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On Fractal Cracks in Micropolar Elastic Solids

In this paper we review the fracture mechanics of smooth cracks in micropolar (Cosserat) elastic solids. Griffith's fracture theory is generalized for cracks in micropolar solids and shown to have two possible forms. The effect of fractality of fracture surfaces on the powers of stress and couple-stress singularity is studied. We obtain the orders of stress and couple-stress singularities at the tip of a fractal crack in a micropolar solid using dimensional analysis and an asymptotic method that we call "method of crack-effect zone." It is shown that orders of stress and couple-stress singularities are equal to the order of stress singularity at the tip of the same fractal crack in a classical solid.
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1 Introduction

Fractal geometry, which has been argued to be a better geometry for modeling natural objects than Euclidean geometry, was introduced by Mandelbrot [1,2]. The term "fractal" was coined by Mandelbrot [2] from the Latin verb *frangere*, "to break," and the corresponding adjective *fractus*, "fragmented and irregular." Fractal geometry has found applications in many fields of science and engineering in recent years. So far fractal geometry's major applications to solid mechanics problems are in contact mechanics and fracture mechanics. Fractal fracture mechanics is a nonclassical fracture mechanics in which cracks are assumed to be fractal curves (surfaces) (Cherepanov et al. [3], Balankin [4]). In classical fracture mechanics it is assumed that cracks are rectifiable curves (surfaces), i.e., curves (surfaces) with finite lengths (areas). Cracks are modeled by smooth curves (surfaces) with probably a finite number of kinks. These simplifying assumptions make fracture mechanics problems mathematically tractable.

Mandelbort et al. [5] experimentally showed that fracture surfaces of steel are fractals. Since that pioneering work many other experimental studies have been done (for example, Brown and Scholz [6]; Power and Tullis [7]; Saouma et al. [8]; Saouma and Barton [9]; Wong et al. [10]). Now we know that cracks can be modeled by fractals in a wide (but finite) range of length scales. A number of theoretical studies have been conducted to date. Mosolov [11] and Gol'dshteĭn and Mosolov [12,13] studied the singularity of stresses at the tip of a mode I self-similar fractal crack showing that the power of stress singularity is a linear function of fractal dimension of the crack. Yavari et al. [14] calculated the orders of stress singularity for mode I, II, and III fractal cracks. Yavari [15], Yavari et al. [16], and Balankin [4] studied HRR singularity for self-similar and self-affine fractal cracks.

Mosolov [17] and Balankin [4] investigated the path independence of *J*-integral for fractal cracks and modified the *J*-integral for fractal cracks. They argued that the modified *J*-integrals are path-independent. This problem was later discussed in Yavari et al. [16]. They mentioned that a fractal *J*-integral should be equal to the potential energy release per unit of a fractal measure. They explained that the modified *J*-integrals defined by Mosolov and by Balankin are only locally path-independent and have no physical meaning. Crack growth in compression was explained by Mosolov and Borodich [18], Mosolov [19], and Balankin [4].

Yavari [15] and Yavari et al. [14] introduced a new mode of fracture in fractal fracture mechanics and called it "the fourth mode" or "the axial mode." They pointed out that the existence of this new mode of fracture could make some single-mode problems of classical fracture mechanics, mixed-mode problems in fractal fracture mechanics. Later, Yavari et al. [16] showed that there are actually three new fractal modes. Xie [20] studied crack branching using a fractal model. Xie and Sanderson [21] explained a paradox in dynamic fracture mechanics using their fractal model. Borodich [22,23] realized that Griffith's criterion must be modified for fractal cracks. He showed that in the modified criterion, the specific surface energy must be defined per unit of a fractal measure (not length or area) of fractal crack growth. Yavari [24] generalized Barenblatt's cohesive fracture theory and developed a fractal cohesive fracture theory.

To our best knowledge, there is no investigation into fractal cracks in micropolar (Cosserat) solids. This paper aims to explore some interesting problems of micropolar fractal fracture mechanics. In Section 2, micropolar elasticity is reviewed and its basic concepts and definitions are explained. Section 3 discusses fracture mechanics of rectilinear cracks in micropolar solids. The effects of couple-stresses in fracture mechanics are reviewed and Griffith's criterion is generalized for both smooth and fractal cracks in micropolar solids in Section 4. Section 5 studies self-similar and self-affine fractal cracks in micropolar solids. Using dimensional analysis and the method of crack-effect zone, it is shown that stresses and couple-stresses at the tip of a fractal crack in a micropolar solid have equal orders of singularity. The Appendix presents some basic definitions and techniques of fractal geometry that are directly relevant to our investigation.

2 Micropolar Elasticity

This section presents a brief introduction to generalized continuum theories and their history. Here we discuss only those aspects of micropolar elasticity theory that are necessary for our investigation of fractal cracks in a micropolar solid. A literature review for fracture mechanics of rectilinear cracks in micropolar solids will be given in the next section.

In classical continuum mechanics, at each point only translational degrees-of-freedom u_i $(i=1,2,3)$ are considered and it is assumed that the interaction between two material points along an arbitrary surface S is completely described by a stress vector $\boldsymbol{\sigma}$ defined on S. These assumptions lead to a mathematically consistent theory of continuum mechanics. Experience has shown that most analytical solutions obtained in the framework of classical continuum mechanics agree very well with the experimental re-

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sults. To data all engineering designs are based on the use of classical continuum mechanics and sometimes even with some more simplifying assumptions.

The curiosity of some distinguished researchers led them to question the above-mentioned hypotheses and to develop generalized continuum theories. It was clear for them that considering only translational degrees-of-freedom might not be enough for continua with microstructure (see [25–37]).

In the original Cosserat brothers' formulation ([26]), rotations $\phi_i\,(i=1,2,3)$ were considered to be independent of displacement components $u_i\,(i=1,2,3)$. However, later most analytical solutions were reported for a special case that is now known as couple-stress theory or constrained Cosserat theory. In couple-stress theory, microrotations are assumed to be equal to macrorotations, i.e., $\phi_i=1/2\varepsilon_{ijk}u_{k,j}$. This is the theory that was developed independently by Grioli [28], by Aero and Kuvshinskii [31], and by Mindlin and Tiersten [32]. Eringen and his co-workers elaborately studied the theory of Cosserat continua and again assumed that microrotations are independent of displacement components. Eringen [36,37] renamed the Cosserat continuum theory and called it micropolar continuum theory. Cowin [38–40] discovered a continuous transition from couple-stress theory to micropolar theory by introducing a coupling number $N\,(0\leqslant N\leqslant 1)$, where $N=1$ corresponds to the couple-stress theory, $N=0$ corresponds to the classical theory, and N between zero and one $(0<N<1)$ corresponds to the micropolar theory. It is known that in couple-stress theory of elasticity two new constants appear and one of them, l has the dimension of length and is called the characteristic length. On the other hand, in micropolar elasticity there are four new material constants and two of them, l_t and l_b have dimensions of length and are called characteristic lengths in torsion and bending, respectively. This means that in generalized continuum theories there is at least one internal length scale and therefore these theories should be able to analytically predict size effects.

Several authors investigated the effects of couple-stresses in different problems of solid mechanics such as stress concentration in the presence of holes and inclusions and the change of size effect in rigidity of different structural members (see [41–62]). Recently, there have been some investigations into strain gradient plasticity (see [63] and references therein). These theories seem to be promising in design of very small structures.

