PERGAMON

International Journal of Solids and Structures 37 (2000) 2377–2403

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

On the kern of a general cross section

Massood Mofid, Arash Yavari*

Department of Civil Engineering, Sharif University of Technology, P.O. Box 11365-9313, Tehran, Iran

Received 6 May 1998; in revised form 24 November 1998

Abstract

The kern of a section is the region in which a compressive point load may be applied without producing any tensile stress on the cross section. Ten theorems describing the characters of the kern of a general cross section are derived. Three types of cross sections are considered: simply connected, multiply connected, and disconnected. It is shown how to obtain the kern of a multiply-connected or disconnected cross section using an auxiliary simply-connected section. Qualitative shapes of the kerns of some cross sections, with known numerically calculated kerns, are obtained using the derived theorems. Kern ratio is defined and its boundedness is discussed. The kern ratio of regular polygonal sections are obtained as a function of the number of vertices and its minimum and maximum are calculated. The paper ends with an analytical derivation of the kern of a general cross section with some examples. © 2000 Elsevier Science Ltd. All rights reserved.

Nomenclature

A, B, C, D, E, F	corner points of the boundary of a cross section
A, B, D, M	points in the domain of a cross section or on its boundary
$A_1, \ldots, A_6$	vertices of a hexagonal cross section
$A'_1, \ldots, A'_6$	vertices of the kern of a hexagonal cross section
A	area of a cross section
A^*	area of the kern of a cross section
$A_{\Delta MNQ}$	area of a triangle with vertices M, N, and Q
AB	the line segment with end points A and B
C	centroid of a cross section

* Corresponding author. Currently at School of Engineering and Applied Science. The George Washington University, Washington, DC 20052, USA.

E-mail address: arashy@seas.gwu.edu (A. Yavari)

0020-7683/00/$ - see front matter
PII: S0020-7683(98)00337-0

M_x, M_y	components of bending moment vector along principal axes
M_1, M_2, M_A, M_B	points on the boundary of the kern of a cross section
M_1M_2	a line segment part of the boundary of the kern of a cross section
N	a point on the concave part of the boundary of a cross section
N'	corresponding point of N on the boundary of the convex hulls of the cross section
$K(\xi,\eta)=0$	a representation of the boundary of the kern of a cross section
I_x, I_y	principal moments of inertia
$KR=A^*/A$	kern ratio
P	the magnitude of a compressive point load
R	radius of a circular cross section
X, Y	coordinates of a point on the convex part of the boundary of a cross section
a, b, c, d, e, f	corner points of the boundary of the kern
$2a$, $2b$	diameters of an elliptical cross section
b	length of the legs of a cruciform cross section
(d_1), (d_2), (d_3)	neutral axes or tangent lines
(d_A), (d_B)	neutral axes corresponding to points A and B
$g(x,y)=0$	a representation for convex part of the boundary of a cross section
m	slope of a tangent line
n	number of vertices of a regular polygonal cross section
r_x, r_y	radii of gyration with respect to principal axes
t	thickness of a hollow cross section or of a cruciform cross section
t_1, t_2	thickness of imaginary strips or cuts
u, v	coordinates of a point on the boundary of the kern of a cross section
x, y	principal axes of a cross section or coordinates of a point
x_0, y_0	coordinates of the points of intersection of a tangent line and principal axes
x_1, y_1	transformed coordinates
α, β, γ	constant multipliers
ξ, η	coordinates of a point
λ	a Lagrange multiplier
σ	normal stress
σ_B^A	stress at point B due to a load applied at point A
τ	a parameter used in parametric representation of the boundary of a cross section
ϕ	angle between a tangent line and the x-axis
Γ	boundary of a cross section
Γ_i	boundary of the ith hole of a multiply-connected cross section
Γ^*	boundary of the kern of a cross section
Γ_cx	convex part of the boundary of a cross section
$\Gamma_\mathrm{cx}^\mathrm{n}$	nonflat part of Γ_cx
Γ_cc	concave part of the boundary of a cross section
Ω	domain of a cross section
Ω_i	domain of subsections of a disconnected cross section
Ω^*	domain of the kern of a cross section
Ω'	domain of the auxiliary cross section
$\Omega^{*\prime}$	domain of the kern of the auxiliary cross section

Mathematical notations

$\mathscr{H}$	set of all cross sections
$\mathbb{N}$	set of positive integers

$\mathbb{R}^+$	set of positive real numbers
$\sim$	conjugacy symbol
$CH(\Omega)$	convex hulls of a cross section with domain Ω
O	Landau's order symbol
Inf	infimum
Sup	supremum

1. Introduction

The kern of a cross section is the area within which any point load applied will produce stresses of the same sign as that of the load throughout the entire cross section. Applying a point load outside the kern produces stresses of different sing. The kern of a section is an old concept for civil and mechanical engineers. The problem was first introduced by the French engineer M. Bresse in 1854 (Todhunter and Pearson, 1960).

The kern concept is widely used in the design of prestressed concrete beams (Connolly, 1960), footings (Boresi and Sidebottom, 1985), and concrete dams (Creager et al., 1945). In the literature, kern has also been called core (Timoshenko, 1955), limit zone (Guyon, 1972), and a special case of the S-polygon (Fuller and Johnston, 1919; McGuire, 1968; Morley, 1948). In almost every elementary or advanced mechanics of materials book, the kern definition with only some simple examples can be found. In some engineering handbooks such as Avallone and Baumeister (1987) and Hicks and Hicks (1985), only a brief description of kern with the kern of some very simple cross sections are presented. Recently, Wilson and Turcotte (1993) established a numerical method for determining the kern for a general cross section. They approximate a general cross section with a polygon and then the kern of this polygon is determined as an approximation of the kern of the original cross section.

The objective of this article is to study analytically the kern of a general cross section. Methods to determine the kern for simple shapes are available. But there is no general analytical method for determining the kern of a general cross section. This article presents some theorems which describe the characters of the kern of a cross section with an arbitrary shape. Using these theorems the qualitative shape of the kern of a general cross section can be obtained. Methods are also presented for quantitative determination of the kern.

This paper is organized as follows. In Section 2, basic formulations for the kern of a general cross section and an equivalent definition are presented. In Section 3, ten theorems regarding the characters of the kern of a general cross section are derived. In the most general case three types of cross sections are considered: simply connected, multiply connected, and disconnected. Methods are proposed for obtaining kerns of multiply-connected and disconnected cross sections using auxiliary simply-connected cross sections. Section 4 discusses the qualitative shape of the kern of the numerical examples solved by Wilson and Turcotte (1993) using the derived theorems. It is shown that their numerical examples have correct qualitative shapes. In Section 5, kern ratio is defined for a general cross section and its boundedness is discussed. Kern ratio for regular polygonal cross sections is obtained as a function of the number of vertices. Section 6 presents an analytical method for deriving a representation for the boundary of the kern of a cross section with a known analytical expression of the boundary. Three examples are solved for demonstrating the capability of the method. A formula is derived for calculating the kern ratio of a general cross section with a known analytical representation of the boundary and then an example is solved. Conclusions are given in Section 7.

2. Basic formulation

A general cross section is shown in Fig. 1. The domain and the boundary of the cross section are denoted by Ω and Γ, respectively. Point C is the centroid of the cross section and x and y are principal axes. A concentrated compressive force is applied at a point $A(\xi, \eta)$. Assuming a linear distribution of normal stress using the method of strength of materials, the stress at a point $M(x, y)$ is expressed as:

$$\sigma(x, y) = -\frac{P}{A} + \frac{M_x y}{I_x} - \frac{M_y x}{I_y} \tag{1a}$$

$$M_x = -P\eta, \quad M_y = P\xi \tag{1b}$$

From eqns (1a) and (1b) we obtain

$$\sigma(x, y) = -\frac{P}{A}\left(1 + \frac{\eta y}{r_x^2} + \frac{\xi x}{r_y^2}\right) \tag{2}$$

where r_x and r_y are the radii of gyration with respect to principal axes. The kern domain and its boundary are denoted by Ω^* and Γ^*, respectively. If $A \in \Omega^*$ we must have $\sigma(x, y) \leqslant 0$, i.e., stress at any point of the section is compressive. Hence,

$$1 + \frac{\eta y}{r_x^2} + \frac{\xi x}{r_y^2} \geqslant 0 \quad \forall (x, y) \in \Omega, \quad (\xi, \eta) \in \Omega^* \tag{3}$$

eqn (3) is symmetric with respect to (x, y) and (ξ, η), and hence they can be interchanged. Thus,

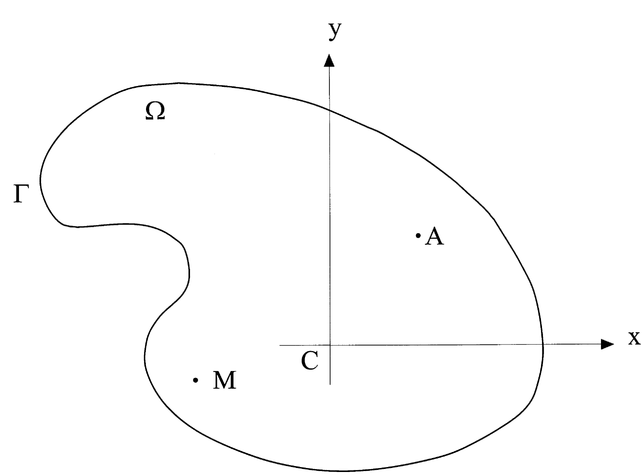

Fig. 1. A general cross section and its principal axes.

$$1+\frac{y\eta}{r_x^2}+\frac{x\xi}{r_y^2}>0 \quad \forall(\xi,\eta)\in\Omega, \quad (x,y)\in\Omega^* \tag{4}$$

This means that if the load P is applied at any point of Ω, the stress at all points of Ω^* are always compressive. This is an equivalent definition of kern which is easier to work with.

