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On a boundary layer phenomenon in Mindlin-Reissner plate theory for laminated circular sector plates

A. Nosier, Tehran, Iran, and **A. Yavari** and **S. Sarkani,** Washington, D.C

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Summary. In this article, the edge-zone equation of Mindlin-Reissner plate theory, for composite plates laminated of transversely isotropic layers is studied. Analytical solutions are obtained for both circular sector and completely circular plates with various boundary conditions. The boundary-layer function and its effect on the stresses are numerically studied. Effects of plate thickness and boundary conditions are investigated. The results for circular and completely circular plates are exactly the same as those of rectangular plates in our previous work. Therefore, this boundary layer phenomenon seems to be geometry independent.

1 Introduction

The classical plate theory put forward by Kirchhoff [1] neglects transverse shear strains and results in a fourth-order equation for the plate transverse deflection. Kirchhoff's theory is plagued by inconsistency between the order of the governing equation and the number of boundary conditions (Stoker [2]). The physical intuition leads one to expect three edge conditions and thus to expect a sixth-order rather than a fourth-order differential equation to govern the plate problem. As a result of attempts to solve this paradox and also because of the inadequacy of classical theory when it is applied in such cases as laminated composites, where shear deformations are not negligible, several shear deformation theories have been proposed to date.

The simplest shear deformation theory is the first-order shear deformation theory. Depending on whether or not the variation of displacement components or stress components with respect to the thickness coordinate is assumed to be known a priori, first-order theories are categorized into two groups: displacement-based or stress-based theories. The displacement-based theory was introduced by Mindlin [3]. Stress-based theories generally take one of the two approaches. In the approach first described by Reissner [4], assumptions are made about the variation of in-plane components of stress, while in the approach attributed to Ambartsumyan [5] the variation of the transverse stress components with respect to the plate thickness coordinate is assumed to be known at the outset. Both displacement-based and stress-based first-order theories result in a system of three differential equations in terms of three dependent variables with a total degree of six. Reissner [4], [6], [7], and [8] was the first to determine that his sixth-order equations can be uncoupled into two equations for a homogeneous isotropic plate. He called these uncoupled equations edge-zone and interior equations. The solution of the edge-zone equation has boundary layer characteristics. For an introduction to asymptotic phenomena in mathematical physics and classical plate theory, the reader may refer to Friedrichs [9] and Schneider [10].

Nosier and Reddy [11] showed that the bending equations of many refined linear theories of symmetric laminated plates made of transversely isotropic layers can be uncoupled into two equations, one in terms of the transverse displacement, w, and the other in terms of a potential function, Φ, called the boundary layer function. They demonstrated that all displacement-based theories except Reddy's [12] can be uncoupled to form a fourth-order interior equation and a second-order edge-zone equation. However, the eighth-order bending equations of Reddy [12], when uncoupled result in a sixth-order interior equation and a second-order edge-zone equation. They also showed that for a simply supported plate the edge-zone solution is identically zero if the rotatory inertia terms are ignored.

Nosier and Reddy [13] considered the tenth-order equations of the nonlinear first-order shear deformation plate theory [14] and the twelfth-order nonlinear equations of the third-order plate theory of Reddy [15] and showed that, for symmetric laminated composites with transversely isotropic layers, the edge-zone equations of both plate theories are second-order equations similar to the one obtained in [11]. Again they demonstrated that, for a simply-supported plate with arbitrary in-plane boundary conditions, and in the absence of rotatory inertia, Φ is identically zero.

Nosier et al. [16] studied the boundary layer function Φ in bending of rectangular Mindlin-Reissner plates with two opposite edges simply-supported. They considered simply-supported, clamped, and free boundary conditions for two other opposite edges of plates. They demonstrated that for simply-supported, clamped, and free edges there exists no boundary layer, a weak boundary layer, and a strong boundary layer effect. They showed that the boundary layer function has boundary layer effects on the stress components. They also observed that the width of the boundary layer function is approximately equal to the thickness of the plate and is larger for free edges. For explaning why the boundary layer phenomenon is not observed in Timoshenko's beam theory, they showed that when the size of a rectangular plate is reduced to that of a beam, the boundary layer phenomenon disappears.

This article utilizes the linear theory to consider the bending problem of a laminated circular sector and completely circular plates with various boundary conditions and demonstrates analytically the contribution to the solution made by the edge-zone or boundary layer equation. All the investigations that were done in [16] for rectangular plates are performed for a circular sector and completely circular plates. It is demonstrated that the boundary-layer function Φ has similar effects in both cases.

2 Governing equations

According to the first-order shear deformation theory (FSDT) in polar coordinates, the bending equations of a plate are obtained from the following displacement fields (see [17]):

$$
\begin{aligned}
u_r(r,\theta,z) &= u(r,\theta) + z\psi_r(r,\theta)\,,\\
u_\theta(r,\theta,z) &= v(r,\theta) + z\psi_\theta(r,\theta)\,, \qquad (1)\\
u_z(r,\theta,z) &= w(r,\theta)\,,
\end{aligned}
$$

where z is the thickness coordinate; u and v are, respectively, the displacements of the middle surface of the plate in the r- and θ-directions; and ψ_r and ψ_θ are known as the rotation functions [18]. Using the strain-displacement relations in polar coordinates [19], it may be shown that:

$$\varepsilon_{rr} = \varepsilon_1{}^0 + z\varkappa_1{}^0\,, \qquad \varepsilon_{\theta\theta} = \varepsilon_2{}^0 + z\varkappa_2{}^0\,, \qquad \varepsilon_{zz} = 0\,,$$
$$2\varepsilon_{r\theta} = \varepsilon_6{}^0 + z\varkappa_6{}^0\,, \qquad 2\varepsilon_{rz} = \varepsilon_5{}^0 + z\varkappa_5{}^0\,, \qquad 2\varepsilon_{\theta z} = \varepsilon_4{}^0 + z\varkappa_4{}^0\,, \tag{2}$$

