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## A Geometric Theory of Nonlinear Morphoelastic Shells

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**Abstract** Many thin three-dimensional elastic bodies can be reduced to elastic shells: two-dimensional elastic bodies whose reference shape is not necessarily flat. More generally, morphoelastic shells are elastic shells that can remodel and grow in time. These idealized objects are suitable models for many physical, engineering, and biological systems. Here, we formulate a general geometric theory of nonlinear morphoelastic shells that describes both the evolution of the body shape, viewed as an orientable surface, as well as its intrinsic material properties such as its reference curvatures. In this geometric theory, bulk growth is modeled using an evolving referential configuration for the shell, the so-called material manifold. Geometric quantities attached to the surface, such as the first and second fundamental forms, are obtained from the metric of the three-dimensional body and its evolution. The governing dynamical equations for the body are obtained from variational consideration by assuming that both fundamental forms on the material manifold are dynamical variables in a Lagrangian field theory. In the case where growth can be modeled by a Rayleigh potential, we

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also obtain the governing equations for growth in the form of kinetic equations coupling the evolution of the first and the second fundamental forms with the state of stress of the shell. We apply these ideas to obtain stress-free growth fields of a planar sheet, the time evolution of a morphoelastic circular cylindrical shell subject to time-dependent internal pressure, and the residual stress of a morphoelastic planar circular shell.

**Keywords** Bulk growth · Morphoelasticity · Shell · Nonlinear elasticity · Geometric mechanics · Residual stress

**Mathematics Subject Classification** 74Axx · 74Fxx · 74Lxx

## 1 Introduction

Growth and remodeling are particularly important processes in many physical and biological systems (Hori et al. 1986; Silberberg et al. 1989; Pollack et al. 1996; Delsanto et al. 2004; Geitmann and Ortega 2009), and their interplay with mechanical stress is a well-established fact (Hsu 1968; Skalak 1982; Skalak et al. 1982; Fung 1983, 1991, 1995; Taber 1995; Helmlinger et al. 1997; Humphrey 2002). Growth and remodeling of a body can happen in such a way (non-uniformly) that a relaxed state may not exist in the physical space, and since the body is constrained to deform in the Euclidean space, this leads to a state of residual stresses. Such stresses are in fact residual as they persist even when all the external loads are removed (Skalak et al. 1996; Takamizawa and Matsuda 1990). Note that the presence of residual stresses in certain biological tissues has been experimentally verified (Chuong and Fung 1986; Omens and Fung 1990; Han and Fung 1991). Also, as highlighted by Fung (1991), residual stresses are of crucial importance to the working conditions and physiological functions of living organs.

In continuum mechanics, stress is related to a measure of strain, e.g., the deformation gradient, with respect to a stress-free reference configuration. However, in some cases, such a configuration may not be realized in the Euclidean three-dimensional space, i.e., the material manifold is not necessarily Euclidean. This issue has been traditionally addressed by assuming a decomposition of the deformation gradient  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_p$  into an elastic part  $\mathbf{F}_e$  and a non-elastic part  $\mathbf{F}_p$ , thus, performing a conceptual local release of stress to a locally stress-free intermediate configuration followed by an elastic deformation to a configuration in the physical space where residual stresses occur (Eckart 1948; Kröner 1959; Stojanović et al. 1964). In the context of growth mechanics, most of the existing formulations rely on this multiplicative decomposition. The non-elastic part of the deformation gradient  $\mathbf{F}_p = \mathbf{F}_g$  characterizes the growth of the reference configuration and transforms it to a locally relaxed intermediate configuration (Kondaurov and Nikitin 1987; Rodriguez et al. 1994; Lubarda and Hoger 2002; Ben Amar and Goriely 2005). See (Lubarda 2004; Ambrosi et al. 2011) for an extensive review and a comprehensive bibliography on the subject and Sadik and Yavari (2016) for a historical perspective on the decomposition of deformation gradient in anelasticity. Recently, Yavari (2010) introduced a geometric theory of growing

nonlinear elastic solids in which the stress-free material configuration is a Riemannian manifold with an evolving geometry. This geometrical approach provides a mathematically precise framework to study bulk growth and the induced residual stresses, and leads to a systematic method to find the stress-free growth distributions in nonlinear elasticity.

Research interest in elastic shells was mainly triggered by the experimental work of [Chladni \(1830\)](#) on the tones of vibrating plates, which led to several attempts to lay down a theoretical framework to explain his findings. In 1809, the French *Académie des sciences* sponsored a prize on the theoretical investigation of the vibration of elastic surfaces and it was won by [Germain \(1821\)](#) whose work proved later on to be partially incorrect (see the historical introduction by [Love \(1892\)](#)). The first attempts to formulate a general theory and obtain the governing equations for the deformation of elastic shells are attributed to [Aron \(1874\)](#) and [Mathieu \(1882\)](#). However, it was [Love \(1888\)](#) who first obtained a consistent general theory for small strains of linear elastic shells based on the work of [Kirchhoff \(1850\)](#) on the vibrations of plates.

The governing equations of elastic shells in terms of stress and couple-stress tensors were first derived in rectangular Cartesian coordinates by [Cosserat and Cosserat \(1909\)](#) (see ([Ericksen and Truesdell 1958](#)) for an extension of Cosserats' work). The coordinate-free expression of these equations was, however, presented in its full generality by [Synge and Chien \(1941\)](#). They localized the integral balance laws of shells valid in a Cartesian coordinate chart to obtain the governing differential equations and then obtained the coordinate-free expressions of these differential equations. Alternative derivations for the governing equations of elastic shells in terms of stress and couple-stress tensors were later proposed by [Novozhilov \(1964\)](#); [Green and Zerna \(1950\)](#); [Naghdi \(1963\)](#) starting from the general three-dimensional equations of equilibrium and by ([Koiter 1966](#)) by means of the principle of virtual work.

While considerable progress has been made in the modeling of three-dimensional growing elastic bodies, a complete continuum theory for morphoelastic shells is not yet available. In particular, there is no general formulation for the computation of residual stresses and couple stresses due to bulk growth in shells. All the aforementioned shell theories are restricted to model shells in the context of mass-conserving elasticity. It is worthwhile, however, to mention the work by [Goriely and Ben Amar \(2005\)](#) on a growing shell embedded in an elastic medium and the ongoing effort on modeling morphoelastic plates ([Efrati et al. 2009](#); [Dervaux et al. 2009](#); [McMahon et al. 2011a, b](#)).

More recently, [Pezzulla et al. \(2015a, b\)](#) investigated geometry-driven growth-like morphing of thin bilayer shells. Remarkably, they were able to predict and experimentally obtain domes or a saddle surface in a shrink-fit problem in thin circular disks ([Pezzulla et al. 2015a](#)). They also showed that a large isotropic expansion of one layer with respect to the other leads to a cylindrical bending of the bilayer sheet ([Pezzulla et al. 2015b](#)).

In this paper, following [Yavari \(2010\)](#)'s approach of an evolving material manifold to model bulk growth, we develop a geometric nonlinear theory of morphoelastic shells. We model bulk growth in orientable surfaces using evolving first and second fundamental forms in the material manifold. In Sect. 2, we discuss the idealization of a thin body to a shell and its evolving referential geometry as an embedded hyperplane to model growth through evolving first and second fundamental forms. In Sect. 3, we

present the kinematics of shells, include a discussion on the compatibility equations, and introduce a systematic method to find those growth fields that leave a stress-free shell stress-free. In Sect. 4, we derive the balance of mass for morphoelastic shells. Following a Lagrangian field theory, we also derive the balance laws for morphoelastic shells and the kinetic equations for the evolution of growth such that the evolution of the first and the second fundamental forms is coupled with the state of stress of the shell. In Sect. 5, we look at a few examples to demonstrate the capability of the proposed geometric theory in the calculation of the time evolution of residual stresses induced by growth. First, we consider a planar shell and look for a family of growth fields that leave the shell stress-free. We numerically find embeddings of the evolving surfaces in  $\mathbb{R}^3$ . We find that these stress-free growth distributions of the initially planar shell can force it to evolve to either another flat sheet (e.g., cylindrical sheets), sheets with positive curvature (e.g., spherical sheets), or sheets with negative curvature (e.g., saddle-like sheets). Next we look at the problems of a morphoelastic infinitely long circular cylindrical shell subject to a time-dependent internal pressure and that of a morphoelastic initially planar circular disk. We numerically solve the kinetic equations for the evolution of growth and observe the coupling between the state of stress of the shell and the evolution of its curvature. We consequently obtain the evolution of the geometry of the shells and the induced residual stress and couple-stress fields.

## 2 Differential Geometry of Shells

In this section, we review a few elements of the differential geometry of shells as embedded surfaces in three-dimensional manifold and discuss the idealization of a thin body as a shell and the evolving geometry of a morphoelastic shell.

### 2.1 Geometry of an embedded surface

In this section, we tersely review some elements of the geometry of two-dimensional embedded surfaces in three-dimensional manifolds (see for example (Hicks 1965; do Carmo 1992) for a detailed account on the subject). Let  $(\mathcal{B}, \bar{\mathbf{G}})$  be an orientable three-dimensional Riemannian manifold and let  $(\mathcal{H}, \mathbf{G})$  be an orientable two-dimensional Riemannian submanifold of  $(\mathcal{B}, \bar{\mathbf{G}})$ , i.e.,  $\mathbf{G} = \bar{\mathbf{G}}|_{\mathcal{H}}$ . Let  $\mathfrak{X}(\mathcal{H})$  be the space of smooth tangent vector fields on  $\mathcal{H}$ . Using the decomposition  $T_X\mathcal{B} = T_X\mathcal{H} \oplus (T_X\mathcal{H})^\perp$ ,  $\forall X \in \mathcal{H}$ , we define the space of smooth normal vector fields  $\mathfrak{X}(\mathcal{H})^\perp \subset \mathfrak{X}(\mathcal{B})$ . Let  $N \in \mathfrak{X}(\mathcal{H})^\perp$  be the smooth unit normal vector field of  $\mathcal{H}$ . The orientation of the unit normal vector field  $N$  is chosen such that the orientation induced by the local coordinate chart of the surface  $\mathcal{H}$  and the unit normal vector field as the last coordinate on  $\mathcal{B}$  is consistent with the orientation of  $\mathcal{B}$ . Let  $\nabla^{\mathcal{H}}$  and  $\bar{\nabla}$  be the Levi-Civita connections of  $(\mathcal{H}, \mathbf{G})$  and  $(\mathcal{B}, \bar{\mathbf{G}})$ , respectively. Note that the Levi-Civita connection  $\nabla^{\mathcal{H}}$  of the metric  $\mathbf{G}$  is precisely the connection induced by the Levi-Civita connection  $\bar{\nabla}$  of the metric  $\bar{\mathbf{G}}$ . The connection  $\nabla^{\mathcal{H}}$  in terms of the connection  $\bar{\nabla}$  is given by

$$\nabla_X^{\mathcal{H}} Y = \bar{\nabla}_{\bar{X}} \bar{Y} - \bar{G}(\bar{\nabla}_{\bar{X}} \bar{Y}, N) N, \quad \forall X, Y \in \mathfrak{X}(\mathcal{H}),$$

where  $\bar{X} \in \mathfrak{X}(\mathcal{B})$  and  $\bar{Y} \in \mathfrak{X}(\mathcal{B})$  are any local extensions of  $X$  and  $Y$ , respectively, i.e.,  $\bar{X}(X) = X(X), \forall X \in \mathcal{H}$ . The second fundamental form of  $\mathcal{H}$  is defined as the symmetric tensor  $B \in \Gamma(S^2 T^* \mathcal{H})$  given by

$$B(X, Y) = \bar{G}(\bar{\nabla}_{\bar{X}} \bar{Y}, N) = -\bar{G}(\bar{\nabla}_{\bar{X}} N, \bar{Y}), \quad \forall X, Y \in \mathfrak{X}(\mathcal{H}). \tag{2.1}$$

The connection  $\nabla$  on  $T\mathcal{H}$  induces a connection on  $S^2 T^* \mathcal{H}$  defined by

$$(\nabla_X A)(Y, Z) = X(A(Y, Z)) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z), \quad \forall A \in \Gamma(S^2 T^* \mathcal{H}).$$

The curvature tensor  $\mathcal{R}$  of a Riemannian manifold  $(M, G)$  is defined as

$$\mathcal{R}(X, Y, Z, W) = G(R(X, Y)Z, W), \quad \forall X, Y, Z, W \in \mathfrak{X}(\mathcal{M}),$$

where  $R$  is given in terms of the Levi-Civita connection  $\nabla^M$  by

$$R(X, Y)Z = \nabla_{[X, Y]}^M Z - \nabla_X^M \nabla_Y^M Z + \nabla_Y^M \nabla_X^M Z.$$

In components, the curvature tensor reads

$$\begin{aligned} \mathcal{R}_{ABCD} &= \mathcal{R}(\partial_A, \partial_B, \partial_C, \partial_D) \\ &= \left( \partial_B \Gamma^K_{AC} - \partial_A \Gamma^K_{BC} + \Gamma^L_{AC} \Gamma^K_{BL} - \Gamma^L_{BC} \Gamma^K_{AL} \right) G_{KD}. \end{aligned}$$

Given the symmetries of the curvature tensor, if  $n$  is the dimension of the manifold  $M$ , its curvature tensor  $\mathcal{R}$  has  $n^2(n^2 - 1)/12$  independent components. In particular, for a two-dimensional surface ( $n = 2$ ), the curvature tensor has one independent component  $\mathcal{R}_{1221}$ .

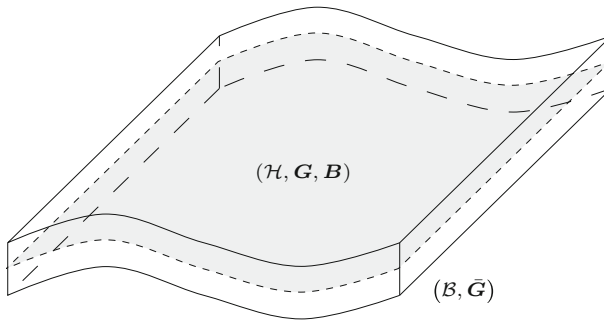
We denote the Riemann curvature tensors of  $\mathcal{H}$  and  $\mathcal{B}$  by  $\mathcal{R}^{\mathcal{H}}$  and  $\bar{\mathcal{R}}$ , respectively. The Gauss equation gives a relation between the Riemann curvature tensor and the second fundamental form of  $\mathcal{H}$ , and the Riemann curvature tensor of  $\mathcal{B}$  as

$$\bar{\mathcal{R}}(X, Y, Z, W) = \mathcal{R}^{\mathcal{H}}(X, Y, Z, W) - B(X, Z) B(Y, W) + B(X, W) B(Y, Z). \tag{2.2}$$

The second fundamental form also satisfies the Codazzi-Mainardi equation that can be written as

$$\bar{\mathcal{R}}(X, Y, Z, N) = \left( \nabla_Y^{\mathcal{H}} B \right)(X, Z) - \left( \nabla_X^{\mathcal{H}} B \right)(Y, Z). \tag{2.3}$$

Let  $(X^1, X^2, X^3)$  be a local coordinate chart for  $\mathcal{B}$  such that at any point of the hypersurface  $\mathcal{H}$ ,  $\{X^1, X^2\}$  is a local coordinate chart for  $\mathcal{H}$  and the normal vector field  $N$  to  $\mathcal{H}$  is tangent to the coordinate curve  $X^3$ . We say that such a chart is compatible with



**Fig. 1** The mid-surface  $(\mathcal{H}, \mathbf{G}, \mathbf{B})$  as a Riemannian submanifold of  $(\mathcal{B}, \bar{\mathbf{G}})$

$\mathcal{H}$ . Note that given the symmetries of the curvature tensor and the second fundamental form, the Gauss and Codazzi–Mainardi equations reduce in components to

$$\bar{\mathcal{R}}_{1212} - \mathcal{R}_{1212}^{\mathcal{H}} = B_{11}B_{22} - B_{12}B_{12}, \tag{2.4a}$$

$$\bar{\mathcal{R}}_{1213} = B_{11|2} - B_{21|1}, \tag{2.4b}$$

$$\bar{\mathcal{R}}_{2123} = B_{22|1} - B_{12|2}, \tag{2.4c}$$

where we denote by a stroke  $|$  the covariant derivative corresponding to the Levi–Civita connection of  $(\mathcal{H}, \mathbf{G})$ , i.e.,  $B_{AB|C} = B_{AB,C} - \Gamma^K_{CA}B_{KB} - \Gamma^K_{CB}B_{AK}$ , where  $\Gamma^C_{AB}$  is the Christoffel symbol of the connection  $\nabla^{\mathcal{H}}$  in the local chart  $\{X^1, X^2\}$ .

The fundamental theorem of surface theory, first proved by [Bonnet \(1865\)](#), implies that the geometry of a surface is fully described by its metric and its second fundamental form ([do Carmo 1976](#); [Ivey and Landsberg 2003](#)).

### 2.2 Idealization of a Thin Body

Let  $B$  be a three-dimensional thin body (i.e., its thickness is negligible compared to the other two dimensions) identified with an orientable three-dimensional Riemannian manifold  $\mathcal{B}$  endowed with the metric  $\bar{\mathbf{G}}$ . Let  $H$ —the mid-surface of  $B$ —be identified with  $(\mathcal{H}, \mathbf{G}, \mathbf{B})$ , a two-dimensional Riemannian submanifold of  $(\mathcal{B}, \bar{\mathbf{G}})$  with first and second fundamental forms  $\mathbf{G}$  and  $\mathbf{B}$  (see Fig. 1). We assume in the following that  $\mathcal{H}$  is an orientable hypersurface of  $\mathcal{B}$ . We show that the natural isometric embedding of  $\mathcal{H}$  in  $\mathcal{B}$  induces independent in-plane and out-of-plane geometries for the hypersurface  $\mathcal{H}$ .

Let  $(X^1, X^2, X^3)$  be a local coordinate chart compatible with  $\mathcal{H}$ . In this coordinate chart, at any point  $X \in \mathcal{B}$ , the metric  $\bar{\mathbf{G}}$  of  $\mathcal{B}$  has the following representation:

$$\bar{\mathbf{G}}(X) = \begin{pmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) & \bar{G}_{13}(X) \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) & \bar{G}_{23}(X) \\ \bar{G}_{13}(X) & \bar{G}_{23}(X) & \bar{G}_{33}(X) \end{pmatrix}.$$



If  $X \in \mathcal{H}$ , we have

$$\bar{G}(X) = \begin{pmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) & 0 \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the metric  $G$  of  $\mathcal{H}$ , referred to as the first fundamental form, has the following representation

$$G(X) = \begin{pmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) \end{pmatrix}, \quad \forall X \in \mathcal{H}, \tag{2.5}$$

and the second fundamental form of  $\mathcal{H}$  has the following components

$$B_{AB}(X) = \bar{\Gamma}^3_{AB}(X), \quad A, B = 1, 2, \quad \forall X \in \mathcal{H},$$

where  $\bar{\Gamma}^C_{AB} = \frac{1}{2} \sum_K \bar{G}^{CK} (\partial_A \bar{G}_{KB} + \partial_B \bar{G}_{KA} - \partial_K \bar{G}_{AB})$  is the Christoffel symbol of the Levi–Civita connection of  $(\mathcal{B}, \bar{G})$ . Therefore

$$B_{AB}(X) = -\frac{1}{2} \left. \frac{\partial \bar{G}_{AB}}{\partial X^3} \right|_{\mathcal{H}}(X), \quad A, B = 1, 2, \quad \forall X \in \mathcal{H}, \tag{2.6}$$

where  $\bar{G}_{AB}$  should be thought of as a function on the coordinate curve  $X^3$  and  $\left. \frac{\partial \bar{G}_{AB}}{\partial X^3} \right|_{\mathcal{H}}$  is evaluated at the point where the curve  $X^3$  meets the hypersurface  $\mathcal{H}$ . Since  $\bar{G}_{AB}$  and  $\left. \frac{\partial \bar{G}_{AB}}{\partial X^3} \right|_{\mathcal{H}}$  can be prescribed independently, Eqs. (2.5) and (2.6) demonstrate that the independent first and second fundamental forms  $G$  and  $B$  of the hypersurface  $\mathcal{H}$  can be obtained from the metric  $\bar{G}$  of the embedding space  $\mathcal{B}$ . Therefore, we only need to specify the components  $\bar{G}_{AB}$  for  $A, B = 1, 2$  to characterize the geometry of  $\mathcal{H}$ . We introduce the following notation

$$\bar{G}_{\mathcal{H}}(X) := \begin{pmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) \end{pmatrix}, \quad \forall X \in \mathcal{B}. \tag{2.7}$$

*Remark 2.1* In the local coordinate chart  $(X^1, X^2, X^3)$ , note that the components  $\bar{G}_{A3}$ ,  $A = 1, 2, 3$  of the metric  $\bar{G}$  do not affect the geometry of  $\mathcal{H}$ . Indeed, from Eqs. (2.5) and (2.6), the geometry of  $\mathcal{H}$  depends only on the restriction of the metric  $\bar{G}$  to  $\mathcal{H}$  (i.e.,  $\bar{G}_{AB}|_{\mathcal{H}}$ ,  $A, B = 1, 2$ ) and its first-order derivative along the normal to  $\mathcal{H}$  (i.e.,  $\left. \frac{\partial \bar{G}_{AB}}{\partial X^3} \right|_{\mathcal{H}}$ ,  $A, B = 1, 2$ ). Therefore, higher-order variations of  $\bar{G}_{\mathcal{H}}$  along the thickness of  $\mathcal{B}$  are not captured by the geometry of  $\mathcal{H}$ . As an example, let  $\mathcal{B}$  be a thin body in  $\mathbb{R}^3$  with the coordinate chart  $(X^1, X^2, X^3)$  such that the hypersurface  $X^3 = 0$  contains the mid-surface  $\mathcal{H}$ . Two different metrics for  $\mathcal{B}$  such that  $\bar{G}_{AB}^1(X) = e^{X^3} \delta_{AB}$ , and  $\bar{G}_{AB}^2(X) = (1 + X^3) \delta_{AB}$ ,  $\forall X \in \mathcal{B}$ ,  $A, B = 1, 2$ , correspond to the same geometry for  $\mathcal{H}$  given by  $G_{AB} = \delta_{AB}$  and  $B_{AB} = -\frac{1}{2} \delta_{AB}$ ,  $A, B = 1, 2$ . Also, if we consider an evolving metric such that  $\bar{G}_{AB}(X, t) = (1 + (X^3)^2 f(t)) \delta_{AB}$ ,  $\forall X \in \mathcal{B}$ ,  $A, B = 1, 2$ , where  $f$  is a given function of time, we find that the geometry of  $\mathcal{H}$  does not capture this evolution as it remains unchanged both in-plane and out-of-plane ( $G = \delta$  and  $B = 0$ ).