3. Some theorems

In this section ten theorems regarding the characters of the kern of a general cross section are derived. A general cross section belongs to one of the following three categories: simply-connected, multiply-connected, or disconnected cross sections. For the sake of simplicity, at first Ω is assumed to be simply connected. Other cases are considered in sequel.

3.1. Simply-connected cross sections

Eight theorems are derived for simply-connected cross sections in this subsection and then in the subsequent subsections it is shown they are also valid for multiply-connected and disconnected cross sections.

Theorem 1. For any cross section with domain Ω, centroid belongs to the kern.

Proof. The inequality (3) always holds for $(\xi,\eta)=(0,0)$. Hence $C\in\Omega^*$.

Theorem 2. If stress at a point M is always compressive due to a load P applied at any point of Γ, the stress at M is also compressive due to the load P applied at any point of Ω.

Proof. The normal stress at a point M due to a load P applied at a point D is denoted by σ_M^D. In Fig. 2, M is a point inside the cross section with compressive stress. We must show that $\sigma_M^B \leqslant 0$ implies $\sigma_M^A \leqslant 0$ for all points A lie on the line segment BC. Assume that $A(\xi_1,\eta_1)$ and $B(\xi_2,\eta_2)$. The point A lies on the line segment BC, hence

$$0<\frac{\xi_1}{\xi_2}=\alpha<1, \quad \frac{\eta_1}{\eta_2}=\alpha \tag{5}$$

and

$$\sigma_M^B=-\frac{P}{A}\left(1+\frac{x\xi_2}{r_y^2}+\frac{y\eta_2}{r_x^2}\right)\leqslant 0 \quad \text{or} \left(1+\frac{x\xi_2}{r_y^2}+\frac{y\eta_2}{r_x^2}\right)\geqslant 0 \tag{6}$$

Thus,

$$\frac{1}{\alpha}\left(-\frac{A}{P}\sigma_M^A\right)+\left(1-\frac{1}{\alpha}\right)\geqslant 0 \tag{7}$$

Therefore,

$$\sigma_M^A\leqslant\frac{P}{A}(\alpha-1)<0 \quad \text{or } \sigma_M^A\leqslant 0 \tag{8}$$

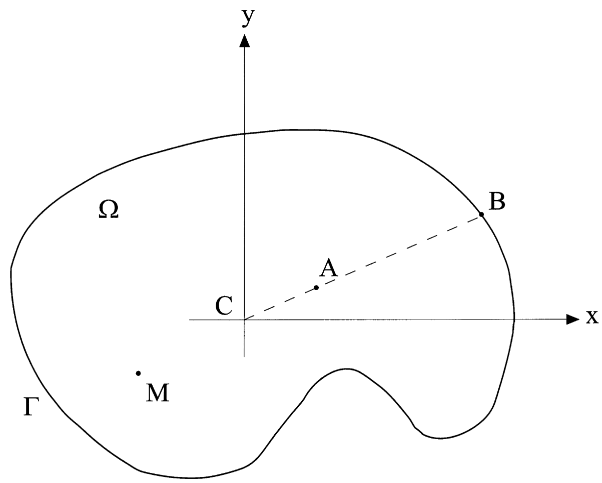

Fig. 2. An internal point and its corresponding point on the boundary of the cross section.

Corollary. A point M belongs to Γ^* if the stress at M is always zero due to a load P applied at any point of Γ.

Definition 1. Convex hulls of a region with domain Ω is the smallest convex region that includes Ω (Luenberger, 1973) and is denoted by $CH(\Omega)$. Therefore if Ω is the domain of a convex cross section, $CH(\Omega) = \Omega$. Obviously for a general cross section $\Omega \subset CH(\Omega)$. In Fig. 3, the convex hulls of the cross section is the shaded area.

For a general cross section, Γ can be partitioned into two sets: the convex part Γ_{cx} and the concave part Γ_{cc}. It is noted that Γ_{cx} and Γ_{cc} might be disconnected sets.

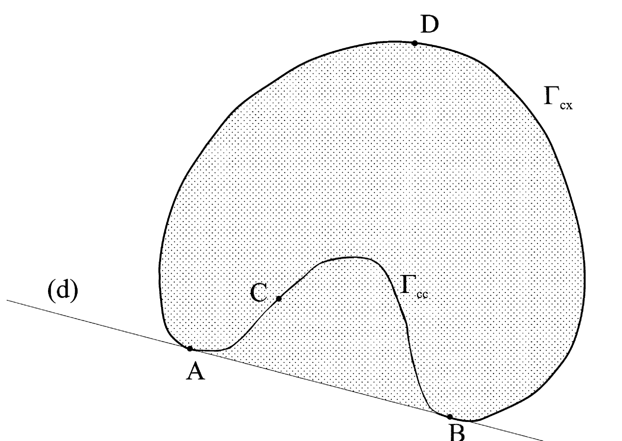

Fig. 3. A nonconvex cross section and its convex hulls, the shaded area is the convex hulls.

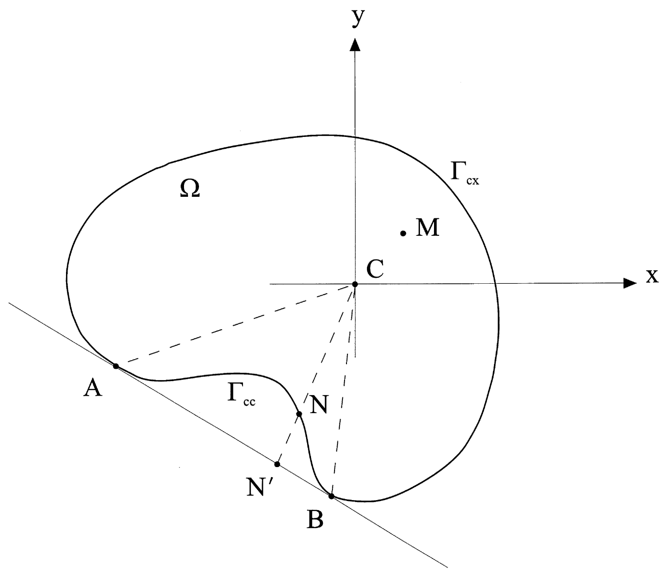

Fig. 4. Convex and concave parts of the boundary of a cross section.

Theorem 3. In Theorem 2, Γ can be replaced by Γ_{cx}.

Proof. Suppose that a point M belongs to Ω^*. We assume that Γ can be partitioned into convex and concave boundaries, Γ_{cx} and Γ_{cc}. In general we have $n(\Gamma_{cx})_i$'s and $(\Gamma_{cc})_i$'s. The generalization to this case is straightforward. In Fig. 4, points A and B are the end points of Γ_{cc} (or Γ_{cx}). We know that for any point D on Γ, $\sigma_M^D \leqslant 0$. It is also noted that $\sigma_M^D \leqslant 0$ implies $\sigma_D^M \leqslant 0$, hence,

$$\sigma_M^A \leqslant 0 \Longrightarrow \sigma_A^M \leqslant 0 \tag{9a}$$

$$\sigma_M^B \leqslant 0 \Longrightarrow \sigma_B^M \leqslant 0 \tag{9b}$$

Now consider an arbitrary point N on Γ_{cc}. Every point of Γ_{cc} has a corresponding point on the line segment AB. In Fig. 4, the point $N' \in AB$ is the corresponding point of $N \in \Gamma_{cc}$. The point N' belongs to the line segment AB, therefore, $\sigma_{N'}^M \leqslant 0$. But the distance between N and the centroid (C) is less than the distance between the centroid and N'. Thus, $\sigma_N^M \leqslant 0$. Therefore if all the points of Γ_{cx} satisfy the requirements of Theorem 2, all the points of Γ_{cc} will automatically satisfy them and hence Γ can be replaced by Γ_{cx} in Theorem 2.

Corollary. If Ω is not a convex set, for obtaining Ω^* one can consider an auxiliary cross section whose domain is $\Omega' = CH(\Omega)$ with the same area and principal axes as the cross section; the added area(s) is (are) virtual and hence A, r_x, and r_y remain unchanged. The kern of this auxiliary cross section is the kern of the original cross section, i.e., $\Omega'^* = \Omega^*$.

Definition 2. A point $M \in \Gamma^*$ is called the conjugate of a point $A \in \Gamma$ if $\sigma_M^A = 0$. We denote this conjugacy by $M \sim A$.

