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A study of the edge-zone equation of Mindlin-Reissner plate theory in bending of laminated rectangular plates

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Summary. The bending equations of the Mindlin-Reissner theory of plates laminated of transversely isotropic layers are reformulated in terms of the boundary-layer and transverse displacement functions. Analytical expressions are obtained for the primary response quantities of rectangular laminates with various boundary conditions. It is found that various edge conditions have boundary-layer effects on the primary and secondary response quantities that can be characterized as nonexistent, weak, or strong.

1 Introduction

Classical plate theory, which neglects transverse shear strains, results in a fourth-order equation in the transverse deflection. On the other hand, in the past two decades numerous refined plate theories that include the transverse shear strains have appeared in the literature (see [1], [2]). The transverse shear deformation is often taken into account by the introduction of two rotation functions into the displacement-based theories. These functions represent rotations of straight lines about the x- and y-axes. Inclusion of the transverse shear strains increases the total order of governing equations by at least two. One such theory is known as Mindlin's theory [3], which is a sixth-order plate theory (see also Reissner [4]). Mindlin-Reissner theory is also referred to as the first-order shear deformation plate theory (FSDT). The three bending equations in this theory are often reformulated in terms of a potential function and the transverse displacement function for symmetric plates laminated of transversely isotropic layers (see [5]–[8]). When this potential function is introduced, the two rotation functions are eliminated, and, therefore, the three coupled governing equations are replaced by two uncoupled equations. One of these two equations is expressed in terms of the transverse deflection function and is known as the interior equation. The second equation is expressed in terms of a potential function and is known as the edge-zone (or boundary-layer) equation. Irschik [9] developed analogies between dynamic shear deformation theories of layered beams and plates and classical theories for homogeneous single-layer structures. This paper studies the linear response of composite plates. For nonlinear plate and shell theories of laminated composites, the reader may refer to [6], [10], and [11].

In the present work, analytical solutions are obtained for the primary response quantities of laminated transversely isotropic rectangular plates in bending with various boundary conditions. Also, through numerical calculations the boundary-layer phenomenon existing in FSDT is studied. The results brought about by use of the FSDT are compared with those due to classical plate theory (CPT).

2 Governing equations

The first-order shear deformation theory assumes that the plane sections originally perpendicular to the longitudinal plane of the plate remain plane, but not necessarily perpendicular to the longitudinal plane; that is, in order to obtain the bending equations, it is assumed that

$$\begin{aligned} u_1(x,y,z) &= u(x,y) + z\psi_x(x,y)\,, \\ u_2(x,y,z) &= v(x,y) + z\psi_y(x,y)\,, \\ u_3(x,y,z) &= w(x,y)\,, \end{aligned} \tag{1}$$

where u and v are, respectively, the displacements of the middle surface in the x- and y-directions (see Fig. 1); ψ_x and ψ_y are known as the rotation functions [12]; and z is the thickness coordinate. The linear strains, in the Cartesian coordinates, associated with the displacement field (1) may be represented as

$$\begin{aligned} &\varepsilon_{11} = {\varepsilon_1}^0 + z{\varkappa_1}^0\,, \qquad \varepsilon_{22} = {\varepsilon_2}^0 + z{\varkappa_2}^0\,, \qquad \varepsilon_{33} = 0\,, \\ &2\varepsilon_{12} = {\varepsilon_6}^0 + z{\varkappa_6}^0\,, \qquad 2\varepsilon_{13} = {\varepsilon_5}^0 + z{\varkappa_5}^0\,, \qquad 2\varepsilon_{23} = {\varepsilon_4}^0 + z{\varkappa_4}^0\,, \end{aligned} \tag{2}$$

where

$$\begin{aligned} &{\varepsilon_1}^0 = u_{,x}\,, \quad {\varepsilon_2}^0 = v_{,y}\,, \quad {\varepsilon_4}^0 = \psi_y + w_{,y}\,, \quad {\varepsilon_5}^0 = \psi_x + w_{,x}\,, \quad {\varepsilon_6}^0 = u_{,y} + v_{,x}\,, \\ &{\varkappa_1}^0 = \psi_{x,x}\,, \quad {\varkappa_2}^0 = \psi_{y,y}\,, \quad {\varkappa_4}^0 = {\varkappa_5}^0 = 0\,, \quad {\varkappa_6}^0 = \psi_{x,y} + \psi_{y,x}\,. \end{aligned} \tag{3}$$

