}, \tag{4.19}$$

where  $\Psi = \Psi(\mathbf{X}, \Theta, \mathbf{F}, \mathbf{G}, \mathbf{g})$  is the material free energy density. In calculating  $\frac{dP^{aA}(\epsilon)}{d\epsilon}$ , we need to consider the changes in both  $\mathbf{F}$  and  $\mathbf{G}$ :

$$\frac{dP^{aA}(\epsilon)}{d\epsilon} = \frac{\partial P^{aA}}{\partial F^b_B} \frac{dF^b_B}{d\epsilon} + \frac{\partial P^{aA}}{\partial G_{CD}} \frac{dG_{CD}}{d\epsilon}. \tag{4.20}$$

Let us define

$$\mathbb{A}^{aA}{}_b{}^B = \frac{\partial P^{aA}}{\partial F^b_B} = g^{ac} \frac{\partial^2 \Psi}{\partial F^b_B \partial F^c_A} \quad \text{and} \quad \mathbb{B}^{aACD} = \frac{P^{aA}}{G_{CD}} = g^{ac} \frac{\partial^2 \Psi}{\partial G_{CD} \partial F^c_A}, \tag{4.21}$$

where the derivatives are to be evaluated at the reference motion  $\epsilon = 0$ . Noting that for the case of an Euclidean ambient space (see (4.4))

$$\frac{dF^a_A}{d\epsilon} \Big|_{\epsilon=0} = \frac{\partial U^a}{\partial X^A} \tag{4.22}$$

we obtain

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} P^{aA}(\epsilon) = \mathbb{A}^{aA}{}_b{}^B U^b{}_{,B} + \mathbb{B}^{aACD} \delta G_{CD}. \tag{4.23}$$

Using

$$\Gamma_{BC}^A = \frac{1}{2} G^{AD} \left( \frac{\partial G_{BD}}{\partial X^C} + \frac{\partial G_{CD}}{\partial X^B} - \frac{\partial G_{BC}}{\partial X^D} \right) \tag{4.24}$$

and

$$\frac{dG^{AB}}{d\epsilon} = -G^{AC} G^{BD} \frac{dG_{CD}}{d\epsilon}, \tag{4.25}$$

we obtain

$$\delta \Gamma_{AB}^A = \frac{d}{d\epsilon} \Big|_{\epsilon=0} \Gamma_{AB}^A(\epsilon)$$

$$= -G^{CD} \delta G_{CD} \Gamma_{AB}^A + \frac{1}{2} G^{AD} \left[ \frac{\partial \delta G_{BD}}{\partial X^C} + \frac{\partial \delta G_{CD}}{\partial X^B} - \frac{\partial \delta G_{BC}}{\partial X^D} \right]. \tag{4.26}$$

In the case of isotropic growth, this is reduced to

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} \Gamma_{AB}^A(\epsilon) = \frac{3}{2} \frac{\partial \beta}{\partial X^B}. \tag{4.27}$$

With these results, the linearized balance of linear momentum (4.16) becomes

$$(\mathbb{A}^{aA}{}_b{}^B U^b{}_{,B})_{,A} + (\mathbb{B}^{aACD} \delta G_{CD})_{,A} + \frac{3}{2} \frac{\partial \beta}{\partial X^B} P^{aB} = 0. \tag{4.28}$$

Assuming that  $\mathbb{A}$  and  $\mathbb{B}$  are independent of  $\mathbf{X}$ , the linearized equilibrium equations are simplified and read

$$\mathbb{A}^{aA}{}_b{}^B \frac{\partial^2 U^b}{\partial X^A \partial X^B} + \mathbb{B}^{aACD} G_{CD} \frac{\partial \beta}{\partial X^A} + \frac{3}{2} \frac{\partial \beta}{\partial X^B} P^{aB} = 0. \tag{4.29}$$

If the initial configuration is stress free, we have

$$\mathbb{A}^{aA}{}_b{}^B \frac{\partial^2 U^b}{\partial X^A \partial X^B} = -\mathbb{B}^{aACD} G_{CD} \frac{\partial \beta}{\partial X^A}. \tag{4.30}$$

Let us now simplify the above linearized equations for a specific class of elastic materials.