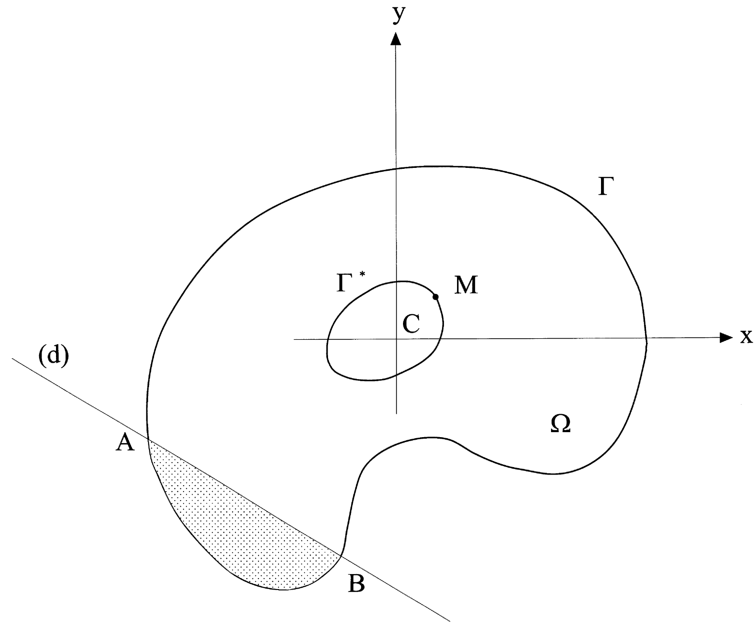

Fig. 5. A general cross section and the neutral axis of bending due to a load applied at a point on the boundary of the kern.

Definition 3. If the boundary of a cross section includes some line segments, each of them is called a flat segment of the boundary. Convex part of the boundary, Γ_{cx}, can be partitioned into two parts: the flat part and the nonflat part. The nonflat part is denoted by Γ_{cx}^{n}.

Theorem 4. Each point of Γ^* is the conjugate of only one point on Γ_{cx}^{n}.

Proof. Consider two points A and B on Γ_{cx} and assume that $M \sim A$ and $M \sim B$. Therefore, stress at the line segment AB is zero. Since A and B belong to Γ_{cx}, the line (d) passing through A and B intersects the section and hence there is a region in which stress is tensile (the shaded area in Fig. 5). This contradiction shows that A and B coincide.

Corollary 1. All points of a flat segment of the boundary of a cross section have the same conjugate point on Γ^*. This point is called the conjugate point of the flat segment.

Corollary 2. If $M \in \Gamma^*$ and $M \sim A$, the neutral axis of bending due to a load applied at M is tangent to Γ at A.

Theorem 5. The kern of a general cross section is always a convex region.

Proof. If a load P is applied at a point A of Γ, the neutral axis of bending cannot intersect Γ^*; it must be tangent to it. Hence, Γ^* cannot be concave. Actually when a point load moves on Γ_{cx}, neutral axes envelope a region which is the kern; obviously this region is a convex set.

Corollary. For a convex cross section without flat segments, conjugacy is a one-to-one relation. In other words, each point A of Γ has only one conjugate point on Γ^* and vice versa. Also $M \sim A$ implies $A \sim M$.

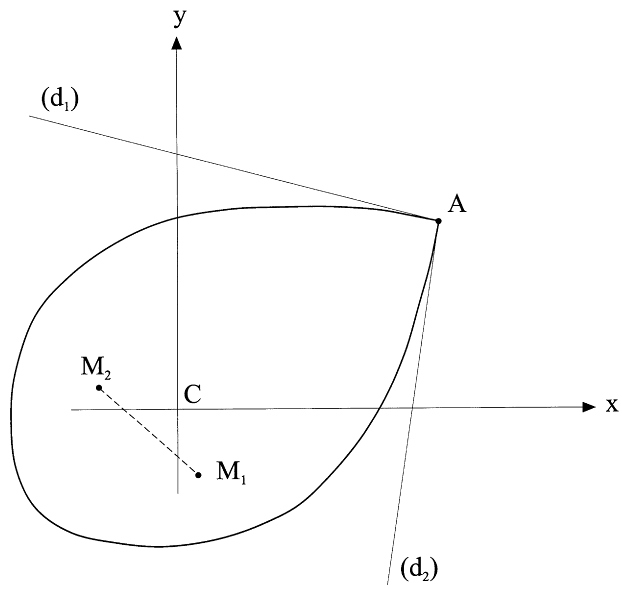

Fig. 6. A cross section with a cusp on the boundary.

Now a cross section with a cusp on the boundary is considered. Most of the sections used in practical applications have several cusps. Therefore this case is of practical importance. In Fig. 6, the two lines (d_1) and (d_2) are tangent to Γ at point A. Suppose that these two lines are the neutral axes of bending due to a load applied at points M_1 and M_2, respectively.

Theorem 6. The line segment M_1M_2 belongs to Γ^*.

Proof. If a load P is applied at the point A, we have

$$A \in d_1 \Longrightarrow \sigma^A_{M_1} = 0 \quad \text{and} \quad A \in d_2 \Longrightarrow \sigma^A_{M_2} = 0 \tag{10}$$

Therefore, stress is zero for all points of the line segment M_1M_2. From Theorem 5 it is known that Γ^* is convex. Thus, $M_1M_2 \in \Gamma^*$.

Now we derive a relationship between the coordinates of conjugate points. In Fig. 7 the point $A(X, Y)$ belongs to Γ_{cx} and $M(u, v)$ is its conjugate point on Γ^*. If a load P is applied at the point M, we have

$$\frac{ux}{r_y^2} + \frac{vy}{r_x^2} + 1 = 0 \tag{11}$$

The slope of this line is

$$m = -\frac{ur_x^2}{vr_y^2} \tag{12}$$

Hence,

$$v = \frac{ur_x^2}{mr_y^2} \tag{13}$$

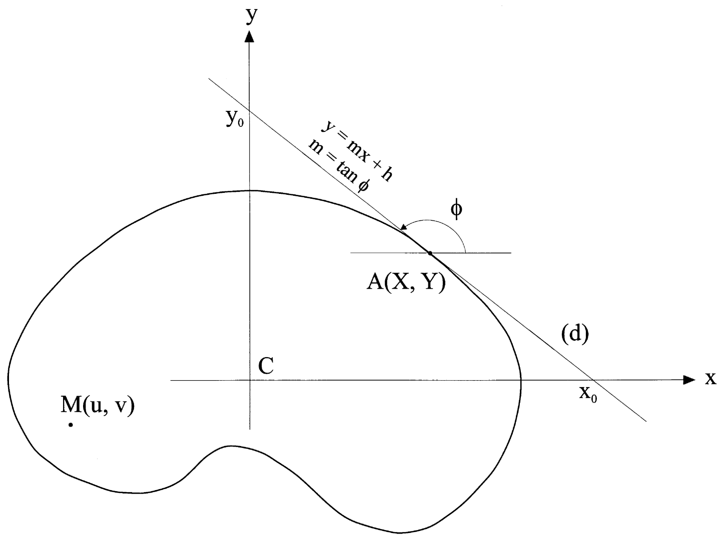

Fig. 7. A general cross section and a tangent line at a boundary point.

If a load P is applied at point A, we have

$$\frac{uX}{r_y^2} + \frac{vY}{r_x^2} + 1 = 0 \tag{14}$$

Substituting (13) into (14) yields

$$u = \frac{mr_y^2}{T - mX}, \quad v = -\frac{mr_x^2}{Y - mX} \tag{15}$$

Suppose that the tangent line at A intersects x- and y-axes at $(x_0, 0)$, $(0, y_0)$, respectively. Thus,

$$m = -\frac{y_0}{x_0} \tag{16}$$

We know that

$$y_0 = Y - \tan \varphi X = Y - mX \tag{17}$$

Substituting (16) and (17) into (15) yields

$$u = -\frac{r_y^2}{x_0}, \quad v = -\frac{r_x^2}{y_0} \tag{18}$$

Theorem 7. Suppose that A and B are two points on Γ_{cx} and M_A and M_B are their conjugate points on Γ^*. If tangent lines at A and B intersect each other on $x(y)$-axis, the line segment $M_A M_B$ is parallel to $y(x)$-axis (Fig. 8).

Proof. Tangent lines at A and B intersect each other on x-axis, thus $x_0^A = x_0^B$. Thus, from (18) we conclude that $u_A = u_B$. Therefore $M_A M_B$ is parallel to y-axis.