In relations (3) a comma followed by an independent variable denotes partial differentiation with respect to that variable. The equilibrium equations of the theory are obtained from the principle of minimum total potential energy [12]:

$$\delta u: \quad N_{11,x} + N_{12,y} = 0\,, \tag{4.1}$$

$$\delta v: \quad N_{12,x} + N_{22,y} = 0\,, \tag{4.2}$$

$$\delta \psi_x: \quad M_{11,x} + M_{12,y} - \hat{Q}_1 = 0\,, \tag{4.3}$$

$$\delta \psi_y: \quad M_{12,x} + M_{22,y} - \hat{Q}_2 = 0\,, \tag{4.4}$$

$$\delta w: \quad \hat{Q}_{1,x} + \hat{Q}_{2,y} + P_z(x,y) = 0\,, \tag{4.5}$$

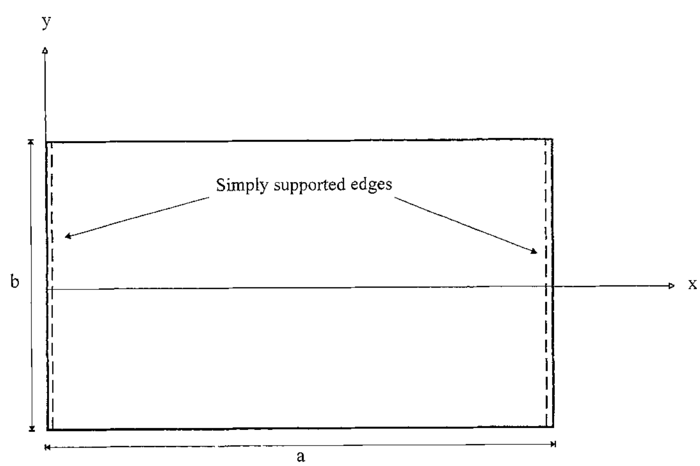

Fig. 1. The geometry of a rectangular plate with simple supports at $x = 0$ and $x = a$

where the symbol δ represents the variational operator and $P_z(x,y)$ denotes the transverse loading function. The stress and moment resultants in Eqs. (4) are defined as:

$$
\begin{aligned}
(N_{11}, N_{22}, N_{12}) &= \int_{-h/2}^{h/2} (\sigma_{11}, \sigma_{22}, \sigma_{12})\, dz\,, \\
(M_{11}, M_{22}, M_{12}) &= \int_{-h/2}^{h/2} (\sigma_{11}, \sigma_{22}, \sigma_{12})\, zdz\,, \\
(\hat{Q}_1, \hat{Q}_2) &= K^2 \int_{-h/2}^{h/2} (\sigma_{13}, \sigma_{23})\, dz\,,
\end{aligned}
\tag{5}
$$

with $K_2 (= 5/6)$ being the shear correction factor and h the thickness of the plate. At any edge of the plate, with the normal vector $\vec{n} = n_1 \vec{e}_1 + n_2 \vec{e}_2$, the following boundary conditions must be specified [12]:

$$\text{either} \quad \delta u = 0 \quad \text{or} \quad N_{11} n_1 + N_{12} n_2 = 0\,, \tag{6.1}$$

$$\text{either} \quad \delta v = 0 \quad \text{or} \quad N_{12} n_1 + N_{22} n_2 = 0\,, \tag{6.2}$$

$$\text{either} \quad \delta \psi_x = 0 \quad \text{or} \quad M_{11} n_1 + M_{12} n_2 = 0\,, \tag{6.3}$$

$$\text{either} \quad \delta \psi_y = 0 \quad \text{or} \quad M_{12} n_1 + M_{22} n_2 = 0\,, \tag{6.4}$$

$$\text{either} \quad \delta w = 0 \quad \text{or} \quad \hat{Q}_1 n_1 + \hat{Q}_2 n_2 = 0\,. \tag{6.5}$$

In the linear theory, when the plate is symmetrically laminated with respect to its middle surface, the first two equations in (4) will be written in terms of u and v and uncoupled from the last three equations. Equations (4.3)–(4.5) are expressed in terms of ψ_x, ψ_y, and w and are known as the bending equations of the plate. Furthermore, for symmetric laminates it can be shown that [13]:

$$
\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} = \begin{Bmatrix} \varkappa_1{}^0 \\ \varkappa_2{}^0 \\ \varkappa_6{}^0 \end{Bmatrix}
\tag{7.1}
$$

and

$$
\begin{Bmatrix} \hat{Q}_2 \\ \hat{Q}_1 \end{Bmatrix} = K^2 \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4{}^0 \\ \varepsilon_5{}^0 \end{Bmatrix}.
\tag{7.2}
$$

If each layer of the symmetric laminate is made of transversely isotropic material, with the plane of isotropy being parallel to the plates middle surface, the rigidity terms A_{45}, D_{16}, and D_{26} will vanish and the remaining ones will be (see [5]–[8]):

$$
\begin{aligned}
A_{44} &= A_{55} = \bar{A} = \sum_{k=1}^{N} (G_z)_k (z_k - z_{k+1})\,, \\
D_{11} &= D_{22} = \bar{D} = \sum_{k=1}^{N} \frac{1}{3} \left(\frac{E}{1 - v^2} \right)_k (z_k{}^3 - z_{k+1}^3)\,, \\
D_{66} &= \bar{C} = \sum_{k=1}^{N} \frac{1}{6} \left(\frac{E}{1 + v} \right)_k (z_k{}^3 - z_{k+1}^3) \quad \text{and} \quad D_{12} = \bar{D} - 2\bar{C}\,,
\end{aligned}
\tag{8}
$$

where E and v are Young's modulus and Poisson's ratio in the x-y plane, and G_z is the shear modulus in the planes normal to the x-y plane. Also, for a single-layer transversely isotropic plate the non-vanishing rigidity terms are given by:

$$\bar{A} = G_z h\,, \qquad \bar{C} = \frac{1}{24}\frac{Eh^3}{1+v}\,, \quad \text{and} \quad \bar{D} = \frac{1}{12}\frac{Eh^3}{1-v^2}\,. \tag{9}$$