*Saint-Venant–Kirchhoff Materials* Saint-Venant–Kirchhoff materials have a constitutive relation that is analogous to the linear isotropic materials, namely, the second Piola–Kirchhoff stress  $\mathbf{S}$  is given in terms of the Lagrangian strain  $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G})$  as (Marsden and Hughes 1983)  $\mathbf{S} = \lambda(\text{tr } \mathbf{E})\mathbf{G}^{-1} + 2\mu\mathbf{E}$  or, in components

$$S^{CD} = \lambda E_{AB} G^{AB} G^{CD} + 2\mu E^{CD} = \frac{\lambda}{2} (C_{AB} G^{AB} - 3) G^{CD} + \mu (C_{AB} G^{AC} G^{BD} - G^{CD}), \tag{4.31}$$

where  $\lambda = \lambda(\mathbf{X})$  and  $\mu = \mu(\mathbf{X})$  are two scalars characterizing the material properties. We can obtain the tensor  $\mathbb{B}^{aCAB}$  from  $\mathbf{S}$  as follows:

$$\mathbb{B}^{aCAB} = \frac{\partial}{\partial G_{AB}} \left( g^{ab} \frac{\partial \psi}{\partial F^b{}_C} \right) = \frac{\partial P^{aC}}{\partial G_{AB}} = F^a{}_D \frac{\partial S^{CD}}{\partial G_{AB}}. \tag{4.32}$$

Using

$$\frac{\partial G^{AB}}{\partial G_{MN}} = -G^{AM} G^{BN} \tag{4.33}$$

we obtain

$$\mathbb{B}^{aACD} G_{CD} = -2C_{MN} F^a{}_B (\lambda G^{AB} G^{MN} + 2\mu G^{AM} G^{BN}) + (3\lambda + 2\mu) F^a{}_B G^{AB}. \tag{4.34}$$

The initial metric is Euclidean; in Cartesian coordinates,  $G_{AB} = \delta_{AB}$ . Since the ambient space is also Euclidean, we can choose a Cartesian coordinate system whose axes coincide with the initial location of the material points along the material Cartesian axis. This will give,  $F^a_A = \delta^a_A$ , where  $a$  and  $A$  both range over 1, 2, 3. Hence

$$\mathbb{B}^{aACD} G_{CD} = -\frac{3\lambda + 2\mu}{2} \delta^{aA}. \tag{4.35}$$

Similarly, for an initially stress-free material manifold, we obtain

$$\mathbb{A}^{aA}{}_b{}^B = F^a_M F^c_N g_{bc} [\lambda G^{AM} G^{BN} + \mu (G^{AB} G^{MN} + G^{AN} G^{BM})]. \tag{4.36}$$

For the case of an initially Euclidean material manifold with Cartesian coordinates we have

$$\mathbb{A}^{aA}{}_b{}^B \frac{\partial^2 U^b}{\partial X^A \partial X^B} = (\lambda + \mu) U_{b,ab} + \mu U_{a,bb}. \tag{4.37}$$

Therefore, (4.30) reads

$$(\lambda + \mu) U_{b,ab} + \mu U_{a,bb} = \frac{3\lambda + 2\mu}{2} \frac{\partial \beta}{\partial x_a}, \tag{4.38}$$

where we have identified the indices  $a$  and  $A$ . In analogy with thermal stresses,  $\beta \delta_{ab}$  can be thought of as an eigenstrain. See Goriely et al. (2008) for a review of the existing linearized growth models.

*Stress-Free Growth Distributions in the Linearized Theory* In this paragraph we show that in dimension three, if  $\beta$  is linear in  $\{X^A\}$ , i.e. if  $\beta = \mathbf{a} \cdot \mathbf{X}$  for some constant vector  $\mathbf{a}$ , then a stress-free body remains stress free after growth. This is very similar to what is already known in classical linear thermoelasticity: temperature distributions linear in Cartesian coordinates leave a stress-free body stress free (Boley and Weiner 1997; Ozakin and Yavari 2010).