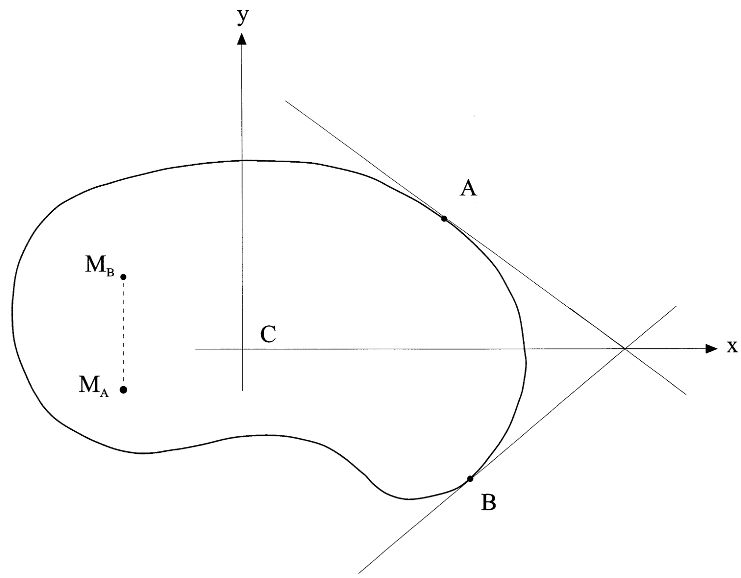

Fig. 8. Tangent lines at two boundary points A and B intersect on x-axis. The line segment connecting their conjugate points is parallel to y-axis.

Corollary. If tangent line at a point A on Γ_{cx} is parallel to $x(y)$-axis, the conjugate point of A on Γ^* lies on $y(x)$-axis.

Theorem 8. Kern of a general cross section has a nonzero area.

Proof. If Γ_{cx} is an empty set, $CH(\Omega)$ is a convex polygon with a nonzero area. If Γ_{cx} is not an empty set, three points of it, say, A, B, and C have three distinct conjugate points M, N, and Q on Γ^*, respectively. According to Theorem 5 Ω^* is a convex set. Hence,

$$A_{\Delta MNQ} \leqslant A^* \Longrightarrow A^* > 0 \tag{19}$$

3.2. Multiply-connected cross sections

In this section a general cross section with several holes is considered. Multiply-connected cross sections are encountered, for example, in voided prestressed slabs and beams (Libby, 1977). The following theorem relates the characters of the kern of multiply-connected cross sections to that of simply-connected cross sections.

Definition 4. The auxiliary simply-connected cross section Ω' of a multiply-connected cross section Ω has the same boundary Γ but all the holes are assumed to be filled. The geometrical properties of the auxiliary cross section, the principal axes, A, I_x, and I_y are the same as those of the multiply-connected cross section.

Theorem 9. The kern of the auxiliary simply-connected cross section Ω' of a multiply-connected cross section Ω are the same, i.e. $\Omega'^* = \Omega^*$.

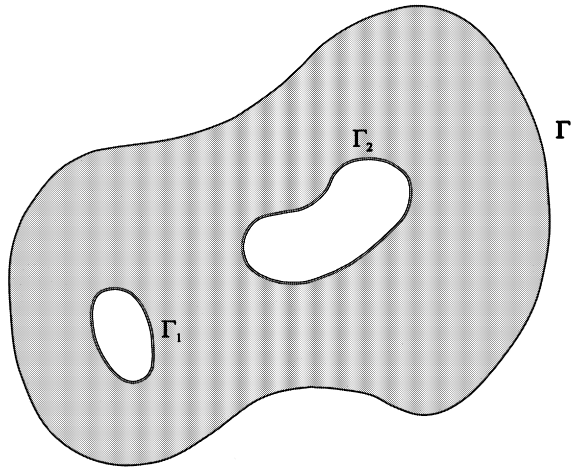

Fig. 9. A multiply-connected cross section.

Proof. For the sake of simplicity a section with two holes is considered, generalization of the proof to the case of a section with n holes is straightforward. Fig. 9 shows a cross section with two holes. The boundary of the section and the holes are denoted by Γ, Γ_1, and Γ_2, respectively. Suppose that we have cut the section along the two line segments AB and CD (Fig. 10). The widths of these cuts are t_1 and t_2, respectively. By these imaginary cuts our multiply-connected region is converted to a nonconvex simply-connected region. Using the corollary of Theorem 3 and approaching t_1 and t_2 to zero completes the proof.

3.3. Disconnected cross sections

This section investigates the kern of disconnected cross sections. While we have openings in prestressed AASHTO girders or box girders for utility pipes to go through we have an example of disconnected cross sections. Footings connected by ground (strap) beams are another practical application of disconnected cross sections. For the sake of simplicity, a disconnected cross section with two disjoint connected subsections is considered. The results can be easily generalized for the general case of a disconnected cross section with n disjoint subsections.

Definition 5. The auxiliary simply-connected cross section of a disconnected cross section is constructed as follows. First each disjoint subsection is assumed to be simply connected. If any of them is multiply connected, the holes are assumed to be filled without changing the geometrical properties. The convex hulls of $\Omega = \bigcup \Omega_i$ is the auxiliary simply-connected cross section of Ω.

The following theorem converts the studying of the kern of a disconnected cross section to that of a simply-connected cross section.

Theorem 10. The kern of a disconnected cross section is the same as that of its auxiliary simply-connected cross section.

Proof. For the sake of simplicity, a disconnected cross section with two disjoint subsections is considered. Generalization of the proof to the case of a cross section with n disjoint subsections is straightforward. In Fig. 11 a disconnected cross section composed of two disjoint subsections is shown.

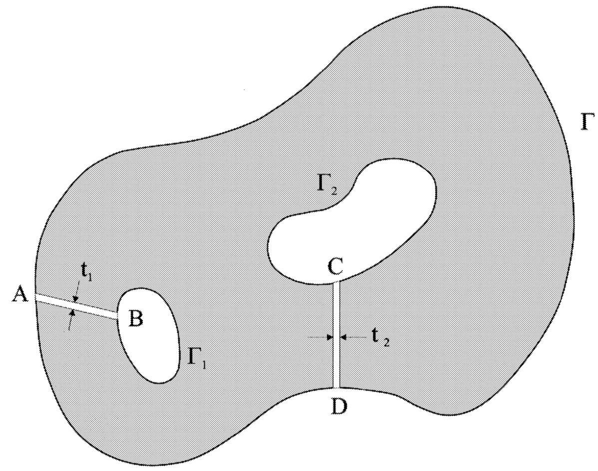

Fig. 10. Converting a multiply-connected cross section to a simply-connected one by imaginary cuts.

By adding the two very thin strips AB and CD, the cross section is converted to a multiply-connected cross section as shown in Fig. 12. The widths of these strips are t_1 and t_2, respectively. Using Theorem 9 and approaching t_1 and t_2 to zero completes the proof.

4. The qualitative shape of the kern of some cross sections

Using our ten theorems, one can check the correctness of a numerical method for obtaining the kern of a general cross section. In this section the qualitative shape of the kern of the numerical examples solved by Wilson and Turcotte (1993) are obtained using Theorems 1–10. It is shown that the qualitative shapes of all the numerical examples they solved are correct.

Example 1. An L-shaped cross section is shown in Fig. 13. The domain of this L-shaped cross section is a nonconvex set. The convex hulls of this section is the five-sided polygon $ABCDE$. The qualitative shape of the boundary of the kern is obtained as follows:

- The line segment DC is parallel to x-axis and hence according to the corollary of Theorem 7 the point a, the conjugate point of the line segment DC on Γ^*, lies on y-axis.
- Points b and e are conjugate points of the flat segments ED and BC, respectively. According to Theorem 6, the line segments ab and ae belong to Γ^*.
- Points c and d are conjugate points of the flat segments AE and AB, respectively and hence, Theorem 6 tells us that the line segment cd belongs to Γ^*.

Therefore, the kern is the shaded convex polygon $abcde$.

Example 2. A channel cross section is show in Fig. 14. The convex hull of this cross section is the rectangle $ABCD$. Using the corollary of Theorem 3 the kern of the channel cross section is the same as that of a rectangle without changing the principal axes. Points a, b, c, and d are conjugate

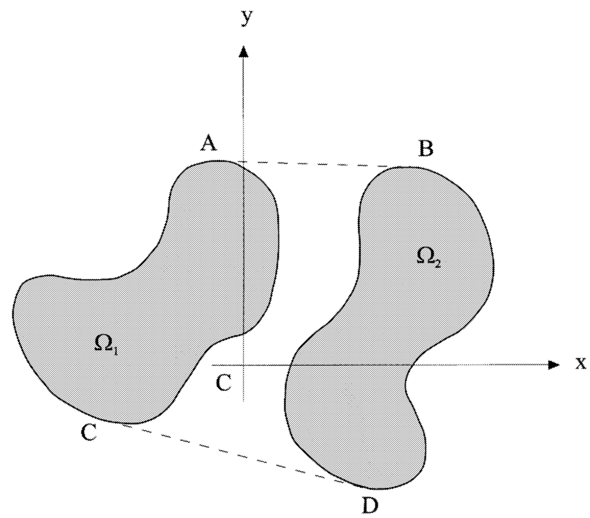

Fig. 11. A disconnected cross section.

points of flat segments AB, BC, CD, and AD, respectively. Using Theorem 6, the kern is the shaded polygon.

Example 3. In this example a square cross section with a triangular hole is considered. The cross section shown in Fig. 15 is a doubly-connected cross section. Since the cutout is symmetric about y-axis the kern must be symmetric with respect to y-axis. According to Theorem 9 the kern of this cross section is the same as that of the auxiliary simply-connected section which is a rectangle in this example. Again points a, b, c and d are conjugate points of flat segments AB, BC, CD, and AD, respectively. Therefore the kern has the shown qualitative shape.