where

$$\varepsilon_1{}^0 = u_{,r}\,, \qquad \varepsilon_2{}^0 = \frac{1}{r}\,(u + v_{,\theta})\,, \qquad \varepsilon_4{}^0 = \frac{1}{r}\,w_{,\theta} + \psi_\theta\,,$$
$$\varepsilon_5{}^0 = w_{,r} + \psi_r\,, \qquad \varepsilon_6{}^0 = \frac{1}{r}\,u_{,\theta} + v_{,r} - \frac{1}{r}\,v\,, \qquad \varkappa_1{}^0 = \psi_{r,r}\,, \tag{3}$$
$$\varkappa_2{}^0 = \frac{1}{r}\,(\psi_r + \psi_{\theta,\theta})\,, \qquad \varkappa_4{}^0 = \varkappa_5{}^0 = 0\,, \qquad \varkappa_6{}^0 = \frac{1}{r}\,\psi_{r,\theta} + \psi_{\theta,r} - \frac{1}{r}\,\psi_\theta\,.$$

In relation (3), a comma followed by a variable denotes partial differentiation with respect to that variable. Using (3) in the integral principle of equilibrium (i.e., the principle of minimum total potential energy) yields the equilibrium equations of the plate and the appropriate boundary conditions at the edges of the plate [19]:

$$\delta u \;:\quad N_{rr,r} + \frac{1}{r}\,N_{r\theta,\theta} + \frac{1}{r}\,(N_{rr} - N_{\theta\theta}) = 0\,, \tag{4.1}$$

$$\delta v \;:\quad N_{r\theta,r} + \frac{2}{r}\,N_{r\theta} + \frac{1}{r}\,N_{\theta\theta,\theta} = 0\,, \tag{4.2}$$

$$\delta\psi_r :\quad M_{rr,r} + \frac{1}{r}\,M_{r\theta,\theta} + \frac{1}{r}\,(M_{rr} - M_{\theta\theta}) - \hat{Q}_r = 0\,, \tag{4.3}$$

$$\delta\psi_\theta :\quad M_{r\theta,r} + \frac{2}{r}\,M_{r\theta} + \frac{1}{r}\,M_{\theta\theta,\theta} - \hat{Q}_\theta = 0\,, \tag{4.4}$$

$$\delta w \;:\quad \hat{Q}_{r,r} + \frac{1}{r}\,\hat{Q}_r + \frac{1}{r}\,\hat{Q}_{\theta,\theta} + P_z(r,\theta) = 0\,, \tag{4.5}$$

where δ represents the variational symbol and P_z denotes the transverse load. The stress and moment resultants in Eqs. (4) are defined as follows:

$$(N_{rr}, N_{\theta\theta}, N_{r\theta}) = \int_{-h/2}^{h/2} (\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})\,dz\,,$$
$$(M_{rr}, M_{\theta\theta}, M_{r\theta}) = \int_{-h/2}^{h/2} (\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})\,z dz\,, \tag{5}$$
$$(\hat{Q}_r, \hat{Q}_\theta) = K^2 \int_{-h/2}^{h/2} (\sigma_{rz}, \sigma_{\theta z})\,dz\,,$$

where h is the total thickness of the plate and K^2 is a shear correction factor, which is taken here to be equal to 5/6. At any edge of the plate, with a normal $\vec{n} = n_r\vec{e}_r + n_\theta\vec{e}_\theta$, the boundary conditions require the specification of [19]:

$$\text{either}\quad \delta u = 0 \quad \text{or} \quad N_{rr}n_r + N_{r\theta}n_\theta = 0\,, \tag{6.1}$$

$$\text{either}\quad \delta v = 0 \quad \text{or} \quad N_{r\theta}n_r + N_{\theta\theta}n_\theta = 0\,, \tag{6.2}$$

$$\text{either}\quad \delta\psi_r = 0 \quad \text{or} \quad M_{rr}n_r + M_{r\theta}n_\theta = 0\,, \tag{6.3}$$

$$\text{either}\quad \delta\psi_\theta = 0 \quad \text{or} \quad M_{r\theta}n_r + M_{\theta\theta}n_\theta = 0\,, \tag{6.4}$$

$$\text{either}\quad \delta w = 0 \quad \text{or} \quad \hat{Q}_r n_r + \hat{Q}_\theta n_\theta = 0\,. \tag{6.5}$$

When the plate is symmetrically laminated with respect to its middle surface, the first two equations in (4) will be uncoupled from the last three equations. The former equations are then known as the stretching equations of the plate and the latter ones are known as the bending equations of the plate. Furthermore, if each layer (or lamina) is made of a transversely isotropic material, with the plane of isotropy parallel to the middle surface, then it may be shown that:

$$M_{rr} = \bar{D}\varkappa_1{}^0 + (\bar{D} - 2\bar{C})\varkappa_2{}^0\,, \qquad M_{\theta\theta} = (\bar{D} - 2\bar{C})\varkappa_1{}^0 + \bar{D}\varkappa_2{}^0\,,$$
$$M_{r\theta} = \bar{C}\varkappa_6{}^0\,, \qquad \hat{Q}_r = K^2\bar{A}\varepsilon_5{}^0\,, \qquad \hat{Q}_\theta = K^2\bar{A}\varepsilon_4{}^0\,, \tag{7}$$

where the rigidity terms in (7) are defined as:

$$\bar{A} = \sum_{k=1}^{N} (G_z)_k\,(z_k - z_{k+1})\,,$$
$$\bar{C} = \sum_{k=1}^{N} \frac{1}{6}\left(\frac{E}{1+v}\right)_k (z_k{}^3 - z_{k+1}^3)\,, \tag{8}$$
$$\bar{D} = \sum_{k=1}^{N} \frac{1}{3}\left(\frac{E}{1-v^2}\right)_k (z_k{}^3 - z_{k+1}^3)\,,$$