### 2.3 Evolving Geometry of a Morphoelastic Shell

In order to model bulk growth of the body  $\mathcal{B}$ , we assume, following Yavari (2010), an evolving metric for the material manifold  $(\mathcal{B}, \bar{\mathbf{G}})$ , i.e., we leave the manifold  $\mathcal{B}$  fixed and endow it with an evolving metric<sup>1</sup>  $\bar{\mathbf{G}}$ , i.e.,  $\bar{\mathbf{G}} = \bar{\mathbf{G}}(X, t)$ , such that at  $t = 0$ , we have  $\bar{\mathbf{G}}(X, 0) = \bar{\mathbf{G}}^0(X)$  the metric of a natural stress-free configuration of  $\mathcal{B}$ . In this paper, however, we are interested in growth of thin bodies, and hence, we consider the manifold  $(\mathcal{B}, \bar{\mathbf{G}})$  with an evolving metric such that, in a local coordinate chart  $(X^1, X^2, X^3)$  as introduced in Sect. 2.2, only  $\bar{\mathbf{G}}_{\mathcal{H}}$  is evolving.<sup>2</sup> Now, we leave the mid-surface manifold  $\mathcal{H}$  fixed and let its evolving first and second fundamental forms  $\mathbf{G}$  and  $\mathbf{B}$  be induced from its natural isometric embedding in  $\mathcal{B}$ . Therefore, in the local coordinate chart  $(X^1, X^2, X^3)$ , the metric reads

$$\mathbf{G}(X, t) = \begin{pmatrix} \bar{G}_{11}(X, t) & \bar{G}_{12}(X, t) \\ \bar{G}_{12}(X, t) & \bar{G}_{22}(X, t) \end{pmatrix}, \quad \forall X \in \mathcal{H}, \quad (2.8)$$

and the second fundamental form of  $\mathcal{H}$  is written as

$$B_{AB}(X, t) = -\frac{1}{2} \left. \frac{\partial \bar{G}_{AB}}{\partial X^3} \right|_{\mathcal{H}}(X, t), \quad A, B = 1, 2, \quad \forall X \in \mathcal{H}. \quad (2.9)$$

We will discuss, in Sect. 4.2, the governing equations for the evolution of the first and the second fundamental forms of shells and see how the evolving geometry of the material manifold, i.e., the growth of the morphoelastic shell, is coupled with its current state of stress. Note that the evolving fundamental forms  $\mathbf{G}$  and  $\mathbf{B}$  are compatible in the manifold  $(\mathcal{B}, \bar{\mathbf{G}})$ , i.e., they satisfy the Gauss and Codazzi–Mainardi equations (2.2) and (2.3).

To illustrate the evolving geometry of a surface, we consider a flat thin body that can be represented by a planar surface  $(\mathcal{H}, \mathbf{G})$  in  $(\mathbb{R}^3, \bar{\mathbf{G}})$ . Let  $(X^1, X^2, X^3)$  be the standard coordinate chart for  $\mathbb{R}^3$  such that the hyperplane  $X^3 = 0$  contains the surface  $\mathcal{H}$ . If we assume that the body undergoes a growth that is uniform through its thickness, then we can model this growth by an evolving metric  $\bar{\mathbf{G}}_{\mathcal{H}}$  such that  $\bar{G}_{AB}(X, t) = \bar{G}_{AB}(X^1, X^2, t)$ ,  $\forall X \in \mathcal{B}$ ,  $A, B = 1, 2$  (i.e.,  $\bar{G}_{AB3}$  do not depend on  $X^3$ ), and then we obtain an evolving geometry for  $\mathcal{H}$  with an evolving metric  $G_{AB}(X, t) = \bar{G}_{AB}(X, t)$ ,  $\forall X \in \mathcal{H}$ , and a vanishing second fundamental form. As an example  $\bar{G}_{AB}(X^1, X^2, t) = f(t)\delta_{AB}$ ,  $A, B = 1, 2$ , for some function  $f$  of time, models a uniform in-plane growth with no out-of-plane geometry change (i.e., a vanishing second fundamental form). However, if we assume that the body undergoes a growth that is not uniform through its thickness, we obtain an evolving geometry for  $\mathcal{H}$  such that the second fundamental form evolves with time:  $B_{AB}(X, t) = -\frac{1}{2} \left. \frac{\partial \bar{G}_{AB}}{\partial X^3} \right|_{\mathcal{H}}(X, t)$ . As an example, we let  $\bar{G}_{AB}(X, t) = f(X^3, t)\delta_{AB}$ ,  $A, B = 1, 2$ , for some function

<sup>1</sup> Other examples of evolving material metrics in mechanics have been introduced in (Ozakin and Yavari 2010; Yavari and Goriely 2012a, b, 2013a, 2015, 2013b, 2014; Sadik and Yavari 2015).

<sup>2</sup> cf. (2.7) where the notation  $\bar{\mathbf{G}}_{\mathcal{H}}$  was introduced.

$f$  of time and  $X^3$  such that  $f(0, t) = 1$ ,  $\frac{\partial f}{\partial X^3}(0, 0) = 0$ , and  $\frac{\partial f}{\partial X^3}(0, t) \neq 0$  for  $t \neq 0$  (e.g.,  $f(X^3, t) = e^{X^3 t}$ ). Then we have an evolving geometry for  $\mathcal{H}$  such that the metric of  $\mathcal{H}$  remains unchanged:  $G_{AB} = \delta_{AB}$ , while its out-of-plane geometry evolves with time:  $B_{AB} = -\frac{1}{2} \frac{\partial f}{\partial X^3}(0, t) \delta_{AB}$ .

*Remark 2.2* Given the fact (discussed in Remark 2.1) that the geometry of  $\mathcal{H}$  depends only on the restriction of the metric  $\bar{\mathbf{G}}$  to  $\mathcal{H}$  and its first-order derivative along the normal to  $\mathcal{H}$ , we are bound to model a restrictive class of material evolutions. We assume that the evolving material manifold  $(\mathcal{H}, \mathbf{G})$  at time  $t$  is diffeomorphic to the reference manifold  $(\mathcal{H}, \mathbf{G}^0)$  at time  $t = 0$ , so that this diffeomorphism can be extended to a neighborhood of  $\mathcal{H}$  in  $\mathcal{B}$  in such a way that the push-forward of the reference normal vector field  $N^0$  of  $\mathcal{H}$  in  $(\mathcal{B}, \bar{\mathbf{G}}^0)$  is precisely the evolving normal vector field  $N$  of  $\mathcal{H}$  in  $(\mathcal{B}, \bar{\mathbf{G}})$ . Note that this implies that during the material evolution of the shell, at any point of  $\mathcal{H}$ , the normal to  $\mathcal{H}$  remains normal.

We can write the evolving metric  $\bar{\mathbf{G}}_{\mathcal{H}}$  in the form

$$\bar{\mathbf{G}}_{\mathcal{H}}(X, t) = \bar{\mathbf{G}}_{\mathcal{H}}^0(X) e^{2\bar{\omega}(X,t)}, \quad \forall X \in \mathcal{B},$$

where  $\bar{\omega}$  is a smooth  $(\uparrow)$ -rank tensor characterizing growth of the thin body  $\mathcal{B}$  such that  $\bar{\omega}(X, 0) = \mathbf{0}$ . Following (2.8), the evolving first fundamental form of  $\mathcal{H}$  is given by

$$\mathbf{G}(X, t) = \bar{\mathbf{G}}_{\mathcal{H}}^0(X) e^{2\omega(X,t)}, \quad \forall X \in \mathcal{H},$$

where  $\omega = \bar{\omega}|_{\mathcal{H}}$ . Following (2.9), for  $X \in \mathcal{H}$ , the evolving second fundamental form of  $\mathcal{H}$  is given by

$$\begin{aligned} \mathbf{B}(X, t) &= -\frac{1}{2} \frac{\partial \bar{\mathbf{G}}_{\mathcal{H}}}{\partial X^3}(X, t) \\ &= -\frac{1}{2} \frac{\partial}{\partial X^3} \left[ \bar{\mathbf{G}}_{\mathcal{H}}^0 e^{2\bar{\omega}} \right](X, t) \\ &= -\frac{1}{2} \frac{\partial \bar{\mathbf{G}}_{\mathcal{H}}^0}{\partial X^3}(X) e^{2\bar{\omega}(X,t)} - \bar{\mathbf{G}}_{\mathcal{H}}^0(X) \frac{\partial \bar{\omega}}{\partial X^3}(X, t) e^{2\bar{\omega}(X,t)}. \end{aligned}$$

For  $X \in \mathcal{H}$ , we introduce the following notations:

$$\begin{aligned} \mathbf{G}^0(X) &:= \bar{\mathbf{G}}_{\mathcal{H}}^0(X), \quad \mathbf{B}^0(X) := -\frac{1}{2} \frac{\partial \bar{\mathbf{G}}_{\mathcal{H}}^0}{\partial X^3}(X), \quad \omega(X, t) := \bar{\omega}(X, t), \\ \text{and } \mathbf{K}(X, t) &= \frac{\partial \bar{\omega}}{\partial X^3}(X, t), \end{aligned}$$

and hence we write the evolving first and second fundamental forms of  $\mathcal{H}$  as

$$\mathbf{G}(X, t) = \mathbf{G}^0(X) e^{2\omega(X,t)}, \quad \mathbf{B}(X, t) = \mathbf{B}^0(X) e^{2\omega(X,t)} - \mathbf{G}^0(X) \mathbf{K}(X, t) e^{2\omega(X,t)}, \tag{2.10}$$

such that  $\omega(X, 0) = \mathbf{0}$  and  $K(X, 0) = \mathbf{0}$ , so that at  $t = 0$ ,  $\mathbf{G}(t = 0) = \mathbf{G}^0$  and  $\mathbf{B}(t = 0) = \mathbf{B}^0$ . For isotropic growth, we assume  $\omega_1 = \omega_2 = \omega$  and  $K_1 = K_2 = K$ , recalling that for  $A = \{1, 2\}$ ,  $K_A = \frac{\partial \omega_A}{\partial X^3}$ . Therefore, we have

$$\mathbf{G}(X, t) = \mathbf{G}^0(X)e^{2\omega(X,t)}, \quad \mathbf{B}(X, t) = \mathbf{B}^0(X)e^{2\omega(X,t)} - K(X, t)\mathbf{G}^0(X)e^{2\omega(X,t)}, \tag{2.11}$$

such that  $\omega(X, 0) = 0$  and  $K(X, 0) = 0$ .

*Remark 2.3* The Riemannian surface form—i.e., the volume form of the surface  $\mathcal{H}$ —associated with the metric  $\mathbf{G}$  is written as

$$dS(X, \mathbf{G}) = \sqrt{\det \mathbf{G}} dX^1 \wedge dX^2 = \sqrt{\det \mathbf{G}^0} e^{\text{tr}(\omega(X,T))} dX^1 \wedge dX^2 = e^{\text{tr}(\omega(X,t))} dS_0(X),$$

where  $dS_0$  is the Riemannian surface form associated with the metric  $\mathbf{G}^0$ . Using the identity  $\det e^A = e^{\text{tr}(A)}$ , the rate of change of the volume element due to the evolving metric is given by

$$\frac{d}{dt} dS(X, \mathbf{G}) = \frac{d}{dt} [\text{tr}(\omega(X, t))] dS(X, \mathbf{G}). \tag{2.12}$$

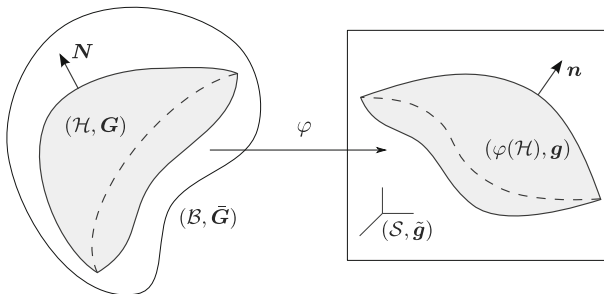
Alternatively, by using the identity  $\frac{d}{dt} [\det A(t)] = \det A(t) \text{tr} \left[ A^{-1}(t) \frac{d}{dt} A(t) \right]$ , we find that

$$\frac{d}{dt} dS(X, \mathbf{G}) = \frac{1}{2} \text{tr} \left( \frac{d\mathbf{G}}{dt} \right) dS(X, \mathbf{G}).$$

### 3 Kinematics of Shells

Let the ambient space be  $\mathcal{S} = \mathbb{R}^3$  endowed with the standard Euclidean metric  $\tilde{\mathbf{g}}$ . Recall that the Riemannian surface  $(\mathcal{H}, \mathbf{G}, \mathbf{B})$  is an orientable two-dimensional Riemannian submanifold of  $(\mathcal{B}, \tilde{\mathbf{G}})$ . A configuration of  $\mathcal{H}$  in  $\mathcal{S}$  is a smooth embedding  $\varphi : \mathcal{H} \rightarrow \mathcal{S}$ . We denote the set of all configurations of  $\mathcal{H}$  in  $\mathcal{S}$  by  $\mathcal{C}$ .

As shown in Fig. 2, the Riemannian manifold  $(\varphi(\mathcal{H}), \mathbf{g}, \boldsymbol{\beta})$ , where  $\mathbf{g} := \tilde{\mathbf{g}}|_{\varphi(\mathcal{H})}$  and  $\boldsymbol{\beta} \in \Gamma(S^2 T^* \varphi(\mathcal{H}))$  is the second fundamental form of  $\varphi(\mathcal{H})$ , is a hypersurface in  $\mathcal{S}$ . Let



**Fig. 2** A configuration  $\varphi : \mathcal{H} \rightarrow \mathcal{S}$  of a Riemannian surface  $(\mathcal{H}, \mathbf{G})$  in the ambient space  $(\mathcal{S}, \tilde{\mathbf{g}})$ . The vector fields  $N$  and  $\mathbf{n}$  are the unit normal vector fields of  $\mathcal{H}$  and  $\varphi(\mathcal{H})$ , respectively

$\nabla$  and  $\tilde{\nabla}$  be the Levi–Civita connections of  $\mathbf{g}$  and  $\tilde{\mathbf{g}}$ , respectively. Let  $\mathbf{n} \in \mathfrak{X}(\varphi(\mathcal{H}))^\perp$  be the smooth unit normal vector field of  $\varphi(\mathcal{H})$  and  $\mathcal{R} \in \Gamma(S^2(\Lambda^2 T^* \varphi(\mathcal{H})))$  be the Riemannian curvature of the surface  $\varphi(\mathcal{H})$ . Since the ambient space  $\mathcal{S} = \mathbb{R}^3$  is flat, the Gauss and Codazzi–Mainardi equations for the Riemannian manifold  $(\varphi(\mathcal{H}), \mathbf{g})$  read

$$\mathcal{R}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}) = \boldsymbol{\beta}(\mathbf{x}, \mathbf{z})\boldsymbol{\beta}(\mathbf{y}, \mathbf{w}) - \boldsymbol{\beta}(\mathbf{x}, \mathbf{w})\boldsymbol{\beta}(\mathbf{y}, \mathbf{z}), \tag{3.1a}$$

$$(\nabla_x \boldsymbol{\beta})(\mathbf{y}, \mathbf{z}) = (\nabla_y \boldsymbol{\beta})(\mathbf{x}, \mathbf{z}). \tag{3.1b}$$

We denote objects and indices by uppercase characters in the material manifold (e.g.,  $X \in \mathcal{H}$  for a material point) and by lowercase characters in the spatial manifold (e.g.,  $x \in \varphi(\mathcal{H})$  for a spatial point). In the remainder of the paper, unless stated otherwise, all indices (material and spatial) take values in the range 1, 2. We adopt the standard Einstein convention of summation over repeated indices.

### 3.1 Strain Measures

We define the deformation gradient  $\mathbf{F}$  as the tangent map of  $\varphi : \mathcal{H} \rightarrow \varphi(\mathcal{H})$ , i.e.,  $\mathbf{F}(X) := T_X \varphi : T_X \mathcal{H} \mapsto T_{\varphi(X)} \varphi(\mathcal{H})$ . The right Cauchy–Green deformation tensor  $\mathbf{C} \in \Gamma(S^2 T^* \mathcal{H})$  is defined as the pull-back of the spatial metric (Marsden and Hughes 1983),  $\mathbf{C}(X) := \varphi^* \mathbf{g}(X) : T_X \mathcal{H} \mapsto T_X \mathcal{H}$ , i.e.,  $\mathbf{C}(X, Y) = \mathbf{g}(\varphi_* X, \varphi_* Y), \forall X, Y \in \mathfrak{X}(\mathcal{H})$ . In components,  $C_{AB} = F^a{}_A F^b{}_B g_{ab}$ . The Jacobian of the motion  $J$  relates the material and spatial Riemannian surface forms  $dS(X, \mathbf{G})$  and  $ds(\varphi(X), \mathbf{g})$  by

$$\varphi^* ds = J dS.$$

It can be shown that (Marsden and Hughes 1983)

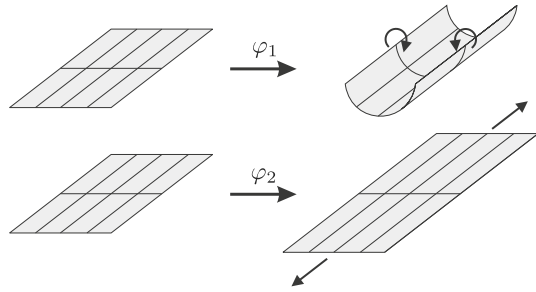
$$J = \sqrt{\frac{\det \varphi^* \mathbf{g}}{\det \mathbf{G}}}. \tag{3.2}$$

The material strain tensor  $\mathbf{E} \in \Gamma(S^2 T^* \mathcal{H})$  is given by  $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G})$ . The spatial strain tensor  $\mathbf{e} \in \Gamma(S^2 T^* \varphi(\mathcal{H}))$  is defined as  $\mathbf{e} = \frac{1}{2}(\mathbf{g} - \mathbf{c})$ , where  $\mathbf{c} = \varphi_* \mathbf{G}$ . Note that  $\mathbf{e} = \varphi_* \mathbf{E}$ . The material and spatial strain tensors are intrinsic in the sense that they are determined by the metrics of the reference and the final configurations of the surface. We introduce extrinsic strain tensors for configurations of surfaces that depend on the second fundamental form as follows. The extrinsic deformation tensor  $\boldsymbol{\Theta} \in \Gamma(S^2 T^* \mathcal{H})$  is defined as the pull-back of the spatial second fundamental form

$$\boldsymbol{\Theta} := \varphi^* \boldsymbol{\beta}.$$

In components,  $\Theta_{AB} = F^a{}_A F^b{}_B \beta_{ab}$ . We define the extrinsic material strain tensor as  $\mathbf{H} := \frac{1}{2}(\boldsymbol{\Theta} - \mathbf{B})$  and the extrinsic spatial strain tensor as  $\boldsymbol{\eta} := \frac{1}{2}(\boldsymbol{\beta} - \boldsymbol{\theta})$ , where  $\boldsymbol{\theta} := \varphi_* \mathbf{B}$ . Note that  $\boldsymbol{\eta} = \varphi_* \mathbf{H}$ . As an example, consider two different configurations of a sheet shown in Fig. 3. The configuration  $\varphi_1$  is an isometry between the sheet and

**Fig. 3** Strains of two different configurations of a sheet: **i** The configuration  $\varphi_1$  maps the sheet to a section of a cylinder with  $E = \mathbf{0}$ , and  $H \neq \mathbf{0}$ , **ii** the configuration  $\varphi_2$  is an in-plane extension of the sheet with  $E \neq \mathbf{0}$ , and  $H = \mathbf{0}$



a section of a cylinder, and therefore,  $E = \mathbf{0}$ . However, note that since the out-of-plane geometry has changed, we have  $H \neq \mathbf{0}$ . On the other hand,  $\varphi_2$  is an in-plane deformation of the sheet with  $E \neq \mathbf{0}$ , and  $H = \mathbf{0}$ .

### 3.2 Compatibility Equations of Shells

The pull-back of the Gauss and the Codazzi equations (3.1) of the surface  $(\varphi(\mathcal{H}), \mathbf{g})$  by  $\varphi$  read (Angoshtari and Yavari 2015)

$$\mathcal{R}^C(X, Y, Z, W) = \Theta(X, Z)\Theta(Y, W) - \Theta(X, W)\Theta(Y, Z), \tag{3.3a}$$

$$\left(\nabla_X^C \Theta\right)(Y, Z) = \left(\nabla_Y^C \Theta\right)(X, Z), \tag{3.3b}$$

where  $\nabla^C$  and  $\mathcal{R}^C$  are, respectively, the Levi–Civita connection and the Riemannian curvature of the Riemannian manifold  $(\mathcal{H}, \mathbf{C})$ . Given a metric  $\mathbf{C} \in \Gamma(S^2T^*\mathcal{H})$  and a symmetric tensor  $\Theta \in \Gamma(S^2T^*\mathcal{H})$ , the relations (3.3) express the compatibility equations for these tensors when  $\mathcal{H}$  is simply connected, i.e., they are the necessary and locally sufficient conditions for the existence of a configuration of  $\mathcal{H}$  with the given deformation tensors that is unique up to isometries of  $\mathcal{S} = \mathbb{R}^3$  when  $\mathcal{H}$  is simply connected (Ivey and Landsberg 2003; Angoshtari and Yavari 2015). Hence, we observe that if  $\varphi_1$  and  $\varphi_2$  are different configurations of the surface  $\mathcal{H} \subset \mathbb{R}^3$  with the same deformation tensors, then  $\varphi_1 \circ \varphi_2^{-1}$  and  $\varphi_2 \circ \varphi_1^{-1}$  are rigid body motions of  $\mathcal{H}$  in  $\mathbb{R}^3$ . Note that similar to (2.4), given the symmetries of the curvature tensor  $\mathcal{R}^C$  and the extrinsic deformation tensor, the compatibility equations reduce in components to

$$\begin{aligned} \mathcal{R}_{12|2}^C &= \Theta_{11}\Theta_{22} - \Theta_{12}\Theta_{12}, \\ \Theta_{11||2} &= \Theta_{12||1}, \\ \Theta_{22||1} &= \Theta_{12||2}, \end{aligned}$$

where we denote by a double stroke  $||$  the covariant derivative corresponding to the Levi–Civita connection of  $\mathbf{C}$ .