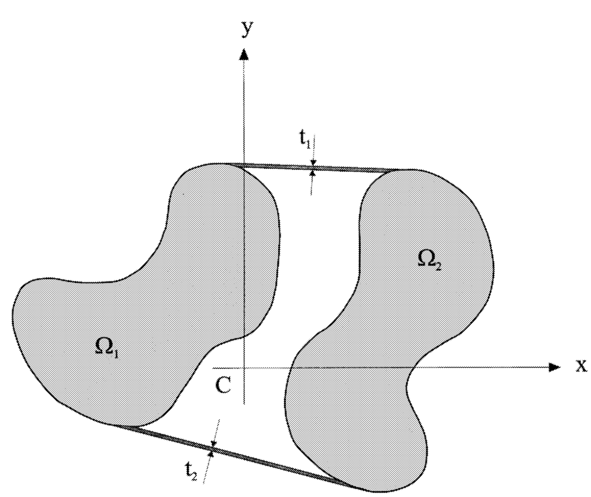

Fig. 12. Converting a disconnected section to a multiply-connected one by imaginary strips.

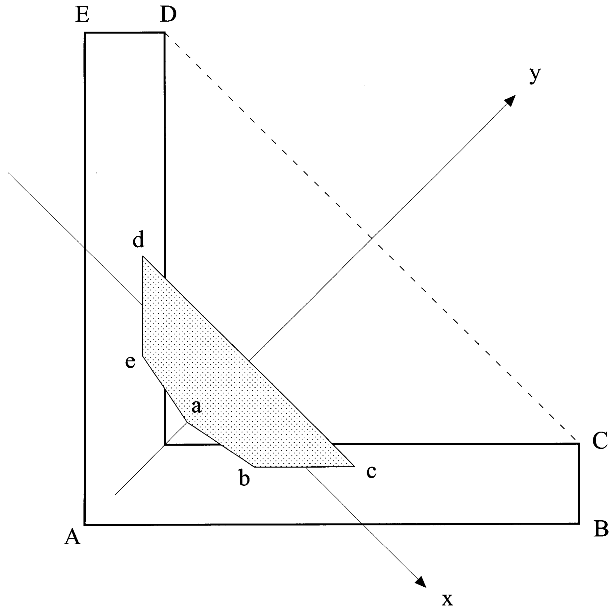

Fig. 13. An L-shaped cross section and its kern.

Example 4. The cross section shown in Fig. 16 is a disconnected cross section composed of two disjoint triangular subsections. The convex hulls of this cross section is the rectangle *ABCD*. The qualitative shape of the kern is, according to Theorem 10, the same as that of the auxiliary simply-connected section which is the rectangle *ABCD* without changing the geometrical properties. Thus the kern is the shaded polygon.

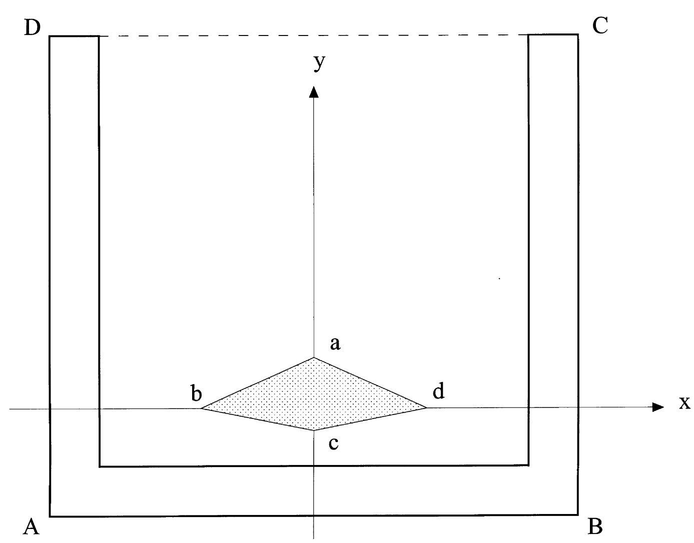

Fig. 14. A channel cross section and its kern.

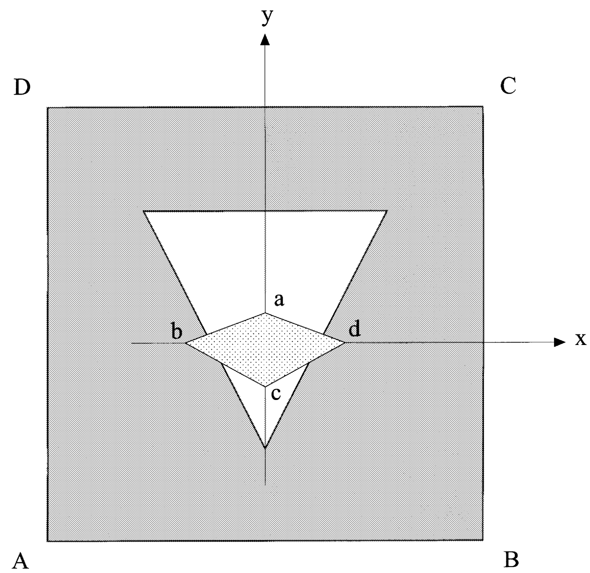

Fig. 15. A square cross section with a triangular hole and its kern.

Example 5. Convex hulls of the Z-shaped cross section shown in Fig. 17 is a six-sided polygon. Hence, according to Theorems 3 and 6, the kern is also a six-sided polygon as shown.

Example 6. The domain of the cross-section shown in Fig. 18 is a concave set. The line segment AB and the arc AB construct the boundary of the convex hulls of the cross section. The qualitiative shape of the boundary of the kern is obtained as follows:

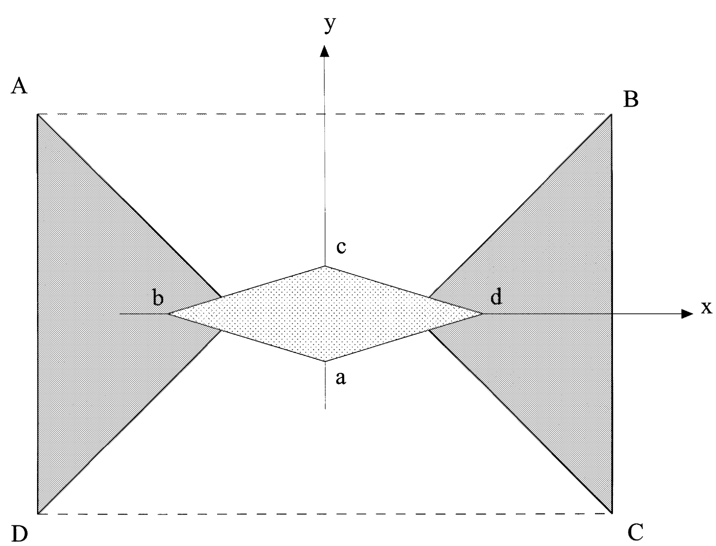

Fig. 16. A disconnected cross section with two disjoint triangular subsections and its kern.

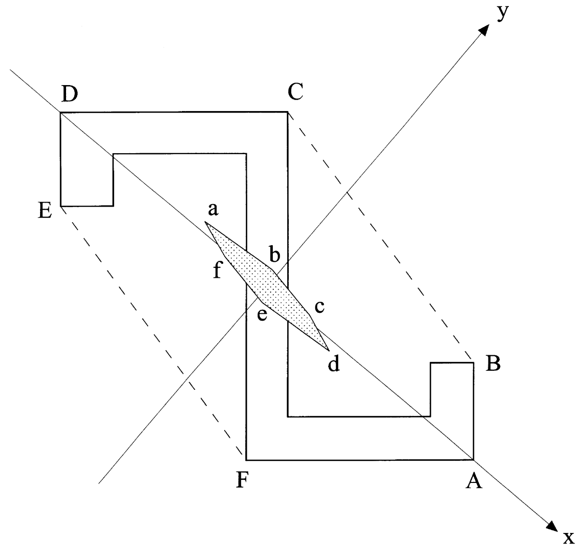

Fig. 17. A Z-shaped cross section and its kern.

- The line segment AB is parallel to x-axis and hence according to the corollary of Theorem 7 the conjugate point on Γ^*, a must lie on y-axis.
- The corresponding points of lines (d_1) and (d_2) on Γ^* are the points b and c, respectively. According to Theorem 6 the line segments ab and ac belong to Γ^*.
- The tangent line at point D, (d_3) is parallel to x-axis and according to Theorem 7 the corresponding point on Γ^*, d lies on y-axis.
- The kern boundary between points b and d and between c and d are two convex curves (Theorem 5). So the kern is the shaded region shown in Fig. 18.