where N is the total number of layers, E and ν are Young's modulus and Poisson's ratio in the plane of isotropy (i.e., the r-θ plane), and G_z is the shear modulus in the plane normal to the plane of isotropy. Substitution of relations (7) into Eqs. (4.3) through (4.5) yields the governing equations of the plate in bending:

$$\delta\psi_r : \bar{D}\left(\psi_{r,rr} + \frac{1}{r}\,\psi_{r,r} - \frac{1}{r^2}\,\psi_r + \frac{1}{r}\,\psi_{\theta,r\theta} - \frac{1}{r^2}\,\psi_{\theta,\theta}\right) + \bar{C}\left(\frac{1}{r^2}\,\psi_{r,\theta\theta} - \frac{1}{r^2}\,\psi_{\theta,\theta} - \frac{1}{r}\,\psi_{\theta,r\theta}\right)$$
$$- K^2\bar{A}\,(\psi_r + w_{,r}) = 0\,, \tag{9.1}$$

$$\delta\psi_\theta : \bar{D}\left(\frac{1}{r}\,\psi_{r,r\theta} + \frac{1}{r^2}\,\psi_{r,\theta} + \frac{1}{r^2}\,\psi_{\theta,\theta\theta}\right) + \bar{C}\left(\frac{1}{r^2}\,\psi_{r,\theta} - \frac{1}{r^2}\,\psi_\theta + \frac{1}{r}\,\psi_{\theta,r} - \frac{1}{r}\,\psi_{r,r\theta} + \psi_{\theta,rr}\right)$$
$$- K^2\bar{A}\left(\psi_\theta + \frac{1}{r}\,w_{,\theta}\right) = 0\,, \tag{9.2}$$

$$\delta w : K^2\bar{A}\left(\psi_{r,r} + \frac{1}{r}\,\psi_r + \frac{1}{r}\,\psi_{\theta,\theta} + w_{,rr} + \frac{1}{r}\,w_{,r} + \frac{1}{r^2}\,w_{,\theta\theta}\right) + P_z = 0\,. \tag{9.3}$$

These are three coupled equations in terms of ψ_r, ψ_θ, and w with a total order of six. Now, by introducing a new function Φ, which will be referred to as the boundary-layer function, such that

$$\Phi = \frac{1}{r}\,\psi_{r,\theta} - \psi_{\theta,r} - \frac{1}{r}\,\psi_\theta \tag{10}$$

and following a procedure as in [11], the bending equations (9) may be recast to yield two uncoupled equations as follows:

$$\bar{C}\nabla^2\Phi - K^2\bar{A}\Phi = 0\,, \tag{11.1}$$

$$\bar{D}\nabla^2\nabla^2 w = P_z - \frac{\bar{D}}{K^2\bar{A}}\,\nabla^2 P_z\,, \tag{11.2}$$

where ∇^2 is the two-dimensional Laplace operator in polar coordinates. Equations (11.1) and (11.2) are known as the edge-zone (or boundary layer) and interior equations of the plate, respectively ([11], [13], [20], and [21]). Also it may be shown that (see [20]):

$$\psi_r = -w_{,r} - \frac{\bar{D}}{K^2\bar{A}} \nabla^2 w_{,r} + \frac{\bar{C}}{K^2\bar{A}} \frac{1}{r} \Phi_{,\theta} - \frac{\bar{D}}{(K^2\bar{A})^2} P_{z,r} , \tag{12.1}$$

$$\psi_\theta = -\frac{1}{r} w_{,\theta} - \frac{\bar{D}}{K^2\bar{A}} \frac{1}{r} \nabla^2 w_{,\theta} - \frac{\bar{C}}{K^2\bar{A}} \Phi_{,r} - \frac{\bar{D}}{(K^2\bar{A})^2} \frac{1}{r} P_{z,\theta} . \tag{12.2}$$

Clearly the boundary conditions corresponding to Eqs. (9) are given by Eqs. (6.3), (6.4), and (6.5).

3 Circular sector plates

The bending of a laminated plate in the form of a sector subjected to a uniformly distributed load $P_z(= P_0)$ will be studied in this section (see. Fig. 1a). To this end, it is assumed that the edges at $\theta = 0$ and $\theta = \theta_0$ have simple supports with boundary conditions (see Eqs. (6.3), (6.4), and (6.5)):

$$\psi_r = M_{\theta\theta} = w = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \theta_0 , \tag{13}$$

where $M_{\theta\theta}$ is given, from Eq. (7), by:

$$M_{\theta\theta} = (\bar{D} - 2\bar{C})\, \psi_{r,r} + \frac{\bar{D}}{r} (\psi_r + \psi_{\theta,\theta}) . \tag{14}$$

Since at these two edges $\psi_r = \psi_{r,r} = 0$, it can be concluded from Eq. (14) that $\psi_{\theta,\theta} = 0$. Therefore, the boundary conditions in Eq. (13) are equivalent to:

$$\psi_r = \psi_{\theta,\theta} = w = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \theta_0 . \tag{15}$$

As far as Eqs. (11) are concerned the boundary conditions in Eq. (15) must be restated in terms of Φ and w. With the help of Eqs. (15), (12.1), and (9.3) it can be shown that these conditions are:

$$\Phi_{,\theta} = \frac{1}{r^2} w_{,\theta\theta} + \frac{P_z}{K^2\bar{A}} = w = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \theta_0 . \tag{16}$$

Next with $\beta_n = n\pi/\theta_0$, the uniformly distributed load may be represented as:

$$P_z(r, \theta) = \sum_{n=1,3,\ldots}^{\infty} \frac{4P_0}{n\pi} \sin \beta_n \theta . \tag{17}$$