### 3.3 Stress-Free Shell Growth

Given a thin body  $\mathcal{B}$  and its idealization, the mid-surface  $\mathcal{H}$ , we want to find those growth fields that leave the shell “stress-free”, by which we mean both stress- and couple-stress-free. As introduced earlier, given the smooth embedding  $\varphi$  of the surface  $(\mathcal{H}, \mathbf{G}, \mathbf{B})$  into the Euclidean space  $\mathcal{S}$  to form a surface  $(\varphi(\mathcal{H}), \mathbf{g}, \boldsymbol{\beta})$ , the tensors  $2\mathbf{E} = \varphi^*\mathbf{g} - \mathbf{G}$  and  $2\mathbf{H} = \varphi^*\boldsymbol{\beta} - \mathbf{B}$ , respectively, provide measures of in-plane and out-of-plane strains. Therefore, the surface is stress-free when these two measures are identically zero, i.e.,  $\varphi^*\mathbf{g} = \mathbf{G}$  and  $\varphi^*\boldsymbol{\beta} = \mathbf{B}$ . Noting that  $\mathbf{C} = \varphi^*\mathbf{g}$  and  $\boldsymbol{\Theta} := \varphi^*\boldsymbol{\beta}$  are uniquely specified by (3.3) when  $\mathcal{H}$  is simply connected, it follows that a simply connected shell  $\mathcal{H}$  is stress-free if and only if  $\mathbf{G}$  and  $\mathbf{B}$  are specified by (3.3), i.e., we have

$$\mathcal{R}^{\mathcal{H}}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) = \mathbf{B}(\mathbf{X}, \mathbf{Z}) \mathbf{B}(\mathbf{Y}, \mathbf{W}) - \mathbf{B}(\mathbf{X}, \mathbf{W}) \mathbf{B}(\mathbf{Y}, \mathbf{Z}), \tag{3.4a}$$

$$\left(\nabla_{\mathbf{Y}}^{\mathcal{H}} \mathbf{B}\right)(\mathbf{X}, \mathbf{Z}) = \left(\nabla_{\mathbf{X}}^{\mathcal{H}} \mathbf{B}\right)(\mathbf{Y}, \mathbf{Z}). \tag{3.4b}$$

These are precisely the necessary and sufficient conditions for the simply-connected surface  $(\mathcal{H}, \mathbf{G}, \mathbf{B})$  to be isometrically embeddable in  $\mathbb{R}^3$ . In components, Eq. (3.4) reduce to

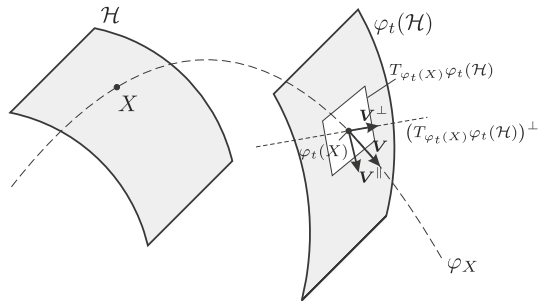
$$\begin{aligned} \mathcal{R}_{12|2}^{\mathcal{H}} &= B_{11} B_{22} - B_{12} B_{12}, \\ B_{11|2} &= B_{12|1}, \\ B_{22|1} &= B_{12|2}. \end{aligned}$$

### 3.4 Velocity and Acceleration

We define a motion to be a smooth curve  $t \mapsto \varphi_t \in \mathcal{C}$ , i.e.,  $\varphi_t : \mathcal{H} \rightarrow \mathcal{S}$  and denote  $\varphi(\cdot, t) := \varphi_t(\cdot)$  and  $\varphi_X(\cdot) := \varphi(X, \cdot)$ . At time  $t$ , the surface  $\varphi_t(\mathcal{H})$  has the metric  $\mathbf{g} := \tilde{\mathbf{g}}|_{\varphi_t(\mathcal{H})}$ , the Levi-Civita connection  $\nabla$ , the unit normal vector field  $\mathbf{n} \in \mathfrak{X}(\varphi_t(\mathcal{H}))^\perp$ , and the second fundamental form  $\boldsymbol{\beta} \in \Gamma(S^2 T^* \varphi_t(\mathcal{H}))$ . The material velocity is the mapping  $\mathbf{V} : \mathcal{H} \times \mathbb{R} \rightarrow T\mathcal{S}$ ,  $(X, t) \mapsto \mathbf{V}(X, t) := T_t \varphi_X [\partial_t]$ ,  $\forall X \in \mathcal{H}$ . We denote for each  $X \in \mathcal{H}$ ,  $\mathbf{V}_X := \mathbf{V}(X, \cdot)$  the vector field along the curve  $\varphi_X$ , i.e.,  $\mathbf{V}_X \in \mathfrak{X}(\varphi_X)$ . Using the decomposition of  $T\mathcal{S}$ , the material velocity can be decomposed as  $\mathbf{V}_X(t) = \mathbf{V}_X^\parallel(t) + \mathbf{V}_X^\perp(t)$ , where  $\mathbf{V}_X^\parallel(t) \in T_{\varphi_t(X)} \varphi_t(\mathcal{H})$  and  $\mathbf{V}_X^\perp(t) \in (T_{\varphi_t(X)} \varphi_t(\mathcal{H}))^\perp$ , i.e.,  $\mathbf{V}^\parallel$  is parallel to  $\varphi_t(\mathcal{H})$  and  $\mathbf{V}^\perp$  is normal to  $\varphi_t(\mathcal{H})$ , see Fig. 4.

The spatial velocity at a fixed time  $t$  is a vector field along  $\varphi_t(\mathcal{H})$  defined as  $\mathbf{v}(x, t) := \mathbf{V}(\varphi_t^{-1}(x), t)$ . Note that even though for a fixed  $t$ , the mapping  $\varphi_t : \mathcal{H} \rightarrow \mathcal{S}$  is a smooth embedding, the mapping  $\varphi : \mathcal{H} \times \mathbb{R} \rightarrow \mathcal{S}$  is, in general, not even an immersion. In fact, it can be seen that  $T_{(X,t)} \varphi$  is not necessarily injective. In  $\{X^A\}$  and  $\{x^a\}$ , in some local coordinate charts for  $\mathcal{H}$  and  $\mathcal{S}$ , respectively,  $T_{(X,t)} \varphi$  reads as follows

**Fig. 4** The decomposition of the material velocity  $\mathbf{V} = \mathbf{V}^{\parallel} + \mathbf{V}^{\perp}$ . The component  $\mathbf{V}^{\parallel}(X, t)$  is an element of  $T_{\varphi(X)}\varphi(\mathcal{H})$ , and  $\mathbf{V}^{\perp}(X, t)$  is normal to  $T_{\varphi(X)}\varphi(\mathcal{H})$



$$T_{(X,t)}\varphi = \begin{pmatrix} \frac{\partial\varphi^1}{\partial X^1} & \frac{\partial\varphi^1}{\partial X^2} & \frac{\partial\varphi^1}{\partial t} \\ \frac{\partial\varphi^2}{\partial X^1} & \frac{\partial\varphi^2}{\partial X^2} & \frac{\partial\varphi^2}{\partial t} \\ \frac{\partial\varphi^3}{\partial X^1} & \frac{\partial\varphi^3}{\partial X^2} & \frac{\partial\varphi^3}{\partial t} \end{pmatrix}.$$

Now, if  $\mathbf{V}(X, t) = \mathbf{0}$  (i.e.,  $\partial\varphi^a/\partial t = 0$  for  $a = 1, 2, 3$ ), or  $\varphi$  is an in-plane motion (i.e., in some coordinate chart for  $\mathcal{S}$  such that  $\partial_3 = \mathbf{n}$  on  $\varphi_t(\mathcal{H})$  we have  $\varphi^3 = 0$ ),  $T_{(X,t)}\varphi$  is clearly not injective. However, if  $T_{(X,t)}\varphi$  is injective, the implicit function theorem tells us that  $\varphi$  is a local diffeomorphism at  $(X, t)$ , and one can construct a local vector field  $\mathcal{V}$  on  $\mathcal{S}$  in a neighborhood of  $\varphi(X, t)$  such that  $\mathcal{V}(\varphi(X, t)) = \mathbf{V}(X, t) = \mathbf{v}(\varphi(X, t), t)$ . Hence, the material acceleration can be in this case unambiguously defined as

$$\mathbf{A}(X, t) = D_{\varphi_X} \mathbf{V}_X := \tilde{\nabla}_{\mathcal{V}} \mathcal{V}(\varphi(X, t)),$$

where  $D_{\varphi_X}$  is the covariant derivative along  $\varphi_X$ . Using the decomposition of the material velocity into parallel and normal components  $\mathcal{V} = \mathcal{V}^{\parallel} + \mathcal{V}^{\perp}$  and assuming that  $\varphi$  is a local diffeomorphism at  $(X, t)$ , one can write

$$\mathbf{A}(X, t) = \tilde{\nabla}_{\mathcal{V}}(\mathcal{V}^{\parallel} + \mathcal{V}^{\perp}) = \tilde{\nabla}_{\mathcal{V}} \mathcal{V}^{\parallel} + \tilde{\nabla}_{\mathcal{V}} \mathcal{V}^{\perp}.$$

Since  $\nabla$  is the Levi–Civita connection on  $T\varphi(\mathcal{H})$  induced by  $\mathbf{g}$ , it is torsion-free and hence

$$\tilde{\nabla}_{\mathcal{V}} \mathcal{V}^{\parallel} = [\mathcal{V}, \mathcal{V}^{\parallel}] + \tilde{\nabla}_{\mathcal{V}^{\parallel}} \mathcal{V} = [\mathcal{V}, \mathcal{V}^{\parallel}] + \tilde{\nabla}_{\mathcal{V}^{\parallel}} \mathcal{V}^{\parallel} + \tilde{\nabla}_{\mathcal{V}^{\parallel}} \mathcal{V}^{\perp}.$$

Note that since  $\mathcal{V} = \mathcal{V}(\tilde{\varphi}(X, t))$  does not explicitly depend on time, hence, denoting the Lie derivative by  $L$ , one can write  $[\mathcal{V}, \mathcal{V}^{\parallel}] = L_{\mathcal{V}} \mathcal{V}^{\parallel}$ , which is tangent to  $\mathcal{S}_t$ .<sup>3</sup> Following the definition of the second fundamental form, we have  $\tilde{\nabla}_{\mathcal{V}^{\parallel}} \mathcal{V}^{\parallel} = \nabla_{\mathcal{V}^{\parallel}} \mathcal{V}^{\parallel} + \beta(\mathcal{V}^{\parallel}, \mathcal{V}^{\parallel})\mathbf{n}$ . We let  $\mathcal{V}^n = \tilde{\mathbf{g}}(\mathcal{V}, \mathbf{n})$ , i.e.,  $\mathcal{V}^{\perp} = \mathcal{V}^n \mathbf{n}$ . The metric compatibility of  $\tilde{\nabla}$

<sup>3</sup> The Lie derivative along the vector field  $\mathcal{V}$  is defined as  $L_{\mathcal{V}} \mathcal{V}^{\parallel} = \left. \frac{d}{dt} \right|_{t=\mathcal{S}} [(\varphi_t \circ \varphi_{\mathcal{S}}^{-1})^* \mathcal{V}^{\parallel}]$ , where  $\varphi_t \circ \varphi_{\mathcal{S}}^{-1}$  is the flow of  $\mathcal{V}$ .



and the fact that  $\tilde{g}(\mathbf{n}, \mathbf{n}) = 1$  imply that for any vector  $\mathbf{W}$  along  $\varphi_t(\mathcal{H})$  in  $\mathcal{S}$ , we have

$$\frac{d}{dt}(\tilde{g}(\mathbf{n}, \mathbf{n})) = 2\tilde{g}(\tilde{\nabla}_{\mathbf{W}}\mathbf{n}, \mathbf{n}) = 0,$$

i.e.,  $\tilde{\nabla}_{\mathbf{W}}\mathbf{n} \in \mathfrak{X}(\varphi(\mathcal{H}))$ . Thus  $\tilde{\nabla}_{\mathbf{V}^{\parallel}}\mathbf{n} = -\mathbf{g}^{\sharp} \cdot \boldsymbol{\beta} \cdot \mathbf{V}^{\parallel}$ . Therefore

$$\tilde{\nabla}_{\mathbf{V}^{\parallel}}\mathbf{V}^{\perp} = \tilde{\nabla}_{\mathbf{V}^{\parallel}}(\mathcal{V}^n\mathbf{n}) = \tilde{\nabla}_{\mathbf{V}^{\parallel}}(\mathcal{V}^n)\mathbf{n} + \mathcal{V}^n\tilde{\nabla}_{\mathbf{V}^{\parallel}}\mathbf{n} = (d\mathcal{V}^n \cdot \mathbf{V}^{\parallel})\mathbf{n} - \mathcal{V}^n\mathbf{g}^{\sharp} \cdot \boldsymbol{\beta} \cdot \mathbf{V}^{\parallel}.$$

On the other hand, we have

$$\tilde{\nabla}_{\mathbf{V}}\mathbf{V}^{\perp} = \tilde{\nabla}_{\mathbf{V}}(\mathcal{V}^n\mathbf{n}) = \frac{d\mathcal{V}^n}{dt}\mathbf{n} + \mathcal{V}^n\tilde{\nabla}_{\mathbf{V}}\mathbf{n}.$$

However, as observed earlier,  $\tilde{\nabla}_{\mathbf{V}}\mathbf{n} \in \mathfrak{X}(\varphi(\mathcal{H}))$ , then, it follows that at  $\varphi(X, t)$ , one can write

$$(\tilde{\nabla}_{\mathbf{V}}\mathbf{V}^{\perp})^{\perp} = \frac{d\mathcal{V}^n}{dt}\mathbf{n}.$$

Let us now compute  $(\tilde{\nabla}_{\mathbf{V}}\mathbf{V}^{\perp})^{\parallel}$ . We consider an arbitrary vector field  $\mathbf{U}$  in  $\mathcal{S}$  such that  $\mathbf{U}$  is tangent to  $\mathcal{H}$  in a neighborhood of  $\varphi(X, t)$ , i.e.,  $\tilde{g}(\mathbf{V}^{\perp}, \mathbf{U}) = 0$ . Hence,  $\tilde{g}(\tilde{\nabla}_{\mathbf{V}}\mathbf{V}^{\perp}, \mathbf{U}) = -\tilde{g}(\mathbf{V}^{\perp}, \tilde{\nabla}_{\mathbf{V}}\mathbf{U})$ . However, at  $\varphi(X, t)$ , we have

$$\begin{aligned} \tilde{\nabla}_{\mathbf{V}}\mathbf{U} &= [\mathbf{V}, \mathbf{U}] + \tilde{\nabla}_{\mathbf{U}}\mathbf{V} = [\mathbf{V}, \mathbf{U}] + \tilde{\nabla}_{\mathbf{U}}\mathbf{V}^{\parallel} + \tilde{\nabla}_{\mathbf{U}}\mathbf{V}^{\perp} \\ &= [\mathbf{V}, \mathbf{U}] + \nabla_{\mathbf{U}}\mathbf{V}^{\parallel} + \boldsymbol{\beta}(\mathbf{V}^{\parallel}, \mathbf{U})\mathbf{n} + (d\mathcal{V}^n \cdot \mathbf{U})\mathbf{n} + \mathcal{V}^n\tilde{\nabla}_{\mathbf{U}}\mathbf{n}. \end{aligned}$$

Hence<sup>4</sup>

$$\tilde{g}(\mathbf{V}^{\perp}, \tilde{\nabla}_{\mathbf{V}}\mathbf{U}) = \mathcal{V}^n\boldsymbol{\beta}(\mathbf{V}^{\parallel}, \mathbf{U}) + \mathcal{V}^n(d\mathcal{V}^n \cdot \mathbf{U}).$$

Thus, it follows from  $\tilde{g}(\tilde{\nabla}_{\mathbf{V}}\mathbf{V}^{\perp}, \mathbf{U}) = -\tilde{g}(\mathbf{V}^{\perp}, \tilde{\nabla}_{\mathbf{V}}\mathbf{U})$  and by arbitrariness of  $\mathbf{U}$  that

$$(\tilde{\nabla}_{\mathbf{V}}\mathbf{V}^{\perp})^{\parallel} = -\mathcal{V}^n\mathbf{g}^{\sharp} \cdot \boldsymbol{\beta} \cdot \mathbf{V}^{\parallel} - \mathcal{V}^n(d\mathcal{V}^n)^{\sharp}.$$

Therefore, the parallel and normal components of the material acceleration read

$$\begin{aligned} \mathbf{A}^{\parallel} &= [\mathbf{V}, \mathbf{V}^{\parallel}] + \nabla_{\mathbf{V}^{\parallel}}\mathbf{V}^{\parallel} - 2\mathcal{V}^n\mathbf{g}^{\sharp} \cdot \boldsymbol{\beta} \cdot \mathbf{V}^{\parallel} - \mathcal{V}^n(d\mathcal{V}^n)^{\sharp}, \\ \mathbf{A}^{\perp} &= \left[\frac{d\mathcal{V}^n}{dt} + \boldsymbol{\beta}(\mathbf{V}^{\parallel}, \mathbf{V}^{\parallel}) + d\mathcal{V}^n \cdot \mathbf{V}^{\parallel}\right]\mathbf{n}. \end{aligned}$$

<sup>4</sup> Note that since the vector  $\mathbf{U}$  is tangent to  $\mathcal{H}$  at  $\varphi(X, t)$ , the vectors  $[\mathbf{V}, \mathbf{U}] = \mathbf{L}_{\mathbf{V}}\mathbf{U}$  and  $\tilde{\nabla}_{\mathbf{U}}\mathbf{n}$  are tangent to  $\mathcal{H}$  as well.

### 4 The Governing Equations of Motion

In this section, we derive the governing equations of motion for morphoelastic shells that include: balance of mass, balance of linear and angular momenta, and the kinetic equations of growth.

#### 4.1 Balance of Mass

We denote the material and spatial surface mass densities (mass per unit area) by  $\rho$  and  $\varrho$ , respectively, and let  $\mathcal{U}$  be any open set in  $\mathcal{H}$  with a smooth boundary. For a growing body, the balance of mass for a motion  $\varphi$  can be written as

$$\frac{d}{dt} \int_{\mathcal{U}} \rho dS = \int_{\mathcal{U}} S_m dS, \tag{4.1}$$

where  $S_m = S_m(X, t)$  is a given scalar field characterizing the material rate of change of mass per unit area. We postulate that the motion  $\varphi$  conserves the mass of the system, i.e.

$$\int_{\varphi_t(\mathcal{U})} \varrho ds = \int_{\mathcal{U}} \rho dS. \tag{4.2}$$

Recalling that  $dS = \sqrt{\det \mathbf{G}} dX^1 \wedge dX^2$ , it follows from (4.1) that  $\rho = \rho(X, \mathbf{G}, t)$ , and we find

$$\frac{1}{\sqrt{\det \mathbf{G}}} \frac{d}{dt} \left( \sqrt{\det \mathbf{G}} \rho \right) = S_m, \tag{4.3}$$

which, using the identity  $\frac{d}{dt} [\det \mathbf{A}(t)] = \det \mathbf{A}(t) \operatorname{tr} \left[ \mathbf{A}^{-1}(t) \frac{d}{dt} \mathbf{A}(t) \right]$ , gives the material local form of the balance of mass for a growing body as

$$\dot{\rho} + \frac{1}{2} \rho \operatorname{tr} \dot{\mathbf{G}} = S_m, \tag{4.4}$$

where the dot denotes total time differentiation. Note that if we write the evolving metric as in (2.10), i.e.,  $\mathbf{G}(X, t) = \mathbf{G}^0(X) e^{2\boldsymbol{\omega}(X,t)}$ , and use the identity  $\det(e^A) = e^{\operatorname{tr}(A)}$ , then (4.3) reads

$$\dot{\rho} + \rho \frac{d}{dt} [\operatorname{tr}(\boldsymbol{\omega})] = S_m.$$

Now, since  $J = \sqrt{\frac{\det \varphi^* \mathbf{g}}{\det \mathbf{G}}} = J(X, \varphi, \mathbf{G}, \mathbf{g})$ , and  $\rho = J\varrho$ , it follows from (4.4) that  $\varrho = \varrho(X, \varphi, \mathbf{G}, \mathbf{g}, t)$ , and we find<sup>5</sup>

$$\dot{\varrho} + \varrho \frac{J}{J} + \frac{1}{2} \varrho \operatorname{tr} \left( \frac{d\mathbf{G}}{dt} \right) = s_m, \tag{4.5}$$

<sup>5</sup> Note that (4.5) can also be obtained from (4.2) and (4.1) by writing

$$\frac{d}{dt} \int_{\varphi_t(\mathcal{U})} \varrho ds = \int_{\varphi_t(\mathcal{U})} s_m ds.$$

where  $s_m(X, t) = \frac{1}{J} S_m(X, t)$  is the spatial rate of change of mass per unit area. Using (3.2), one can write

$$j = \frac{1}{2} \frac{d}{dt} (\det \varphi^* \mathbf{g}) \frac{1}{\sqrt{\det \mathbf{G} \det \varphi^* \mathbf{g}}} - \frac{1}{2} \frac{d}{dt} (\det \mathbf{G}) \sqrt{\frac{\det \varphi^* \mathbf{g}}{(\det \mathbf{G})^3}}.$$

Therefore

$$\frac{j}{J} = \frac{1}{2} \text{tr}_C \left( \frac{d\varphi^* \mathbf{g}}{dt} \right) - \frac{1}{2} \text{tr} \left( \frac{d\mathbf{G}}{dt} \right),$$

where  $\text{tr}_C$  is the trace taken with respect to the metric  $C$ . Recalling the decomposition  $\mathbf{v} = v^\parallel + v^n \mathbf{n}$ , we have (Marsden and Hughes 1983; Verpoort 2008; Kadianakis and Travlopanos 2013)

$$\varphi_* \frac{d\varphi^* \mathbf{g}}{dt} = L_v \mathbf{g} = L_{v^\parallel} \mathbf{g} - 2v^n \boldsymbol{\beta},$$

and it follows that

$$\text{tr}_C \left( \frac{d\varphi^* \mathbf{g}}{dt} \right) = 2\text{div} v^\parallel - 2v^n \text{tr} \boldsymbol{\beta},$$

where  $\text{div}$  denotes the divergence on the surface  $\varphi_t(\mathcal{H})$ . Therefore, (4.5) gives the spatial local form of the balance of mass for a growing shell as  $\dot{\rho} + \rho \text{div} v^\parallel - \rho v^n \text{tr} \boldsymbol{\beta} = s_m$ .