Let us consider a one-parameter family of material metrics  $\mathbf{G}_\epsilon$  and assume that the initial material metric is Euclidean, i.e.  $\mathbf{G}_{\epsilon=0} = \delta$ . The corresponding curvature tensor is  $\mathcal{R}_\epsilon$ . We need to calculate the linearized curvature, i.e.

$$\delta \mathcal{R} = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{R}_\epsilon. \tag{4.39}$$

This will give the solution to stress-free growth distributions. Note that  $\delta \mathbf{G} = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathbf{G}_\epsilon$  corresponds to a linearized growth and is stress free if and only if  $\delta \mathcal{R}$  vanishes. To calculate the curvature variation, we follow Hamilton (1982) and denote the derivative with respect to  $\epsilon$  by  $'$ . Following the definition of the curvature tensor, we can write

$$\begin{aligned} \mathcal{R}'_{ABCD} = & -\frac{1}{2} \left( \frac{\partial^2 G'_{BD}}{\partial X^A \partial X^C} - \frac{\partial^2 G'_{BC}}{\partial X^A \partial X^D} - \frac{\partial^2 G'_{AD}}{\partial X^B \partial X^C} + \frac{\partial^2 G'_{AC}}{\partial X^B \partial X^D} \right) \\ & + \frac{1}{2} G^{PQ} (\mathcal{R}_{ABCP} G'_{QD} + \mathcal{R}_{ABPD} G'_{QC}). \end{aligned} \tag{4.40}$$

In the case of Ricci curvature

$$R'_{AB} = G^{CD} \mathcal{R}'_{ACBD} + (G')^{CD} \mathcal{R}_{ACBD} = G^{CD} \mathcal{R}'_{ACBD} - G^{CP} G^{DQ} G'_{PQ} \mathcal{R}_{ACBD}. \tag{4.41}$$

Similarly, for the scalar curvature we have

$$R' = g^{AB} R'_{AB} + (G')^{AB} R_{AB} = G^{AB} R'_{AB} - G^{AP} G^{BQ} G'_{PQ} R_{AB}. \tag{4.42}$$

If the initial material manifold is Euclidean, i.e. if  $\mathcal{R}_{ACBD} = 0$  and  $R_{AB} = 0$ , we have

$$\delta \mathcal{R}_{ABCD} = -\frac{1}{2} \left( \frac{\partial^2 \delta G_{BD}}{\partial X^A \partial X^C} - \frac{\partial^2 \delta G_{BC}}{\partial X^A \partial X^D} - \frac{\partial^2 \delta G_{AD}}{\partial X^B \partial X^C} + \frac{\partial^2 \delta G_{AC}}{\partial X^B \partial X^D} \right), \tag{4.43}$$

$$\delta R_{AB} = -\frac{1}{2} \left( \frac{\partial^2 \delta G_{CD}}{\partial X^A \partial X^B} - \frac{\partial^2 \delta G_{BC}}{\partial X^A \partial X^D} - \frac{\partial^2 \delta G_{AD}}{\partial X^B \partial X^C} + \frac{\partial^2 \delta G_{AB}}{\partial X^C \partial X^D} \right) \delta^{CD}, \tag{4.44}$$

$$\delta R = \frac{\partial^2 \delta G_{BC}}{\partial X^A \partial X^D} \delta^{AB} \delta^{CD} - \frac{\partial^2 \delta G_{AB}}{\partial X^C \partial X^D} \delta^{AB} \delta^{CD}. \tag{4.45}$$

In the case of isotropic growth we have  $\delta G_{AB} = \beta \delta_{AB}$ . In dimension three, vanishing of the Ricci curvature is equivalent to vanishing of the curvature tensor. Thus,  $\delta R_{AB} = 0$  reduces to

$$\frac{\partial^2 \beta}{\partial X^A \partial X^B} + \frac{\partial^2 \beta}{\partial X^C \partial X^D} \delta^{CD} \delta_{AB} = 0. \tag{4.46}$$