It is also seen that the solution representations

$$\Phi(r, \theta) = \sum_{n=1,3,\ldots}^{\infty} \Phi_n(r) \cos \beta_n \theta , \tag{18.1}$$

$$w(r, \theta) = \sum_{n=1,3,\ldots}^{\infty} w_n(r) \sin \beta_n \theta \tag{18.2}$$

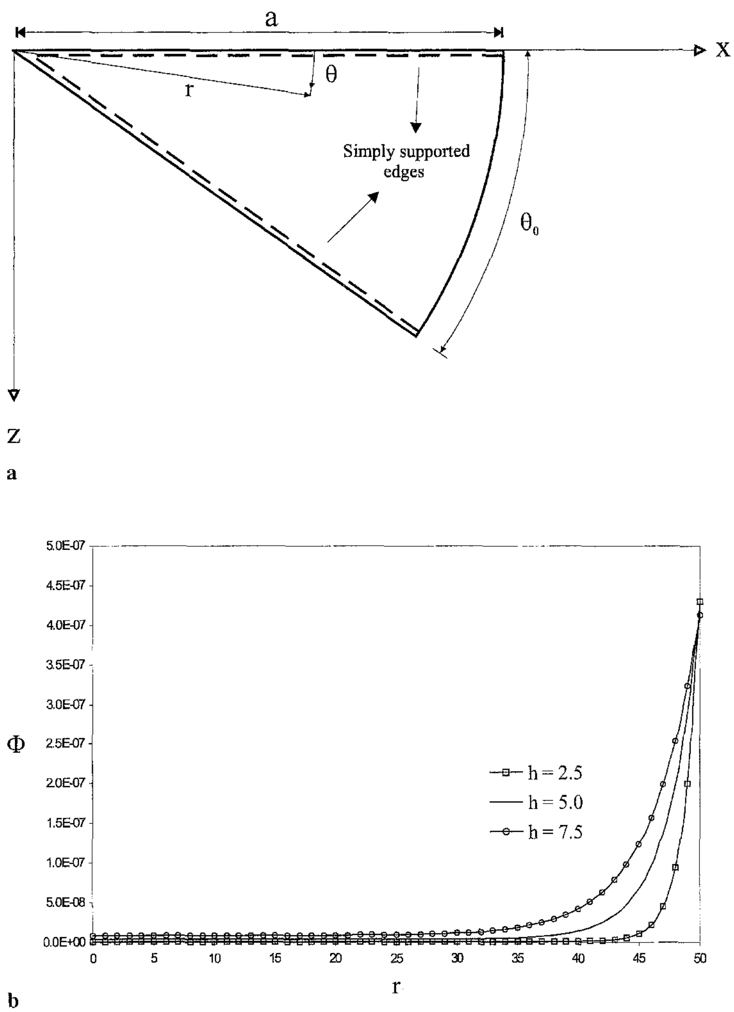

Fig. 1. **a** A circular sector plate with simple supports at $\theta = 0$ and $\theta = \theta_0$; **b** Variation of the boundary layer function in a clamped circular plate ($\theta_0 = 2\pi$ and $\theta = \pi/2$)

satisfy identically the boundary conditions at $\theta = 0$ and $\theta = \theta_0$. Substitution of Eq. (18.1) into Eq. (11.1) yields:

$$r^2 \frac{d^2\Phi_n(r)}{dr^2} + r\frac{d\Phi_n(r)}{dr} - \left({\beta_n}^2 + \frac{K^2\bar{A}}{\bar{C}} r^2\right)\Phi_n(r) = 0\,, \tag{19}$$

which is the modified Bessel equation with the general solution:

$$\Phi_n(r) = C_{n1} I_{\beta_n}(\mu r) + C_{n2} K_{\beta_n}(\mu r)\,, \tag{20}$$

where $I_{\beta n}$ and $K_{\beta n}$ are the modified Bessel functions of the first and second kinds, respectively, and

$$\mu^2 = \frac{K^2\bar{A}}{\bar{C}}\,. \tag{21}$$

Since Φ must be finite at $r = 0$, it should be concluded that $C_{n2} = 0$. Thus:

$$\Phi_n(r) = C_{n1} I_{\beta_n}(\mu r)\,. \tag{22}$$

Also, substitution of Eqs. (17) and (18.2) into (11.2) yields:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{{\beta_n}^2}{r^2}\right)\left(\frac{d^2 w_n}{dr^2}+\frac{1}{r}\frac{dw_n}{dr}-\frac{{\beta_n}^2}{r^2}w_n\right)=\frac{4P_0}{n\pi\bar{D}}\,,\quad n=1,3,\ldots \tag{23}$$

whose general solution may be represented as

$$w_n(r) = w_c(r) + w_p(r)\,, \tag{24}$$

where w_c and w_p represent, respectively, the complementary and particular solutions of Eq. (23). When ${\beta_n}^2 \neq 4$, ${\beta_n}^2 \neq 16$, and $\theta_0 < 2\pi$, these solutions are given by:

$$w_c(r) = C_{n3} r^{\beta_n} + C_{n4} r^{-\beta_n} + C_{n5} r^{\beta_n+2} + C_{n6} r^{-\beta_n+2}\,, \tag{25.1}$$

$$w_p(r) = C r^4\,, \tag{25.2}$$

where

$$C = \frac{\lambda^2}{({\beta_n}^2-4)({\beta_n}^2-16)} \quad \text{and} \quad \lambda^2 = \frac{4P_0}{n\pi\bar{D}}\,. \tag{26}$$

When ${\beta_n}^2 = 4$ or ${\beta_n}^2 = 16$, relation (25.1) still remains valid but the particular solution will be given by:

$$w_p = -\frac{\lambda^2}{96} r^4 \ln r \quad \text{when} \quad \theta_0 = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\,, \tag{27.1}$$

$$w_p = \frac{\lambda^2}{48} r^4 \ln r \quad \text{when} \quad \theta_0 = \frac{2\pi}{4}, \frac{6\pi}{4}\,. \tag{27.2}$$