### 4.2 Balance Laws

Given the right Cauchy–Green deformation tensor  $C$  and the extrinsic deformation tensor  $\Theta$ , the geometry of the deformed surface is uniquely defined (See Sect. 3.2). However, in order to specify the evolution of an element of the deformed surface, we need to know its position  $\varphi$  and its orientation by means of the normal vector field  $\mathcal{N} = \mathbf{n} \circ \varphi$ . Therefore, in the classical theory of nonlinear elasticity of shells, we define the action functional as the map  $S : \mathcal{C} \mapsto \mathbb{R}$

$$S(\varphi) = \int_{t_0}^{t_1} \int_{\mathcal{H}} \mathcal{L}(X, \varphi(X, t), \mathcal{N}(X, t), \dot{\varphi}(X, t), \tilde{\mathbf{g}} \circ \varphi(X, t), C(X, t), \Theta(X, t), \mathbf{G}(X)) dS(X) dt,$$

where  $\mathcal{L} = \mathcal{L}(X, \varphi, \mathcal{N}, \dot{\varphi}, \tilde{\mathbf{g}}, C, \Theta, \mathbf{G})$  is the Lagrangian density per unit surface area.<sup>6</sup> The governing equations of motion follow from Hamilton’s principle of least action, which states that the physical motion  $\varphi$  of  $\mathcal{H}$  between  $t_0$  and  $t_1$  is a critical point for the action functional, i.e.,

$$\delta S(\varphi) = 0.$$

<sup>6</sup> Since the Lagrangian density is a scalar, it depends on the metrics  $\mathbf{G}$  and  $\tilde{\mathbf{g}}$ .

In the present geometric theory of morphoelastic shells, the material first and second fundamental forms are dynamical variables that vary independently of the motion. Therefore, the action functional is modified to read

$$S(\varphi, \mathbf{G}, \mathbf{B}) = \int_{t_0}^{t_1} \int_{\mathcal{H}} \mathcal{L}(X, \varphi(X, t), \mathcal{N}(X, t), \dot{\varphi}(X, t), \tilde{\mathbf{g}} \circ \varphi(X, t), \mathbf{C}(X, t), \Theta(X, t), \mathbf{G}(X, t), \mathbf{B}(X, t)) dS(X, t) dt.$$

We recall that  $dS(X, t) = \sqrt{\det \mathbf{G}(X, t)} dX^1 \wedge dX^2$  and define the Lagrangian density  $\mathcal{L}$  by

$$\mathcal{L}(X, \varphi, \mathcal{N}, \dot{\varphi}, \tilde{\mathbf{g}}, \mathbf{C}, \Theta, \mathbf{G}, \mathbf{B}) = \frac{1}{2} \rho \tilde{\mathbf{g}}(\dot{\varphi}, \dot{\varphi}) - \mathcal{W}(X, \mathbf{C}, \Theta, \mathbf{G}, \mathbf{B}) - \mathcal{V}(X, \varphi, \mathcal{N}, \tilde{\mathbf{g}}),$$

where  $\mathcal{W} = \mathcal{W}(X, \mathbf{C}, \Theta, \mathbf{G}, \mathbf{B})$  is the elastic energy density per unit surface area (related to the elastic deformation of a surface element), and  $\mathcal{V} = \mathcal{V}(X, \varphi, \mathcal{N}, \tilde{\mathbf{g}})$  is the potential energy density per unit surface area (related to the position—respectively orientation—of a surface element in the body force—respectively moment—fields). Similar to the coordinate chart  $(X^1, X^2, X^3)$  previously defined for  $\mathcal{B}$ , let  $(x^1, x^2, x^3)$  be a local coordinate chart for  $\mathcal{S}$  such that at any point of the hypersurface  $\varphi(\mathcal{H})$ ,  $\{x^1, x^2\}$  is a local coordinate chart for  $\varphi(\mathcal{H})$  and the normal vector field  $\mathbf{n}$  to  $\varphi(\mathcal{H})$  is tangent to the coordinate curve  $x^3$ . Therefore, the Lagrangian density in this coordinate chart reads

$$\mathcal{L}(X, \varphi, \mathcal{N}, \dot{\varphi}, \tilde{\mathbf{g}}, \mathbf{C}, \Theta, \mathbf{G}, \mathbf{B}) = \frac{1}{2} \rho g_{ab} \dot{\varphi}^a \dot{\varphi}^b + \frac{1}{2} \rho (\dot{\varphi}^n)^2 - \mathcal{W}(X, \mathbf{C}, \Theta, \mathbf{G}, \mathbf{B}) - \mathcal{V}(X, \varphi, \mathcal{N}, \tilde{\mathbf{g}}). \tag{4.6}$$

Also, because growth is, in general, a non-conservative process, we use the Lagrange-d’Alembert’s principle for non conservative processes. Given non-conservative forces  $\mathbf{F}_\varphi, \mathbf{F}_\mathcal{N}, \mathbf{F}_\mathbf{G}$ , and  $\mathbf{F}_\mathbf{B}$  associated with the variations of the position, the orientation, and the first and second fundamental forms, respectively, the Lagrange-d’Alembert’s principle states that (Marsden and Ratiu 1994)

$$\delta S(\varphi, \mathbf{G}, \mathbf{B}) + \int_{t_0}^{t_1} \int_{\mathcal{H}} (\mathbf{F}_\varphi \cdot \delta\varphi + \mathbf{F}_\mathcal{N} \cdot \delta\mathcal{N} + \mathbf{F}_\mathbf{G} : \delta\mathbf{G} + \mathbf{F}_\mathbf{B} : \delta\mathbf{B}) dS dt = 0.$$

The sources of these forces depend on the particular underlying biological, biochemical, or physical processes leading to growth. Here, we assume the existence of a Rayleigh potential  $\mathcal{R} = \mathcal{R}(\dot{\varphi}, \dot{\mathcal{N}}, \tilde{\mathbf{g}}, \dot{\mathbf{G}}, \dot{\mathbf{B}}, \mathbf{G})$  such that

$$\mathbf{F}_\varphi = -\frac{\partial \mathcal{R}}{\partial \dot{\varphi}}, \quad \mathbf{F}_\mathcal{N} = -\frac{\partial \mathcal{R}}{\partial \dot{\mathcal{N}}}, \quad \mathbf{F}_\mathbf{G} = -\frac{\partial \mathcal{R}}{\partial \dot{\mathbf{G}}}, \quad \text{and} \quad \mathbf{F}_\mathbf{B} = -\frac{\partial \mathcal{R}}{\partial \dot{\mathbf{B}}}.$$

In the context of our theory, we disregard non-conservative forces due to the variations of position and orientation and assume that it is only due to growth, i.e., we assume that  $\mathcal{R} = \mathcal{R}(\dot{\mathbf{G}}, \dot{\mathbf{B}}, \mathbf{G})$ .

In order to take variations, we let  $\varphi_\epsilon$  be a 1-parameter family of motions such that  $\varphi_{0,t} = \varphi_t$ .<sup>7</sup> For fixed  $X$  and  $t$ , we consider the curve  $\varphi_{t,X} : \epsilon \mapsto \varphi_{t,X}(\epsilon) := \varphi_{\epsilon,t}(X)$  and define the variation of motion as the spatial vector field given by

$$\delta\varphi(X, t) = T_\epsilon\varphi_{t,X} [\partial_\epsilon] \Big|_{\epsilon=0} \in T_{\varphi_{\epsilon,t}(X)}\mathcal{S}.$$

Similarly, we let  $\mathbf{G}_\epsilon$  be a 1-parameter family of material metrics such that  $\mathbf{G}_{\epsilon=0} = \mathbf{G}$  and for fixed  $X$  and  $t$ , we define the variation of the metric by the material tensor given by

$$\delta\mathbf{G}(X, t) = \frac{d\mathbf{G}_\epsilon}{d\epsilon} \Big|_{\epsilon=0} (X, t).$$

Also, let  $\mathbf{B}_\epsilon$  be a 1-parameter family of second fundamental forms such that  $\mathbf{B}_{\epsilon=0} = \mathbf{B}$  and for fixed  $X$  and  $t$ , we define the variation of the second fundamental form by the material tensor given by

$$\delta\mathbf{B}(X, t) = \frac{d\mathbf{B}_\epsilon}{d\epsilon} \Big|_{\epsilon=0} (X, t).$$

It follows from Lagrange–d’Alembert’s principle that

$$\begin{aligned} & \int_{t_0}^{t_1} \int_{\mathcal{H}} \left( \frac{\partial \mathcal{L}}{\partial \varphi} \cdot \delta\varphi + \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \cdot \delta\mathcal{N} + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \cdot \delta\dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \tilde{\mathbf{g}}} : \delta\tilde{\mathbf{g}} \circ \varphi + \frac{\partial \mathcal{L}}{\partial \mathbf{C}} : \delta\mathbf{C} + \frac{\partial \mathcal{L}}{\partial \Theta} : \delta\Theta \right. \\ & \quad \left. + \frac{1}{\sqrt{\det \mathbf{G}}} \frac{\partial \sqrt{\det \mathbf{G}} \mathcal{L}}{\partial \mathbf{G}} : \delta\mathbf{G} + \frac{\partial \mathcal{L}}{\partial \mathbf{B}} : \delta\mathbf{B} \right) dSdt \\ & = \int_{t_0}^{t_1} \int_{\mathcal{H}} \left( \frac{\partial \mathcal{R}}{\partial \dot{\mathbf{G}}} : \delta\mathbf{G} + \frac{\partial \mathcal{R}}{\partial \dot{\mathbf{B}}} : \delta\mathbf{B} \right) dSdt. \end{aligned} \tag{4.7}$$

*Remark 4.1* Note that in taking the variation of the action, the variations of  $\mathbf{C}$  and  $\Theta$  must be such that they satisfy the compatibility Eq. (3.3). Since the variation of the action is taken by considering the variation of the deformation mapping  $\varphi$ , the resulting variations of  $\mathbf{C}$  and  $\Theta$  (cf. (4.10) and (4.11)) are trivially compatible, i.e., they trivially satisfy the compatibility Eq. (3.3). Hence these compatibility equations are not constraints.

If we vary  $\epsilon$ , for fixed time  $t$  and  $X \in \mathcal{H}$ , the material velocity  $\dot{\varphi}_\epsilon$  and the unit normal  $\mathcal{N}$  lie in  $T_{\varphi_{\epsilon,t}(X)}\mathcal{S}$ . Therefore, their variations are given by their covariant derivatives along the curve  $\varphi_{t,X}$  in  $\mathcal{S}$  evaluated at  $\epsilon = 0$ . By the symmetry lemma (see (do Carmo

<sup>7</sup> For fixed  $X$  and  $t$ , we let  $\varphi_{\epsilon,t}(X) := \varphi_\epsilon(X, t)$ .

1992; Nishikawa 2002)), we find the variation of the velocity as

$$\begin{aligned} \delta\dot{\varphi} &= D_{\varphi_t, X(\epsilon)} \dot{\varphi}_\epsilon \Big|_{\epsilon=0} = D_{\varphi_t, X(\epsilon)} [T_t \varphi_{\epsilon, X} [\partial_t] \Big|_{\epsilon=0}] = D_{\varphi_X(t)} [T_\epsilon \varphi_{t, X} [\partial_\epsilon]] \Big|_{\epsilon=0} \\ &= D_{\varphi_X(t)} \delta\varphi =: \frac{D\delta\varphi}{dt}. \end{aligned} \tag{4.8}$$

Following (Kadianakis and Travlopanos 2013), the variation of the unit normal vector field is given by

$$\delta\mathcal{N} = D_{\varphi_t, X(\epsilon)} \mathcal{N}_\epsilon \Big|_{\epsilon=0} = \tilde{\nabla}_{\delta\varphi} \mathcal{N} - (d(\delta\varphi^n))^\sharp,$$

where  $\sharp$  denotes the operation of raising indices (sharp operator). In components

$$\delta\mathcal{N}^a = \delta\varphi^b \tilde{\nabla}_{\partial_b} \mathcal{N} - \frac{\partial(\delta\varphi^n)}{\partial x^b} g^{ab} \partial_a = - \left( \delta\varphi^b \beta^a{}_b + \frac{\partial(\delta\varphi^n)}{\partial x^b} g^{ab} \right) \partial_a. \tag{4.9}$$

For any fixed time  $t$  and  $X \in \mathcal{H}$ , the right Cauchy–Green deformation tensor  $C_\epsilon$  lies in  $S^2 T_X^* \mathcal{H}$ . Therefore, the variation of  $C$  is given by its total derivative with respect to  $\epsilon$  evaluated at  $\epsilon = 0$ :

$$\delta C = \frac{dC_\epsilon}{d\epsilon} \Big|_{\epsilon=0} = \frac{d}{d\epsilon} (\varphi_\epsilon^* \mathbf{g}_\epsilon) \Big|_{\epsilon=0} = \frac{d}{d\epsilon} (\varphi^* \varphi_* \varphi_\epsilon^* \mathbf{g}_\epsilon) \Big|_{\epsilon=0} = \varphi^* L_{\delta\varphi} \mathbf{g}.$$

Note that (Marsden and Hughes 1983; Verpoort 2008; Kadianakis and Travlopanos 2013)  $L_{\delta\varphi} \mathbf{g} = L_{\delta\varphi} \mathbf{g} - 2\delta\varphi^n \boldsymbol{\beta}$ . Hence, we have

$$\delta C = \varphi^* L_{\delta\varphi} \mathbf{g} - 2\delta\varphi^n \varphi^* \boldsymbol{\beta}.$$

In components, it reads

$$\delta C_{AB} = F^a{}_A g_{ac} \delta\varphi^c|_B + F^b{}_B g_{bc} \delta\varphi^c|_A - 2\delta\varphi^n F^a{}_A F^b{}_B \beta_{ab}. \tag{4.10}$$

The extrinsic deformation tensor  $\Theta_\epsilon$  lies in the same space  $S^2 T_X^* \mathcal{H}$  for fixed time  $t$  and  $X \in \mathcal{H}$ . Hence, the variation of  $\Theta$  is given by its total derivative with respect to  $\epsilon$  evaluated at  $\epsilon = 0$ :

$$\delta\Theta = \frac{d\Theta_\epsilon}{d\epsilon} \Big|_{\epsilon=0} = \frac{d}{d\epsilon} (\varphi_\epsilon^* \boldsymbol{\beta}_\epsilon) \Big|_{\epsilon=0} = \frac{d}{d\epsilon} (\varphi^* \varphi_* \varphi_\epsilon^* \boldsymbol{\beta}_\epsilon) \Big|_{\epsilon=0} = \varphi^* L_{\delta\varphi} \boldsymbol{\beta}.$$

The variation in terms of the Lie derivative of the second fundamental form is given by (Verpoort 2008; Kadianakis and Travlopanos 2013)

$$L_{\delta\varphi} \boldsymbol{\beta} = L_{\delta\varphi} \boldsymbol{\beta} - \delta\varphi^n \mathcal{C} + \text{Hess}_{\delta\varphi^n},$$

where  $\mathcal{C}$  denotes the third fundamental form of the surface  $\varphi_t(\mathcal{H})$  and is defined for  $\mathbf{x}, \mathbf{y} \in \mathfrak{X}(\varphi_t(\mathcal{H}))$  by

$$\mathcal{C}(\mathbf{x}, \mathbf{y}) := \mathbf{g} \left( \tilde{\nabla}_x \mathbf{n}, \tilde{\nabla}_y \mathbf{n} \right),$$

and  $\text{Hess}_f$  denotes the Hessian of the scalar-valued function  $f$  and is defined for  $\mathbf{x}, \mathbf{y} \in \mathfrak{X}(\varphi_t(\mathcal{H}))$  by

$$\text{Hess}_f(\mathbf{x}, \mathbf{y}) := \mathbf{g} \left( \tilde{\nabla}_{\mathbf{x}} (df)^\sharp, \mathbf{y} \right).$$

Therefore, we have

$$\delta \Theta = \varphi^* L_{\delta\varphi} \boldsymbol{\beta} - \delta\varphi^n \varphi^* \mathbf{C} + \varphi^* \text{Hess}_{\delta\varphi^n},$$

or, in components

$$\begin{aligned} \delta\Theta_{AB} &= F^a{}_A F^b{}_B \beta_{ab|c} \delta\varphi^c + F^a{}_A \beta_{ac} \delta\varphi^c|_B + F^b{}_B \beta_{bc} \delta\varphi^c|_A \\ &\quad - \delta\varphi^n F^a{}_A F^b{}_B \beta_{ac} \beta_{bd} g^{cd} + F^b{}_A \left( \frac{\partial \delta\varphi^n}{\partial x^b} \right) |_B. \end{aligned} \tag{4.11}$$

The variation of the ambient space metric vanishes identically since it is compatible with the connection, i.e.,

$$\delta \tilde{\mathbf{g}} \circ \varphi = D_{\varphi_t, \mathbf{x}} \tilde{\mathbf{g}} \circ \varphi = \tilde{\nabla}_{\delta\varphi} \tilde{\mathbf{g}} = \mathbf{0}. \tag{4.12}$$

In order to obtain the balance laws, we fix the first and the second fundamental forms, i.e.,  $\delta \mathbf{G} = \mathbf{0}$  and  $\delta \mathbf{B} = \mathbf{0}$  and vary  $\varphi$ . Therefore, following (4.7) and by arbitrariness of  $\delta\varphi$  and  $\nabla^{\mathcal{H}} \delta\varphi^n$ , we find the following Euler–Lagrange equations<sup>8</sup>

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \varphi^a} - \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b \beta^b{}_a - \frac{1}{\sqrt{\det \mathbf{G}}} \frac{d}{dt} \left[ \sqrt{\det \mathbf{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^a} \right] - 2 \left[ \frac{\partial \mathcal{L}}{\partial C_{AB}} F^b{}_A g_{ab} \right] |_B \\ - \left[ \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^b{}_A \beta_{ab} \right] |_B - \left[ \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^b{}_A \right] |_B \beta_{ab} = 0, \end{aligned} \tag{4.13a}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \varphi^n} + \left( \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b g^{ab} F^{-A}{}_a \right) |_A - \frac{1}{\sqrt{\det \mathbf{G}}} \frac{d}{dt} \left[ \sqrt{\det \mathbf{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^n} \right] - 2 \frac{\partial \mathcal{L}}{\partial C_{AB}} F^b{}_A F^a{}_B \beta_{ab} \\ - \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^b{}_A F^a{}_B \beta_{ac} \beta_{bd} g^{cd} + \left[ \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^b{}_A \right) |_B F^{-D}{}_b \right] |_D = 0, \end{aligned} \tag{4.13b}$$

along with the following equations prescribing a vanishing initial velocity vector field and vanishing boundary conditions for the loading on the boundary  $\partial \mathcal{H}$

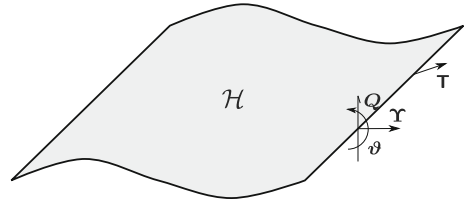
$$\left. \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \right|_{t=t_0} = \mathbf{0}, \tag{4.14a}$$

$$\left( 2 \frac{\partial \mathcal{L}}{\partial C_{AB}} F^c{}_A g_{ac} + 2 \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A \beta_{ac} \right) \mathbb{T}_B = 0, \tag{4.14b}$$

$$\left[ \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AC}} F^b{}_A \right) |_C F^{-B}{}_b + \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b g^{ab} F^{-B}{}_a \right] \mathbb{T}_B = 0, \tag{4.14c}$$

<sup>8</sup> We denote by  $F^{-A}{}_a$  the components of  $\mathbf{F}^{-1}$ , the inverse of  $\mathbf{F}$ . See Appendix 7 for the details of the derivation of (4.13) and (4.14) following (4.7).

**Fig. 5** Shell boundary loads:  $\Upsilon$ ,  $\vartheta$ , and  $\mathbf{Q}$  are the surface traction, the moment, and the shear, respectively.  $\mathbf{T}$  denotes the outward in-plane normal



$$\frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^a{}_A \mathbf{T}_B = 0, \tag{4.14d}$$

where  $\mathbf{T}$  is the outward in-plane vector field normal to the boundary  $\partial\mathcal{H}$ .