Now upon substitution of (8) into (7) it is readily seen that:

$$M_{11} = \bar{D}\varkappa_1{}^0 + (\bar{D} - 2\bar{C})\,\varkappa_2{}^0\,, \tag{10.1}$$

$$M_{22} = (\bar{D} - 2\bar{C})\,\varkappa_1{}^0 + \bar{D}\varkappa_2{}^0\,, \tag{10.2}$$

$$M_{12} = \bar{C}\varkappa_6{}^0\,, \tag{10.3}$$

$$\hat{Q}_1 = K^2\bar{A}\varepsilon_5{}^0 \quad \text{and} \quad \hat{Q}_2 = K^2\bar{A}\varepsilon_4{}^0\,. \tag{10.4}$$

Substitution of relations (10) into the last three equations in (4) yields the bending equations of the plate:

$$\delta\psi_x: \quad \bar{D}\psi_{x,xx} + \bar{C}\psi_{x,yy} + (\bar{D} - \bar{C})\,\psi_{y,xy} - K^2\bar{A}(\psi_x + w_{,x}) = 0\,, \tag{11.1}$$

$$\delta\psi_y: \quad \bar{D}\psi_{y,yy} + \bar{C}\psi_{y,xx} + (\bar{D} - \bar{C})\,\psi_{x,xy} - K^2\bar{A}(\psi_y + w_{,y}) = 0\,, \tag{11.2}$$

$$\delta w: \quad K^2\bar{A}(\psi_{x,x} + \psi_{y,y}) + K^2\bar{A}\nabla^2 w + P_z(x,y) = 0\,, \tag{11.3}$$

where ∇^2 is the Laplace operator. Next by introducing a new function Φ such that:

$$\Phi = \psi_{x,y} - \psi_{y,x} \tag{12}$$

and following the procedure described in [5], the three coupled equations in (11) may be replaced by the following two uncoupled equations:

$$\bar{C}\nabla^2\Phi - K^2\bar{A}\Phi = 0\,, \tag{13.1}$$

$$\bar{D}\nabla^2\nabla^2 w = P_z - \frac{\bar{D}}{K^2\bar{A}}\,\nabla^2 P_z\,. \tag{13.2}$$

Equation (13.1) is known as the edge-zone (or boundary-layer) equation of the plate, and the function Φ is referred to here as the boundary-layer function. Also, Eq. (13.2) is called the interior equation of the plate. Upon solving these equations the rotation functions ψ_x and ψ_y can be obtained from the relations [7]:

$$\psi_x = \frac{\bar{D}}{K^2\bar{A}}\left(-\nabla^2 w_{,x} - \frac{P_{z,x}}{K^2\bar{A}}\right) + \frac{\bar{C}}{K^2\bar{A}}\,\Phi_{,y} - w_{,x} \tag{14.1}$$

and

$$\psi_y = \frac{\bar{D}}{K^2\bar{A}}\left(-\nabla^2 w_{,y} - \frac{P_{z,y}}{K^2\bar{A}}\right) - \frac{\bar{C}}{K^2\bar{A}}\,\Phi_{,x} - w_{,y}\,. \tag{14.2}$$

With w, ψ_x, and ψ_y derived, the stress components in the kth layer can be computed by using the relations:

$$\sigma_{xx} = \frac{E}{1-v^2}\,(\psi_{x,x} + v\psi_{y,y})\,z\,, \qquad \sigma_{yy} = \frac{E}{1-v^2}\,(\psi_{y,y} + v\psi_{x,x})\,z\,, \tag{15.1}$$

$$\sigma_{xy} = \frac{E}{2(1+v)}\,(\psi_{x,y} + \psi_{y,x})\,z\,, \qquad \sigma_{xz} = G_z(\psi_x + w_{,x})\,, \qquad \sigma_{yz} = G_z(\psi_y + w_{,y})\,, \tag{15.2}$$

where E, v, and G_z are the properties of that layer.

3 Application to rectangular plates

Next the bending of a rectangular plate $(a \times b)$ with simple supports at $x = 0$ and $x = a$ is considered (see Fig. 1). If the plate is subjected to a uniformly distributed load $P_z = P_0$, with $\lambda_n = n\pi/a$, the loading function may be expanded as:

$$P_z(x, y) = \sum_{n=1,3,\ldots}^{\infty} \frac{4P_0}{n\pi} \sin \lambda_n x \, . \tag{16}$$