For $\theta_0 = \pi$, w_{c1} is given by:

$$w_{c1}(r) = C_{13} r + C_{14} r^{-1} + C_{15} r^3 + C_{16} r \ln r\,. \tag{28}$$

And for $n \geq 2$ Eq. (25.1) still holds. Also, since w and Q_r must be finite at $r = 0$, it is concluded that $C_{n4} = C_{n6} = 0$. The remaining unknown constants of integration C_{n1}, C_{n3}, and C_{n5} must be determined by imposing three boundary conditions at $r = a$. For example, when the edge of the plate at $r = a$ is simply supported, the following conditions are imposed:

$$w = M_{rr} = \psi_\theta = 0 \quad \text{at} \quad r = a\,. \tag{29}$$

Alternatively, in terms of w and Φ these conditions may be shown to be:

$$w = w_{,rr} + \frac{P_z}{K^2 A} = \Phi_{,r} = 0 \quad \text{at} \quad r = a\,. \tag{30}$$

In order to impose the boundary conditions in (29), the general solutions for ψ_r and ψ_θ must be known. These solutions, on the other hand, are readily obtained by substituting the general solutions of w and Φ into Eqs. (12). In either case, whether the boundary conditions in (29) or in (30) are imposed, a set of three nonhomogeneous algebraic equations of the form:

$$\sum_{j=1}^{3} a_{ij} C_{n\,2j-1} = b_i\,, \quad i = 1,2,3 \tag{31}$$

will be obtained whose solutions yield the integration constants C_{n1}, C_{n3}, C_{n5}. Here, however, it should be noted that the imposition of the last condition in (30) yields:

$$\Phi(r,\theta) = 0\,. \tag{32}$$

That is, there is no boundary layer effect when the sector plate is completely simply supported. This is exactly what was found in [16] for rectangular plates. When the edge at $r = a$ is clamped, the following boundary conditions will be imposed:

$$w = \psi_r = \psi_\theta \quad \text{at} \quad r = a\,, \tag{33}$$

where, again, the expressions for the rotation functions ψ_r and ψ_θ are obtained from Eqs. (12). When the boundary conditions in (33) are imposed, three relations will be obtained as was done in (31) whose solution will yield the integration constants. The expressions for a_{ij} and b_i are given in Appendix A. Finally, when the sector plate has a free edge at $r = a$, the boundary conditions will be:

$$M_{rr} = \hat{Q}_r = M_{r\theta} = 0 \quad \text{at} \quad r = a \tag{34}$$

with M_{rr}, $\hat{Q}_r$, and $M_{r\theta}$ being given in (7). The constants a_{ij} and b_i appearing in Eqs. (31) are given in Appendix B. Once the expressions for w, ψ_r, and ψ_θ have been obtained, the stress components in any layer of the plate at (r, θ, z) are obtained from the relations:

$$\begin{aligned} \sigma_{rr} &= \frac{E}{1-v^2}\,(\varkappa_1{}^0 + v\varkappa_2{}^0)z\,, \qquad \sigma_{\theta\theta} = \frac{E}{1-v^2}\,(\varkappa_2{}^0 + v\varkappa_1{}^0)z\,, \\ \sigma_{r\theta} &= \frac{E}{2(1+v)}\,\varkappa_6{}^0 z\,, \qquad \sigma_{rz} = G_z\varepsilon_5{}^0\,, \qquad \sigma_{\theta z} = G_z\varepsilon_4{}^0\,, \end{aligned} \tag{35}$$

where E and v are, respectively, Young's modulus and Poisson's ratio in the $r-\theta$ plane, and G_z is the shear modulus in the plane normal to the $r-\theta$ plane.

4 Completely circular plates

Here a completely circular plate (i.e., $\theta_0 = 2\pi$) is considered. If the plate is subjected to a uniformly distributed load $P_z(= P_0)$, then the bending will be axisymmetric in which case it may readily be shown that:

$$w(r,\theta) = w(r) = A_1 + A_2 r^2 + \frac{P_0 r^4}{64\bar{D}}\,, \tag{36.1}$$

$$\Phi(r,\theta) = \Phi(r) = A_3\left[1 + \sum_{n=2,4,\ldots}^{\infty} \left(\frac{K^2\bar{A}}{n^2\bar{C}}\right)^{n/2} r^n\right], \tag{36.2}$$

where A_1, A_2, and A_3 are three unknown constants of integration. Also, from Eq. (12.2) it is seen that:

$$\psi_\theta = -\frac{\bar{C}}{K^2\bar{A}}\,\Phi_{,r}\,. \tag{37}$$

Now, when the edge of the plate at $r = a$ is either simply supported or clamped, the conditions

$$\psi_\theta = 0 \quad \text{at} \quad r = a \tag{38}$$

will be one of the boundary conditions. It is clear from Eqs. (36.2) and (37) that the imposition of the boundary conditions in (38) will yield $A_3 = 0$; that is:

$$\Phi(r, \theta) = \Phi(r) = 0\,. \tag{39}$$

Therefore, in the axisymmetric bending problem there exists no boundary layer effect.