*Remark 4.2* Note that we can modify the Lagrange–d’Alembert’s principle in order to prescribe non-vanishing initial and boundary conditions on  $\partial\mathcal{H}$ . Let  $\Upsilon$  be the boundary surface traction,  $\vartheta$  be the boundary moment,  $\mathbf{Q}$  be the boundary shear, and  $\mathbf{V}_{t_0}$  be the initial velocity vector field (see Fig. 5). We write the Lagrange–d’Alembert’s principle as

$$\begin{aligned} &\delta S(\varphi, \mathbf{G}, \mathbf{B}) + \int_{t_0}^{t_1} \int_{\mathcal{H}} (\mathbf{F}_G : \delta \mathbf{G} + \mathbf{F}_B : \delta \mathbf{B}) \, dS dt \\ &+ \int_{t_0}^{t_1} \int_{\partial\mathcal{H}} \left( \mathbf{J} \Upsilon^a g_{ab} \delta \varphi^b + \mathbf{J} \vartheta^a \delta \varphi^n{}_{,A} F^{-A}{}_a + \mathbf{J} \mathbf{Q} \delta \varphi^n \right) \, dL dt \\ &+ \int_{\mathcal{H}} \rho \mathbf{V}_{t_0} \cdot \delta \varphi_{t_0} \, dS = 0, \end{aligned}$$

and from (4.14) we have

$$\begin{aligned} &\left. \frac{\partial \mathcal{L}}{\partial \varphi} \right|_{t=t_0} = \rho \mathbf{V}_{t_0}, \\ &\left( 2 \frac{\partial \mathcal{L}}{\partial C_{AB}} F^c{}_A g_{ac} + 2 \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A \beta_{ac} \right) \mathbf{T}_B = \mathbf{J} g_{ab} \Upsilon^b, \\ &\left[ \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AC}} F^b{}_A \right) \Big|_C F^{-B}{}_b + \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b g^{ab} F^{-B}{}_a \right] \mathbf{T}_B = \mathbf{J} \mathbf{Q}, \\ &\frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^a{}_A \mathbf{T}_B = \mathbf{J} \vartheta^a. \end{aligned}$$

We introduce the following surface tensors:

Second Piola-Kirchhoff stress tensor:  $\mathbf{S} = 2 \frac{\partial \mathcal{W}}{\partial \mathbf{C}}$ , in components,  $S^{AB} = 2 \frac{\partial \mathcal{W}}{\partial C_{AB}}$ ;

First Piola-Kirchhoff stress tensor:  $\mathbf{P} = 2 \mathbf{F} \frac{\partial \mathcal{W}}{\partial \mathbf{C}}$ , in components,

$$P^{bB} = 2 \frac{\partial \mathcal{W}}{\partial C_{AB}} F^b{}_A;$$



Cauchy stress tensor:  $\sigma = \frac{2}{J} F \frac{\partial \mathcal{W}}{\partial C} F^T$ , in components,  $\sigma^{ab} = \frac{2}{J} \frac{\partial \mathcal{W}}{\partial C_{AB}} F^a{}_A F^b{}_B$ ;

Material couple-stress tensor:  $M = \frac{\partial \mathcal{W}}{\partial \Theta}$ , in components,  $M^{AB} = \frac{\partial \mathcal{W}}{\partial \Theta_{AB}}$ ;

Two-point couple-stress tensor:  $\mathcal{M} = F \frac{\partial \mathcal{W}}{\partial \Theta}$ , in components,  $\mathcal{M}^{bB} = \frac{\partial \mathcal{W}}{\partial \Theta_{AB}} F^b{}_A$ ;

Spatial couple-stress tensor:  $\mu = \frac{1}{J} F \frac{\partial \mathcal{W}}{\partial \Theta} F^T$ , in components,

$$\mu^{ab} = \frac{1}{J} \frac{\partial \mathcal{W}}{\partial \Theta_{AB}} F^a{}_A F^b{}_B.$$

We further introduce the following notations for the external loads

$$\text{External body forces : } \mathcal{B} = -\frac{1}{\rho} \frac{\partial \mathcal{V}}{\partial \varphi},$$

$$\text{External body moments : } \mathcal{L} = -\frac{1}{\rho} \frac{\partial \mathcal{V}}{\partial \mathcal{N}}.$$

Recalling the balance of mass (4.3), we have for the Lagrangian density (4.6)

$$\frac{1}{\sqrt{\det G}} \frac{d}{dt} \left[ \sqrt{\det G} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \right] = \rho \tilde{g} A + S_m \tilde{g} V.$$

Therefore, the Euler–Lagrange equations (4.13) read

$$\left( P^{aB} + \beta^a{}_b \mathcal{M}^{bB} \right)_{|B} + \beta^a{}_b \left( \mathcal{M}^{bB} \right)_{|B} + \rho \mathcal{B}^a - \rho \beta^a{}_b \mathcal{L}^b - S_m \dot{\varphi}^a = \rho A^a, \tag{4.15a}$$

$$\left( P^{aB} + \beta^a{}_b \mathcal{M}^{bB} \right) F^c{}_B \beta_{ac} - \left( \mathcal{M}^{bB} \right)_{|B} F^{-D}{}_b \Big|_D + \rho \mathcal{B}^n + \left( \rho \mathcal{L}^a F^{-A}{}_a \right)_{|A} - S_m \dot{\varphi}^n = \rho A^n, \tag{4.15b}$$

and the vanishing initial and boundary conditions

$$V|_{t=t_0} = \mathbf{0}, \tag{4.16a}$$

$$\left( g_{ab} P^{bB} + 2\beta_{ab} \mathcal{M}^{bB} \right) \Upsilon_B = 0, \tag{4.16b}$$

$$\left[ \mathcal{M}^{bC} \right]_C F^{-B}{}_b + \rho \mathcal{L}^a F^{-B}{}_a \Big|_B \Upsilon_B = 0, \tag{4.16c}$$

$$\mathcal{M}^{aB} \Upsilon_B = 0. \tag{4.16d}$$

We apply the Piola transform<sup>9</sup> to (4.15) and (4.16) and obtain the spatial version of the balance of linear momenta as

$$\left( \sigma^{ac} + \beta^a{}_b \mu^{bc} \right)_{|c} + \beta^a{}_b \mu^{bc} \Big|_c + \varrho \mathcal{B}^a - \varrho \beta^a{}_b \mathcal{L}^b - s_m \dot{\varphi}^a = \varrho A^a, \tag{4.17a}$$

<sup>9</sup> Recall the Piola identity  $(JF^{-A}{}_a)_{|A} = 0$ .

$$\left(\sigma^{ac} + \beta^a{}_b \mu^{bc}\right) \beta_{ac} - \mu^{ab}{}_{|ab} + \varrho \mathcal{B}^n + (\varrho \mathcal{L}^a)_{|a} - s_m \dot{\varphi}^n = \varrho A^n, \quad (4.17b)$$

and the vanishing initial and boundary conditions

$$\mathbf{V}|_{t=t_0} = \mathbf{0}, \quad (4.18a)$$

$$\left(\sigma^{ac} + 2\beta^a{}_b \mu^{bc}\right) \mathbf{t}_c = 0, \quad (4.18b)$$

$$\left(\mu^{ab}{}_{|b} + \varrho \mathcal{L}^a\right) \mathbf{t}_a = 0, \quad (4.18c)$$

$$\mu^{ab} \mathbf{t}_b = 0, \quad (4.18d)$$

where  $\mathbf{t}$  is the outward in-plane vector field normal to the boundary  $\partial\varphi_t(\mathcal{H})$ . By pulling back the system of Eq. (4.17) with the mapping  $\varphi$ , we obtain the Euler–Lagrange equations in the convected manifold  $(\mathcal{H}, \mathbf{C})$ <sup>10</sup> in terms of the convected stress tensor  $\Sigma = \varphi^* \sigma = S/J$  and the convected couple-stress tensor  $\Lambda = \varphi^* \mu = M/J$ . If we denote by a double stroke  $\parallel$  the covariant derivative corresponding to the Levi–Civita connection of  $(\mathcal{H}, \mathbf{C})$ , the convected Euler–Lagrange equation reads<sup>11</sup>

$$\begin{aligned} &\left(\Sigma^{AB} + C^{-AC} \Theta_{CD} \Lambda^{DB}\right) \parallel_B + C^{-AC} \Theta_{CD} \Lambda^{DB} \parallel_B + \varrho F^{-A}{}_a \mathcal{B}^a \\ &\quad - \varrho C^{-AC} \Theta_{CD} F^{-D}{}_a \mathcal{L}^a - s_m F^{-A}{}_a \dot{\varphi}^a = \varrho F^{-A}{}_a A^a, \end{aligned} \quad (4.19a)$$

$$\left(\Sigma^{AB} + C^{-AC} \Theta_{CD} \Lambda^{DB}\right) \Theta_{AB} - \Lambda^{AB} \parallel_{AB} + \varrho \mathcal{B}^n + \left(\varrho \mathcal{L}^a F^{-A}{}_a\right) \parallel_A - s_m \dot{\varphi}^n = \varrho A^n, \quad (4.19b)$$

and the vanishing initial and boundary conditions

$$\mathbf{V}|_{t=t_0} = \mathbf{0}, \quad (4.20a)$$

$$\left(\Sigma^{AB} + 2C^{-AC} \Theta_{CD} \Lambda^{DB}\right) \mathbf{T}_B = 0, \quad (4.20b)$$

$$\left(\Lambda^{AB} \parallel_B + \varrho F^{-A}{}_a \mathcal{L}^a\right) \mathbf{T}_A = 0, \quad (4.20c)$$

$$\Lambda^{AB} \mathbf{T}_B = 0. \quad (4.20d)$$

Recall that in terms of the deformation mapping  $\varphi : \mathcal{H} \rightarrow \mathbb{R}^3$ , we can write the components of  $\mathbf{C}$  and  $\Theta$  in a local chart  $\{X, Y\}$  of  $\mathcal{H}$  as follow

<sup>10</sup> We define the convected manifold to be the material manifold  $\mathcal{H}$  equipped with the right Cauchy–Green deformation tensor  $\mathbf{C}$ .

<sup>11</sup> The components of  $\mathbf{C}^{-1}$ , the inverse of  $\mathbf{C}$ , are denoted by  $C^{-AB}$ .

$$C_{AB} = \varphi_{,A} \cdot \varphi_{,B},$$

$$\Theta_{AB} = \varphi_{,AB} \cdot \frac{\varphi_{,X} \times \varphi_{,Y}}{\|\varphi_{,X} \times \varphi_{,Y}\|},$$

where  $\cdot$ ,  $\times$ , and  $\|\cdot\|$ , respectively, denote the dot product, the cross product, and the standard norm in  $\mathbb{R}^3$ .

Given a constitutive relation, the stress and the couple-stress tensors can be written in terms of the first and the second fundamental forms of the deformed surface. On the other hand, the first and the second fundamental forms of the deformed surface can be written in terms of the motion  $\varphi$  such that the compatibility Eq. (3.3) are trivially satisfied. Therefore, the system of Eq. (4.15) (or (4.17), or (4.19)) is a set of three equations for three unknowns (the three components of the motion), and together with the initial and boundary conditions (4.16) (or (4.18), or (4.20)), they form the complete set of governing equations for the morphoelastic shell problem.

*Remark 4.3* Note that both systems of Eqs. (4.15) and (4.17) reduce to the elastic shell equilibrium equations for a zero-acceleration motion in the absence of growth ( $s_m = 0$ ) and dissipation ( $\mathcal{R} = 0$ ). See (Chien 1943) (Equations 2.8), (Sanders Jr 1961) (Equations 55 and 56), (Naghdi 1963) (Equations 5.36), and (Koiter 1966) (Equations 6.3 and 6.4) where the shell problem is described by a system of three equations involving six stress and couple-stress components. Note that an alternative description is provided by a system of six equations involving ten stress and couple-stress components, see (Green and Zerna 1950) (Equations 3.6 and 3.11) and (Eriksen and Truesdell 1958) (Equations 26.6, 26.7 and 26.10).

*Remark 4.4* Following the definitions of the surface tensors and based on the symmetry of the right Cauchy–Green tensor and the extrinsic deformation tensor, we have the following symmetries for the stress tensors, which are the local forms of the balance of angular momenta

$$\begin{aligned} S^T &= S, & \Sigma^T &= \Sigma, & P F^T &= P^T F, & \sigma^T &= \sigma, \\ M^T &= M, & \Lambda^T &= \Lambda, & \mathcal{M} F^T &= \mathcal{M}^T F, & \mu^T &= \mu. \end{aligned} \tag{4.21}$$

### 4.3 Kinetic Equations of Growth

To obtain the kinetic equations governing the evolution of growth, we fix the motion, i.e.,  $\delta\varphi = 0$ , and vary the first and the second fundamental forms. From the arbitrariness of  $\delta\mathbf{G}$  and  $\delta\mathbf{B}$ , we obtain from (4.7) the following kinetic equations for the evolution of the first and the second fundamental forms of  $\mathcal{H}$ :

$$\frac{1}{\sqrt{\det \mathbf{G}}} \frac{\partial(\sqrt{\det \mathbf{G}} \mathcal{L})}{\partial \mathbf{G}} = \frac{\partial \mathcal{R}}{\partial \dot{\mathbf{G}}},$$

$$\frac{\partial \mathcal{L}}{\partial \mathbf{B}} = \frac{\partial \mathcal{R}}{\partial \dot{\mathbf{B}}}.$$

Therefore, we find the following for the Lagrangian density (4.6)

$$\frac{\partial \mathcal{R}}{\partial \dot{\mathbf{G}}} = \frac{1}{2} \left( \frac{1}{2} \rho \tilde{\mathbf{g}}(\dot{\varphi}, \dot{\varphi}) - \mathcal{W} - \mathcal{V} \right) \mathbf{G}^\sharp - \frac{\partial \mathcal{W}}{\partial \mathbf{G}}, \tag{4.22a}$$

$$\frac{\partial \mathcal{R}}{\partial \dot{\mathbf{B}}} = - \frac{\partial \mathcal{W}}{\partial \mathbf{B}}. \tag{4.22b}$$

Assuming the existence of a Rayleigh potential  $\mathcal{R} = \mathcal{R}(\dot{\mathbf{G}}, \dot{\mathbf{B}}, \mathbf{G})$ , we introduce a variational characterization for the variation of energy in the shell due to growth. The system of Eq. (4.22) provides a coupling of the rate of change of the first and the second fundamental forms of  $\mathcal{H}$  with its current state of deformation through the elastic energy density  $\mathcal{W}$  of the material. Therefore, the evolution of the geometry of the shell, i.e., the growth of the morphoelastic shell, is governed by (4.22).

*Remark 4.5* Yavari (2010) discussed the kinetic equation for the evolving metric in the case of bulk growth for three-dimensional nonlinear elasticity. Note, however, that there was a missing term in equation (2.179), which should be corrected to read

$$\delta \int_{t_0}^{t_1} \int_B \mathcal{L} dV dt = \int_{t_0}^{t_1} \int_B \left[ \delta \mathcal{L} + \frac{1}{2} \mathcal{L} \text{tr}(\delta \mathbf{G}) \right] dV dt.$$

The kinetic equation (2.181) in (Yavari 2010) should also be corrected to read

$$\frac{\partial \mathcal{R}}{\partial \dot{\mathbf{G}}} = \frac{\partial \mathcal{L}}{\partial \mathbf{G}} + \frac{1}{2} \mathcal{L} \mathbf{G}^\sharp. \tag{4.23}$$

Ignoring inertial forces and in the absence of body forces, (4.23) reads

$$\frac{\partial \mathcal{R}}{\partial \dot{\mathbf{G}}} = - \frac{\partial \mathcal{W}}{\partial \mathbf{G}} - \frac{1}{2} \mathcal{W} \mathbf{G}^\sharp.$$

Following the material covariance of nonlinear elasticity, Lu and Papadopoulos (2000); Yavari et al. (2006) proved the following relation:

$$\frac{1}{2} \mathbf{S}_o \mathbf{G} = \frac{\partial \mathcal{W}}{\partial \mathbf{G}} \mathbf{G} + \frac{\partial \mathcal{W}}{\partial \mathbf{C}} \mathbf{C},$$

where  $\mathbf{S}_o$  is a stress-like tensor conjugate to  $\dot{\mathbf{G}}$ .  $\mathbf{S}_o$  is associated with the material evolution and is a measure of anisotropy of the medium:  $\mathbf{S}_o = \mathbf{0}$  for an isotropic material. Therefore, (4.23) can be rewritten as

$$\frac{\partial \mathcal{R}}{\partial \dot{\mathbf{G}}} \mathbf{G} = - \frac{1}{2} \mathcal{W} \mathbf{G}^\sharp \mathbf{G} + \frac{1}{2} \mathbf{S} \mathbf{C} - \frac{1}{2} \mathbf{S}_o \mathbf{G}. \tag{4.24}$$

In the context of the multiplicative decomposition of the deformation gradient  $\mathbf{F} = \mathbf{F}_e \mathbf{F}_g$ , the kinetic equation coupling the evolution of growth and stress is written in terms of the growth tensor  $\mathbf{F}_g$ . Fusi et al. (2006) derived it using the so-called principle

of maximum entropy production rate, and [Ambrosi and Guana \(2007\)](#); [Olsson and Klarbring \(2008\)](#) used the Clausius-Duhem inequality. Note that these equations are both equivalent and similar in form to (4.24).

*Example 4.1* As an example, we consider the following Rayleigh potential

$$\mathcal{R}(\dot{\mathbf{G}}, \dot{\mathbf{B}}, \mathbf{G}) = \alpha_1 \text{tr}(\dot{\mathbf{G}}) + \alpha_2 \text{tr}(\dot{\mathbf{G}}^2) + \beta_1 \text{tr}(\dot{\mathbf{B}}) + \beta_2 \text{tr}(\dot{\mathbf{B}}^2). \tag{4.25}$$

In components

$$\begin{aligned} \mathcal{R}(\dot{\mathbf{G}}, \dot{\mathbf{B}}, \mathbf{G}) = & \alpha_1 \dot{G}_{AB} G^{AB} + \alpha_2 \dot{G}_{AB} \dot{G}_{CD} G^{CA} G^{DB} + \beta_1 \dot{B}_{AB} G^{AB} \\ & + \beta_2 \dot{B}_{AB} \dot{B}_{CD} G^{CA} G^{DB}. \end{aligned}$$

Therefore, if we assume a static shell in the absence of body forces and moments, the kinetic equations (4.22) read

$$\dot{\mathbf{G}} = -\frac{\alpha_1}{2\alpha_2} \mathbf{G} - \frac{1}{4\alpha_2} \mathcal{W} \mathbf{G} - \frac{1}{2\alpha_2} \mathbf{G} \frac{\partial \mathcal{W}}{\partial \mathbf{G}} \mathbf{G}, \tag{4.26a}$$

$$\dot{\mathbf{B}} = -\frac{\beta_1}{2\beta_2} \mathbf{G} - \frac{1}{2\beta_2} \mathbf{G} \frac{\partial \mathcal{W}}{\partial \mathbf{B}} \mathbf{G}. \tag{4.26b}$$

In components

$$\begin{aligned} \dot{G}_{AB} = & -\frac{\alpha_1}{2\alpha_2} G_{AB} - \frac{1}{4\alpha_2} \mathcal{W} G_{AB} - \frac{1}{2\alpha_2} G_{AC} \frac{\partial \mathcal{W}}{\partial G_{DC}} G_{DB}, \\ \dot{B}_{AB} = & -\frac{\beta_1}{2\beta_2} G_{AB} - \frac{1}{2\beta_2} G_{AC} \frac{\partial \mathcal{W}}{\partial B_{DC}} G_{DB}. \end{aligned}$$

We assume a Saint Venant–Kirchhoff constitutive model, for which the strain energy density  $\mathcal{W}$  is given by<sup>12</sup>

$$\begin{aligned} \mathcal{W} = & \frac{h}{4} \left\{ \mu \text{tr}[(\mathbf{C} - \mathbf{G})^2] + \frac{\mu\lambda}{2\mu + \lambda} [\text{tr}(\mathbf{C} - \mathbf{G})]^2 \right\} \\ & + \frac{h^3}{12} \left\{ \mu \text{tr}[(\mathbf{\Theta} - \mathbf{B})^2] + \frac{\mu\lambda}{2\mu + \lambda} [\text{tr}(\mathbf{\Theta} - \mathbf{B})]^2 \right\} \\ = & \frac{Eh}{8(1 + \nu)} \left\{ \text{tr}[(\mathbf{C} - \mathbf{G})^2] + \frac{\nu}{1 - \nu} [\text{tr}(\mathbf{C} - \mathbf{G})]^2 \right\} \\ & + \frac{Eh^3}{24(1 + \nu)} \left\{ \text{tr}[(\mathbf{\Theta} - \mathbf{B})^2] + \frac{\nu}{1 - \nu} [\text{tr}(\mathbf{\Theta} - \mathbf{B})]^2 \right\}, \end{aligned} \tag{4.27}$$

<sup>12</sup> For details on the derivation of the Saint Venant–Kirchhoff shell model, see [\(Fox et al. 1993; Le Dret and Raoult 1993; Lods and Miara 1995; Miara 1998; Lods and Miara 1998; Friesecke et al. 2002a, b, 2003\)](#).

where  $\lambda$  and  $\mu$  are, respectively, Lamé’s first and second parameters,  $E$  is Young’s modulus, and  $\nu$  is Poisson’s ratio. Therefore, for a Saint Venant–Kirchhoff constitutive model, the kinetic equations (4.26) read

$$\begin{aligned} \dot{G}_{AB} &= -\frac{\alpha_1}{2\alpha_2} G_{AB} - \frac{h}{16\alpha_2} \left[ \mu (C_{KL} - G_{KL}) (C^{KL} - G^{KL}) + \frac{\mu\lambda}{2\mu + \lambda} (C^K_K - 2)^2 \right] G_{AB} \\ &\quad - \frac{h^3}{48\alpha_2} \left[ \mu (\Theta_{KL} - B_{KL}) (\Theta^{KL} - B^{KL}) + \frac{\mu\lambda}{2\mu + \lambda} (\Theta^K_K - B^K_K)^2 \right] G_{AB} \\ &\quad + \frac{h}{4\alpha_2} \left[ \mu C_{AK} (C_{LB} - G_{LB}) G^{KL} + \frac{\mu\lambda}{2\mu + \lambda} (C^K_K - 2) C_{AB} \right] \\ &\quad + \frac{h^3}{12\alpha_2} \left[ \mu (\Theta_{AK} - B_{AK}) (\Theta_{LB} - B_{LB}) G^{KL} + \frac{\mu\lambda}{2\mu + \lambda} (\Theta^K_K - B^K_K) (\Theta_{AB} - B_{AB}) \right], \\ \dot{B}_{AB} &= -\frac{\beta_1}{2\beta_2} G_{AB} + \frac{h^3}{12\beta_2} \left[ \mu (\Theta_{AB} - B_{AB}) + \frac{\mu\lambda}{2\mu + \lambda} (\Theta^K_K - B^K_K) \right] G_{AB}. \end{aligned}$$

### 5 Examples

As applications of the proposed geometric theory of morphoelastic shells, we study in this section some examples of growth and the induced residual stress fields. We look at the stress-free growth of an initially planar sheet and study the residual stress and geometry evolution of a morphoelastic infinitely long circular cylindrical shell subject to an internal pressure and a morphoelastic initially planar circular disk.