As was mentioned at the beginning of this section, generalized continuum theories attracted theoreticians because of their beauty. To date these theories have not been applied to practical problems. Here we have an example of a field in which experimental studies are far behind the theory. There are several experimental investigations into the mechanical properties of micropolar elastic materials. What we have at this time are just some ranges of these material constants (Schijve [64] and Lakes [60]). So far we have only some qualitative sense of the influences of couple stresses. We are hopeful that future advances in experimental mechanics will make these elegant theories applicable to real engineering problems.

It is worth mentioning that there is a recent interest in generalized continuum theories because of the superiority they have in localization analyses. These studies are beyond the scope of this section and will not be mentioned here. Now we present the basic concepts, definitions, and balance equations of the theory of micropolar elasticity. Here we mainly follow Eringen [37].

In a continuous medium with microstructure each material element contains several micromaterial elements. In micropolar continuum mechanics only microrotations are considered for microelements. Therefore, for each material point, in addition to the three displacements, three microrotations are considered. Microrotations are assumed to be different from macrorotations. Displacement components are denoted by u_i, microrotations by ϕ_i, and macrorotations by r_i. Macrorotations have the following relations with displacements:

$$r_i=\frac{1}{2}\varepsilon_{ijk}u_{k,j} \tag{1}$$

where ε_{ijk} is the permutation symbol. Macrostrains e_{ij} and microstrains ε_{ij} are defined as

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}) \tag{2a}$$

$$\varepsilon_{ij}=e_{ij}+e_{ijk}(r_k-\phi_k). \tag{2b}$$

Curvature tensor is defined by

$$\chi_{ij}=\phi_{j,i}\,. \tag{3}$$

As a consequence of the assumption that each point has six degrees-of-freedom, in a micropolar continuum both stresses and couple-stresses exist and Cauchy's theorem holds for them, i.e.,

$$\sigma_i=\sigma_{ji}n_j \tag{4a}$$

$$m_i=m_{ji}n_j \tag{4b}$$

where σ_i and m_i are components of stress and couple-stress vectors, respectively, n_i is the unit normal vector to an arbitrary surface S, and σ_{ij} and m_{ij} are respective stress and couple-stress tensors. Stress and couple-stress tensors are in general asymmetric. The equilibrium equations are

$$\sigma_{ji,j}=0 \tag{5a}$$

$$m_{ji,j}+\varepsilon_{ijk}\sigma_{j,k}=0. \tag{5b}$$

For a centrosymmetric isotropic micropolar material the stress-strain relations are

$$\sigma_{ij}=\lambda e_{kk}\delta_{ij}+(2\mu+\kappa)e_{ij}+\kappa\varepsilon_{ijk}(r_k-\phi_k) \tag{6a}$$

$$m_{ij}=\alpha\phi_{k,k}\delta_{ij}+\beta\phi_{i,j}+\gamma\phi_{j,i} \tag{6b}$$

where λ and μ are the classical Lamé constants and α, β, γ, and κ are new micropolar constants with the following dimensions:

$$[\alpha]=[\beta]=[\gamma]=F=\frac{M}{L}\ \text{and}\ [\kappa]=\frac{F}{L^2}=\frac{M}{L^3} \tag{7}$$

where F, M, and L are dimensions of force, moment, and length, respectively. The strain energy density has the following form:

$$\begin{aligned}W=\frac{1}{2}(\sigma_{ij}\varepsilon_{ij}+m_{ij}\chi_{ij})=\frac{1}{2}[&\lambda e_{kk}e_{mm}+(2\mu+\kappa)e_{ij}e_{ij}]\\ &+\kappa(r_k-\phi_k)(r_k-\phi_k)+\frac{1}{2}(\alpha\phi_{k,k}\phi_{m,m}\\ &+\beta\phi_{i,j}\phi_{j,i}+\gamma\phi_{i,j}\phi_{i,j}).\end{aligned} \tag{8}$$

The following technical elastic constants have clearer physical meanings ([59]):

$$E=\frac{(2\mu+\kappa)(3\lambda+2\mu+\kappa)}{2\lambda+2\mu+\kappa},\quad G=\frac{2\mu+\kappa}{2},\quad \nu=\frac{\lambda}{2\lambda+2\mu+\kappa} \tag{9a}$$

$$l_t=\sqrt{\frac{\beta+\gamma}{2\mu+\kappa}},\quad l_b=\sqrt{\frac{\gamma}{2(2\mu+\kappa)}} \tag{9b}$$

$$N=\sqrt{\frac{\kappa}{2(\mu+\kappa)}},\quad \psi=\frac{\beta+\gamma}{\alpha+\beta+\gamma} \tag{9c}$$

where E, G, ν, l_t, l_b, N, and ψ are Young's modulus, Poisson's ratio, the characteristic length in torsion, the characteristic length in bending, coupling number, and polar ratio, respectively. These constants have the following dimensions:

$$\begin{aligned}[][E]=[G]=FL^{-2},\quad [\nu]=1\\ [l_t]=[l_b]=L\end{aligned} \tag{10}$$

$$[N]=[\psi]=1.$$

It is seen that two internal characteristic lengths exist. Therefore, this theory is capable of analytically predicting size effects. It is worth mentioning that these characteristic lengths appear in the stress field solutions even for force control loading conditions. Therefore, in dimensional analysis formulations these characteristic lengths must be taken into account.

3 Fracture Mechanics of Smooth Cracks in Micropolar Solids

In this section, fracture mechanics of smooth cracks in a micropolar solid is reviewed. Here, the effects of couple-stresses on the stress distribution around the tip of a smooth crack are discussed. In 1960s and 1970s when generalized continuum theories were rediscovered and elaborately developed, several researchers became interested in examining the influence of couple-stresses in problems in which classical theory predicts infinite stresses. One such problem with great practical importance was the stress distribution near the tip of a smooth crack. It was known that stresses and strains around the tip of a crack are unbounded and have an $r^{-1/2}$ singularity. Researchers were hopeful not to see this pathological problem in higher-order continuum theories. Unfortunately, higher-order continuum theories could not solve this pathological problem; both stresses and couple-stresses were observed to be unbounded at the crack tip. There is a limited number of investigations in micropolar fracture mechanics, which will be reviewed in this section.