If the bending of the plate is no longer axisymmetric, the first boundary condition in (16) still holds when the plate is simply supported at the edge $r = a$. With this conditions it may be shown that, regardless of the loading function $P_z(r, \theta)$, the boundary-layer function $\Phi(r, \theta)$ is identically equal to zero. Also, a free boundary at $r = a$ is not admissible because the plate will not be globally in equilibrium. Thus, there remains only the case where the plate is clamped at $r = a$. In order to study the effect of the boundary-layer function on the response quantities of the plate it is assumed that:

$$P_z(r, \theta) = \frac{P_0 r}{a} \cos\theta\,. \tag{40}$$

That is, a load that has a linear variation in the r-direction and a sinusoidal in the θ-direction is assumed. With the loading function in (40), it may be shown that the general solutions of Eqs. (11) are given by:

$$\Phi(r, \theta) = C_1 I_1(\mu r) \sin\theta\,, \tag{41.1}$$

$$w(r, \theta) = \left(C_2 r + C_3 r^3 + \frac{P_0 r^5}{192 a \bar{D}}\right) \cos\theta\,, \tag{41.2}$$

where C_1, C_2, and C_3 are three unknown constants of integration. By imposing the boundary conditions (33), these constants are found to be:

$$C_1 = \frac{P_0 \bar{D}}{a\mu K^2 \bar{A} \bar{C} I_1'(\mu a)}\,, \qquad C_2 = \frac{(3a^2 + 8m_1)D_1 - a^2 D_2}{D_3}\,, \qquad C_3 = \frac{D_2 - D_1}{D_3}\,, \tag{42.1}$$

where

$$D_1 = -\frac{P_0 a^3}{192 \bar{D}}\,, \qquad D_2 = -\frac{5 P_0 a^3}{192 \bar{D}} - \frac{3 m_1 P_0 a}{8 \bar{D}} - \frac{m_1 P_0}{a} + \frac{m_1 P_0 I_1(\mu a)}{\mu a^2 I_1'(\mu a)}\,,$$
$$D_3 = 2a^2 + 8m_1\,, \qquad m_1 = \frac{\bar{D}}{(K^2 \bar{A})^2}\,, \tag{42.2}$$

and I_1' denotes the derivative of I_1 with respect to r. With $w(r, \theta)$ and $\Phi(r, \theta)$ known, the rotation functions ψ_r and ψ_θ are obtained from the relations (12).

5 Numerical results

In our numerical examples, the laminated plate is assumed to be a single-layer transversely isotropic plate with the following properties:

$$E = 5 \times 10^5\,, \quad G_z = 0.15E\,, \quad v = 0.25\,. \tag{43}$$

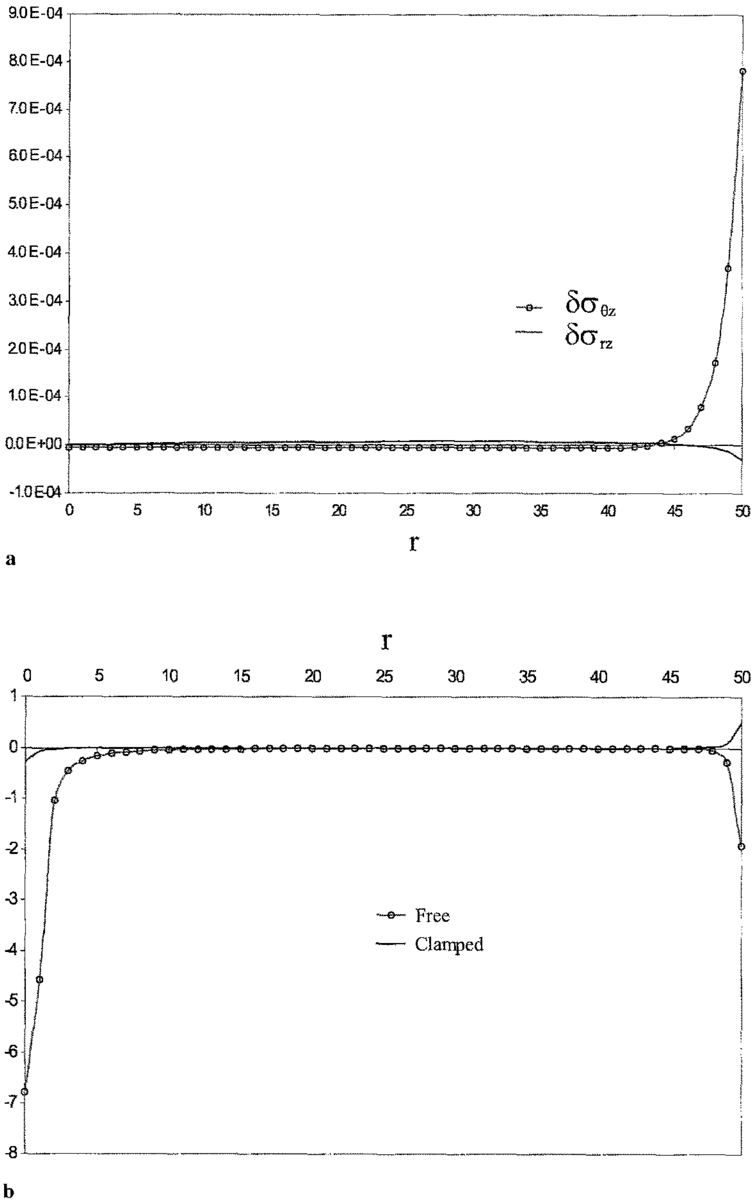

Fig. 2. **a** Effect of boundary layer function on the interlaminar stresses in a clamped circular plate ($h = 1$, $\theta_0 = 2\pi$, and $\theta = \pi/4$); **b** Effect of boundary condition at $r = a$ on the stress components $\sigma_{\theta Z}$ in a circular sector plate ($h = 1$, $\theta_0 = \pi/3$, and $\theta = \pi/6$)

Also, it is assumed that $K^2 = 5/6$, $a = 50$, and $P_0 = 1$. It should be noted that the units must be consistent. For example, if E is given in Psi, then a will be in inches.