#### 5.1 Stress-Free Growth Fields for an Initially Flat Simply Connected Shell

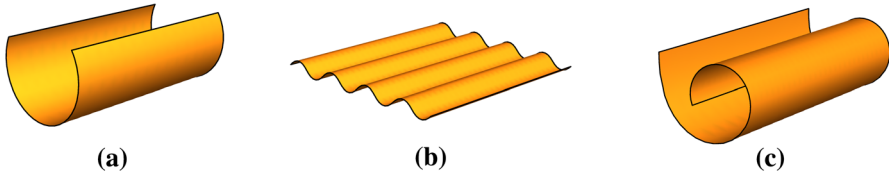
We consider an initially flat thin shell  $\mathcal{B}$  such that its mid-surface  $\mathcal{H}$  is simply connected. Let  $(X, Y, Z)$  be the standard coordinate chart for  $\mathbb{R}^3$  such that the hyperplane  $X^3 = 0$  contains  $\mathcal{H}$ . We assume that the morphoelastic shell is undergoing a growth field that is modeled by the following evolving metric for  $\mathcal{B}$ :

$$\bar{G} = \begin{pmatrix} e^{2\bar{\omega}_X(X,Y,Z,t)} & 0 & 0 \\ 0 & e^{2\bar{\omega}_Y(X,Y,Z,t)} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which corresponds to the following evolving first and second fundamental forms for the mid-surface  $\mathcal{H}$ :

$$\begin{aligned} G &= \begin{pmatrix} e^{2\omega_X(X,Y,t)} & 0 \\ 0 & e^{2\omega_Y(X,Y,t)} \end{pmatrix}, \\ B &= \begin{pmatrix} -K_X(X, Y, t)e^{2\omega_X(X,Y,t)} & 0 \\ 0 & -K_Y(X, Y, t)e^{2\omega_Y(X,Y,t)} \end{pmatrix}, \end{aligned}$$

where  $\omega_A(X, Y, t) = \bar{\omega}_A(X, Y, 0, t)$  and  $K_A(X, Y, t) = \frac{\partial \bar{\omega}_A}{\partial Z}(X, Y, 0, t)$  for  $A = X, Y$ . Following Sect. 3.3, the growth of a simply-connected shell is stress-free if and



**Fig. 6** Visualization of a few stress-free material evolutions of an initially planar sheet with the prescribed evolving fundamental forms such that the in-plane growth is uniform, i.e.,  $\omega_A = \omega_A(t)$  for  $A = X, Y$ , the Gaussian curvature is vanishing, i.e.,  $K_X K_Y = 0$ , and the nonzero principal curvature is such that  $K_X = K_X(X, t)$  or  $K_Y = K_Y(Y, t)$ . We assume for these figures that  $K_Y = 0$  and **a**  $K_X = K_X(t)$  to grow to a cylindrical portion, **b**  $K_X(X, t) = k_1(t) \sin(k_2(t)X)$ , where  $k_1 = k_1(t)$  and  $k_2 = k_2(t)$  are some arbitrary functions of time resulting in a sheet with sinusoidal rippling, and **c**  $K_X(X, t) = k(t)\sqrt{X}$ , for  $X > 0$ , where  $k = k(t)$  is some arbitrary function of time

only if the relations (3.4) hold. In the case of an initially flat morphoelastic simply-connected shell, the growth is stress-free if and only if

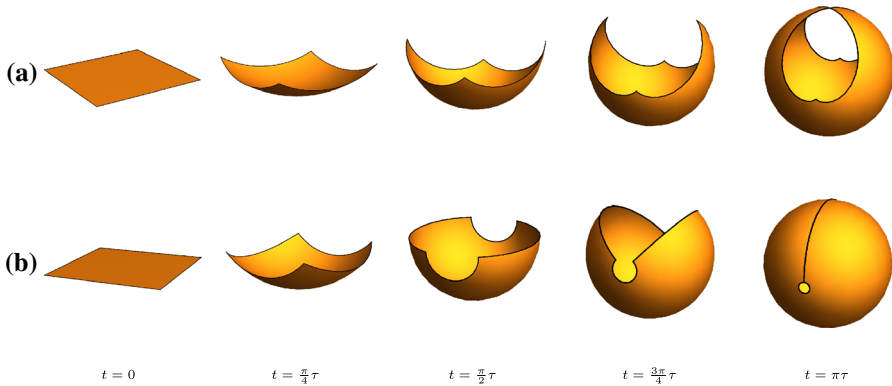
$$\begin{aligned}
 & e^{2\omega_X} \left[ \left( \frac{\partial \omega_Y}{\partial Y} - \frac{\partial \omega_X}{\partial Y} \right) \frac{\partial \omega_X}{\partial Y} - \frac{\partial^2 \omega_X}{\partial Y^2} \right] \\
 & + e^{2\omega_Y} \left[ \left( \frac{\partial \omega_X}{\partial X} - \frac{\partial \omega_Y}{\partial X} \right) \frac{\partial \omega_Y}{\partial X} - \frac{\partial^2 \omega_Y}{\partial X^2} \right] = K_X K_Y e^{2\omega_X} e^{2\omega_Y}, \\
 & \frac{\partial K_X}{\partial Y} = (K_Y - K_X) \frac{\partial \omega_X}{\partial Y}, \\
 & \frac{\partial K_Y}{\partial X} = (K_X - K_Y) \frac{\partial \omega_Y}{\partial X}.
 \end{aligned}$$

Now we consider the following simplifying assumptions:

- If we assume that the in-plane growth is uniform, i.e.,  $\omega_A = \omega_A(t)$  for  $A = X, Y$ , we find that the growth is stress-free if and only if  $K_X = K_X(X, t)$ ,  $K_Y = K_Y(Y, t)$ , and  $K_X K_Y = 0$ . This case includes the stress-free growth of a planar sheet into a cylindrical portion. See Fig. 6 for examples of evolutions of planar sheets into flat surfaces with stress-free growth.
- If we assume that the evolving curvatures  $K_X$  and  $K_Y$  are uniform, i.e.,  $K_A = K_A(t)$  for  $A = X, Y$ , we distinguish the following cases:
  - If  $K_X \neq K_Y$ , then the growth is stress-free if and only if  $\omega_X$  and  $\omega_Y$  are uniform and  $K_X K_Y = 0$ . This is precisely the case of a planar sheet evolving to a cylindrical portion with a stress-free growth (see Fig. 6a).
  - If  $K = K_X = K_Y$ , then the growth is stress-free if and only if

$$\begin{aligned}
 & e^{2\omega_X} \left[ \left( \frac{\partial \omega_Y}{\partial Y} - \frac{\partial \omega_X}{\partial Y} \right) \frac{\partial \omega_X}{\partial Y} - \frac{\partial^2 \omega_X}{\partial Y^2} \right] \\
 & + e^{2\omega_Y} \left[ \left( \frac{\partial \omega_X}{\partial X} - \frac{\partial \omega_Y}{\partial X} \right) \frac{\partial \omega_Y}{\partial X} - \frac{\partial^2 \omega_Y}{\partial X^2} \right] = K^2 e^{2\omega_X} e^{2\omega_Y}.
 \end{aligned}$$

- If we assume that the in-plane growth is isotropic, i.e.,  $\omega = \omega_X = \omega_Y$ , we distinguish the following cases:



**Fig. 7** Example 5.1: Visualization of the stress-free material evolution of an initially planar sheet with the prescribed evolving fundamental forms (5.2) and (5.3), shown, respectively, in **a**, **b**, at different times. Note that the change of shape of the shell is due to growth and not stretch; such an evolution is stress-free

- If  $K = K_X = K_Y$ , then the growth is stress-free if and only if  $K$  is uniform and  $\frac{\partial^2 \omega}{\partial Y^2} + \frac{\partial^2 \omega}{\partial X^2} = -K^2 e^{2\omega}$ . In particular, if  $K_X = K_Y = 0$ , then the growth is stress-free if and only if  $\omega$  is harmonic. See Example 5.1 and Fig. 7 for examples of such a stress-free growth.
- If  $K_X \neq K_Y$ , then the growth is stress-free if and only if

$$\begin{aligned} \frac{\partial^2 \omega}{\partial Y^2} + \frac{\partial^2 \omega}{\partial X^2} &= -K_X K_Y e^{2\omega}, \\ \frac{\partial K_X}{\partial Y} &= (K_Y - K_X) \frac{\partial \omega}{\partial Y}, \\ \frac{\partial K_Y}{\partial X} &= (K_X - K_Y) \frac{\partial \omega}{\partial X}. \end{aligned}$$

See Example 5.2 and Fig. 8 for examples of such a stress-free growth assuming that  $K_X = -K_Y$ .

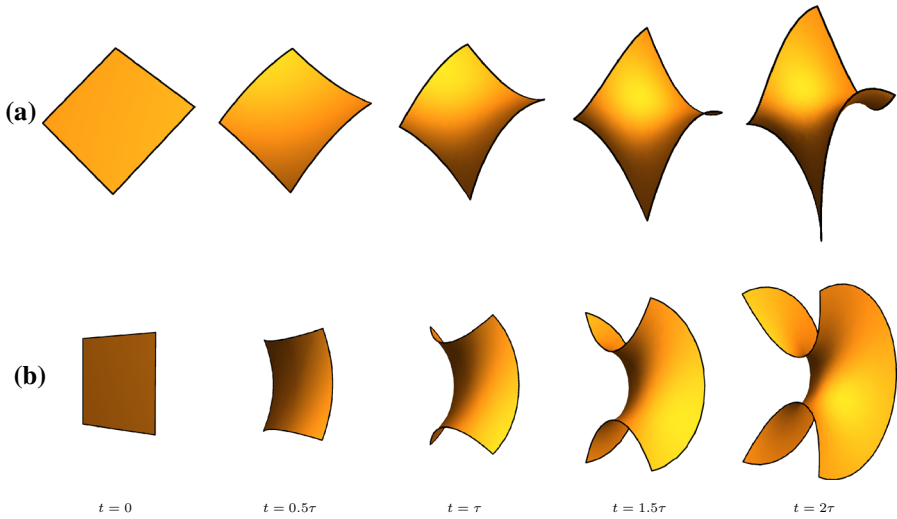
*Example 5.1* In this example, we consider a morphoelastic initially planar square sheet in the  $XY$ -plane such that center of the shell coincides with the origin of the coordinate system and the sides of the shell are parallel to the  $X$  and  $Y$  axes. We assume that both the in-plane and the out-of-plane growths are isotropic, i.e.,  $\omega = \omega_X = \omega_Y$ , and  $K = K_X = K_Y$ . Therefore, the growth is stress-free if and only if  $K = K(t)$  is a uniform arbitrary function of time and  $\omega$  is such that

$$\frac{\partial^2 \omega}{\partial Y^2} + \frac{\partial^2 \omega}{\partial X^2} = -K^2 e^{2\omega}. \tag{5.1}$$

Following Polyanin and Zaitsev (2004), a solution of (5.1) is given by

$$\omega(X, Y, t) = \frac{1}{2} \ln \left( \frac{A^2(t) + B^2(t)}{K^2(t) \cosh^2 [C(t) + A(t)X + B(t)Y]} \right),$$





**Fig. 8** Example 5.2: Visualization of the stress-free material evolution of an initially planar sheet with the prescribed evolving fundamental forms (5.6) and (5.7), shown, respectively, in **a**, **b** at different times. Note that the change of shape of the shell is due to growth and not stretch; such an evolution is stress-free

for some arbitrary functions of time  $A = A(t)$ ,  $B = B(t)$ ,  $C = C(t)$ , and  $K = K(t)$ . Therefore, the first and the second fundamental forms read

$$G = \frac{A^2(t) + B^2(t)}{K^2(t) \cosh^2 [C(t) + A(t)X + B(t)Y]} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B = -\frac{A^2(t) + B^2(t)}{K(t) \cosh^2 [C(t) + A(t)X + B(t)Y]} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is readily seen that every point of the surface is an umbilical point (the principal curvatures are equal to  $K(t)$ ). Therefore, at a given time  $t$ , we have a surface of constant nonnegative curvature  $K^2(t)$ , and hence it is either a planar ( $K = 0$ ) or a spherical ( $K > 0$ ) surface of radius  $1/K(t)$  (see Figure 7).

The functions  $A = A(t)$ ,  $B = B(t)$ ,  $C = C(t)$ , and  $K = K(t)$  define the time evolution of the first and the second fundamental forms. Given a constitutive equation for the material, their evolution can subsequently be obtained from the kinetic equations (4.22) governing the evolution of growth. As an example, and for the purpose of illustrating the non-trivial evolution of the initially planar shell as a result of a stress-free growth, we consider the following cases:

- We assume that  $A(t) = t/\tau$ ,  $B(t) = t/\tau$ ,  $C(t) = 0$ , and  $K_0(t) = \sqrt{2}t/\tau$ , where  $\tau$  is some growth characteristic time. It follows that  $\omega(X, Y, t) = -\ln \{ \cosh [(X + Y)t/\tau] \}$ , such that at  $t = 0$  they satisfy  $\omega(X, Y, 0) = 0$  and  $K(0) = 0$ . Therefore, we have the following evolving first and second fundamental forms:

$$\mathbf{G} = \frac{1}{\cosh^2 [(X + Y)t/\tau]} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = -\frac{\sqrt{2}t/\tau}{\cosh^2 [(X + Y)t/\tau]} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.2}$$

- We assume that  $A(t) = 2t/\tau$ ,  $B(t) = 0$ ,  $C(t) = 0$  and  $K_0(t) = 2t/\tau$ . It follows that  $\omega(X, Y, t) = -\ln \{\cosh [2Xt/\tau]\}$ , such that at  $t = 0$  they satisfy  $\omega(X, Y, 0) = 0$  and  $K(X, Y, 0) = 0$ . Therefore, we have the following evolving first and second fundamental forms:

$$\mathbf{G} = \frac{1}{\cosh^2 [2Xt/\tau]} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = -\frac{2t/\tau}{\cosh^2 [2Xt/\tau]} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.3}$$

We visualize in Fig. 7, the evolution of the initially planar sheet with the prescribed fundamental forms (5.2) and (5.3).

*Example 5.2* In this example, we consider a morphoelastic initially flat square sheet in the  $XY$ -plane such that the center of the shell coincides with the origin of the coordinate system and the sides of the shell are parallel to the  $X$  and  $Y$  axes. We assume that the in-plane growth is isotropic, i.e.,  $\omega = \omega_X = \omega_Y$ , and assume that  $K = K_X = -K_Y \neq 0$ . We look for  $\omega$  and  $K$  such that the growth is stress-free, i.e., such that

$$\frac{\partial^2 \omega}{\partial Y^2} + \frac{\partial^2 \omega}{\partial X^2} = K^2 e^{2\omega}, \tag{5.4a}$$

$$\frac{\partial K}{\partial Y} = -2K \frac{\partial \omega}{\partial Y}, \tag{5.4b}$$

$$\frac{\partial K}{\partial X} = -2K \frac{\partial \omega}{\partial X}. \tag{5.4c}$$

It follows from (5.4b) and (5.4c) that  $K(X, Y, t) = K_o(t)e^{-2\omega(X, Y, t)}$ , for some arbitrary function of time  $K_o = K_o(t)$ . Therefore, (5.4a) now reads

$$\frac{\partial^2 \omega}{\partial Y^2} + \frac{\partial^2 \omega}{\partial X^2} = K_o^2 e^{-2\omega}. \tag{5.5}$$

Following Polyanin and Zaitsev (2004), a solution for (5.5) is given by

$$\omega(X, Y, t) = -\frac{1}{2} \ln \left( \frac{A^2(t) + B^2(t)}{K_o^2(t) \cosh^2 (C(t) + A(t)X + B(t)Y)} \right),$$

for some arbitrary functions of time  $A = A(t)$ ,  $B = B(t)$ ,  $C = C(t)$ , and  $K_o(t)$ . As an example, and for the purpose of illustrating the non-trivial form the initially flat shell could adopt as a result of a stress-free growth, we consider the following cases:

- We assume that  $A(t) = t/\tau$ ,  $B(t) = t/\tau$ ,  $C(t) = 0$ , and  $K_0(t) = \sqrt{2}t/\tau$ . It follows that

$$\omega(X, Y, t) = \ln \{\cosh [(X + Y)t/\tau]\}, \quad K(X, Y, t) = \frac{\sqrt{2}t/\tau}{\cosh^2 [(X + Y)t/\tau]},$$

such that at  $t = 0$  they satisfy  $\omega(X, Y, 0) = 0$  and  $K(X, Y, 0) = 0$ . Therefore, we have the following evolving first and second fundamental forms:

$$G = \cosh^2 [(X + Y)t/\tau] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \sqrt{2}t/\tau \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.6)$$

- We assume that  $A(t) = 2t/\tau$ ,  $B(t) = 0$ ,  $C(t) = 0$ , and  $K_0(t) = 2t/\tau$ . It follows that

$$\omega(X, Y, t) = \ln \{ \cosh [2Xt/\tau] \}, \quad K(X, Y, t) = \frac{2t/\tau}{\cosh^2 [2Xt/\tau]},$$

such that at  $t = 0$  they satisfy  $\omega(X, Y, 0) = 0$  and  $K(X, Y, 0) = 0$ . Therefore, we have the following evolving first and second fundamental forms:

$$G = \cosh^2 [2Xt/\tau] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = 2t/\tau \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.7)$$

We visualize in Fig. 8, the evolution of the initially planar sheet with the prescribed fundamental forms (5.6) and (5.7).

*Remark 5.1* In the previous examples, we obtained the first and the second fundamental forms for stress-free growth fields. Recall that a growth field leaves the surface stress-free if and only if it is embeddable in  $\mathbb{R}^3$ . Therefore, given a surface  $(\mathcal{H}, G, B)$  with a stress-free growth field, we can find an isometric embedding of it in  $\mathbb{R}^3$  by integrating for the  $\mathbb{R}^3$ -valued function  $f$  the following system of partial differential equations written in a local chart  $\{X, Y\}$  of  $\mathcal{H}$ :

$$f_{,AB} = \Gamma^C_{AB} f_{,C} + B_{AB} N, \quad (5.8)$$

where  $N = \frac{f_{,X} \times f_{,Y}}{\|f_{,X} \times f_{,Y}\|}$ ,  $\times$  and  $\|\cdot\|$ , respectively, denote the cross product and the standard norm in  $\mathbb{R}^3$ ,  $\Gamma^C_{AB}$  is the Christoffel symbol of the Levi–Civita connection  $\nabla^{\mathcal{H}}$  in the local chart  $\{X^1, X^2\}$ . The integrability conditions for (5.8) are the equality of the mixed partial for  $f$ , i.e.,  $f_{,XY} = f_{,YX}$ , which is equivalent to the stress-free growth compatibility conditions (or the embeddability conditions) (3.4). Figures 6, 7, and 8 are obtained by plotting  $f$  in  $\mathbb{R}^3$  following the numerical integration of (5.8). We fix the rigid body motion of the surface by assuming  $f(0, 0) = \mathbf{0}$ ,  $f_{,X}(0, 0) = (\sqrt{G_{11}}(0, 0), 0, 0)^T$ , and  $f_{,Y}(0, 0) = (0, \sqrt{G_{22}}(0, 0), 0)^T$ , where  $\top$  denotes transpose of a vector in  $\mathbb{R}^3$ .

### 5.2 An Infinitely Long Morphoelastic Circular Cylindrical Shell

We consider an infinitely long morphoelastic thin hollow circular cylinder  $\mathcal{B}$  under uniform internal pressure  $p_i = p_i(t)$ , with thickness  $h$  and mid-radius  $R_o$ , made of a homogeneous isotropic material. Let  $\mathcal{B}$  undergo a circumferential radially symmetric but non-uniform growth through its thickness. In the cylindrical coordinates  $(R, \Phi, Z)$ ,

such that  $R \geq 0$ ,  $0 \leq \Phi \leq 2\pi$ , and  $Z \in \mathbb{R}$ , we represent growth by the following evolving material metric

$$\bar{\mathbf{G}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 e^{2\bar{\omega}(R,t)} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which corresponds to the following evolving first and second fundamental forms for the mid-cylinder  $\mathcal{H}$  of radius  $R_o$  in the coordinate system  $(\Phi, Z)$ :

$$\mathbf{G} = \begin{pmatrix} R_o^2 e^{2\omega(t)} & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -[1 + R_o K(t)] R_o e^{2\omega(t)} & 0 \\ 0 & 0 \end{pmatrix},$$

where  $\omega(t) = \bar{\omega}(R_o, t)$  and  $K(t) = \frac{\partial \bar{\omega}}{\partial R}(R_o, t)$ .

Let  $(r, \phi, z)$  be the cylindrical coordinate system for the Euclidean ambient space. Based on the symmetry of the problem, and in order to find the growth-induced residual stress field, we embed the material manifold into the Euclidean ambient space to form a circular cylindrical shell of radius  $r = r(t)$  such that  $(\phi, z) = (\Phi, Z)$ . The spatial first and second fundamental forms for the cylindrical shell in the cylindrical coordinate system  $(\phi, \xi)$  can be written as

$$\mathbf{g} = \begin{pmatrix} r^2(t) & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} -r(t) & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the deformation tensors read

$$\mathbf{C} = \begin{pmatrix} r^2(t) & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\Theta} = \begin{pmatrix} -r(t) & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that since the cylinder is made of a homogeneous and isotropic material, and because of the radial symmetry of the problem, we have

$$\sigma^{\phi\phi}(t) = \sigma^{\phi\phi}(r(t), \omega(t), K(t)), \quad \sigma^{z\phi}(t) = 0, \quad \sigma^{zz}(t) = \sigma^{zz}(r(t), \omega(t), K(t)), \tag{5.9a}$$

$$\mu^{\phi\phi}(t) = \mu^{\phi\phi}(r(t), \omega(t), K(t)), \quad \mu^{z\phi}(t) = 0, \quad \mu^{zz}(t) = \mu^{zz}(r(t), \omega(t), K(t)). \tag{5.9b}$$

It follows that the only non-trivial equilibrium equation is (4.17b), which is simplified to read<sup>13</sup>

$$\left( \sigma^{\phi\phi} - \frac{1}{r} \mu^{\phi\phi} \right) r - p_i = 0. \tag{5.11}$$

<sup>13</sup> Following (4.17b), there are three equilibrium equations

$$\left( \sigma^{ac} + \beta^a_b \mu^{bc} \right)_{|c} + \beta^a_b \mu^{bc}_{|c} = 0, \quad \text{for } a = r, \phi, \tag{5.10a}$$

$$\left( \sigma^{ac} + \beta^a_b \mu^{bc} \right) \beta_{ac} - \mu^{ab}_{|ab} + p_i = 0. \tag{5.10b}$$

We assume a Saint Venant–Kirchhoff constitutive model, for which the strain energy density  $\mathcal{W}$  is given by (4.27). Therefore, the nonzero components of the material stress and couple stress tensors read

$$\begin{aligned} \sigma^{\phi\phi} &= h \frac{E}{2(1-\nu^2)} \left( \frac{r^2(t)}{R_o^2} e^{-2\omega(t)} - 1 \right) \frac{1}{r^2(t)}, \\ \sigma^{zz} &= h \frac{E\nu}{2(1-\nu^2)} \left( \frac{r^2(t)}{R_o^2} e^{-2\omega(t)} - 1 \right) \frac{1}{r^2(t)}, \end{aligned} \tag{5.12a}$$

$$\begin{aligned} \mu^{\phi\phi} &= \frac{h^3}{6} \frac{E}{2(1-\nu^2)} \left[ (1 + R_o K(t)) R_o e^{2\omega(t)} - r(t) \right] \frac{e^{-2\omega(t)}}{R_o^2 r^2(t)}, \\ \mu^{zz} &= \frac{h^3}{6} \frac{E\nu}{2(1-\nu^2)} \left[ (1 + R_o K(t)) R_o e^{2\omega(t)} - r(t) \right] \frac{e^{-2\omega(t)}}{R_o^2 r^2(t)}. \end{aligned} \tag{5.12b}$$

Then, it follows from (5.11) that

$$\begin{aligned} \frac{r^3(t)}{R_o^3 e^{3\omega(t)}} - \frac{p_i(t) R_o e^{\omega(t)}}{Y_c h} \frac{r^2(t)}{R_o^2 e^{2\omega(t)}} + \left( \frac{1}{6} \frac{h^2}{R_o^2 e^{2\omega(t)}} - 1 \right) \frac{r(t)}{R_o e^{\omega(t)}} \\ - \frac{1}{6} \frac{h^2}{R_o^2 e^{\omega(t)}} (1 + R_o K(t)) = 0, \end{aligned} \tag{5.13}$$

where  $Y_c = \frac{E}{2(1-\nu^2)} = \mu + \frac{\mu\lambda}{2\mu+\lambda}$ . We introduce the following dimensionless quantities:

$$\hat{h} = \frac{h}{R_o}, \quad \hat{K} = R_o K, \quad \hat{r} = \frac{r}{R_o}, \quad \hat{p}_i = \frac{p_i}{Y_c},$$

and rewrite (5.13) as

$$\frac{\hat{r}^3(t)}{e^{3\omega(t)}} - \frac{\hat{p}_i(t) e^{\omega(t)}}{\hat{h}} \frac{\hat{r}^2(t)}{e^{2\omega(t)}} + \left( \frac{\hat{h}^2}{6e^{2\omega(t)}} - 1 \right) \frac{\hat{r}(t)}{e^{\omega(t)}} - \frac{\hat{h}^2}{6e^{\omega(t)}} (1 + \hat{K}(t)) = 0. \tag{5.14}$$

Equation (5.14) is a cubic equation in  $\hat{r}(t)e^{-\omega(t)}$  that has at least one real solution, which is calculated in closed form using the method of Cardano–Tartaglia (See (Cox 2012) for more details on the method). To obtain closed form solutions for (5.14), we use the following change of variable

$$\zeta = \frac{\hat{r}(t)}{e^{\omega(t)}} - \frac{\hat{p}_i(t) e^{\omega(t)}}{3h}.$$

Footnote 13 continued

Because of the symmetry of the problem and the isotropy of the material, the stresses take the form (5.9). This implies that Eq. (5.10a) are trivially satisfied and the terms containing derivatives in (5.10b) vanish. Therefore, we are left with Eq. (5.11) as the only non-trivial equilibrium equation.