This is equivalent to

$$\beta_{,12} = \beta_{,13} = \beta_{,23} = 0, \tag{4.47}$$

$$2\beta_{,11} + \beta_{,22} + \beta_{,33} = 0, \tag{4.48}$$

$$\beta_{,11} + 2\beta_{,22} + \beta_{,33} = 0, \tag{4.49}$$

$$\beta_{,11} + \beta_{,22} + 2\beta_{,33} = 0. \tag{4.50}$$

The three relations (4.47) imply that  $\beta = f(X^1) + g(X^2) + h(X^3)$  for arbitrary functions  $f, g$ , and  $h$ . The next three relations, (4.48)–(4.50), imply that  $\beta_{,11} = \beta_{,22} = \beta_{,33} = 0$  and therefore  $f''(X^1) = g''(X^2) = h''(X^3) = 0$ ; and hence  $\beta$  is linear in Cartesian coordinates of the initial material manifold.

In dimension two,  $\delta R = 0$  reduces to

$$\frac{\partial^2 \beta}{\partial X^A \partial X^B} \delta_{AB} = 0. \tag{4.51}$$

This means that  $\beta$  has to be a harmonic function to represent a stress-free growth distribution. Again, this is very similar to what we know from classical linear thermoelasticity (Boley and Weiner 1997; Ozakin and Yavari 2010).

## 5 Concluding Remarks

In this paper, we presented a geometric theory of elastic solids with bulk growth. We assumed that the material points are preserved but density and “shape” are time dependent. We modeled a body with bulk growth by a Riemannian material manifold with an evolving metric tensor. The time dependency of material metric is such that the growing body is always stress free in the material manifold. We showed that the energy balance needs to be modified when the material metric is time dependent. Covariance of the energy balance then gives all the balance laws. We also showed that entropy production inequality has a non-standard form when the material manifold has an evolving metric. We showed that a more general notion of covariance of energy balance that includes temperature rescalings, in addition to giving all the balance laws, gives the constitutive restrictions imposed by the Clausius–Duhem inequality. We then showed how the principle of maximum entropy production can be used to obtain thermodynamically-consistent evolution equations for the material metric.

We showed how analytical solutions for the residual stress field can be obtained in three examples of growing bodies with radial symmetries. We showed that even if mass is conserved, i.e. when growth results in only shape changes, still one may see residual stresses. In the case of isotropic growth, we studied stress-free growth distributions using the material curvature tensor in both two and three dimensions.

A concrete connection was made between our geometric theory and the conventional decomposition of the deformation gradient into elastic and growth parts. We showed that, in a special coordinate basis,  $\mathbf{F}_e$  is our  $\mathbf{F}$ . The present geometric theory is more natural and does not introduce a mysterious intermediate configuration. We linearized the nonlinear theory about a reference motion. Assuming that both the ambient space and the initial material manifold are Euclidean, we showed that growth results in eigenstrains very similar to those of classical linear thermoelasticity. We found those growth distributions that are stress free in the linearized framework in both dimensions two and three.

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## References

- Ambrosi, D., Guana, F.: Stress-modulated growth. *Math. Mech. Solids* **12**(3), 319–342 (2007)
- Ambrosi, D., Mollica, F.: The role of stress in the growth of a multicell spheroid. *J. Math. Biol.* **48**, 477–499 (2004)
- Ben Amar, M., Goriely, A.: Growth and instability in elastic tissues. *J. Mech. Phys. Solids* **53**, 2284–2319 (2005)
- Berger, M.: *A Panoramic View of Riemannian Geometry*. Springer, New York (2003)
- Bilby, B.A., Bullough, R., Smith, E.: Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry. *Proc. R. Soc. Lond. A* **231**(1185), 263–273 (1955)
- Bilby, B.A., Gardner, L.R.T., Stroh, A.N.: Continuous distribution of dislocations and the theory of plasticity. In: *Proceedings of the Ninth International Congress of Applied Mechanics, Brussels, 1956*, pp. 35–44. Université de Bruxelles, Brussels (1957)
- Boley, B.A., Weiner, J.H.: *Theory of Thermal Stresses*. Dover, New York (1997)
- Brethert, F.P.: A note on Hamilton’s principle for perfect fluids. *J. Fluid Mech.* **44**, 19–31 (1970)