Also, the solution representations:

$$w(x, y) = \sum_{n=1,3,\ldots}^{\infty} w_n(y) \sin \lambda_n x \, , \tag{17.1}$$

$$\Phi(x, y) = \sum_{n=1,3,\ldots}^{\infty} \Phi_n(y) \cos \lambda_n x \tag{17.2}$$

will identically satisfy the boundary conditions at $x = 0$ and $x = a$. Solution (17) is referred to as the generalized Levy solution [14]. Substitution of Eqs. (16) and (17) into Eqs. (13) yields two ordinary differential equations whose solutions, along with (17), result in:

$$w(x, y) = \sum_{n=1,3,\ldots}^{\infty} \Bigg[A_{n4} \sinh \lambda_n y + A_{n1} \cosh \lambda_n y + A_{n2} y \sinh \lambda_n y + A_{n5} y \cosh \lambda_n y + \frac{4P_0}{n\pi \bar{D} {\lambda_n}^4} \left(1 + \frac{\bar{D}}{K^2 \bar{A}} {\lambda_n}^2 \right) \Bigg] \sin \lambda_n x \tag{18.1}$$

and

$$\Phi(x, y) = \sum_{n=1,3,\ldots}^{\infty} (A_{n3} \sinh \gamma_n y + A_{n6} \cosh \gamma_n y) \cos \lambda_n x \, , \tag{18.2}$$

where

$${\gamma_n}^2 = {\lambda_n}^2 + \frac{K^2 \bar{A}}{\bar{C}} \tag{19}$$

and A_{n1} through A_{n6} are six unknown integration constants.

As far as boundary conditions at the edges at $y = \pm b/2$ are concerned, three cases will be considered. In the first case, these edges are assumed to be free, and they are denoted by F-F. In the second case, these edges are assumed to be clamped and will be denoted by C-C. Lastly, these edges are assumed to have simple supports and they are denoted by S-S. Due to the existing symmetry in the bending of the plate, the constants of integration A_{n4}, A_{n5}, and A_{n6} may be set equal to zero. Therefore all that is needed is to impose the boundary conditions at one edge of the plate, say $y = +b/2$. It should be noted that in symmetrical bending A_{n6} must be set equal to zero because, for such a bending, $\Phi(x, y) = -\Phi(x, -y)$, which can be observed from Eq. (12).

Now, in the F-F case the boundary conditions that must be imposed at $y = b/2$ are:

$$M_{22} = M_{12} = \hat{Q}_2 = 0 \quad \text{at} \quad y = \frac{b}{2} \, , \tag{20}$$

where, from Eqs. (10) and (3):

$$M_{22} = (\bar{D} - 2\bar{C})\,\psi_{x,x} + \bar{D}\psi_{y,y}\,, \tag{21.1}$$

$$M_{12} = \bar{C}(\psi_{x,y} + \psi_{y,x})\,, \tag{21.2}$$

$$\hat{Q}_2 = K^2\bar{A}(\psi_y + w_{,y})\,. \tag{21.3}$$

Imposition of the conditions (20) on the solutions (18) yields three non-homogeneous algebraic equations in the form:

$$\sum_{j=1}^{3} a_{ij}A_{nj} = b_i\,, \quad i = 1,2,3 \tag{22}$$

whose solution will yield the integration constants A_{n1}, A_{n2}, and A_{n3}. The constants a_{ij} and b_i are listed in Appendix A.

When the edges of the plate at $y = \pm b/2$ are clamped (i.e., in the C-C case), the boundary conditions will be:

$$w = \psi_x = \psi_y = 0 \quad \text{at} \quad y = +\frac{b}{2}\,. \tag{23}$$

When these conditions are imposed on the solution (18), three equations of the form (22) will again be obtained. The constants a_{ij} and b_i for this case are listed in Appendix B.

Lastly, if the edges at $y = \pm b/2$ have simple supports (i.e., in the S-S case) the boundary conditions at $y = b/2$ will be:

$$w = 0\,, \tag{24.1}$$

$$\psi_x = 0\,, \tag{24.2}$$

$$M_{22} = 0\,, \tag{24.3}$$

where the expression for ψ_x is given by Eq. (14.1) and that for M_{22} is given by Eq. (21.1). Now because of (24.1) and (24.2) it may be argued that

$$w_{,x} = w_{,xx} = \psi_{x,x} = 0 \quad \text{at} \quad y = \frac{b}{2}\,. \tag{25}$$

Substituting Eqs. (24) into (21.1) and taking (25) into account, it follows that

$$M_{22} = -\bar{D}\left(\nabla^2 w + \frac{P_z}{K^2\bar{A}}\right) = 0 \quad \text{at} \quad y = \frac{b}{2}\,. \tag{26}$$

Finally, substituting (26) and (14.1) into (24.2), it follows that

$$\Phi_{,y} = 0 \quad \text{at} \quad y = \frac{b}{2}\,. \tag{27}$$

It is now readily seen from Eqs. (27) and (18.2), with $A_{n6} = 0$, that

$$\Phi(x,y) = 0\,. \tag{28}$$

That is, in the S-S case there exists no boundary-layer effect.