In terms of  $\zeta$ , (5.14) reads

$$\zeta^3 + m\zeta + n = 0, \tag{5.15}$$

where  $m = \frac{\hat{h}^2}{6e^{2\omega(t)}} - 1 - \frac{\hat{p}_i^2(t)e^{2\omega(t)}}{3\hat{h}^2}$  and  $n = -\frac{\hat{p}_i(t)e^{\omega(t)}}{27\hat{h}} \left( \frac{2\hat{p}_i(t)^2e^{2\omega(t)}}{\hat{h}^2} - \frac{3\hat{h}^2}{2e^{2\omega(t)}} + 9 \right) - \frac{\hat{h}^2}{6e^{\omega(t)}} \left( 1 + \hat{K}(t) \right)$ .

The discriminant of (5.15) reads  $\Delta = -4m^3 - 27n^2$ . We distinguish three cases:

- (i) When  $\Delta > 0$ , (5.13) has three real solutions, of which we pick the positive one (the physically meaningful one). For  $i \in \{0, 1, 2\}$ , these three solutions are

$$\hat{r}(t) = \frac{\hat{p}_i(t)e^{2\omega(t)}}{3\hat{h}} + 2e^{\omega(t)}\sqrt{\frac{-m}{3}} \cos \left[ \frac{1}{3} \cos^{-1} \left( \frac{-n}{2} \sqrt{\frac{27}{-m^3}} \right) + \frac{2i\pi}{3} \right]. \tag{5.16}$$

- (ii) When  $\Delta = 0$ , (5.13) has two real solutions, of which we pick the positive one. These two solutions are

$$\hat{r}(t) = \frac{\hat{p}_i(t)e^{2\omega(t)}}{3\hat{h}} + \frac{3ne^{\omega(t)}}{m}, \quad \hat{r}(t) = \frac{\hat{p}_i(t)e^{2\omega(t)}}{3\hat{h}} - \frac{3ne^{\omega(t)}}{2m}. \tag{5.17}$$

- (iii) When  $\Delta < 0$ , (5.13) has a unique real solution, which reads

$$\hat{r}(t) = \frac{\hat{p}_i(t)e^{2\omega(t)}}{3\hat{h}} + e^{\omega(t)} \left( \frac{-n + \sqrt{\frac{-\Delta}{27}}}{2} \right)^{\frac{1}{3}} + e^{\omega(t)} \left( \frac{-n - \sqrt{\frac{-\Delta}{27}}}{2} \right)^{\frac{1}{3}}. \tag{5.18}$$

Now we only need to find the evolution of the first and the second fundamental forms of the material manifold to characterize the growth of the cylindrical shell. We assume the following Rayleigh potential

$$\mathcal{R}(\dot{\mathbf{G}}, \dot{\mathbf{B}}, \mathbf{G}) = \alpha_1 \text{tr}(\dot{\mathbf{G}}) + \alpha_2 \text{tr}(\dot{\mathbf{G}}^2) + \beta_1 \text{tr}(\dot{\mathbf{B}}) + \beta_2 \text{tr}(\dot{\mathbf{B}}^2). \tag{5.19}$$

Therefore, Eq. (4.22) yield the following kinetic equations for  $\omega$  and  $K$

$$\begin{aligned} \dot{\omega} = & -\frac{\alpha_1}{4\alpha_2} - \frac{Y_c h}{32\alpha_2} \left( \frac{r^2}{R_o^2} e^{-2\omega} - 1 \right)^2 + \frac{Y_c h}{8\alpha_2} \left( \frac{r^2}{R_o^2} e^{-2\omega} - 1 \right) \frac{r^2}{R_o^2} e^{-2\omega} \\ & + \frac{Y_c h^3}{32\alpha_2} \left( K + \frac{1}{R_o} - \frac{r}{R_o^2} e^{-2\omega} \right)^2, \end{aligned} \tag{5.20a}$$

$$\dot{K} = -2\dot{\omega} \left( K + \frac{1}{R_o} \right) + \frac{\beta_1}{2\beta_2} - \frac{Y_c h^3}{12\beta_2} \left( K + \frac{1}{R_o} - \frac{r}{R_o^2} e^{-2\omega} \right). \tag{5.20b}$$

The system of ordinary differential Eq. (5.20) along with the initial conditions  $\omega(0) = 0$  and  $K(0) = 0$  (see (2.11)) is an initial-value problem. Following the existence and uniqueness theorems of the initial-value problems (Coddington and Levinson 1955),

$\exists T > 0$  such that (5.20) has a unique solution  $(\omega, K)$  for  $t \in [0, T]$  with  $\omega(0) = 0$  and  $K(0) = 0$ .

*Example 5.3* In this example, we assume that the growth through the thickness of the hollow cylinder  $\mathcal{B}$  is non-uniform and symmetric with respect to the mid-cylinder  $\mathcal{H}$ . In particular, we assume that  $K(t) = \frac{\partial \bar{\omega}}{\partial R}(R_o, t) \neq 0$  and  $\omega(t) = \bar{\omega}(R_o, t) = 0$ . Therefore, the kinetic equations reduce to (5.20b),<sup>14</sup> giving the evolution of  $K$ , which further simplifies to read<sup>15</sup>

$$\dot{K} = \frac{\beta_1}{2\beta_2} + \frac{Y_c h^3}{12\beta_2} \left( \frac{r}{R_o^2} - K - \frac{1}{R_o} \right). \tag{5.21}$$

We define the following characteristic time

$$\tau = \frac{12\beta_2}{Y_c h^3}. \tag{5.22}$$

We differentiate with respect to  $\hat{t} = t/\tau$  and rewrite (5.21) using dimensionless quantities  $\hat{K}$  and  $\hat{r}$

$$\dot{\hat{K}}(\hat{t}) = \frac{6\beta_1 R_o}{Y_c h^3} + \left( \hat{r}(\hat{t}) - \hat{K}(\hat{t}) - 1 \right), \tag{5.23}$$

where  $\hat{r}$  is given by (5.16), (5.17), or (5.18) depending on the sign of the discriminant  $\Delta$ . Following (5.12), the physical components of the residual stresses read

$$\begin{aligned} \hat{\sigma}^{\phi\phi} &= hY_c \left( \hat{r}^2(t) - 1 \right), & \hat{\sigma}^{zz} &= hY_s \left( \hat{r}^2(t) - 1 \right), \\ \hat{\mu}^{\phi\phi} &= h^2 Y_c \frac{\hat{h}}{6} \left[ 1 + \hat{K}(t) - \hat{r}(t) \right], & \hat{\mu}^{zz} &= h^2 Y_s \frac{\hat{h}}{6} \left[ 1 + \hat{K}(t) - \hat{r}(t) \right], \end{aligned}$$

where  $Y_s = \frac{\mu\lambda}{2\mu+\lambda}$  and recall that  $Y_c = \mu + \frac{\mu\lambda}{2\mu+\lambda}$ .

In the linear theory of elasticity, it can be shown that the circumferential stress in a thin-walled cylinder of radius  $R_o$  and thickness  $h$ , subject to an internal pressure  $p_i$ , is given by

$$\hat{\sigma}_{\text{lin}}^{\phi\phi} = \frac{pR_o}{h}.$$

Starting from our circumferential shell stress  $\hat{\sigma}^{\phi\phi}$ , we want to prove that for  $p$  and  $h$  small,  $\hat{\sigma}^{\phi\phi}$  approaches  $h\hat{\sigma}_{\text{lin}}^{\phi\phi}$ . For  $p$  and  $h$  small enough, the discriminant  $\Delta > 0$  and the dimensionless radius is written as

$$r(p, h, t) = \frac{p_i R_o^2}{3Y_c h} + 2R_o \sqrt{\frac{-m}{3}} \cos \left[ \frac{1}{3} \cos^{-1} \left( \frac{-n}{2} \sqrt{\frac{27}{-m^3}} \right) \right],$$

<sup>14</sup> When  $\omega = 0$ , the first fundamental form  $\mathbf{G}$  is not a dynamical variable anymore, and hence, the kinetic equation (5.20a) should be discarded.

<sup>15</sup> Recall that  $r$  depends on  $K$  as can be seen in (5.16), (5.17), or (5.18) depending on the value of the discriminant  $\Delta$ .

where  $m = \frac{h^2}{6R_o^2} - 1 - \frac{p_i^2 R_o^2}{3Y_c^2 h^2}$  and  $n = -\frac{p_i R_o}{27Y_c h} \left( \frac{2p_i^2 R_o^2}{Y_c^2 h^2} - \frac{3h^2}{2R_o^2} + 9 \right) - \frac{h^2}{6R_o^2} (1 + R_o K(t))$ . For small  $p$  and  $h$ , we have the following asymptotic expansion

$$r(p, h, t) = R_o + \left[ \frac{R_o^2}{2h} + o(h) \right] \frac{p}{Y_c} + o(h) + o(p^2).$$

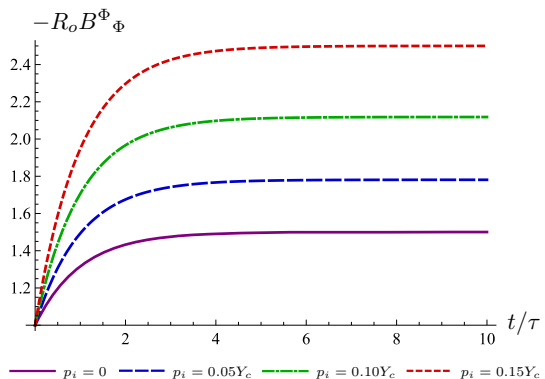
It follows that

$$\frac{\hat{\sigma}^{\phi\phi}}{h} = Y_c \left( \frac{r^2(t)}{R_o^2} - 1 \right) = \frac{pR_o}{h} + o(h) p + o(h^2) + o(p^2).$$

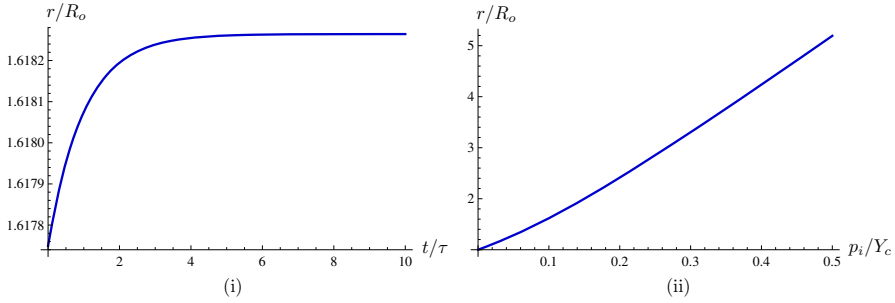
For a shell of thickness  $h = 0.1R_o$ , a growth such that  $\beta_1 = Y_c h^3 / (12R_o)$ , and for different values of a constant internal pressure  $p_i$ , we numerically solve (5.23) for the evolution of  $K$  (see Fig. 9 for the evolution of the material curvature  $B^\Phi_\phi = -\frac{1}{R_o} - K(t)$ ). We consequently obtain the evolution of its radius and the corresponding circumferential stress and couple stress (see Figs. 10, 11, 12). We also consider a time-dependent internal pressure and numerically solve (5.23) for the evolution of  $K$ . We assume in particular a time-dependent profile of internal pressure given in Fig. 13. We show in Fig. 14 the evolution of the material curvature  $B^\Phi_\phi$  and the resulting couple stress induced by the time-dependent internal pressure shown in Fig. 13.

We note that the cylindrical shell expands and contracts following the internal pressure level, through the explicit dependence of the spatial radius  $r$  on the internal pressure  $p_i$ . Also, the circumferential stress follows the profile of expansion and contraction of the cylinder and has a value close to that of the thin-walled cylinders, i.e.,  $\frac{p_i r}{h}$ . Note that we have omitted the plots of evolution of the radius and the circumferential stress as their profiles are similar to that of the internal pressure. We note that  $\hat{\mu}^{\phi\phi} = \beta_1 - h^2 Y_c \frac{\hat{h}}{6} \hat{K}(\hat{t}) = \beta_1 + h^2 Y_c \frac{\hat{h}}{6} \hat{B}^\Phi_\phi(\hat{t})$ , and we can hence interpret, in this case, the circumferential couple stress as the rate of change of the material curvature augmented by the growth-induced residual couple stress  $\beta_1$ . We observe that at large times, the rate of change of the curvature  $\hat{K}$  tends to zero (suggesting that the system (5.23) asymptotically tends to its attracting fixed point). Therefore, in the case of the

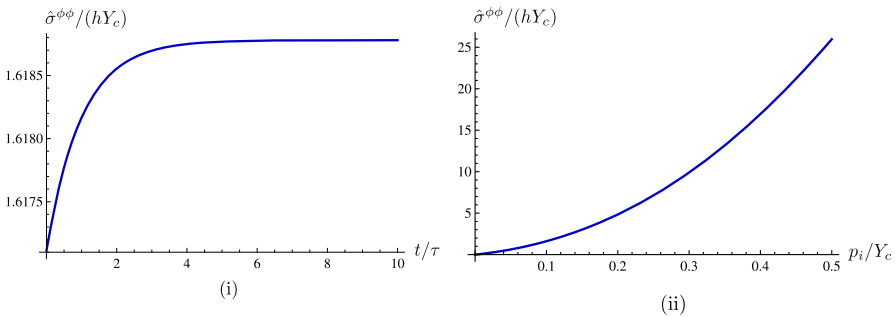
**Fig. 9** The evolution of the material curvature  $B^\Phi_\phi$  of a morphoelastic circular cylindrical shell with  $h = 0.1R_o$  for different values of the constant internal pressure  $p_i$





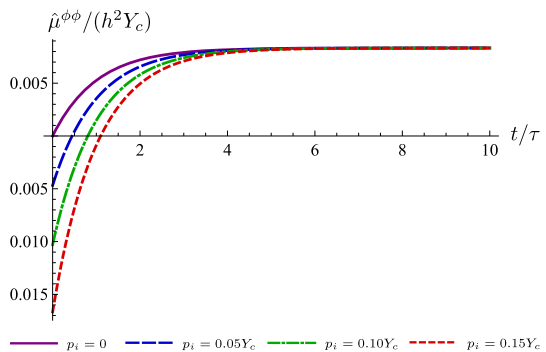


**Fig. 10** The evolution of the radius  $r$  of a morphoelastic circular cylindrical shell with  $h = 0.1R_o$ . **i** shows the time evolution for a constant internal pressure  $p_i = 0.1Y_c$ , and **ii** shows the internal pressure dependence at time  $t = 10\tau$



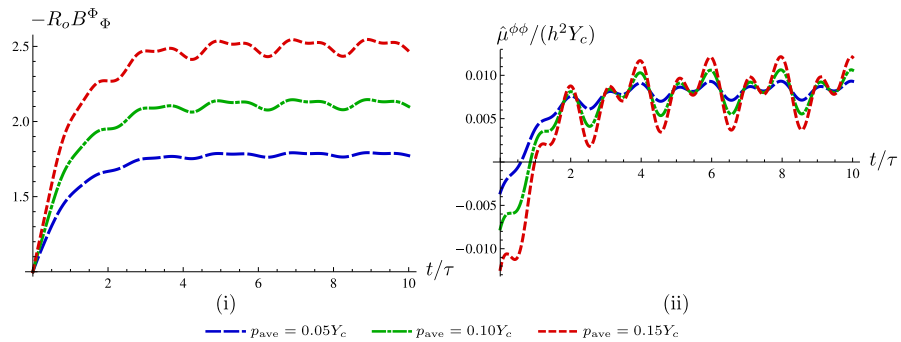
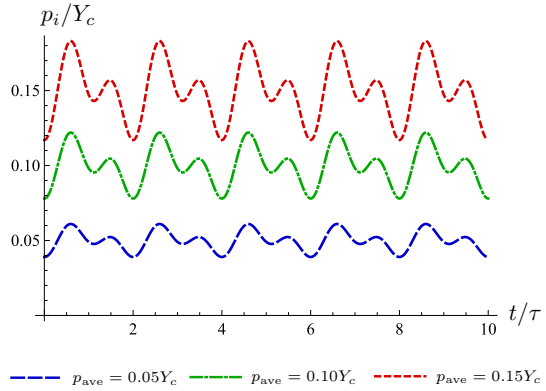
**Fig. 11** The evolution of the circumferential stress of a morphoelastic circular cylindrical shell with  $h = 0.1R_o$ . **i** shows the time evolution for a constant internal pressure  $p_i = 0.1Y_c$ , and **ii** shows the internal pressure dependence at time  $t = 10\tau$

**Fig. 12** The evolution of the circumferential couple stress of a morphoelastic circular cylindrical shell with  $h = 0.1R_o$  for different values of the constant internal pressure  $p_i$



constant internal pressure, the circumferential stress tends to the residual couple stress  $\beta_1$ , and in the case of a time-dependent pressure, the mean value of the circumferential stress tends to the residual couple stress  $\beta_1$ . Note that the plots of the longitudinal stress and couple stress are omitted because they are, respectively, proportional to the circumferential stress and couple stress by a factor of  $Y_s/Y_c = \nu$ .

**Fig. 13** The time-dependent profiles for internal pressure  $p_i$ . Each profile corresponds to its average pressure value  $p_{ave}$



**Fig. 14** The evolution of **i** the material curvature  $B^\Phi_\Phi$ , and **ii** the circumferential couple stress of a morphoelastic circular cylindrical shell with  $h = 0.1R_o$  for the time-dependent internal pressure plotted in Fig. 13

### 5.3 A Morphoelastic Circular Shell

In the absence of body forces, we consider an initially planar thin morphoelastic circular disk  $\mathcal{B}$  with vanishing boundary loads. We assume that the disk is undergoing radially symmetric but non-uniform growth through its thickness such that the radial and circumferential curvatures are evolving, while the intrinsic metric of the shell remains unchanged. Let  $(R, \Phi, Z)$  be the standard cylindrical coordinate system for  $\mathbb{R}^3$  such that initially the mid-surface  $\mathcal{H}$  of the shell lies in the hyperplane  $Z = 0$  and the origin of the coordinate system coincides with the center of the circular mid-surface. Let  $R_o$  be the radius of the circular disk. For time  $t \geq 0$ , we represent the aforementioned growth by the following evolving material metric:

$$\bar{\mathbf{G}} = \begin{pmatrix} e^{2\omega_R(R,Z,t)} & 0 & 0 \\ 0 & e^{2\omega_\Theta(R,Z,t)} R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.24}$$

such that  $\omega_A$  for  $A = R, \Theta$  are symmetric with respect to  $Z$ , i.e.,  $\omega_A(R, Z, t) = -\omega_A(R, -Z, t)$ , which implies that  $\omega_A(R, 0, t) = 0$ .<sup>16</sup> Therefore, the first and the second fundamental forms of  $\mathcal{H}$  read

$$G = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}, \quad B = \begin{pmatrix} -K_R(R, t) & 0 \\ 0 & -R^2 K_\Theta(R, t) \end{pmatrix},$$

where  $K_A(R, t) = \frac{\partial \omega_A}{\partial Z}(R, 0, t)$  for  $A = R, \Theta$ .

We endow the ambient Euclidean space  $\mathbb{R}^3$  with the standard cylindrical coordinate system  $(r, \phi, z)$ . In order to study the growth of the shell and obtain the growth-induced residual stresses, we embed the shell into the Euclidean ambient space and look for solutions of the form  $(r, \phi, z) = (r(R, t), \Phi, z(R, t))$ . We fix the rigid body motion of the embedded surface by assuming  $r(0, t) = 0, z(0, t) = 0$ , and  $z'(0, t) = 0$ . Therefore, the deformation tensors read

$$C = \begin{pmatrix} r'^2 + z'^2 & 0 \\ 0 & r^2 \end{pmatrix}, \quad \Theta = \frac{1}{(r'^2 + z'^2)^{1/2}} \begin{pmatrix} r'z'' - r''z' & 0 \\ 0 & rz' \end{pmatrix},$$

where  $r' = \frac{\partial r}{\partial R}$  and  $r'' = \frac{\partial^2 r}{\partial R^2}$ . We introduce the function  $\chi = \chi(R, t)$  defined such that  $z' = \chi r'$ . Hence, the deformation tensors in terms of  $r$  and  $\chi$  read

$$C = \begin{pmatrix} r'^2(1 + \chi^2) & 0 \\ 0 & r^2 \end{pmatrix}, \quad \Theta = \frac{1}{(1 + \chi^2)^{1/2}} \begin{pmatrix} \chi' r' & 0 \\ 0 & \chi r \end{pmatrix}.$$

We assume that the shell is made of a homogeneous and isotropic material. Because of the symmetry of the problem, the second Piola–Kirchhoff stress and material couple-stress components have the following forms

$$S^{RR} = S^{RR}(R, t), \quad S^{R\Phi} = 0, \quad S^{\Phi\Phi} = S^{\Phi\Phi}(R, t), \\ M^{RR} = M^{RR}(R, t), \quad M^{R\Phi} = 0, \quad M^{\Phi\Phi} = M^{\Phi\Phi}(R, t).$$

In the convected manifold  $(\mathcal{H}, C)$ , the only nonzero Christoffel symbols of the Levi–Civita connection are

$$\check{\Gamma}^R_{RR} = \frac{\chi \chi'}{1 + \chi^2} + \frac{r''}{r'}, \quad \check{\Gamma}^R_{\Phi\Phi} = -\frac{r}{r'(1 + \chi^2)}, \quad \check{\Gamma}^{\Phi}_{R\Phi} = \frac{r'}{r}.$$

<sup>16</sup> We let for example  $\omega_A = ZK_A(R, t)$  for  $A = R, \Theta$  in (5.24).