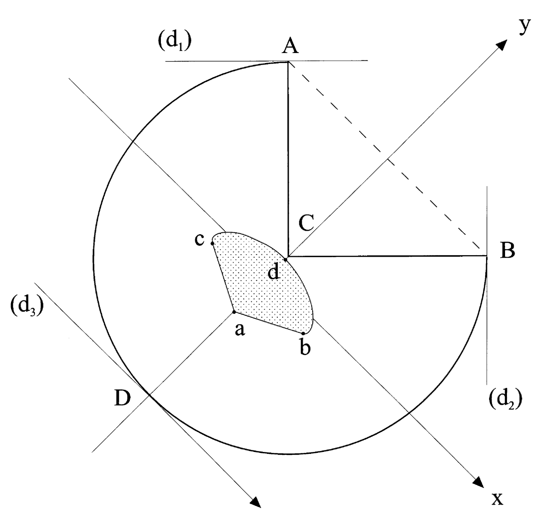

Fig. 18. A 270° sectorial cross section and its kern.

Clearly the qualitative shapes of the kerns of the numerical examples of Wilson and Turcotte (1993) are correct.

5. Kern ratio of a general cross section

Definition 6. The kern ratio of a cross section is denoted by KR and is defined as follows

$$KR = \frac{A^*}{A} \tag{20}$$

where A and A^* are areas of the cross section and the kern, respectively. In engineering applications it is desirable to have a cross section with the largest possible kern ratio. A cross section with a larger kern ratio has less possibility of occurrence of tensile stresses due to moving loads. In the following it is demonstrated that the kern ratio has no maximum for concave, multiply connected, and disconnected cross sections.

Fig. 19 shows a cruciform cross section and the qualitative shape of its kern. It can be easily shown that for this cross section, with a fixed area ($4bt + t^2 = \text{constant}$), we have

$$A^* = O\left(\frac{l}{t}\right) \Longrightarrow \lim_{(t,b)\to(0,\infty)} KR = \infty \tag{21}$$

It is to be noted that in (21) t and b tend to zero and infinity, respectively, in such a way that A is always a constant. Also it can be easily shown that for a circular ring with radius R and thickness t and a square box with edge length a and thickness t, $KR = R/8t, a/72t$, respectively. Therefore, for a constant area, approaching t to zero, KR tends to infinity for both cross sections. Therefore, for multiply-connected cross sections kern ratio is unbounded. Clearly this is also the case for disconnected cross sections. For example consider the cross section shown in Fig. 16. By moving away the two disjoint triangles along x-axis, the kern ratio increases unboundedly.

Denoting the set of all cross sections by $\mathscr{H}$, KR: $\mathscr{H} \to \mathbb{R}^+$ is a function which assigns a positive real number to a cross section. From (21) we know that Sup $(KR) = \infty$ and according to Theorem 8, Inf $(KR) = 0$. As we will see later, KR: $\mathscr{H} \to \mathbb{R}$ is not a one-to-one function even for cross sections with the same area.

Lemma. The area of the kern of a general cross section is always less than that of the convex hulls of the cross section.

Proof. Obviously all points of Ω^* lie inside the boundary of $CH(\Omega)$. Therefore the area of the kern cannot be larger than or equal to that of the convex hulls of the cross section.

The above lemma tells us that for a simply-connected cross section kern ratio is bounded by unity. It should be mentioned that we were unable to find a convex simply-connected cross section with $KR > 1/16$.

Now, as a simple example, regular polygonal cross sections are considered. An explicit expression for the kern ratio of a regular polygonal cross section is obtained and its maximum and minimum are discussed. For an n-sided polygonal cross section, kern is also n-sided polygon. Fig. 20 shows a regular hexagonal cross section and its kern. Using the fact that a regular polygon can be partitioned into n identical isosceles triangles, the moments of inertia with respect to principal axes x and y can be expressed as

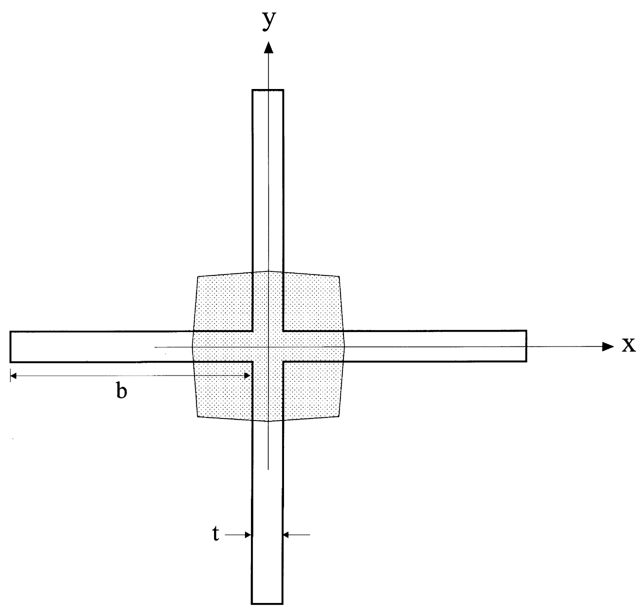

Fig. 19. A cruciform cross section and its kern.

$$I_x = I_y = \frac{na^4}{192} \cot\left(\frac{\pi}{n}\right)\left[1 + 3\cot^2\left(\frac{\pi}{n}\right)\right] \tag{22}$$

where a is the length of the edges of the polygon. It can be easily shown that the length of the edges of the kern, a' can be written as

$$a' = \frac{a}{4}\left[\frac{1}{3}\sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right) + \cos\left(\frac{\pi}{n}\right)\right] \tag{23}$$

Therefore,

$$KR(n) = \frac{1}{16}\left[\frac{1}{3}\sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right) + \cos\left(\frac{\pi}{n}\right)\right]^2 \tag{24}$$

In Fig. 21 a graph of KR versus n is shown. We know that $n \in \mathbb{N}$ ($n \geqslant 3$). But for the sake of clarity a continuous curve is shown for $n \geqslant 4$. Clearly kern ratio is minimum for square cross sections and is maximum for equilateral triangular and circular cross sections. That is,

$$\begin{cases} KR_{\max} = \dfrac{1}{16} & \text{for } n = 3, n \to \infty \\ KR_{\min} = \dfrac{1}{18} & \text{for } n = 4 \end{cases} \tag{25}$$

As mentioned before $KR{:}\mathscr{H} \to \mathbb{R}$ is not one-to-one even for cross sections with a constant area; it has the same value for equilateral triangular and circular cross sections. Also as we will see later all elliptical cross sections have $KR = 1/16$.

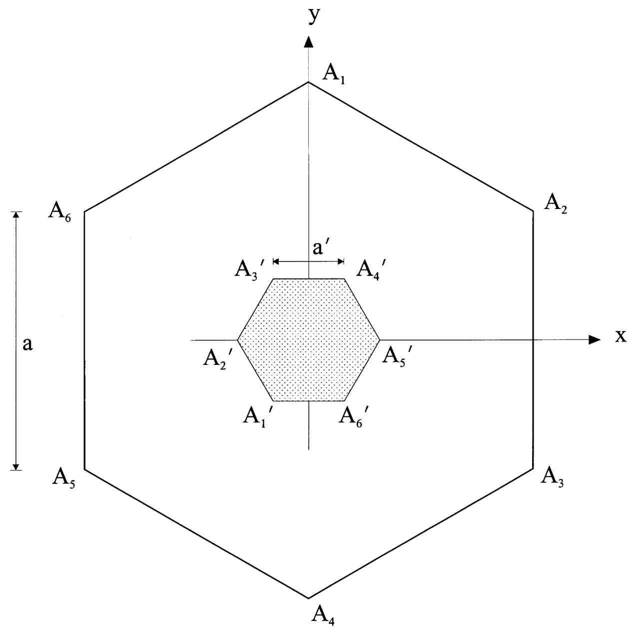

Fig. 20. A hexagonal cross section and its kern.

In the next section, for a convex cross section with a known parametric representation of the boundary Γ, a parametric representation for the kern boundary Γ^* is obtained.

6. Analytical derivation of kern

Consider a point $M(\xi, \eta)$ on Γ^*. If a load P is applied at M, we have

$$\frac{1}{A} + \frac{P\xi x}{I_y} + \frac{P\eta y}{I_x} \geqslant 0 \quad \forall (x, y) \in \Omega \tag{26}$$

Assume that ξ and η are fixed. The conjugate point of M on Γ minimizes the following function and the minimum is zero ($f_{\min} = 0$),

$$f(x, y) = \frac{1}{A} + \frac{\xi}{I_y}x + \frac{\eta}{I_x}y \tag{27}$$

For finding the conjugate point of M, the function $f(x, y)$ must be minimized with the constraint $g(x, y) = 0$, where $g(x, y) = 0$ is a representation of Γ. For this extremization problem with a constraint, the method of Lagrange multipliers is used; the function $F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$ must be minimized without any constraint where λ is a Lagrange multiplier. Hence,