4 Discussion of numerical results

To study the effects of the boundary-layer equation and the shear deformation, a single-layer transversely isotropic plate is considered here. The material properties (in either SI or British units) are assumed to be:

$$E = 5 \times 10^5\,, \qquad G_z = 0.15E\,, \qquad v = 0.25\,.$$

It is also assumed that $a = b = 50$ and $P_0 = 1$. Note that for a homogeneous isotropic plate G_z is simply replaced by G. There are two methods for obtaining numerical results within the framework of classical plate theory (CPT). In the first method the bending equation of the plate in CPT is solved. In the second method, a large value is assumed for $K^2\bar{A}$ in the results of FSDT. In fact, what then is done, it is readily seen from Eqs. (14):

$$\psi_x = -w_{,x} \quad \text{and} \quad \psi_y = -w_{,y} \tag{29}$$

which are the conditions under which plane sections remain plane and also perpendicular to the middle surface of the plate after deformation. On the other hand, from Eq. (13.2) one obtains

$$\bar{D}\nabla^2\nabla^2 w = P_z \tag{30}$$

which is the governing equation of plate according to CPT. Furthermore, by dividing Eq. (13.1) by $K^2\bar{A}$ and letting $K^2\bar{A}$ approach infinity, one obtains

$$\Phi = 0 \tag{31}$$

which correctly indicates that there is no boundary-layer phenomenon within the framework of CPT. In the present work both methods are used for obtaining the results of CPT, although, for brevity, the analysis of the first method is not presented here. The transverse displacement of the plate at $x = a/2$ and along the positive y-axis according to FSDT and CPT is shown in Fig. 2. In all the plots presented here, the C-C and F-F cases are denoted by "Clamped" and "Free," respectively. As can be seen from Fig. 2, the transverse displacement is underestimated by CPT. The variation of σ_{yy} at $z = h/2$ and along the positive y-axis is shown in Fig. 3. It is seen that at the free edge the stress component σ_{yy} vanishes. This can

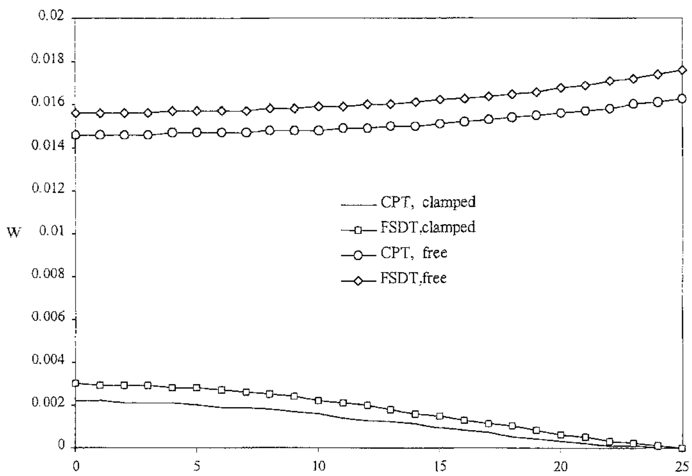

Fig. 2. Comparison of transverse displacement according to CPT and FSDT ($h = 5$)

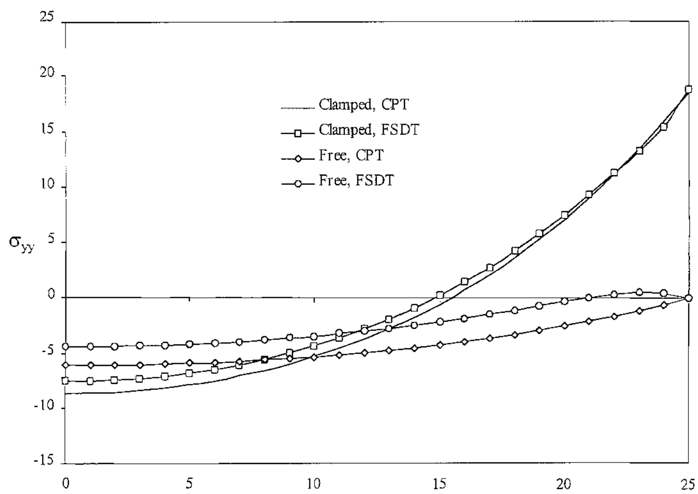

Fig. 3. Variation of σ_{yy} in the C-C and F-F cases according to CPT and FSDT ($x = a/2$ and $h = 7.5$)

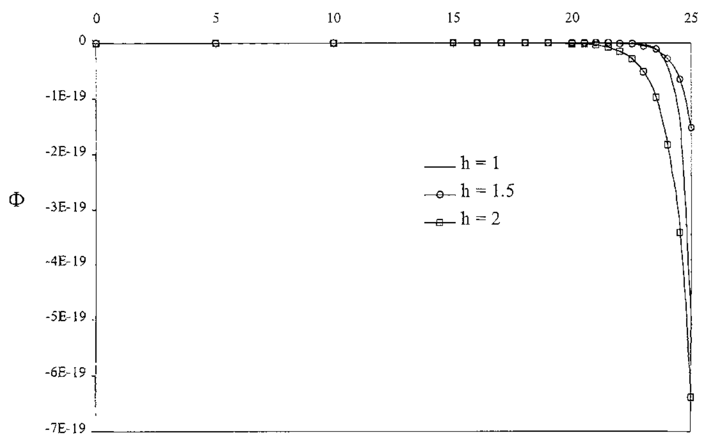

Fig. 4. Variation of the boundary-layer function along the positive y-axis in the F-F case ($x = a/2$)

also be verified analytically. One of the boundary conditions at the free edge is $M_{22} = 0$ (see Eq. (20)). For a single-layer, transversely isotropic plate, this condition from Eq. (21.1) yields

$$v\psi_{x,x} + \psi_{y,y} = 0\,. \tag{32}$$

From Eqs. (32) and (15.1) it is seen that $\sigma_{yy} = 0$ at the free edge. For a laminated, transversely isotropic plate the last two boundary conditions at the free edge in Eq. (20) are

$$\psi_{x,y} + \psi_{y,x} = 0 \quad \text{and} \quad \psi_y + w_{,y} = 0\,. \tag{33}$$