Therefore, the equilibrium equations follow from (4.19) and read<sup>17</sup>

$$\begin{aligned} & \frac{1}{R} \left[ R \left( S^{RR} + 2 \frac{\chi'}{r'(1+\chi^2)^{3/2}} M^{RR} \right) \right]_{,R} \\ & + 2 \left( \frac{\chi\chi'}{1+\chi^2} + \frac{r''}{r'} \right) \left( S^{RR} + 2 \frac{\chi'}{r'(1+\chi^2)^{3/2}} M^{RR} \right) \\ & - \frac{r}{r'(1+\chi^2)} \left( S^{\Phi\Phi} + 2 \frac{\chi}{r(1+\chi^2)^{1/2}} M^{\Phi\Phi} \right) - \left( \frac{\chi'}{r'(1+\chi^2)^{3/2}} \right)_{,R} M^{RR} \\ & - \left( \frac{\chi'}{r'(1+\chi^2)^{3/2}} - \frac{\chi}{r(1+\chi^2)^{1/2}} \right) \frac{r}{r'(1+\chi^2)} M^{\Phi\Phi} = 0, \end{aligned} \tag{5.25a}$$

$$\begin{aligned} & \frac{\chi'r'}{(1+\chi^2)^{1/2}} \left( S^{RR} + \frac{\chi'}{r'(1+\chi^2)^{3/2}} M^{RR} \right) \\ & + \frac{\chi r}{(1+\chi^2)^{1/2}} \left( S^{\Phi\Phi} + \frac{\chi}{r(1+\chi^2)^{1/2}} M^{\Phi\Phi} \right) \\ & - \frac{r}{R} \left( \frac{r'}{r} + \frac{\chi\chi'}{1+\chi^2} + \frac{r''}{r'} \right) \left[ \frac{1}{r} (RM^{RR}) \right]_{,R} \\ & + 2 \frac{R}{r} \left( \frac{\chi\chi'}{1+\chi^2} + \frac{r''}{r'} \right) M^{RR} - \frac{R}{r'(1+\chi^2)} M^{\Phi\Phi} \Big] \\ & - \frac{r}{R} \left[ \frac{1}{r} (RM^{RR}) \right]_{,R} + 2 \frac{R}{r} \left( \frac{\chi\chi'}{1+\chi^2} + \frac{r''}{r'} \right) M^{RR} - \frac{R}{r'(1+\chi^2)} M^{\Phi\Phi} \Big]_{,R} = 0. \end{aligned} \tag{5.25b}$$

The boundary conditions (4.20) for zero surface load, zero moment load, and zero shear load at  $R = R_o$  read

$$S^{RR} = 0, \quad M^{RR} = 0, \quad \frac{1}{R} (RM^{RR})' - r r' M^{\Phi\Phi} = 0. \tag{5.26}$$

We assume a Saint Venant–Kirchhoff constitutive model, for which the strain energy density  $\mathcal{W}$  is given by (4.27). Therefore, the nonzero components of the material stress and couple stress tensors read

$$S^{RR} = Y_c h \left\{ \left[ r'^2 (1 + \chi^2) - 1 \right] + \nu \left( \frac{r^2}{R^2} - 1 \right) \right\},$$

<sup>17</sup> Note that  $(\Sigma^{AB} + C^{-AC} \Theta_{CD} \Lambda^{DB})_{||B} + C^{-AC} \Theta_{CD} \Lambda^{DB}_{||B} = (\Sigma^{AB} + 2C^{-AC} \Theta_{CD} \Lambda^{DB})_{||B} - (C^{-AC} \Theta_{CD})_{||B} \Lambda^{DB}$ . Also, we have  $J = \frac{r}{R}$ . Therefore, the convected stress and couple-stress tensors read  $\Sigma = \frac{R}{r} S$  and  $\Lambda = \frac{R}{r} M$ .

$$\begin{aligned}
 S^{\Phi\Phi} &= Y_c h \left\{ \nu \left[ r'^2 (1 + \chi^2) - 1 \right] + \left( \frac{r^2}{R^2} - 1 \right) \right\} \frac{1}{R^2}, \\
 M^{RR} &= Y_c h^3 \left\{ \left[ \frac{\chi' r'}{(1 + \chi^2)^{1/2}} + K_R \right] + \nu \left[ \frac{\chi r}{R^2 (1 + \chi^2)^{1/2}} + K_\Theta \right] \right\}, \\
 M^{\Phi\Phi} &= Y_c h^3 \left\{ \nu \left[ \frac{\chi' r'}{(1 + \chi^2)^{1/2}} + K_R \right] + \left[ \frac{\chi r}{R^2 (1 + \chi^2)^{1/2}} + K_\Theta \right] \right\} \frac{1}{R^2},
 \end{aligned}$$

where  $Y_c = \frac{E}{2(1-\nu^2)}$ . Therefore, we have the governing Eq. (5.25) and the boundary conditions (5.26) in terms of  $r$  and  $\chi$ . However, we still need the kinetic equations for the evolution of growth in order to fully solve the problem. We consider the Rayleigh potential  $\mathcal{R}(\dot{\mathbf{B}}, \mathbf{G}) = \beta_1 \text{tr}(\dot{\mathbf{B}}) + \beta_2 \text{tr}(\dot{\mathbf{B}}^2)$ . Therefore, we find the following evolution equations for the principal curvatures:

$$\dot{K}_R = \frac{\beta_1}{2\beta_2} - \frac{Y_c h^3}{12\beta_2} \left\{ \left[ \frac{\chi' r'}{(1 + \chi^2)^{1/2}} + K_R \right] + \nu \left[ \frac{\chi r}{R^2 (1 + \chi^2)^{1/2}} + K_\Theta \right] \right\}, \tag{5.27a}$$

$$\dot{K}_\Theta = \frac{\beta_1}{2\beta_2} - \frac{Y_c h^3}{12\beta_2} \left\{ \nu \left[ \frac{\chi' r'}{(1 + \chi^2)^{1/2}} + K_R \right] + \left[ \frac{\chi r}{R^2 (1 + \chi^2)^{1/2}} + K_\Theta \right] \right\}. \tag{5.27b}$$

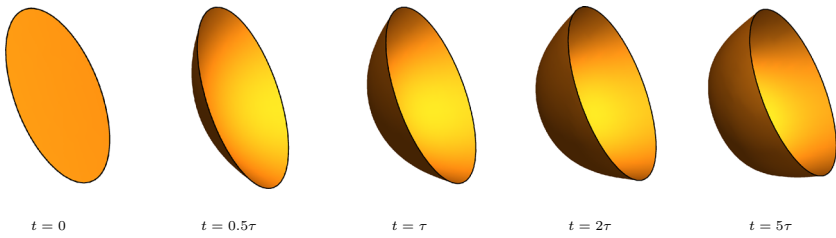
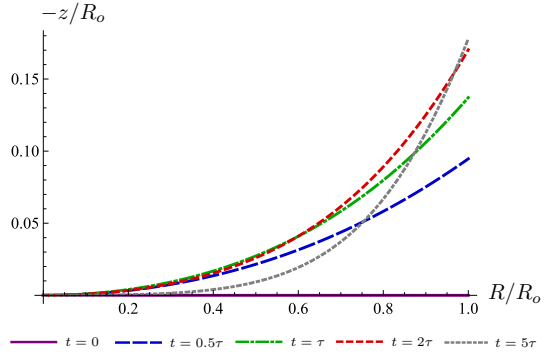
Using the time differentiation with respect to  $\hat{t} = t/\tau$  where  $\tau$  is defined by (5.22) and the spatial differentiation with respect to  $\hat{R} = R/R_o$ , (5.27) reads

$$\begin{aligned}
 \dot{\hat{K}}_R &= \frac{6\beta_1 R_o}{Y_c h^3} - \left\{ \left[ \frac{\chi' \hat{r}'}{(1 + \chi^2)^{1/2}} + \hat{K}_R \right] + \nu \left[ \frac{\chi \hat{r}}{\hat{R}^2 (1 + \chi^2)^{1/2}} + \hat{K}_\Theta \right] \right\}, \\
 \dot{\hat{K}}_\Theta &= \frac{6\beta_1 R_o}{Y_c h^3} - \left\{ \nu \left[ \frac{\chi' \hat{r}'}{(1 + \chi^2)^{1/2}} + \hat{K}_R \right] + \left[ \frac{\chi \hat{r}}{\hat{R}^2 (1 + \chi^2)^{1/2}} + \hat{K}_\Theta \right] \right\},
 \end{aligned}$$

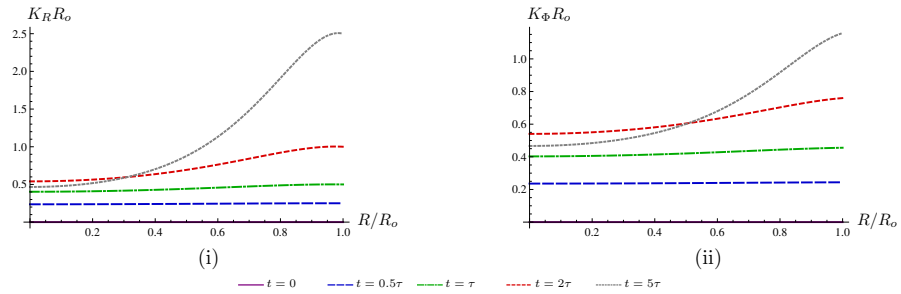
where  $\hat{K}_A = K_A R_o$  for  $A = R, \Theta$ ,  $\hat{r} = \frac{r}{R_o}$ ,  $\hat{R} = \frac{R}{R_o}$ , and  $\hat{h} = \frac{h}{R_o}$ .

We numerically solve the governing Eq. (5.25) and the boundary conditions (5.26) along with the kinetic equations (5.27) for a shell of thickness  $h = 0.1 R_o$  made of an isotropic and homogeneous material with  $\nu = 0.5$ , and undergoing a growth such that  $\beta_1 = Y_c h^3 / (12 R_o)$ . In Fig. 15, we show the evolution of a radial fiber by plotting its radius  $r$  and elevation  $z$  as a function of  $R$ . We observe that the radius  $r$  remains almost unchanged from  $R$  and is almost time-independent, while the out-of-plane elevation changes the fiber from its original configuration on the plane  $z = 0$  to adopt a curved configuration. In Fig. 16, we show the evolution of the spatial embedding of the disk from its initial planar configuration to a non-trivial curved disk. In Fig. 17, we show the evolution of the radial and the circumferential curvatures. In Fig. 18, we show the evolutions of the residual stresses and couple stresses in the growing disk.

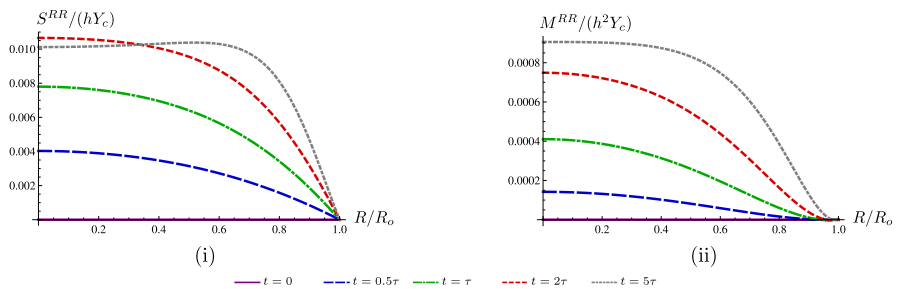
**Fig. 15** Evolution of the out-of-plane spatial elevation profile  $z = z(R, t)$  of a radial fiber of the initially planar growing disk



**Fig. 16** Visualization of the evolution of an initially planar growing disk. The out-of-plane elevation is scaled to a factor of 5



**Fig. 17** Evolution of the radial **i** and circumferential **ii** principal curvatures on a radial fiber of the initially planar growing disk



**Fig. 18** Evolution of **i** the radial stress and **ii** the radial couple-stress on a radial fiber of the initially planar growing disk

## 6 Concluding Remarks

In this paper, we presented within the framework of nonlinear elasticity, a geometric theory of morphoelastic shells to study the coupling of bulk growth with elasticity. We looked at how bulk growth affects the geometry of the shell and eventually leads to residual stresses and couple stresses in the shell. We identified the thin body with a three-dimensional Riemannian manifold and modeled the shell by the mid-surface of the thin body as an orientable two-dimensional Riemannian submanifold. The shell geometry is hence characterized by its first and second fundamental forms that, respectively, represent the in-plane and out-of plane geometries. The nonlinear shell compatibility equations were related to the Gauss and Codazzi–Mainardi equations. We modeled the growth of a shell by considering a referential evolution such that the material points remain fixed, while the fundamental forms evolve to account for growth. The evolution of the fundamental forms is such that the shell always remains stress-free in the material manifold, which is not necessarily Euclidean. We discussed a systematic method to obtain those growth fields that leave a simply-connected shell both stress-free and couple-stress-free. Using a Lagrangian field theory, we derived the governing equations of motion. We obtained the balance laws for morphoelastic shells and a kinetic equation governing the evolution of the growth through imposing a coupling between the state of stress of the shell and the time evolution of the fundamental forms. As an example, we considered a planar sheet and found a family of stress-free growth fields. We observed that stress-free growth can evolve a planar shell into another flat shell, a positively curved, or a negatively curved one. We studied the growth of a morphoelastic infinitely long circular cylindrical shell subject to a time-dependent internal pressure. We also considered a growing circular shell that evolves to a curved cap. In both cases, we numerically obtained the evolution of the evolving curvatures and the induced residual stresses and couple stresses.

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## 7 Appendix: Derivation of the Euler–Lagrange Equations

In this appendix, we work out in detail the derivation of the Euler–Lagrange equations first assuming that  $\delta \mathbf{G} = \delta \mathbf{B} = \mathbf{0}$ . We substitute (4.8), (4.9), (4.10), (4.11), and (4.12) into (4.7) to obtain

$$\int_{t_0}^{t_1} \int_{\mathcal{H}} \left[ \frac{\partial \mathcal{L}}{\partial \varphi^a} \delta \varphi^a + \frac{\partial \mathcal{L}}{\partial \varphi^n} \delta \varphi^n - \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b \left( \delta \varphi^a \beta^b{}_a + \frac{\partial (\delta \varphi^n)}{\partial x^a} g^{ab} \right) + \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \cdot \frac{D \delta \varphi}{dt} \right. \\ \left. + \frac{\partial \mathcal{L}}{\partial C_{AB}} \left( 2F^b{}_B g_{bc} \delta \varphi^c|_A - 2\delta \varphi^n F^a{}_A F^b{}_B \beta_{ab} \right) + \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} \left( F^a{}_A F^b{}_B \beta_{ab|c} \delta \varphi^c \right) \right]$$

$$\begin{aligned}
 & + 2F^a{}_A \beta_{ac} \delta\varphi^c|_B - \delta\varphi^n F^a{}_A F^b{}_B \beta_{ac} \beta_{bd} g^{cd} + F^b{}_A \left( \frac{\partial \delta\varphi^n}{\partial x^b} \right) \Big|_B \Big] \\
 & \sqrt{\det \mathbf{G}} dX^I dt = 0.
 \end{aligned}
 \tag{7.1}$$

Hence, we have

$$\begin{aligned}
 & \int_{t_0}^{t_1} \int_{\mathcal{H}} \left\{ \frac{\partial \mathcal{L}}{\partial \varphi^a} \delta\varphi^a + \frac{\partial \mathcal{L}}{\partial \varphi^n} \delta\varphi^n + \frac{1}{\sqrt{\det \mathbf{G}}} \frac{d}{dt} \left( \sqrt{\det \mathbf{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \cdot \delta\varphi \right) \right. \\
 & - \frac{1}{\sqrt{\det \mathbf{G}}} \frac{d}{dt} \left( \sqrt{\det \mathbf{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \right) \cdot \delta\varphi \\
 & - \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b \beta^b{}_a \delta\varphi^a - \left( \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b \delta\varphi^n g^{ab} F^{-A}{}_a \right) \Big|_A \\
 & + \left( \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b g^{ab} F^{-A}{}_a \right) \Big|_A \delta\varphi^n \\
 & + 2 \left( \frac{\partial \mathcal{L}}{\partial C_{AB}} F^c{}_A g_{ac} \delta\varphi^a \right) \Big|_B - 2 \left( \frac{\partial \mathcal{L}}{\partial C_{AB}} F^c{}_A g_{ac} \right) \Big|_B \delta\varphi^a \\
 & - 2 \frac{\partial \mathcal{L}}{\partial C_{AB}} F^a{}_A F^b{}_B \beta_{ab} \delta\varphi^n \\
 & + \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A F^b{}_B \beta_{bc|a} \delta\varphi^a + 2 \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A \beta_{ac} \delta\varphi^a \right) \Big|_B \\
 & - 2 \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A \beta_{ac} \right) \Big|_B \delta\varphi^a \\
 & - \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^a{}_A F^b{}_B \beta_{ac} \beta_{bd} g^{cd} \delta\varphi^n + \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^b{}_A \frac{\partial \delta\varphi^n}{\partial x^b} \right) \Big|_B \\
 & - \left[ \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^b{}_A \right) \Big|_B \delta\varphi^n F^{-D}{}_b \right] \Big|_D \\
 & + \left[ \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^b{}_A \right) \Big|_B F^{-D}{}_b \right] \Big|_D \delta\varphi^n \Big] \sqrt{\det \mathbf{G}} dX^I dt = 0.
 \end{aligned}
 \tag{7.2}$$

We can rewrite (7.2) as

$$\begin{aligned}
 & \int_{t_0}^{t_1} \int_{\mathcal{H}} \left\{ \frac{1}{\sqrt{\det \mathbf{G}}} \frac{d}{dt} \left( \sqrt{\det \mathbf{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \cdot \delta\varphi \right) + 2 \left( \frac{\partial \mathcal{L}}{\partial C_{AB}} F^c{}_A g_{ac} \delta\varphi^a \right) \Big|_B \right. \\
 & + 2 \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A \beta_{ac} \delta\varphi^a \right) \Big|_B - \left[ \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^b{}_A \right) \Big|_B \delta\varphi^n F^{-D}{}_b \right] \Big|_D \\
 & - \left( \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b \delta\varphi^n g^{ab} F^{-A}{}_a \right) \Big|_A + \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} \frac{\partial \delta\varphi^n}{\partial X^A} \right) \Big|_B \\
 & + \left[ \frac{\partial \mathcal{L}}{\partial \varphi^a} - \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b \beta^b{}_a - \frac{1}{\sqrt{\det \mathbf{G}}} \frac{d}{dt} \left( \sqrt{\det \mathbf{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^a} \right) - 2 \left( \frac{\partial \mathcal{L}}{\partial C_{AB}} F^c{}_A g_{ac} \right) \Big|_B \right.
 \end{aligned}$$



$$\begin{aligned}
 & + \left. \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A F^b{}_B \beta_{bc|a} - 2 \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A \beta_{ac} \right) \right|_B \Big] \delta \varphi^a \\
 & + \left[ \frac{\partial \mathcal{L}}{\partial \varphi^n} + \left( \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b g^{ab} F^{-A}{}_a \right) \right|_A - \frac{1}{\sqrt{\det \mathbf{G}}} \frac{d}{dt} \left( \sqrt{\det \mathbf{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^n} \right) \right. \\
 & - 2 \frac{\partial \mathcal{L}}{\partial C_{AB}} F^a{}_A F^b{}_B \beta_{ab} - \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^a{}_A F^b{}_B \beta_{ac} \beta_{bd} g^{cd} \\
 & \left. + \left[ \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^b{}_A \right) \right|_B F^{-D}{}_b \right]_{|D} \Big] \delta \varphi^n \Big\} dS dt = 0. \tag{7.3}
 \end{aligned}$$

At  $t = t_1$ , we assume that  $\varphi_{\epsilon, t_1} = \varphi_{t_1}$  so that  $\delta \varphi_{t_1} = 0$ . Therefore, by integrating the first term in (7.3) in the time domain, we obtain only one term at  $t = t_0$  giving the initial condition on the velocity at  $t = t_0$ . By applying Stokes' theorem to the following five terms, if we denote by  $\mathbf{T}$  the outward in-plane vector field normal to the boundary curve  $\partial \mathcal{H}$ , we obtain

$$\begin{aligned}
 & - \int_{\mathcal{H}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \cdot \delta \varphi dS \Big|_{t=t_0} + \int_{t_0}^{t_1} \int_{\partial \mathcal{H}} \left\{ \left( 2 \frac{\partial \mathcal{L}}{\partial C_{AB}} F^c{}_A g_{ac} + 2 \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A \beta_{ac} \right) \mathbf{T}_B \delta \varphi^a \right. \\
 & - \left[ \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AC}} F^b{}_A \right) \right|_C F^{-B}{}_b + \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b g^{ab} F^{-B}{}_a \Big] \mathbf{T}_B \delta \varphi^n + \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} \mathbf{T}_B \frac{\partial \delta \varphi^n}{\partial X^A} \Big\} dL dt \\
 & \int_{t_0}^{t_1} \int_{\mathcal{H}} \left\{ \left[ \frac{\partial \mathcal{L}}{\partial \varphi^a} - \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b \beta^b{}_a - \frac{1}{\sqrt{\det \mathbf{G}}} \frac{d}{dt} \left( \sqrt{\det \mathbf{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^a} \right) - 2 \left( \frac{\partial \mathcal{L}}{\partial C_{AB}} F^c{}_A g_{ac} \right) \right]_{|B} \right. \\
 & + \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A F^b{}_B \beta_{bc|a} - 2 \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A \beta_{ac} \right) \Big]_{|B} \delta \varphi^a \\
 & + \left[ \frac{\partial \mathcal{L}}{\partial \varphi^n} + \left( \left( \frac{\partial \mathcal{L}}{\partial \mathcal{N}} \right)_b g^{ab} F^{-A}{}_a \right) \right]_{|A} - \frac{1}{\sqrt{\det \mathbf{G}}} \frac{d}{dt} \left( \sqrt{\det \mathbf{G}} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^n} \right) \\
 & - 2 \frac{\partial \mathcal{L}}{\partial C_{AB}} F^c{}_A F^b{}_B \beta_{bc} - \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^a{}_A F^b{}_B \beta_{ac} \beta_{bd} g^{cd} \\
 & \left. + \left[ \left( \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^b{}_A \right) \right|_B F^{-D}{}_b \right]_{|D} \Big] \delta \varphi^n \Big\} dS dt = 0. \tag{7.4}
 \end{aligned}$$

By arbitrariness of  $\delta \varphi^{\parallel}$ ,  $\delta \varphi^n$ , and  $d(\delta \varphi^n)$ , the Euler–Lagrange equations for shells (4.13) together with the initial and boundary conditions (4.14) follow from (7.4). Note that following Codazzi's equation (2.3), we have  $\beta_{bc|a} = \beta_{ac|b}$ . Therefore  $\frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A F^b{}_B \beta_{bc|a} = \frac{\partial \mathcal{L}}{\partial \Theta_{AB}} F^c{}_A \beta_{ac|B}$ .

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