The first investigation into fracture mechanics of smooth cracks in Cosserat continua was performed by Sternberg and Muki [65]. They solved the problem of an infinite two-dimensional plane-strain linear couple-stress medium with a finite crack under a uniform tensile stress state perpendicular to the crack axis at infinity. They showed that both stresses and couple-stresses have an $r^{-1/2}$ singularity at the crack tip. They observed that couple-stresses only change the angular variation of stresses around the crack tip; the form of the radial variation of stresses remains unchanged. They found the following asymptotic expressions for stresses and couple-stresses:

$$\sigma_{xx}(r,\theta)=-(1-2\nu)\frac{K_I}{\sqrt{2r}}\left[\cos\frac{\theta}{2}-\frac{1}{2}\sin\theta\sin\frac{3\theta}{2}\right]+O(r^0) \tag{11a}$$

$$\sigma_{yy}(r,\theta)=\frac{K_I}{\sqrt{2r}}\left[(3-2\nu)\cos\frac{\theta}{2}-\frac{1}{2}(1-2\nu)\sin\theta\sin\frac{3\theta}{2}\right]+O(r^0) \tag{11b}$$

$$\sigma_{xy}(r,\theta)=-\frac{K_I}{\sqrt{2r}}\left[4(1-\nu)\sin\frac{\theta}{2}+\frac{1}{2}(1-2\nu)\times\sin\theta\cos\frac{3\theta}{2}\right]+O(r^0) \tag{11c}$$

$$\sigma_{yx}(r,\theta)=-(1-2\nu)\frac{K_I}{\sqrt{2r}}\left[\frac{1}{2}\sin\theta\cos\frac{3\theta}{2}\right]+O(r^0) \tag{11d}$$

and

$$m_{xz}(r,\theta)=-\frac{\hat{K}_I}{\sqrt{2r}}\left[\frac{a}{2}\sin\frac{\theta}{2}\right]+O(r^0) \tag{12a}$$

$$m_{xz}(r,\theta)=\frac{\hat{K}_I}{\sqrt{2r}}\left[\frac{a}{2}\cos\frac{\theta}{2}\right]+O(r^0) \tag{12b}$$

where σ_{xx}, σ_{yy}, σ_{xy}, σ_{yx} are (force-) stresses and m_{xz} and m_{yz} are couple-stresses and

$$K_I=f\left(\frac{l}{a},\nu\right)\sigma^\infty\sqrt{a}\quad\text{and}\quad\hat{K}_I=\hat{f}\left(\frac{l}{a},\nu\right)\sigma^\infty\sqrt{a} \tag{13}$$

where K_I and $\hat{K}_I$ are stress- and couple-stress intensity factors, ν is Poisson's ratio, l is the characteristic length of the couple-stress material, and $2a$ is the crack length. Later Sih and Liebowitz [66] found the asymptotic expressions of the displacement and rotation components as shown below:

$$u_r(r,\theta)=\frac{K_I\sqrt{2r}}{8\mu}\left[3(1-2\nu)\cos\frac{\theta}{2}-(7-6\nu)\cos\frac{3\theta}{2}\right]+O(r)$$
$$u_\theta(r,\theta)=\frac{K_I\sqrt{2r}}{8\mu}\left[-(1-2\nu)\sin\frac{\theta}{2}+(7-6\nu)\sin\frac{3\theta}{2}\right]+O(r) \tag{14}$$
$$\omega_z(r,\theta)=\frac{\hat{K}_I\sqrt{2r}}{8\mu l^2}\left[a\sin\frac{\theta}{2}\right]+O(r).$$

For a crack in a Cosserat continuum, strain energy release may be calculated as

$$G=\lim_{\delta a\to 0}\frac{1}{\delta a}\int_0^{\delta a}[\sigma_{yy}(\delta a-\xi,0)u_y(\xi,\pi)+m_{yz}(\delta a-\xi,0)\omega_z(\xi,\pi)]d\xi. \tag{15}$$

Using the above formula, Sih and Liebowitz [66] found the strain energy release rate.

$$G=\frac{\pi}{2\mu}\left[(1-\nu)(3-2\nu)K_I^2+\frac{1}{16}\left(\frac{a}{l}\right)^2\hat{K}_I^2\right] \tag{16}$$

There are some other interesting investigations into fracture mechanics of cracks in micropolar solids (see [67–73]). Now it is known that the classical theory underestimates the value of K_I and overestimates the energy release rate G.

Another interesting investigation into micropolar fracture mechanics was conducted by Atkinson and Leppington [74]. They analyzed two problems: (1) a semi-infinite crack under an internal stress acting on the crack faces and (2) a finite crack in an infinite solid under a uniform stress at infinity. They solved the second problem only for cases in which l/a is very small ($l/a \ll 1$). They showed that both stresses and couple-stresses at the tip of a crack in a couple-stress or micropolar medium have an $r^{-1/2}$ singularity. They also demonstrated that the angular variations of stresses and couple-stresses in couple-stress and micropolar continua are a little different but have a similar form. Atkinson and Leppington defined the *J*-integral for both couple-stress and micropolar theories and showed that *J*-integral is path-independent. Recently, Lubarda and Markenscoff [75] studied some conservation integrals for linear couple-stress elasticity.

4 Micropolar Griffith's Criterion

For finding the orders of stress and couple-stress singularity at the tip of a fractal crack, we utilize an energy approach. The fractal crack is in equilibrium and hence the virtual work of all forces in a virtual displacement, which is an infinitesimal crack growth, is zero. For a cracked body, the principle of virtual work must be modified to take into account the work done in a crack propagation and strain energy release due to a crack growth. Griffith's [76,77] criterion is actually a modified energy balance for cracked bodies. In this section we generalized Griffith's theory for smooth and fractal cracks in micropolar solids.

4.1 Griffith's Theory for a Smooth Crack in a Micropolar Solid. When a crack propagates, new free surfaces are created. For creating these new free surfaces some amount of surface energy is needed to overcome the cohesive forces. This amount of energy is provided by an equal amount of strain energy release. This is Griffith's criterion [76,77], which was originally stated for

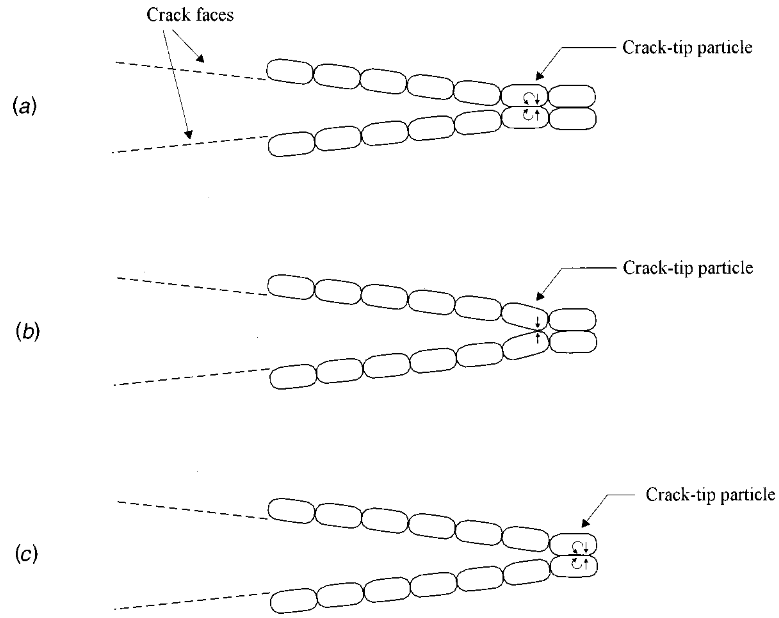

Fig. 1 Mechanism of crack propagation in a micropolar continuum: (*a*) crack-tip particles withstand rotation and separation, (*b*) the first step in crack propagation—crack-tip particles rotate with respect to each other, and (*c*) the second step in crack propagation—crack-tip particles move apart and neighboring particles become the next crack-tip particles

a rectilinear crack in a classical continuum. Mosolov [11] used this criterion for fractal cracks assuming that the specific surface energy per unit length remains unchanged and only the length of the crack increases in the case of a fractal crack. Later, Borodich [22,23] noticed that Griffith's criterion must be modified and in the modified criterion the specific surface energy must be defined per unit of a fractal measure. To our best knowledge, there is no discussion on Griffith's criterion for cracks in micropolar solids. This theory can be easily generalized for smooth and fractal cracks in a micropolar solid, as we show below.