Substituting (33) into (15.2) yields also the additional results that

$$\sigma_{yz} = \sigma_{xy} = 0\,. \tag{34}$$

It is to be noted that (34) holds even for laminated plates, but σ_{yy} vanishes only at the free edge of a single-layer plate. The variation of the boundary-layer function Φ for the F-F case is shown in Fig. 4. Note that the boundary-layer effect becomes significant only near the edge zone of the plate. Furthermore, the width of the edge zone is approximately equal to the thickness of the plate. In order to study the effect of b (i.e., the length of the plate in the y-direction) on the variation of Φ, several plots are depicted in Fig. 5 for different values of b. It should be

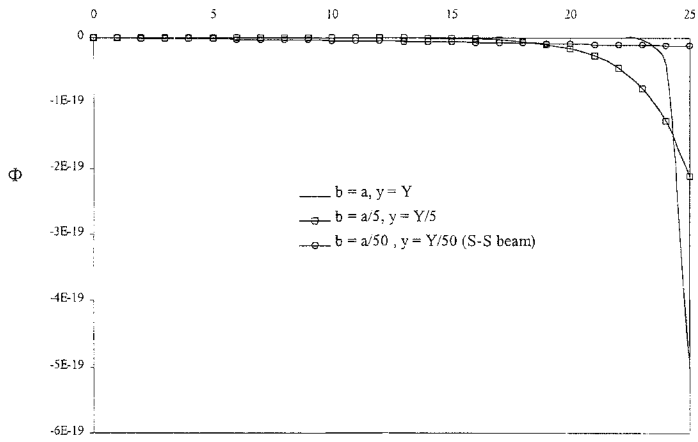

Fig. 5. The effect of the width b of the plate on the boundary-layer function, F-F case ($a = 50, x = a/2$, and $h = 1$)

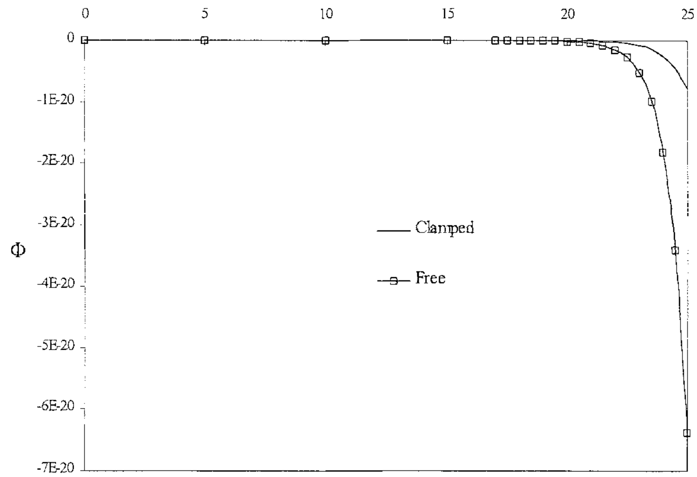

Fig. 6. The effect of boundary conditions on the magnitude of the boundary-layer function ($a = b = 50$, $x = a/2$, and $h = 2$)

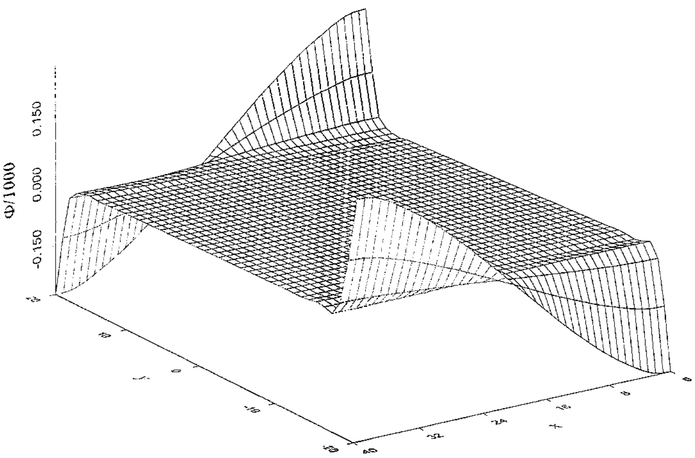

Fig. 7. A three-dimensional plot of the boundary-layer function in the C-C case ($a = b = 50$ and $h = 2$)

noted that for small values of b the plate reduces to a beam with simple supports at $x = 0$ and $x = a$. As b decreases, the boundary-layer phenomenon disappears. This disappearance explains why the boundary-layer phenomenon is not observed in the Timoshenko beam theory. The variations of Φ in the C-C and F-F cases are compared in Fig. 6. There it can be seen that the F-F case shows a stronger boundary-layer effect than the C-C case. A three-dimensional plot in the C-C case showing the variation of Φ is also presented in Fig. 7.