- Chen, Y.C., Hoger, A.: Constitutive functions of elastic materials in finite growth and deformation. *J. Elast.* **59**(1–3), 175–193 (2000)
- Coleman, B.D., Noll, W.: The thermodynamics of elastic materials with heat conduction and viscosity. *Arch. Ration. Mech. Anal.* **13**, 167–178 (1963)
- Cowin, S.C., Hegedus, D.H.: Bone remodeling 1. Theory of adaptive elasticity. *J. Elast.* **6**, 313–326 (1976)
- DiCarlo, A., Quiligotti, S.: Growth and balance. *Mech. Res. Commun.* **29**, 449–456 (2002)
- Eckart, C.: The thermodynamics of irreversible processes. 4. The theory of elasticity and anelasticity. *Phys. Rev.* **73**(4), 373–382 (1948)
- Efrati, E., Sharon, E., Kupferman, R.: Elastic theory of unconstrained non-Euclidean plates. *J. Mech. Phys. Solids* **57**(4), 762–775 (2009)
- Eisenhart, L.P.: *Riemannian Geometry*. Princeton University Press, Princeton (1926)
- Eisenhart, L.P.: *Non-Riemannian Geometry*. Dover, New York (1927)
- Epstein, M., Elżanowski, M.: *Material Inhomogeneities and their Evolution*. Springer, New York (2007)
- Epstein, M., Maugin, G.A.: Thermomechanics of volumetric growth in uniform bodies. *Int. J. Plast.* **16**, 951–978 (2000)
- Fung, Y.C.: On the foundations of biomechanics. *J. Appl. Mech.* **50**, 1003–1009 (1983)
- Fusi, L., Farina, A., Ambrosi, D.: Mathematical modeling of a solid-liquid mixture with mass exchange between constituents. *Math. Mech. Solids* **11**(6), 575–595 (2006)
- Garikipati, K., Arruda, E.M., Gosh, K., Narayanan, H., Calve, S.: A continuum treatment of growth in biological tissue: the coupling of mass transport and mechanics. *J. Mech. Phys. Solids* **52**(7), 1595–1625 (2004)
- Goriely, A., Robertson-Tessi, M., Tabor, M., Vandiver, R.: Elastic growth models. In: Mondaini, R. (ed.) *Mathematical Modelling of Biosystems*. Springer, Berlin (2008)
- Green, A.E., Naghdi, P.M.: On thermodynamics and nature of second law. *Proc. R. Soc. Lond. Ser. A* **357**, 253–270 (1977)
- Green, A.E., Naghdi, P.M.: A demonstration of consistency of an entropy balance with balance of energy. *Z. Angew. Math. Phys.* **42**, 159–168 (1991)
- Green, A.E., Rivlin, R.S.: On Cauchy's equations of motion. *Z. Angew. Math. Phys.* **15**, 290–293 (1964)
- Hamilton, R.S.: 3-manifolds with positive Ricci curvature. *J. Differ. Geom.* **17**, 255–306 (1982)
- Hoger, A.: Virtual configurations and constitutive equations for residually stressed bodies with material symmetry. *J. Elast.* **48**, 125–144 (1997)
- Holm, D.D., Marsden, J.E., Ratiu, T.S.: The Euler–Poincaré equations and semidirect products with applications to continuum theories. *Adv. Math.* **137**(1), 1–81 (1998)
- Hsu, F.H.: The influences of mechanical loads on the form of a growing elastic body. *J. Biomech.* **1**, 303–313 (2003)
- Humphrey, J.D.: Continuum biomechanics of soft biological tissues. *Proc. R. Soc. Lond. Ser. A* **459**, 3–46 (2003)
- Klarbring, A., Olsson, T., Stalhand, J.: Theory of residual stresses with application to an arterial geometry. *Arch. Mech.* **59**, 341–364 (2007)
- Kondaurov, V.I., Nikitin, L.V.: Finite strains of viscoelastic muscle tissue. *PMM J. Appl. Math. Mech.* **51**, 346–353 (1987)
- Kondo, K.: Geometry of elastic deformation and incompatibility. In: Kondo, K. (ed.) *Memoirs of the Unifying Study of the Basic Problems in Engineering Science by Means of Geometry*, vol. 1, pp. 5–17. Division C, Gakujutsu Bunken Fukyo-Kai (1955a)
- Kondo, K.: Non-Riemannian geometry of imperfect crystals from a macroscopic viewpoint. In: Kondo, K. (ed.) *Memoirs of the Unifying Study of the Basic Problems in Engineering Science by Means of Geometry*, vol. 1, pp. 6–17. Division D-I, Gakujutsu Bunken Fukyo-Kai (1955b)
- Kondo, K.: Non-Riemannian and Finslerian approaches to the theory of yielding. *Int. J. Eng. Sci.* **1**, 71–88 (1963)
- Kondo, K.: On the analytical and physical foundations of the theory of dislocations and yielding by the differential geometry of continua. *Int. J. Eng. Sci.* **2**, 219–251 (1964)
- Kröner, E.: Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen. *Arch. Ration. Mech. Anal.* **4**, 273–334 (1960)
- Lee, E.H.: Elastic-plastic deformation at finite strains. *J. Appl. Mech.* **36**, 1–6 (1967)
- Lee, J.M.: *Riemannian Manifold. An Introduction to Curvature*. Springer, New York (1997)
- Lee, E.H., Liu, D.T.: Finite-strain elastic-plastic theory with application to plane-wave analysis. *J. Appl. Phys.* **38**, 19–27 (1967)
- Loret, B., Simoes, F.M.F.: A framework for deformation, generalized diffusion, mass transfer and growth in multi-species multi-phase biological tissues. *Eur. J. Mech. A-Solids* **24**(5), 757–781 (2005)