In a micropolar continuum each material point can rotate and translate independently. Now suppose that there is a finite crack of length 2a in a micropolar solid. Figure 1(*a*) shows a crack and some particles (material points) on the crack surfaces. When the crack propagates, crack-tip particles separate from each other. Unlike a crack in a classical solid, this separation of crack-tip particles is a two-step process as shown in Figs. 1(*b*) and 1(*c*). In the first step crack-tip particles rotate with respect to each other but do not move, i.e.,

$$\Delta\phi=\phi_2-\phi_1\neq 0 \quad \text{and} \quad \Delta u=u_2-u_1=0. \tag{17}$$

In the next step, crack-tip particles move apart but do not rotate, i.e.,

$$\Delta u=u_2-u_1\neq 0 \quad \text{and} \quad \Delta\phi=\phi_2-\phi_1=0. \tag{18}$$

After this step, these particles are no longer crack-tip particles; they belong to two free surfaces (see Fig. 1(*c*)). Obviously, the surface energy δU_s needed for creating the new free surfaces has two parts, δU_s^{ϕ} and δU_s^{u}, where δU_s^{ϕ} is the surface energy spent on rotating particles in the path of crack growth and δU_s^{u} is the surface energy spent on separating these particles from each other. Figure 2 shows a crack and the dashed line is the crack propagation path. Crack-tip particles on the path of crack propagation are shown in this figure. Similar to the surface energy release rate, strain energy release rate is composed of two parts: stress part δU_e^{σ}, and couple-stress part δU_e^{m}.

Griffith's criterion for a crack in a micropolar solid may be stated in two different forms, depending on whether the effects of stresses and couple-stresses are considered uncoupled or coupled.

(I) Uncoupled Micropolar Griffith's Criterion. This form of Griffith's criterion states that a crack propagates by an amount δa if the following conditions are satisfied simultaneously:

$$\delta U_e^{\sigma}=\delta U_s^{u} 2t\gamma_u\delta a \tag{19a}$$

$$\delta U_e^{m}=\delta U_s^{\phi}=2t\gamma_{\phi}\delta a \tag{19b}$$

where t, γ_u, and γ_{ϕ} are thickness, displacement, and rotation specific surface energies, respectively. Dimensions of these two surface energies are $[\gamma_u]=[\gamma_{\phi}]=FL^{-1}$.

(II) Coupled Micropolar Griffith's Criterion. In this form of the Griffith's criterion effects of stresses and couple-stresses are

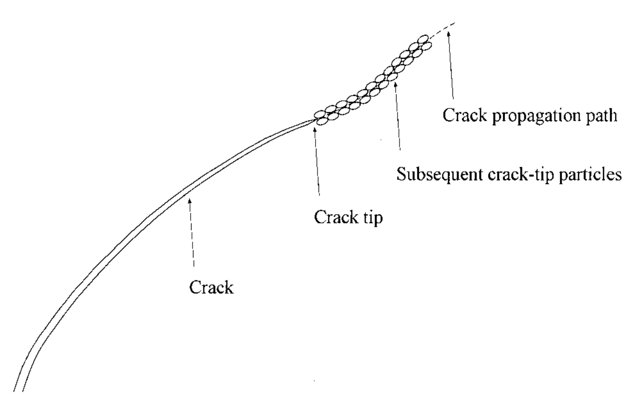

Fig. 2 A crack in a micropolar solid and its propagation path. The particles shown are the particles on the subsequent free surfaces.

assumed to be coupled. Coupled micropolar Griffith's criterion states that the crack propagates by an amount δa if

$$\delta U_e=\delta(U_e^\sigma+U_e^m)=\delta U_s=2t\gamma_m\delta a=2t(\gamma_u+\gamma_\phi)\delta a. \quad (20)$$

It should be noted that the micropolar specific surface energy γ_m is generally different from the classical specific energy γ. Obviously, if (19*a*) and (19*b*) are satisfied, (20) is automatically satisfied. In other words, the uncoupled criterion is stronger than the coupled criterion.

4.2 Griffith's Criterion for a Fractal Crack in a Micropolar Solid. For a smooth crack, surface energy required for crack propagation is proportional to the length (area) of the newly created free surfaces. In the case of a fractal crack the true length (area) of new free surfaces should be considered. Because the true length (area) of a fractal curve (surface) is infinity, a fractal measure should be utilized. The surface energy required to create a fractal crack in a classical solid is

$$U_s=2t\gamma_f(D)m_D \quad (21)$$

where t is the plate thickness, $\gamma_f=\gamma_f(D)$ is the specific surface energy per unit of a fractal measure, and m_D is the corresponding fractal measure and is proportional to a^D (see the Appendix). Specific surface energy per unit of a fractal measure was introduced by Borodich [22,23] and has the dimension $[\gamma_f]=FL^{-D}$, where F and L are dimensions of force and length, respectively. There are two important issues arising from Borodich's generalization of Griffith's criterion that should be explained: (1) It should be noted that γ_f is not a material property. In general, it is possible to have cracks with different fractal dimensions in the same material. Therefore, in Eq. (21) γ_f cannot be a material property; it depends on both the material and the fractal dimensions of the fractal crack. (2) "Fractal measure" is an ambiguous term; there are different definitions of dimension and consequently these different dimensions have different corresponding measures. For self-similar fractals all different dimensions have the same value and hence the corresponding measures they define are identical. Therefore, for self-similar fractals "fractal measure" is not an ambiguous term. However, this is not the case for self-affine fractals; different definitions of dimension give completely different dimensions for the same self-affine fractal set. Obviously, the relevant fractal dimension for calculating the surface energy of a fractal crack is the divider (latent) fractal dimension. Therefore, the specific surface energy should be defined per divider fractal measure, although it can be defined for other fractal measures as well.

For a fractal crack in a micropolar solid, Griffith's criterion again has uncoupled and coupled forms and only the surface energies should be modified as

$$\delta U_s^u=2t\gamma_u^f(D_{D_D})\delta m_{D_D} \quad \text{and} \quad \delta U_s^\phi=2t\gamma_\phi^f(D_D)\delta m_{D_D} \quad (22a)$$

$$\delta U_s=\delta U_s^u+\delta U_s^\phi=2t[\gamma_u^f(D_D)+\gamma_\phi^f(D_D)]\delta m_{D_D} \quad (22b)$$

where γ_u^f and γ_ϕ^f are fractal specific surface energies per unit of latent fractal measure and m_{D_D} is the latent fractal measure. Thus we have the following two forms of fractal micropolar Griffith's criterion.

(I) Uncoupled Fractal Micropolar Griffith's Criterion. A fractal crack with divider fractal dimension D_D propagates by an amount δm_{D_D} if the following two conditions are satisfied simultaneously:

$$\delta U_e^\sigma=\delta U_s^u=2t\gamma_u^f\delta m_{D_D} \quad (23a)$$

$$\delta U_e^m=\delta U_s^\phi=2t\gamma_\phi^f\delta m_{D_D}. \quad (23b)$$

(II) Coupled Fractal Micropolar Griffith's Criterion. A fractal crack with divider fractal dimension D_D propagates by an amount δm_{D_D} if the following condition is satisfied:

$$\delta U_e=\delta(U_e^\sigma+U_e^m)=\delta U_s=2t\gamma_m^f\delta m_{D_D}=2t(\gamma_u^f+\gamma_\phi^f)\delta m_{D_D}. \quad (24)$$

In the next section we use both forms of micropolar Griffith's criterion for calculating the orders of stress and couple-stress singularity at the tip of a fractal crack. We will show that both criteria give equal orders of stress and couple-stress singularity.