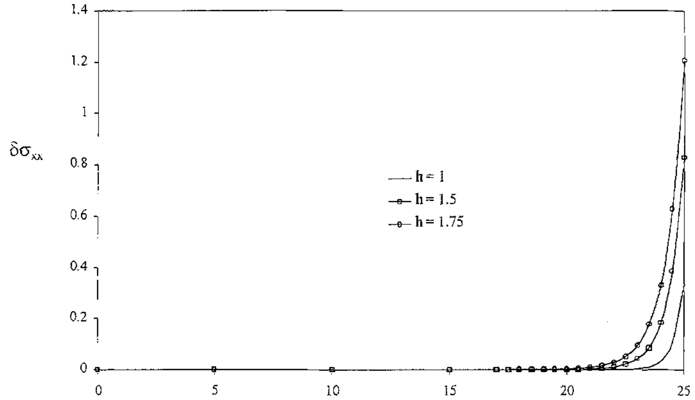

Fig. 8. Effect of the plate's thickness and boundary-layer function on $\delta\sigma_{xx}$ in the C-C case ($a = b = 50$, and $x = a/2$)

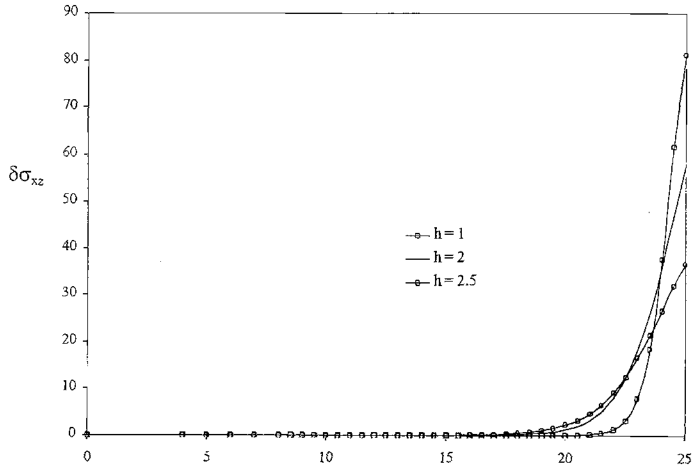

Fig. 9. Effect of the plate's thickness and boundary-layer function on $\delta\sigma_{xz}$ in the F-F case ($a = b = 50$, and $x = a/2$)

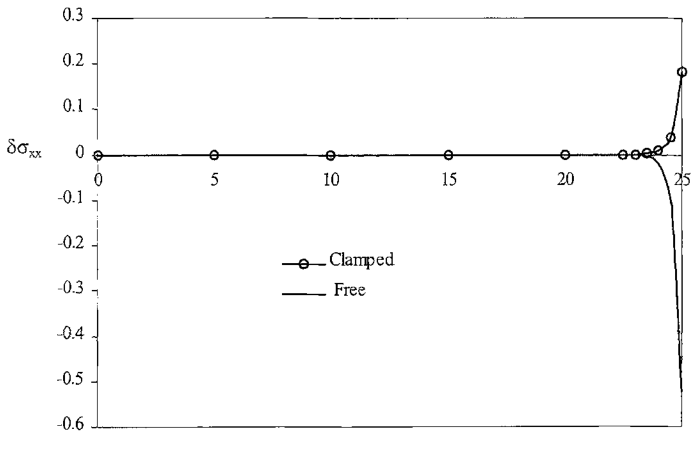

Fig. 10. Effect of the boundary conditions and boundary-layer function on $\delta\sigma_{xx}$ ($a = b = 50$, and $h = .75$)

Next, in order to study the effect of Φ on the stress components, the following quantity is introduced:

$$\delta\sigma_{xx} = \frac{\sigma_{xx} - \sigma_{xx}(\Phi = 0)}{\sigma_{xx}} \times 100\,, \tag{35}$$

where by $\sigma_{xx}(\Phi = 0)$ is meant that in calculating σ_{xx} the function Φ is set equal to zero. The variation of $\delta\sigma_{xx}$ in the C-C case is shown in Fig. 8, where the boundary-layer effect is seen to be confined to the edge zone of the plate. A similar plot is depicted for $\delta\sigma_{xz}$ in Fig. 9 for the F-F case; the width of the boundary-layer is increased in this case. Note that the definition of $\delta\sigma_{xz}$ is similar to the definition of $\delta\sigma_{xx}$. Finally, a comparison is made in Fig. 10 between the C-C and F-F cases. Here, the F-F case shows a stronger boundary-layer effect than the C-C case.