The variation of Φ at $\theta = \pi/2$ in a completely circular plate with a clamped edge and subjected to the loading represented in (40) is depicted in Fig. 1b. By assuming different values for the thickness of the plate, it is seen that the boundary layer effect becomes significant only in the edge zone of the plate. This is why Φ is called boundary-layer function. It is seen that the width of this boundary layer is approximately equal to the plate thickness. This is exactly what was observed in [16] for rectangular plates. This conclusion is correct for circular sector plates as well. But, for the sake of brevity, we do not present any more numerical results.

Next, the effect of the boundary layer funtion Φ on the stress components is studied. To this end, the following quantity is defined:

$$\delta\sigma_{\theta z} = \frac{\sigma_{\theta z} - \sigma_{\theta z}(\Phi = 0)}{\sigma_{\theta z}} \times 100\,, \tag{44}$$

where by $\sigma_{\theta z}(\Phi = 0)$ it is meant that in calculating $\sigma_{\theta z}$ the function Φ is set equal to zero. The quantities $\delta\sigma_{\theta\theta}$, $\delta\sigma_{rr}$, $\delta\sigma_{r\theta}$, and $\delta\sigma_{rz}$ are defined similar to the definition shown in (44). The variations of $\delta\sigma_{rz}$ and $\delta\sigma_{\theta z}$ in a clamped completly circular plate with the loading function as in Eq. (40) are shown in Fig. 2a. It is seen that the effect of Φ is confined to the edge zone of the plate. Therefore, Φ has a boundary layer effect on the stresses. The effect made by the boundary conditions may be studied with the help of Fig. 2b. It is seen that for a clamped sector plate the boundary layer effect is weaker than it is when the edge is free.

6 Conclusions

The bending problem for laminated circular sector plates and completely circular plates is considered in the present work. It is assumed that each lamina is made of a transversely isotropic material. The theory considered is called the first-order shear deformation plate theory (also known as the Mindlin-Reissner plate theory). Analytical expressions are obtained for primary response quantities of laminated circular sector plates and completely circular plates with various boundary conditions and loading functions. It is analytically shown that when the edge of the completely circular plate and the edge of circular sector plates are simply supported, the boundary layer effect disappears. Numerical results indicate that the boundary-layer function and its influence on the stress components are confined to the edge zone of the plate. It is also seen that the boundary layer effect is stronger in the presence of a free edge than it is near a clamped edge. It is seen that all the characteristics of Φ and its influence on the stresses in circular sector plates are exactly the same as those in rectangular plates studied in [16]. Therefore, it can be concluded that the boundary layer characteristics of Φ are geometry independent.

The numerical calculations consider a single-layer plate with transversely isotropic material; however, the conclusions drawn here are also valid both for isotropic plates and for laminated transversely isotropic plates.

Appendix A

The constants a_{ij} and b_i appearing in Eq. (31) when the edge of the circular sector at $r = a$ is clamped are:

$$a_{11} = 0\,, \qquad a_{12} = 1\,, \qquad a_{13} = a^2\,,$$

$$a_{21} = \frac{\bar{C}}{K^2\bar{A}}\,\frac{\beta_n}{a}\,I_{\beta_n}(\mu a)\,, \qquad a_{22} = \beta_n a^{n-1}\,, \qquad a_{23} = a^{\beta_n - 1}\left[4\beta_n(\beta_n + 1)\,\frac{\bar{D}}{K^2\bar{A}} + a^2(\beta_n + 2)\right],$$

$$a_{31} = \frac{\bar{C}}{K^2\bar{A}}\,\mu I'_{\beta_n}(\mu a)\,, \qquad a_{32} = \beta_n a^{\beta_n - 1}\,, \qquad a_{33} = \beta_n a^{\beta_n - 1}\left[a^2 + 4(\beta_n + 1)\frac{\bar{D}}{K^2\bar{A}}\right],$$

$$b_1 = -Ca^{4-\beta_n}\,, \qquad b_2 = 2Ca\left[\frac{\bar{D}}{K^2\bar{A}}\,(\beta_n{}^2 - 16) - 2a^2\right], \qquad b_3 = a\beta_n C\left[\frac{\bar{D}}{K^2\bar{A}}\,(\beta_n{}^2 - 16) - a^2\right],$$

where $\beta_n = n\pi/\theta_0$ and the constant C is as defined in Eq. (26). For the special values of the sector angle θ_0 discussed in Eq. (27) the a_{ij}'s remain the same as in the above; however, the constant b_i's are given by:

$$b_1 = -Da^{4-\beta_n}\ln a\,,$$

$$b_2 = 2aD\left[\beta_n{}^2 m_1 - 2(8m_1 + a^2)\right]\ln a + aD(\beta_n{}^2 m_1 - 32m_1 - a^2)\,,$$

$$b_3 = -8m_1 aD\beta_n + a\beta_n D(\beta_n{}^2 m_1 - 16m_1 - a^2)\ln a\,,$$

where $m_1 = \bar{D}/(K^2\bar{A})^2$ and

$$D = -\frac{\lambda^2}{96} \quad \text{when} \quad \theta_0 = \pi/4,\ 3\pi/4,\ 7\pi/4 \quad \text{and}$$

$$D = \frac{\lambda^2}{48} \quad \text{when} \quad \theta_0 = 2\pi/4 \quad \text{and} \quad 6\pi/4\,.$$