5 Fractal Cracks in Micropolar Elastic Solids

In this section radial variations of stresses and couple-stresses around the tip of a fractal crack in a micropolar solid are investigated. To the best of our knowledge, there is no investigation into this problem in the literature. Without loss of generality, a mode I problem is solved. Consider an infinite medium made of a micropolar material with a finite crack of nominal length $2a$. It is assumed that the cracked solid is under a uniform tensile stress σ^∞ perpendicular to the crack axis applied at infinity (see Fig. 3(*a*)).

One major difference between this problem and the similar problem of a fractal crack in a classical solid is that a micropolar material has two internal characteristic length scales, l_b and l_t. Here l_b and l_t are characteristic lengths in bending and torsion, respectively. On the other hand, a couple-stress material has only one characteristic length l. For a micropolar material in a two-dimensional problem only one of the characteristic lengths appears in the equilibrium equations. Therefore, it is assumed that the medium has a characteristic length and it is denoted by l. It is known that even for a force control loading this characteristic length appears in the stress solutions in the form of l/a, where "a" is a geometric characteristic length of the problem, for example hole radius or crack length.

The method of crack-effect zone, which was introduced by Yavari et al. [16], is utilized. When the system shown in Fig. 3(*a*) is uncracked, only one of stress components is nonzero and has a uniform distribution; all other stresses and all couple-stresses are identically zero. When the crack is formed, stresses and couple-stresses are perturbed. This stress perturbation is significant only in a finite zone around the crack. For the cracked system all stresses and couple-stresses are nonzero in the crack-effect zone. The crack-effect zone may be covered by a disk $\mathfrak{R}_c$, as shown in Fig. 3(*a*). We assume that the micropolar material is centrosymmetric isotropic and homogeneous. The strain energy of the system may be written as

$$U_e=U_e^\sigma+U_e^m=\int_{\mathfrak{R}}\left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij}+\frac{1}{2}m_{ij}\chi_{ij}\right)dV \quad (25)$$

where σ_{ij}, m_{ij}, ε_{ij}, and χ_{ij} are stresses, couple-stresses, strains, and curvatures, respectively. The strain energy can be decomposed into two parts as follows:

$$U_e=\int_{\mathfrak{R}-\mathfrak{R}_c}\left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij}+\frac{1}{2}m_{ij}\chi_{ij}\right)dV+\int_{\mathfrak{R}_c}\left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij}+\frac{1}{2}m_{ij}\chi_{ij}\right)dV. \quad (26)$$

When the crack propagates by an infinitesimal amount δa, the change of the strain energy in $\mathfrak{R}_c$ is dominant, hence

$$\delta U_e\cong\delta\int_{\mathfrak{R}_c}\left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij}+\frac{1}{2}m_{ij}\chi_{ij}\right)dV. \quad (27)$$

For a centrosymmetric material stress-strain and couple-stress-curvature relations are uncoupled, i.e., stresses are not functions of curvatures and couple-stresses are not functions of strains. Therefore, the constitutive equations may be written as

$$\sigma_{ij}=C_{ijkl}\varepsilon_{kl} \quad \text{and} \quad m_{ij}=\hat{C}_{ijkl}\chi_{kl} \quad (28)$$

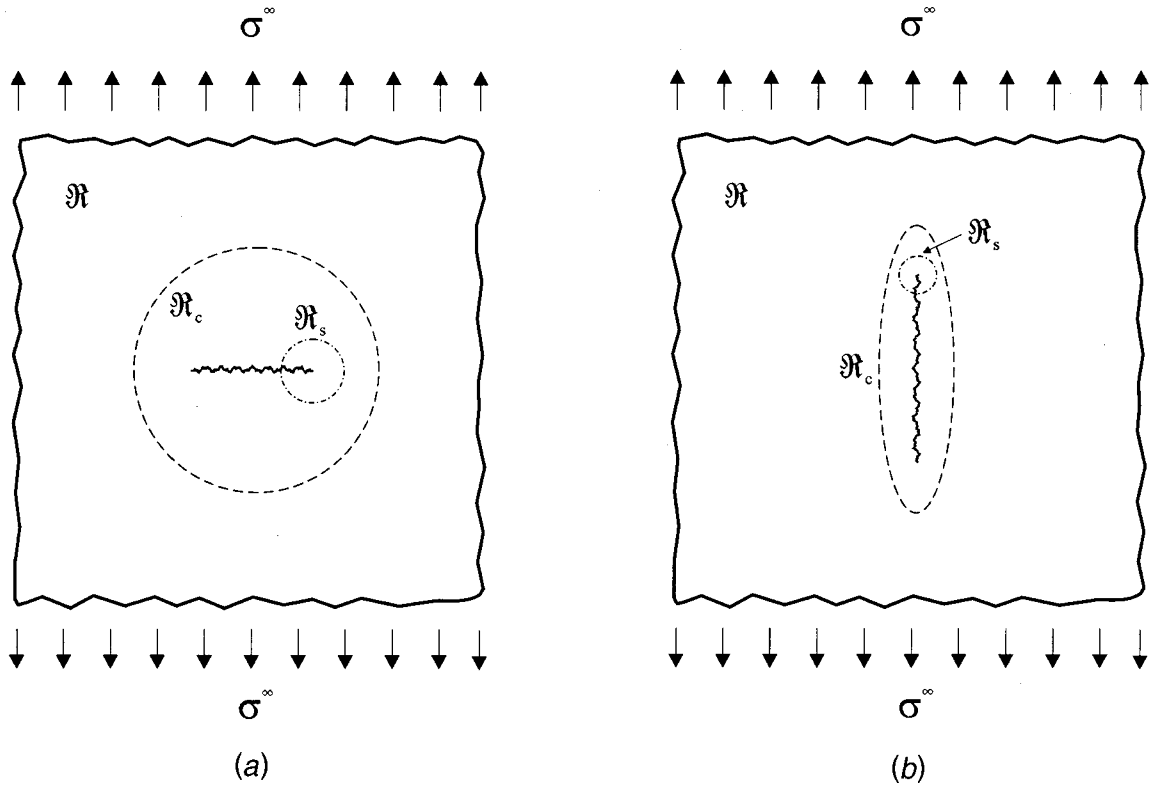

Fig. 3 (*a*) A two-dimensional micropolar solid with a finite fractal crack perpendicular to the applied stresses, (*b*) an infinite micropolar solid with a finite fractal crack parallel to the applied stresses

where C_{ijkl} and $\hat{C}_{ijkl}$ are fourth-order tensors and are mechanical properties of the material. The following asymptotic stresses and couple-stresses are assumed at the crack tip

$$\sigma_{ij}(r,\theta)=K_I r^{-\alpha_1} f_{ij}\left(\theta,\nu,\frac{l}{a},H\right) \tag{29a}$$

$$m_{ij}(r,\theta)=\hat{K}_I r^{-\alpha_2} \hat{f}_{ij}\left(\theta,\nu,\frac{l}{a},H\right) \tag{29b}$$