5 Conclusions

The governing equations of the first-order shear deformation theory of laminated transversely isotropic plates are recast in terms of the boundary-layer function and the transverse displacement function. Analytical expressions are obtained for the primary response quantities of rectangular plates with various boundary conditions. For simply-supported, clamped, and free edges it is shown that there exists no boundary-layer, a weak boundary-layer, and a strong boundary-layer effect, respectively. The effects of the boundary-layer function on stress components are also studied. The width of the boundary-layer is observed to be approximately equal to the thickness of the plate and to become larger at the free edges. It is further shown that when the size of a plate is reduced to that of a beam, the boundary-layer phenomenon disappears, and in doing so makes clear why the boundary-layer phenomenon is not predicted in Timoshenko's beam theory.

In the present work the numerical results pertain to a single-layer, transversely isotropic plate; however, the conclusions reached here are also valid for laminated transversely isotropic and homogeneous isotropic plates. For further research into this subject, it may be instructive to compare the results of the present work with those of Levinson's and Reddy's third-order shear deformation theories.

It is worth mentioning that we have investigated this boundary-layer phenomenon for circular sector plates and completely circular plates. We reached the same conclusions made in this article for rectangular plates. These results will appear in a future communication.

Appendix A

When the edges of the laminate at $y = \pm b/2$ are free, the constants a_{ij} and b_i are given by:

$$a_{11} = 2m\lambda_n{}^2\bar{D}\cosh\left(\frac{\lambda_n b}{2}\right)\left[(\bar{C}-\bar{D})+\lambda_n{}^2(\lambda_n{}^4-1)\right],$$

$$a_{12} = \bar{D}m\lambda_n\left[b\lambda_n(\bar{C}-\bar{D}+\lambda_n{}^6-\lambda_n{}^2)\sinh\left(\frac{\lambda_n b}{2}\right)+2(2\lambda_n{}^6-2\lambda_n{}^2+2m\lambda_n{}^2\bar{C}-\bar{D})\cosh\left(\frac{\lambda_n b}{2}\right)\right],$$

$$a_{13} = -2mp\lambda_n\gamma_n\bar{C}\bar{D}\cosh\left(\frac{\gamma_n b}{2}\right), \qquad a_{21} = -2\lambda_n{}^2\sinh\left(\frac{\lambda_n b}{2}\right),$$

$$a_{22} = -2\lambda_n(1+2m\lambda_n{}^2)\sinh\left(\frac{\lambda_n b}{2}\right) - \lambda_n{}^2 b\cosh\left(\frac{\lambda_n b}{2}\right),$$

$$a_{23} = p(\lambda_n{}^2+\gamma_n{}^2)\sinh\left(\frac{\gamma_n b}{2}\right),$$

$$a_{31} = 0, \qquad a_{32} = 2\bar{D}\lambda_n{}^2\sinh\left(\frac{\lambda_n b}{2}\right), \qquad a_{33} = -\bar{C}\sinh\left(\frac{\gamma_n b}{2}\right)\lambda_n,$$

$$b_1 = m\bar{D}\lambda_n{}^2\left[\lambda_n{}^2+2(m\lambda_n{}^2-1)\bar{C}+\bar{D}\right]K_n, \qquad b_2 = 0, \qquad b_3 = 0,$$

where

$$m = \frac{\bar{D}}{K^2 \bar{A}}, \qquad p = \frac{\bar{C}}{K^2 \bar{A}}, \qquad K_n = \frac{4P_0}{n\pi \bar{D} \lambda_n^4} \left(1 + \frac{\bar{D}}{K^2 \bar{A}} \lambda_n^2\right).$$

Appendix B

When the edges of the laminate at $y = \pm b/2$ are clamped, the constants a_{ij} and b_i are given by:

$$a_{11} = \cosh\left(\frac{\lambda_n b}{2}\right), \qquad a_{12} = \frac{b}{2} \sinh\left(\frac{\lambda_n b}{2}\right), \qquad a_{13} = 0, \qquad a_{21} = -\bar{D}\lambda_n^3 \cosh\left(\frac{\lambda_n b}{2}\right),$$

$$a_{22} = -\bar{D}\lambda_n^2 \left[2\cosh\left(\frac{\lambda_n b}{2}\right) + \frac{\lambda_n b}{2} \sinh\left(\frac{\lambda_n b}{2}\right)\right], \qquad a_{23} = \bar{C}\gamma_n \cosh\left(\frac{\gamma_n b}{2}\right),$$

$$a_{31} = \lambda_n \sinh\left(\frac{\lambda_n b}{2}\right), \qquad a_{32} = 2m\lambda_n^2 \sinh\left(\frac{\lambda_n b}{2}\right) + \sinh\left(\frac{\lambda_n b}{2}\right) + \frac{b\lambda_n}{2} \cosh\left(\frac{\lambda_n b}{2}\right),$$

$$a_{33} = -p\lambda_n \sinh\left(\frac{\gamma_n b}{2}\right), \qquad b_1 = -\frac{4P_0}{n\pi \bar{D}\lambda_n^4}\left(1 + \frac{\bar{D}}{K^2 \bar{A}} \lambda_n^2\right), \qquad b_2 = b_3 = 0,$$

where m and p are given in Appendix A.

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