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \tag{28a}$$

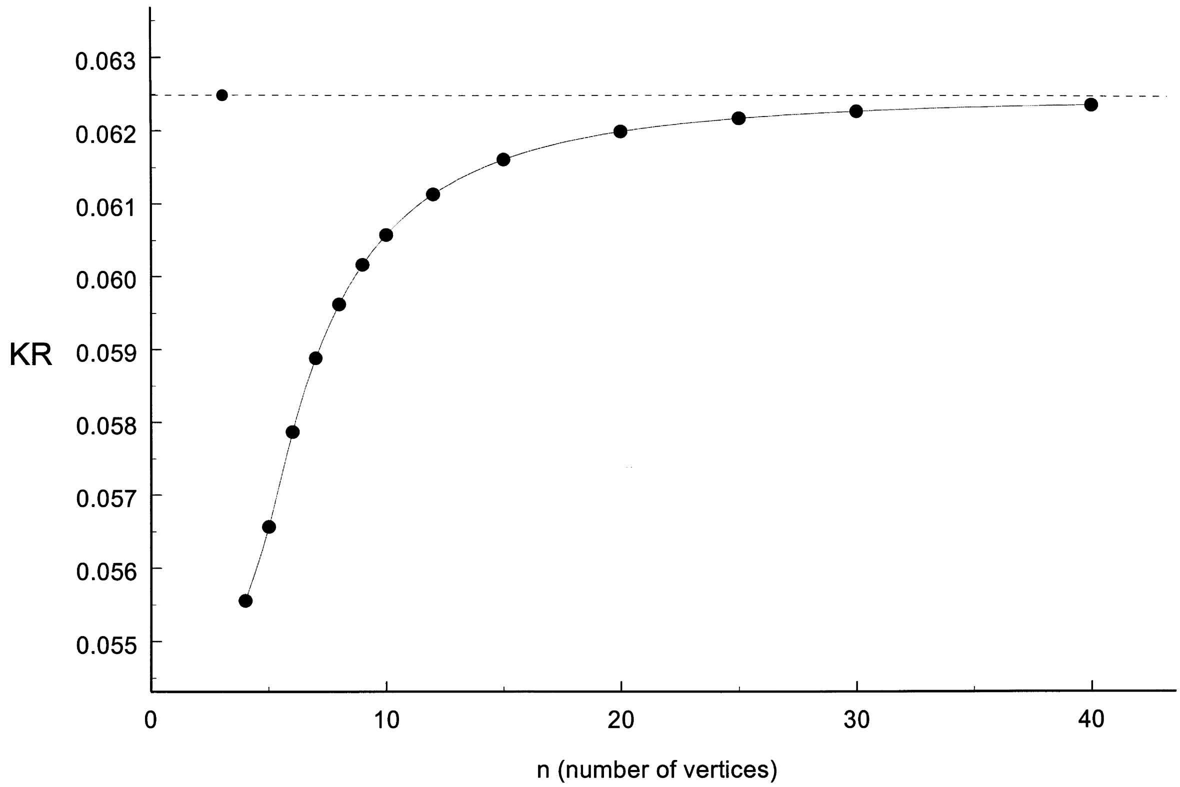

Fig. 21. Kern ratio of regular polygonal cross sections versus the number of vertices.

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \tag{28b}$$

$$\frac{\partial F}{\partial \lambda} = g = 0 \tag{28c}$$

Substituting (27) into (28) yields

$$\frac{\xi}{I_y} = -\lambda \frac{\partial g}{\partial x}, \quad \frac{\eta}{I_x} = -\lambda \frac{\partial g}{\partial y} \tag{29}$$

Thus,

$$y' = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} = -\frac{\xi I_x}{\eta I_y} \tag{30}$$

where $y' = \mathrm{d}y/\mathrm{d}x$ is the implicit derivative of g with respect to x. From (30), we obtain

$$\eta = -\frac{\xi I_x}{y' I_y} \tag{31}$$

The minimum of $f(x, y)$ is zero, hence

$$\xi = -\frac{y' I_y}{A(xy' - y)}, \quad \eta = \frac{I_x}{A(xy' - y)} \tag{32}$$

In summary, eliminating x and y from the following three equations yields $K(\xi, \eta) = 0$ which is a representation of Γ^*.

$$\xi = -\frac{y' I_y}{A(xy' - y)} \tag{33a}$$

$$\eta = \frac{I_x}{A(xy' - y)} \tag{33b}$$

$$g(x, y) = 0 \tag{33c}$$

If Γ has a parametric representation $x = x(\tau)$, $y = y(\tau)$, $\alpha \leqslant \tau \leqslant \beta$, the above equations can be rewritten as

$$\xi = -\frac{y' I_y}{A(xy' - x'y)} \tag{34a}$$

$$\eta = -\frac{x' I_x}{A(xy' - x'y)} \tag{34b}$$

where x' and y' are derivatives with respect to τ. Eliminating τ in (34a) and (34b) yields a representation for Γ^*. In order to clarify the technique described above, three examples are considered.

Example 7. An elliptical cross section with diameters $2a$ and $2b$ is shown in Fig. 22. For this cross section

$$A = \pi ab, \quad I_x = \frac{\pi}{4} ab^3, \quad I_y = \frac{\pi}{4} a^3 b$$

$$g(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \tag{35}$$

Thus,

$$y' = -\frac{\dfrac{\partial g}{\partial x}}{\dfrac{\partial g}{\partial y}} = -\lambda \frac{\xi I_x}{\eta I_y} \tag{36}$$

Substituting (35) into (32) we have

$$\xi = -\frac{x}{4}, \quad \eta = -\frac{y}{4} \quad \text{or } x = -4\xi, \quad y = -4\eta \tag{37}$$

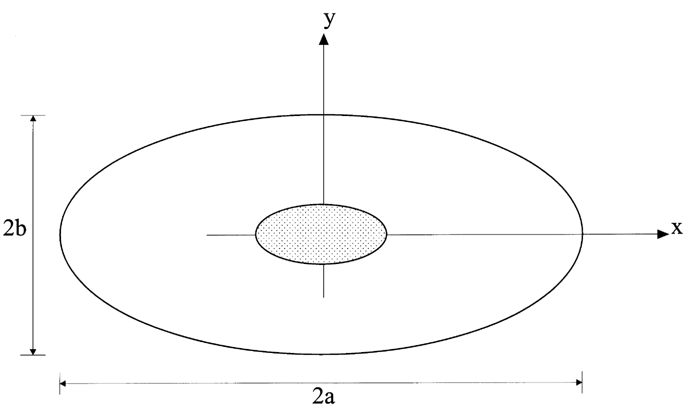

Fig. 22. An elliptical cross section and its kern.

Substituting (37) into $g(x, y) = 0$ yields

$$\frac{\xi^2}{\left(\frac{a}{4}\right)^2} + \frac{\eta^2}{\left(\frac{b}{4}\right)^2} = 1 \tag{38}$$

Therefore, the kern is an ellipse with diameters $a/2$ and $b/2$. Therefore, for all elliptic sections, regardless of a/b ratio, $KR = 1/16$.

Example 8. Consider an elliptical cross section with a circular hole. The center of the hole coincides with the centroid of the ellipse (Fig. 23). For this cross section,

$$A = \pi(ab - c^2), \quad I_x = \frac{\pi}{4}(ab^3 - c^4), \quad I_y = \frac{\pi}{4}(a^3b - c^4)$$

$$g(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{39}$$

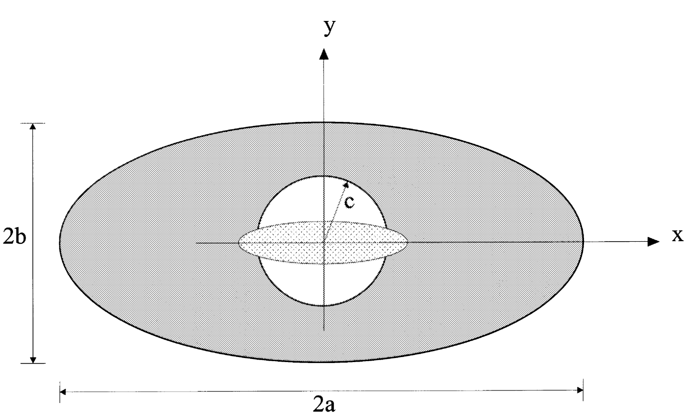

Fig. 23. An elliptical cross section with a circular hole and its kern.

From (32) we have

$$\xi = -\alpha x, \quad \eta = -\beta y$$

$$\alpha = \frac{a^3 b - c^4}{4a^2(ab - c^2)}, \quad \beta = \frac{ab^3 - c^4}{4a^2(ab - c^2)} \tag{40}$$

Substituting (40) into (39) yields

$$\frac{\xi^2}{(\alpha a)^2} + \frac{\eta^2}{(\beta b)^2} = 1 \tag{41}$$

which is an ellipse with diameters $2\alpha a$ and $2\beta b$.