where K_I^f and $\hat{K}_I^f$ are fractal stress and couple-stress intensity factors, respectively, and H is the Hurst exponent (see the Appendix). We will calculate α_1 and α_2 using the method of crack-effect zone. The above asymptotic stresses and couple-stresses are dominant for $r \leqslant r_{s_1}$ and $r \leqslant r_{s_2}$, respectively. Therefore, Eqs. (29*a*) and (29*b*) are valid in a disk $\mathfrak{R}_s$ with radius $r_s=\min(r_{s_1},r_{s_2})$. Here, the method of crack-effect zone should be used very carefully. Because the change of U_e in $\mathfrak{R}_s$ is dominant, the change of strain energy may be expressed as

$$\delta U_e=\delta\int_{\mathfrak{R}_c-\mathfrak{R}_s}\left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij}+\frac{1}{2}m_{ij}\chi_{ij}\right)dV+\delta\int_{\mathfrak{R}_s}\left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij}+\frac{1}{2}m_{ij}\chi_{ij}\right)dV\cong\delta\int_{\mathfrak{R}_s}\left(\frac{1}{2}\sigma_{ij}\varepsilon_{ij}+\frac{1}{2}m_{ij}\chi_{ij}\right)dV. \tag{30}$$

From (29) and (30) we have

$$\delta U_e=\delta U_e^\sigma+\delta U_e^m \tag{31a}$$

$$\delta U_e^\sigma \propto \delta(r_s^{-\alpha_1} r_s^{-\alpha_1} r_s^2) \quad \text{and} \quad \delta U_e^m \propto \delta(r_s^{-\alpha_2} r_s^{-\alpha_2} r_s^2) \tag{31b}$$

where "$\propto$" means "proportional to." For a rectilinear or fractal crack in a micropolar continuum r_s is not necessarily proportional to "*a*." Here r_s is a function of *a*, *l*, ν, and *H*, i.e., $r_s=f(a,l,\nu,H)$. Using Buckingham's ([78,79]) Π theorem we must have

$$\frac{r_s}{a}=\hat{\Phi}\left(\frac{l}{a},\nu,H\right) \quad \text{or} \quad r_s=a\hat{\Phi}\left(\frac{l}{a},\nu,H\right). \tag{32}$$

As it is seen from (32), r_s is not necessarily proportional to "*a*." The functional form of $\hat{\Phi}$ cannot be found using dimensional analysis and this makes the use of crack-effect zone method very difficult. But we know that for most engineering materials *l* is very small ($l/a \ll 1$). We also know that the following limit exists:

$$\lim_{l/a\to 0}\hat{\Phi}\left(\frac{l}{a},\nu,H\right)=\hat{\Phi}(0,\nu,H)=\Phi(\nu,H) \tag{33}$$

because when l/a tends to zero for a constant "*a*" we approach the classical theory and obviously $\hat{\Phi}$ is defined for the classical theory and is finite. Thus we have a complete similarity or a similarity of the first kind (see, for example, the excellent book of Barenblatt [80]). Therefore according to dimensional analysis for very small $l/a(l/a\ll 1)\hat{\Phi}$ can be considered independent of l/a and replaced by its limit Φ. Therefore, $r_s \sim a$ for $l/a \ll 1$. As a matter of fact, we do not need to limit ourselves to the case $l/a \ll 1$. We can show that $\hat{\Phi}$ is not a function of l/a as we see in the following. We know that for a smooth crack both stresses and couple-stresses have an $r^{-1/2}$ singularity regardless of the size of the characteristic length(s) of the cracked micropolar material. Suppose that the radius of the dominant zone of stress and couple-stress singularity is r_s. Thus, $\delta U_s \propto \delta(r_s^{2-2\alpha})=\delta(r_s)$ and $\delta U_s \propto \delta(a)$. Therefore, according to Griffith's criterion we must have $r_s \propto a$. Thus

$$\hat{\Phi}\left(\frac{l}{a},\nu,H=1\right)=f(\nu). \quad (34)$$

For a fractal crack H is a local parameter while l/a is a global parameter. The reason is that H is defined only for the fractal crack, which has 2-measure (area) zero but l/a is defined for all points of the domain other than the crack. Therefore, we expect $\hat{\Phi}$ to be separable, i.e.,

$$\hat{\Phi}\left(\frac{l}{a},\nu,H\right)=\hat{\Phi}_1\left(\frac{l}{a},\nu\right)\hat{\Phi}_2(\nu,H). \quad (35)$$

From (34) and (35) we obtain

$$\hat{\Phi}\left(\frac{l}{a},\nu,H\right)=\tilde{\Phi}(H,\nu). \quad (36)$$

Therefore, r_s is always proportional to "a" regardless of the value of l/a.

From (31b) we have

$$\delta U_e^{\sigma}\propto\delta(a^{2-2\alpha_1}) \quad \text{and} \quad \delta U_e^{m}\propto\delta(a^{2-2\alpha_2}). \quad (37)$$

The next thing we need is the asymptotic form of surface energies. From (22) we have

$$\delta U_s^{u}\propto\delta U_s^{\phi}\propto\delta U_s\propto\begin{cases}\delta(a^{1/H}) & \frac{1}{2}\leq H<1\\ \delta(a^2) & 0<H\leq\frac{1}{2}\end{cases}. \quad (38)$$

We use both forms of Griffith's criterion and show that they give us the same result.

Using uncoupled micropolar Griffith's criterion is easier and yields

$$\delta(a^{2-2\alpha_1})\propto\begin{cases}\delta(a^{1/H}) & \frac{1}{2}\leq H<1\\ \delta(a^2) & 0<H\leq\frac{1}{2}\end{cases} \quad (39a)$$

$$\delta(a^{2-2\alpha_2})\propto\begin{cases}\delta(a^{1/H}) & \frac{1}{2}\leq H<1\\ \delta(a^2) & 0<H\leq\frac{1}{2}\end{cases}. \quad (39b)$$

Thus

$$\alpha_1=\alpha_2=\begin{cases}\frac{2H-1}{2H} & \frac{1}{2}\leq H<1\\ 0 & 0<H\leq\frac{1}{2}\end{cases}. \quad (40)$$

Using coupled Griffith's criterion is tricky. From (20), (38), and (39) we obtain

$$C_1\delta(a^{2-2\alpha_1})+C_2\delta(a^{2-2\alpha_2})\propto\begin{cases}\delta(a^{1/H}) & \frac{1}{2}\leq H<1\\ \delta(a^2) & 0<H\leq\frac{1}{2}\end{cases} \quad (41)$$

where C_1 and C_2 are not functions of a. We prove by contradiction that α_1 and α_2 must be equal. Suppose that $\alpha_1\neq\alpha_2$ and for example $\alpha_1>\alpha_2$. Notice that "$\propto$" means "proportional to" and that Eq. (41) holds for an arbitrary crack length "a." For a very large "a" ($a\gg 1$) we can write

$$a^{2-2\alpha_2}\gg a^{2-2\alpha_1}. \quad (42)$$

Hence

$$C_1\delta(a^{2-2\alpha_1})+C_2\delta(a^{2-2\alpha_2})\cong C_2\delta(a^{2-2\alpha_2})\propto\begin{cases}\delta(a^{1/H}) & \frac{1}{2}\leq H<1\\ \delta(a^2) & 0<H\leq\frac{1}{2}\end{cases} \quad (43)$$