Appendix B

The constants a_{ij} and b_i appearing in Eq. (31) when the edge of the circular sector at $r = a$ is free are:

$$a_{11} = \frac{m_2\beta_n}{a}\,I_{\beta_n}(\mu a)\,, \qquad a_{12} = 0\,, \qquad a_{13} = 4m_2\beta_n(\beta_n + 1)a^{\beta_n - 1}\,,$$

$$a_{21} = \frac{m_2\beta_n}{a}\left[\frac{1-v}{a}\,I_{\beta_n}(\mu a) - \mu(1-v)I'_{\beta_n}(\mu a)\right], \qquad a_{22} = \beta_n(\beta_n - 1)\,(v-1)a^{\beta_n - 2}\,,$$

$$a_{23} = a^{\beta_n - 2}\left\{4\beta_n{}^3 m_1(v-1) + \beta_n{}^2 a^2(v-1) - \beta_n\left[4m_1(v-1) + a^2(v+3)\right] - 2a^2(v+1)\right\},$$

$$a_{31} = \frac{1}{2}\,I_{\beta_n}(\mu a) - m_2\mu^2 I''_{\beta_n}(\mu a)\,, \qquad a_{32} = -\beta_n(\beta_n - 1)\,a^{\beta_n - 2}\,,$$

$$b_1 = 2Cm_1 a(\beta_n{}^2 - 16)\,,$$

$$b_2 = C\left\{\beta_n{}^4 m_1 v - \beta_n{}^2\left[2m_1(9v+1) + a^2 v\right] + 4\left[8m_1(v+1) + a^2(v+3)\right]\right\},$$

$$b_3 = -\beta_n C(\beta_n{}^2 m_1 - 16m_1 - 3a^2)\,,$$

where $\beta_n = n\pi/\theta_0$, $m_1 = \bar{D}/K^2\bar{A}$, $m_2 = \bar{C}/K^2\bar{A}$, and C is defined in Eq. (26). For the special values of θ_0 discussed in Eq. (27) the a_{ij} remain the same as in the above; however, the constant b_i's are given by:

$$b_1 = 2aD(\beta_n{}^2 - 16)\ln a + aD(\beta_n{}^2 - 32)\,,$$

$$b_2 = -\,D\left\{\beta_n{}^4 m_1 v - \beta_n{}^2\left[2m_1(9v+1) + a^2 v\right] + 4\left[8m_1(v+1) + a^2(v+3)\right]\ln a\right\}$$
$$+\,D\left[\beta_n{}^2 m_1(9v+3) - 32m_1(v+2) - a^2(v+7)\right],$$

$$b_3 = -\beta_n D(\beta_n{}^2 m_1 - 16m_1 - 3a^2)\ln a - \beta_n D(\beta_n{}^2 m_1 - 24m_1 - a^2)\,,$$

where

$$D = -\frac{\lambda^2}{96} \quad \text{when} \quad \theta_0 = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4 \quad \text{and}$$

$$D = \frac{\lambda^2}{48} \quad \text{when} \quad \theta_0 = 2\pi/4 \quad \text{and} \quad 6\pi/4\,,$$

with λ^2 defined in Eq. (26).

References

[1] Kirchhoff, G.: Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. J. Reine Angew. Math. **40**, 51–58 (1850).
[2] Stocker, J. J.: Mathematical problems connected with the bending and buckling of elastic plates. Bull. Am. Math. Soc. **48**, 247–261 (1942).
[3] Mindlin, R. D.: Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates. J. Appl. Mech. **18**, 31–38 (1951).
[4] Reissner, E.: The effect of transverse shear deformation on the bending of elastic plates. J. Appl. Mech. **12**, 69–77 (1945).
[5] Ambartsumyan, S. A.: Theory of anisotropic plates. Stanford, Conn: Technomic 1970.
[6] Reissner, E.: On the theory of bending of elastic plates. J. Math. Phys. **23**,184–191 (1944).
[7] Reissner, E.: On bending of elastic plates. Quart. Appl. Math. **5**, 55–68 (1947).
[8] Reissner, E.: Reflections on the theory of elastic plates. Appl. Mech. Rev. **38**, 1453–1464 (1985).
[9] Friedrichs, K. O.: Asymptotic phenomena in mathematical physics. Bull. Am. Math. Soc. **61**, 485–504 (1955).
[10] Schneider W.: Einfluß einer kleinen Biegesteifigkeit auf die Querschwingungen einer eingespannten rechteckigen Membran. Acta Mech. **13**, 293–302 (1972).
[11] Nosier, A., Reddy. J. N.: On boundary layer and interior equations for higher-order theories of plates. ZAMM **72**, 657–666 (1992).
[12] Reddy, J. N.: A simple higher-order theory for laminated composite plates. J. Appl. Mech. **51**, 745–752 (1984).
[13] Nosier, A., Reddy, J. N.: A study of non-linear dynamics equation of higher order shear deformation plate theories. Int. J. Non-Linear Mech. **22**, 233–249 (1991).
[14] Reddy, J. N.: Geometrically nonlinear transient analysis of laminated composite plates. AIAA J. **21**, 621–629 (1983).
[15] Reddy, J. N.: A refined nonlinear theory of plates with transverse shear deformation. Int. J. Solids Struct. **29**, 881–896 (1984).
[16] Nosier, A., Yavari, A., Sarkani, S.: A study of the edge-zone equation of Mindlin-Reissner plate theory in bending of laminated rectangular plates. Acta Mech. (in press).
[17] Mindlin, R. N., Deresiewicz, H.: Thickness-shear and flexural vibrations of a circular disk. J. Appl. Phys., **25**, 1329–1332 (1954).
[18] Reddy, J. N.: Energy and variational methods in applied mechanics. New York: John Wiley 1984.
[19] Reismann, H.: Elastic plates, theory and applications. New York: John Wiley 1988.
[20] Nosier, A., Reddy, J. N.: On vibration and buckling of symmetric laminated plates according to shear deformations theories. Part I. Acta Mech. **94**, 123–144 (1992).
[21] Nosier, A., Reddy, J. N.: On vibration and buckling of symmetric laminated plates according to shear deformation theories. Part II. Acta Mech. **94**, 145–169 (1992).

Authors' addresses: A. Nosier, Department of Mechanical Engineering, Sharif University of Technology, Azadi Ave., Tehran, Iran; A. Yavari and S. Sarkani, School of Engineering and Applied Science, The George Washington University, Washington, DC 20052, U.S.A.