Example 9. Consider the semi-circular cross section shown in Fig. 24. For this cross section

$$A = \frac{\pi R^2}{2}, \quad I_x = \frac{9\pi^2 + 64}{72\pi} R^4, \quad I_y = \frac{\pi R^4}{8} \tag{42}$$

Points b and c correspond to lines (d_1) and (d_2), respectively and the point a is the conjugate point of the flat segment AB. Therefore according to Theorem 6 two line segments ab and ac belong to Γ^*. Now we obtain a representation for the rest of the kern boundary. The circular are AB has the following representation in xy system:

$$g(x, y) = x^2 + \left(y + \frac{4R}{3\pi}\right)^2 - R^2 = 0 \tag{43}$$

Substituting (43) into (32) yields

$$\xi = -\frac{x I_y}{A\left(x^2 + y^2 + \frac{4R}{3\pi} y\right)}, \quad \eta = -\frac{\left(y + \frac{4R}{3\pi}\right) I_x}{A\left(x^2 + y^2 + \frac{4R}{3\pi} y\right)} \tag{44}$$

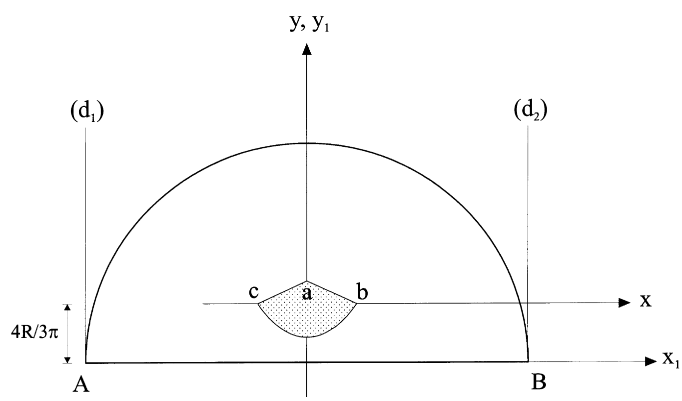

Fig. 24. A semi-circular cross section, the kern consists of two line segments and an elliptical arc.

Eliminating x and y from (44), we get

$$\frac{\xi^2}{\alpha^2}+\frac{(\eta+\gamma)^2}{\beta^2}=1 \tag{45}$$

where

$$\alpha^2=\frac{I_y^2}{A^2(R^2-a^2)},\quad \beta^2=\frac{R^2I_y^2}{A^2(R^2-a^2)^2},$$

$$\gamma=\frac{aI_y}{A(R^2-a^2)} \tag{46}$$

Thus, the kern consists of two line segments and an elliptical arc.

Another method for analytical derivation of the kern is as follows. We know that when a point load moves on the convex part of the boundary of a cross section the neutral axes envelope the boundary of the kern. Consider two points $A(x, y)$ and $B(x_1, y_1)$ on Γ_{cx}. Neutral axes corresponding to these points are:

$$(d_A):\quad \frac{x}{r_y^2}\,\xi+\frac{y}{r_x^2}\,\eta+1=0 \tag{47a}$$

$$(d_B):\quad \frac{x_1}{r_y^2}\,\xi+\frac{y_1}{r_x^2}\,\eta+1=0 \tag{47b}$$

The intersection of these two lines is a point $M(\xi, \eta)$ with the following coordinates:

$$\xi=r_y^2\,\frac{y_1-y}{xy_1-yx_1},\quad \eta=r_x^2\,\frac{x_1-x}{xy_1-yx_1} \tag{48}$$

Now let B approach A. The point of intersection of d_A and d_B approaches the conjugate point of A. Substituting $x_1=x+\mathrm{d}x$, $y_1=y+dy$ in (48) yields

$$\xi=-r_y^2\,\frac{dy}{x\,\mathrm{d}y-y\,\mathrm{d}x},\quad \eta=r_x^2\,\frac{\mathrm{d}x}{x\,\mathrm{d}y-y\,\mathrm{d}x} \tag{49}$$

If Γ has a parametric representation $x=x(\tau)$, $y=y(\tau)$, $\alpha\leqslant\tau\leqslant\beta$, eqn (49) can be written as

$$\xi=-r_y^2\frac{y'\tau}{x(\tau)y'(\tau)-y(\tau)x'(\tau)} \tag{50a}$$

$$\eta=r_x^2\frac{x'\tau}{x(\tau)y'(\tau)-y(\tau)x'(\tau)} \tag{50b}$$

which is exactly the same as (34).

6.1. Direct calculation of kern ratio for convex cross sections

Kern ratio for a convex cross section with a known boundary representation can be calculated as follows

$$A^* = \oint_{\Gamma^*} \eta \, \mathrm{d}\xi \tag{51}$$

Substituting (32) into (51) yields

$$A^* = \frac{I_x I_y}{A^2} \oint_{\Gamma} \frac{y y''}{(x y' - y)^3} \, \mathrm{d}x$$

$$= \frac{I_x I_y}{A^2} \oint_{\Gamma} \frac{g_{,xx} g_{,y}^2 - 2 g_{,xy} g_{,y} + g_{,yy} g_{,x}^2}{(x g_{,x} - y g_{,y})^3} y \, \mathrm{d}x \tag{52}$$

Where a comma followed by a variable means differentiation with respect to that variable. Therefore

$$KR = \frac{I_x I_y}{A^3} \oint_{\Gamma} \frac{G_{,xx} g_{,y}^2 - 2 g_{,xy} g_{,y} + g_{,yy} g_{,x}^2}{(x g_{,x} + y g_{,y})^3} y \, \mathrm{d}x \tag{53}$$

Example 10. Consider the elliptical cross section of Fig. 22. Substituting (35) into (53) yields

$$KR = \frac{1}{16\pi ab} \oint_{\Gamma} y \, \mathrm{d}x = \frac{A}{16\pi ab} = \frac{1}{16} \tag{54}$$

which is the same result as obtained in Example 7.

7. Conclusions

Ten theorems regarding the characters of the kern of simply-connected, multiply-connected, and disconnected cross sections were derived. Methods were proposed for obtaining the kern of a multiply-connected or disconnected cross section using an auxiliary simply-connected cross section. Capabilities of the derived theorems in describing the qualitative shape of the kern of a general cross section were demonstrated by obtaining the qualitative shapes of the kerns of the numerical examples solved by Wilson and Turcotte (1993).

Kern ratio was defined for a general cross section. It was shown that kern ratio is bounded by unity for convex simply-connected cross sections. However examples of concave simply-connected, multiply-connected and disconnected cross sections with unbounded kern ratios were found. The kern ratio of regular polygonal cross sections was calculated as a function of the number of vertices of the polygon. It was observed that the kern ratio takes its minimum for square and its maximum for circular and equilateral triangular cross sections, respectively. It was mentioned that two different cross sections can have equal kern ratios; for example all elliptical cross sections have the same kern ratio.

A method proposed for determining a representation of the boundary of the kern of a cross section with a known analytical representation of the boundary. The capability of the method was demonstrated through some examples. Also a formula was given for direct calculation of the kern ratio of a convex cross section with a known analytic representation of the boundary.

Our ten theorems can be used to obtain the qualitative shape of the kern of an arbitrary cross section which is very useful for preliminary designs and engineering judgements.

It should be mentioned that from a geometrical point of view the boundary of the kern of a cross section is the antipolar reciprocal of the convex part of the boundary of the cross section with regard to the ellipse of inertia. A similar definition was given in Todhunter and Pearson (1960). However, here we presented a more precise definition. For an introduction to polar transformations in geometry, the reader can refer to Alshiller-Court (1925), Baker (1943), Eves (1963), Zwikker (1963).

References

Altshiller-Court, N., 1925. College Geometry. Johnson Publishing Company, Richmond.
Avallone, E.A., Baumeister III, T., 1987. Standard Handbook for Mechanical Engineers. McGraw-Hill, New York.
Baker, H.F., 1943. An Introduction to Plane Geometry. Cambridge University Press, London.
Boresi, A.P., Sidebottom, O.M., 1985. Advanced Mechanics of Materials. John Wiley and Sons, New York.
Connolly, W.H., 1960. Design of Prestressed Concrete Beams. F.W. Dodge Corporation, New York.
Creager, W.P., Justin, J., Hinds, H., 1945. Engineering for Dams, vol. II: Concrete Dams. John Wiley and Sons, New York.
Eves, H., 1963. A Survey of Geometry., vol. I. Allyn and Bacon, Boston.
Flugge, W., 1962. Handbook of Engineering Mechanics. McGraw-Hill, New York.
Fuller, C.E., Johnston, W.A., 1919. Applied Mechanics., vol. II. John Wiley and Sons, New York.
Guyon, Y., 1972. Limit-State Design of Prestressed Concrete., vol. 1. John Wiley and Sons, New York.
Hicks, T.G., Hicks, S.D., 1985. Standard Handbook of Engineering Calculations. McGraw-Hill, New York.
Libby, J.R., 1997. Modern Prestressed Concrete, 2nd ed. Von Nostrand Reinhold Co, New York.
Luenberger, D., 1973. Introduction to Linear and Nonlinear Programming. Addision-Wesley, Reading, Mass.
McGuire, W., 1968. Steel Structures. Prentice-Hall, New Jersey.
Morley, A., 1948. Theory of Structures. Longmans, Green and Co, London.
Nilson, A.H., 1978. Design of Prestressed Concrete. John Wiley and Sons, New York.
Timoshenko, S., 1955. Strength of Materials—Part I, 3rd ed. Van Nostrand, Princeton, New Jersey.
Todhunter, I., Pearson, K., 1960. A History of the Theory of Elasticity and of the Strength of Materials, vol. II. Dover Publications, pp. 352–353, pp. 879–880.
Wilson, H.B., Turcotte, L.H., 1993. Determining the kern for a compression member of general cross section. Advances in Engineering Software. 17, 113–123.
Zwikker, C., 1963. The Advanced Geometry of Plane Curves and Their Applications. Dover Publications, New York.