Thus

$$\alpha_2=\begin{cases}\frac{2H-1}{2H} & \frac{1}{2}\leq H<1\\ 0 & 0\leq H\leq\frac{1}{2}\end{cases} \quad (44)$$

On the other hand, for a very small "a" ($a\ll 1$) we can write

$$a^{2-2\alpha_1}\gg a^{2-2\alpha_2}. \quad (45)$$

Thus

$$C_1\delta(a^{2-2\alpha_1})+C_2\delta(a^{2-2\alpha_2})\cong C_1\delta(a^{2-2\alpha_1})\propto\begin{cases}\delta(a^{1/H}) & \frac{1}{2}\leq H<1\\ \delta(a^2) & 0<H\leq\frac{1}{2}\end{cases}. \quad (46)$$

Hence

$$\alpha_2=\begin{cases}\frac{2H-1}{2H} & \frac{1}{2}\leq H<1\\ 0 & 0<H\leq\frac{1}{2}\end{cases}. \quad (47)$$

From (44) and (47) we see that $\alpha_1=\alpha_2$, which is a contradiction. Therefore our assumption was false and α_1 and α_2 must be equal, i.e.,

$$\alpha_1=\alpha_2=\begin{cases}\frac{2H-1}{2H} & \frac{1}{2}\leq H<1\\ 0 & 0<H\leq\frac{1}{2}\end{cases}. \quad (48)$$

Therefore

$$\sigma_{ij}\sim r^{\frac{-2H-1}{2H}},\quad m_{ij}\sim r^{\frac{-2H-1}{2H}} \quad \text{as} \quad r\to 0 \quad \frac{1}{2}\leq H<1 \quad (49a)$$

$$\sigma_{ij}\sim r^0,\quad m_{ij}\sim r^0 \quad \text{as} \quad r\to 0 \quad 0<H\leq\frac{1}{2}. \quad (49b)$$

It is seen that both forms of Griffith's criterion yield the same result: stresses and couple-stresses have equal orders of singularity and this order of singularity is the same as that of stresses at the tip of a fractal crack in a classical continuum. This result is similar to that reached by Sternberg and Muki [65]: that in a couple-stress medium at the tip of a smooth crack both stresses and couple-stresses have $r^{-1/2}$ singularities. This is also true for self-similar cracks; orders of stress and couple-stress singularity are equal.

A similar result can be reached for mode IV self-affine cracks. A mode IV fractal crack is shown in Fig. 3(b). This new mode of fractal fracture was introduced by Yavari [15] and Yavari et al. [14]. As was done for a mode I fractal crack, the orders of stress and couple-stress singularity can be calculated. The only modification in the analysis is to change Eq. (37) to read ([14,16])

$$\delta U_e^{\sigma}\propto\delta(a^{1+H-2\alpha_1 H}) \quad \text{and} \quad \delta U_e^{m}\propto\delta(a^{1+H-2\alpha_2 H}) \quad (50)$$

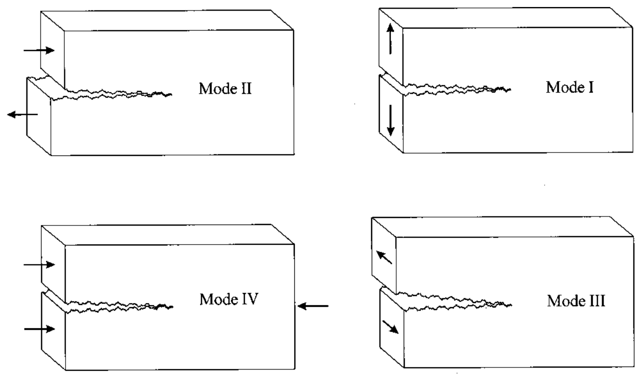

Fig. 4 The four modes of fractal fracture: mode I (opening mode), mode II (shearing mode), mode III (tearing mode), and mode IV (axial mode)

The stresses and couple-stresses have the following asymptotic forms:

$$\sigma_{ij}\sim r\,\frac{-H^2-H+1}{2H^2},\quad m_{ij}\sim r\,\frac{-H^2-H+1}{2H}\quad \text{as}\quad r\to 0\quad \frac{1}{g}\leqslant H<1 \tag{51a}$$

$$\sigma_{ij}\sim r^0,\quad m_{ij}\sim r^0\quad \text{as}\quad r\to 0\quad 0<H\leqslant\frac{1}{g} \tag{51b}$$

where $g=(\sqrt{5}+1)/2$ is the Golden ratio. All four modes of fracture are shown in Fig. 4. (Actually, there are six modes. We found the fifth and sixth modes very recently ([16])).

For three-dimensional cracked bodies made of a couple-stress material or a micropolar material a similar conclusion can be reached.

6 Conclusions

Fracture mechanics of smooth cracks in micropolar continua is reviewed. Griffith's fracture theory is generalized for rectilinear and fractal cracks in micropolar continua. It is seen that Griffith's criterion can have two forms: uncoupled micropolar Griffith's criterion and coupled micropolar Griffith's criterion. Using dimensional analysis and the method of crack-effect zone it is shown that both forms of Griffith's criterion predict that stresses and couple-stresses have the same order of singularity. This order of stress and couple-stress singularity is shown to be equal to that of stresses at the tip of a fractal crack in a classical continuum.

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Appendix

Fractal Geometry. This Appendix presents some concepts and definitions of fractal geometry. Here we discuss only those aspects of fractal geometry that are directly relevant to our investigation. For more details the reader may refer to Mandelbrot [2,81–83], Feder [84], Vicsek [85], and Falconer [86,87].

Suppose that $U\neq\emptyset$ is a subset of $\mathbb{R}^n$. The diameter of U is defined as

$$\mathrm{diam}(U)=\sup\{|x-y|:x,y\in U\}. \tag{A1}$$

An ε-cover of S is a countable or finite collection of sets $\{U_i\}$ such that

1. $0<\mathrm{diam}(U_i)\leqslant\varepsilon$,
2. $S\subset\cup_{i=1}^{\infty}U_i$.

Consider a set $S\subset\mathbb{R}^n$. An affine transformation of real scaling ratios $r_1,r_2,\ldots,r_n$ $(0<r_i<1)$ transforms each $x=(x_1,x_2,\ldots,x_n)\in S$ into $r(x)=(r_1x_1,r_2r_2,\ldots,r_nx_n)\in r(S)$. The set S is self-affine if it is composed of N nonoverlapping subsets congruent to $r(S)$. If the above property holds for S when $r_1=r_2=\ldots=r_n=r$, it is called a self-similar set. A self-similar fractal is invariant under an isotropic length-scale transformation while a self-affine fractal is invariant under a transformation with different length scales in different directions.

Roughly speaking, measure of a set $S\subset\mathbb{R}^n$ tells us about the size of the set and is denoted by $\mu(S)$. In other words, measure is a generalized size. μ is a measure on $\mathbb{R}^n$ if it assigns a non-negative real number (possibly $+\infty$) to each subset of $\mathbb{R}^n$ and satisfies the following requirements:

1. $\mu(\emptyset)=0$
2. $\mu(A)\leqslant\mu(B)$ if $A\subset B$
3. If $A_1,A_2,\ldots$ is a finite or countable sequence of subsets of $\mathbb{R}^n$ then

$$\mu\left(\bigcup_{i=1}^{\infty}A_i\right)\leqslant\sum_{i=1}^{\infty}\mu(A_i) \tag{A2}$$

with equality if A_i's are disjoint subsets of $\mathbb{R}^n$.