- Lubarda, V.A., Hoger, A.: On the mechanics of solids with a growing mass. *Int. J. Solids Struct.* **39**, 4627–4664 (2002)
- Lubarda, V.A.: Constitutive theories based on the multiplicative decomposition of deformation gradient: Thermoelasticity, elastoplasticity, and biomechanics. *Appl. Mech. Rev.* **57**(2), 95–108 (2004)
- Marsden, J.E., Hughes, T.J.R.: *Mathematical Foundations of Elasticity*. Dover, New York (1983)
- Marsden, J.E., Ratiu, T.: *Introduction to Mechanics and Symmetry*. Springer, New York (2003)
- Martyushev, L.M., Seleznev, V.D.: Maximum entropy production principle in physics, chemistry and biology. *Phys. Rep.* **426**(1), 1–45 (2006)
- Mazzucato, A.L., Rachele, L.V.: Partial uniqueness and obstruction to uniqueness in inverse problems for anisotropic elastic media. *J. Elast.* **83**, 205–245 (2006)
- Miehe, C.: A constitutive frame of elastoplasticity at large strains based on the notion of a plastic metric. *Int. J. Solids Struct.* **35**, 3859–3897 (1998)
- Naumov, V.E.: Mechanics of growing deformable solids—a review. *J. Eng. Mech.* **120**, 207–220 (1994)
- Nishikawa, S.: *Variational Problems in Geometry*. American Mathematical Society, Providence (2002)
- Ozakin, A., Yavari, A.: A geometric theory of thermal stresses. *J. Math. Phys.* **51**, 032902 (2010)
- Perelman, G.: The entropy formula for the Ricci flow and its geometric applications (2002). <http://arXiv.org/math.DG/0211159v1>
- Peterson, P.: *Riemannian Geometry*. Springer, New York (1997)
- Rajagopal, K.R., Srinivasa, A.R.: On thermomechanical restrictions of continua. *Proc. R. Soc. A* **460**(2042), 631–651 (2004a)
- Rajagopal, K.R., Srinivasa, A.R.: On the thermomechanics of materials that have multiple natural configurations—Part I: Viscoelasticity and classical plasticity. *Z. Angew. Math. Phys.* **55**(5), 861–893 (2004b)
- Riewe, F.: Nonconservative Lagrangian and Hamiltonian mechanics. *Phys. Rev. E* **53**(2), 1890–1899 (1997)
- Rodriguez, E.K., Hoger, A., McCulloch, A.D.: Stress-dependent finite growth in soft elastic tissues. *J. Biomech.* **27**, 455–467 (1994)
- Senan, N.A.F., O’Reilly, O.M., Treslerras, T.N.: Modeling the growth and branching of plants: a simple rod-based model. *J. Mech. Phys. Solids* **56**, 3021–3036 (2008)
- Simo, J.C., Marsden, J.E.: On the rotated stress tensor and the material version of the Doyle–Ericksen formula. *Arch. Ration. Mech. Anal.* **86**, 213–231 (1984)