Now suppose that $S\subset\mathbb{R}^n$ and $D\in\mathbb{R}^+\cup\{0\}$. The D-dimensional Hausdorff measure of S is denoted by $\mathcal{H}^D(S)$ and is defined as

$$H^D(S)=\lim_{\varepsilon\to 0}H_\varepsilon^D(S) \tag{A3}$$

where

$$H_\varepsilon^D(S)=\inf\left\{\sum_{i=1}^{\infty}\mathrm{diam}(U_i)^D:\{U_i\}\ \text{is an}\ \varepsilon-\text{cover of}\ S\right\} \tag{A4}$$

It can be shown that $\mathcal{H}^D$ has all the properties of a measure. It can be proved that for any set S, $\mathcal{H}^D(S)$ has a jump from $+\infty$ to 0 for one and only one value of D, which is called the Hausdorff dimension of S, i.e.,

$$D_H=\inf\{D:H^D(S)=0\}=\sup\{D:H^D(S)=+\infty\}. \tag{A5}$$

Therefore

$$H^D(S)=\begin{cases}+\infty & D<\mathrm{Dim}_H S\\ 0 & D>\mathrm{Dim}_H S\end{cases}. \tag{A6}$$

There are many other definitions of dimension. One disadvantage of Hausdorff dimension is the difficulty of calculating it, which makes it impractical. Here we discuss two other important dimensions, namely box dimension and divider dimension. All different dimensions somehow measure the complexity of irregularity of a set. It should be emphasized that dimension provides only limited information about a fractal set. In most definitions there is a measurement at scale ε. For each ε irregularities below this scale are ignored and the behavior of measurements as $\varepsilon\to 0$ is studied.

Box dimension: Let $S\neq\emptyset$ be a subset of $\mathbb{R}^n$ and let $N_\varepsilon^B(S)$ be the smallest number of sets of diameter at most ε which can cover S. Box dimension of S is D_B if

$$N_\varepsilon^B(S)=0(\varepsilon^{-D_B})\quad \text{as}\quad \varepsilon\to 0\quad \text{or}\quad D_B=\lim_{\varepsilon\to 0}\frac{\log N_\varepsilon^B(S)}{-\log\varepsilon} \tag{A7}$$

where O is Landau's order symbol. It can be shown that always $D_H\leqslant D_B$. For self-similar fractals the equality holds. Box measure m_{D_B} is defined as

$$m_{D_B}^{\varepsilon}=N_{\varepsilon}^{B}(S)\varepsilon^{D_B}$$

$$=\inf\left\{\sum_i \varepsilon^{D_B}:\{U_i\} \text{ in a finite } \varepsilon-\text{cover of } S\right\},$$

$$m_{D_B}=\lim_{\varepsilon\to 0}m_{D_B}^{\varepsilon}. \tag{A8}$$

In calculating Hausdorff measure different weights $|U_i|^s$ are assigned to covering sets U_i while in box measure the same weight ε^{D_B} is used for all covering sets. It should be noted that m_{D_B} is not a mathematical measure on subsets of $\mathbb{R}^n$ because it is not σ-additive. (It is actually a "content.")

Divider dimension: This is the most important dimension in applications to fractal fracture mechanics problems. Consider a Jordan curve C (a curve that does not intersect itself) $f:[a,b]\to\mathbb{R}^n$, where f is a bijection (a one-to-one and onto function). For $\varepsilon>0$ define $N_{\varepsilon}^{D}(C)$ to be the maximum number of points $x_0,x_1,x_2,\ldots,x_m$ on C such that $|x_k-x_{k-1}|=\varepsilon$ for $k=1,2,\ldots,m$. Therefore, the approximate length of the curve $L_{\varepsilon}(C)$ is $L_{\varepsilon}(C)=O[(N_{\varepsilon}^{D}(C)-1)\varepsilon]$. The divider dimension of C is D_D if

$$L_{\varepsilon}(C)=O(\varepsilon^{1-D_D}) \text{ as } \varepsilon\to 0 \text{ or } N_{\varepsilon}^{D}(C)=O(\varepsilon^{-D_D}) \text{ as } \varepsilon\to 0. \tag{A9}$$

We know that $N_{\varepsilon}^{D}(C)$ is dimensionless while ε has dimension of length. Therefore from ($A9$) we conclude that

$$N_{\varepsilon}^{D}(C)\sim\left(\frac{\varepsilon}{L_0}\right)^{-D_D} \text{ or } N_{\varepsilon}^{D}(C)\sim\varepsilon^{-D_D}L_0^{D_D} \tag{A10}$$

where L_0 is the nominal length of C. It can be shown that for any Jordan curve C, $D_D\geqslant D_B$. For self-similar curves the equality holds. Divider measure m_{D_D} is defined as

$$m_{D_D}^{\varepsilon}=N_{\varepsilon}^{D}(C)\varepsilon^{D_D}, \quad m_{D_D}=\lim_{\varepsilon\to 0} m_{D_D}^{\varepsilon}. \tag{A11}$$

From ($A10$) and ($A11$) we can write

$$m_{D_D}\sim L_0^{D_D} \quad \text{or} \quad m_{D_D}=\eta L_0^{D_D}. \tag{A12}$$

Like box measure, divider measure is not a mathematical measure because it is not σ-additive.

Consider a topologically one-dimensional set. Suppose that this set can be expressed as the graph of a single-valued function $F(t)$ embedded in $\mathbb{R}^2$. Then $F(t)$ is a self-affine function if

$$F(t)=r^{-H}F(rt) \quad \forall r,t\in R \tag{A13}$$

where $H(0<H<1)$ is the Hurst (roughness) exponent. Weierstrass-Mandelbrot function is an example of a self-affine function and is defined as

$$WM(x)=\sum_{-\infty}^{+\infty} a^{-nH}[1-\cos(a^n x)] \quad a>1, \quad 0<H<1. \tag{A14}$$

This function satisfies the invariance relation ($A13$). It can be shown that for a self-affine curve locally ($\varepsilon\leqslant\varepsilon_x$) we have

$$D_B=2-H \text{ and } D_D=\begin{cases}\dfrac{1}{H} & \dfrac{1}{2}\leqslant H<1\\[2mm] 2 & 0<H\leqslant\dfrac{1}{2}\end{cases} \tag{A15a}$$

and globally ($\varepsilon\geqslant\varepsilon_x$)

$$D_B=D_D=1. \tag{A15b}$$

The length scale ε_x is called the crossover length ($\varepsilon_x^H\cong\varepsilon_x$). In general, for a self-affine fractal (with Hurst exponent H) embedded in $\mathbb{R}^n$ the divider and box dimensions are locally related to roughness exponent by

$$D_D=\begin{cases}\dfrac{n-1}{H} & \dfrac{n-1}{n}\leqslant H<1\\[2mm] n & 0<H\leqslant\dfrac{n-1}{n}\end{cases} \tag{A16a}$$

$$D_B=n-H. \tag{A16b}$$

And globally, $D_D=D_B=n-1$.

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