- Skalak, R., Dasgupta, G., Moss, M., Otten, E., Dullemeijer, P., Vilmann, H.: Analytical description of growth. *J. Theor. Biol.* **94**, 555–577 (1982)
- Skalak, R., Zargaryan, S., Jain, R.K., Netti, P.A., Hoger, A.: Compatibility and the genesis of residual stress by volumetric growth. *J. Math. Biol.* **34**, 889–914 (1996)
- Spivak, M.: *A Comprehensive Introduction to Differential Geometry*, vol. III. Publish or Perish, Houston (1999)
- Stojanović, R.: On the stress relation in non-linear thermoelasticity. *Int. J. Non-Linear Mech.* **4**, 217–233 (1969)
- Stojanović, R., Djurić, S., Vujošević, L.: On finite thermal deformations. *Arch. Mech. Stosow.* **16**, 103–108 (1964)
- Takamizawa, K.: Stress-free configuration of a thick-walled cylindrical model of the artery—an application of Riemann geometry to the biomechanics of soft tissues. *J. Appl. Mech.* **58**, 840–842 (1991)
- Takamizawa, K., Matsuda, T.: Kinematics for bodies undergoing residual stress and its applications to the left ventricle. *J. Appl. Mech.* **57**, 321–329 (1990)
- Topping, P.: *Lectures on the Ricci Flow*. Cambridge University Press, New York (2006)
- Vujošević, L., Lubarda, V.A.: Finite-strain thermoelasticity based on multiplicative decomposition of deformation gradient. *Theor. Appl. Mech.* **28–29**, 379–399 (2002)
- Wald, R.M.: *General Relativity*. The University of Chicago Press, Chicago (1984)
- Wanas, M.I.: Absolute parallelism geometry: developments, applications and problems (2008). [arXiv:gr-qc/0209050v1](http://arXiv:gr-qc/0209050v1)
- Yavari, A.: On geometric discretization of elasticity. *J. Math. Phys.* **49**, 022901 (2008)
- Yavari, A., Marsden, J.E.: Covariant balance laws in continua with microstructure. *Rep. Math. Phys.* **63**(1), 1–42 (2009a)
- Yavari, A., Marsden, J.E.: Energy balance invariance for interacting particle systems. *Z. Angew. Math. Phys.* **60**(4), 723–738 (2009b)

- Yavari, A., Ozakin, A.: Covariance in linearized elasticity. *Z. Angew. Math. Phys.* **59**(6), 1081–1110 (2008)
- Yavari, A., Marsden, J.E., Ortiz, M.: On the spatial and material covariant balance laws in elasticity. *J. Math. Phys.* **47**, 042903 (2006). 85–112
- Youssef, N.L., Sid-Ahmed, A.M.: Linear connections and curvature tensors in the geometry of parallelizable manifolds. *Rep. Math. Phys.* **60**, 39–53 (2007)
- Ziegler, H.: *An Introduction to Thermomechanics*. North-Holland, Amsterdam (1983)