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On the stress singularities generated by anisotropic eigenstrains and the hydrostatic stress due to annular inhomogeneities

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ABSTRACT

The problems of singularity formation and hydrostatic stress created by an inhomogeneity with eigenstrain in an incompressible isotropic hyperelastic material are considered. For both a spherical ball and a cylindrical bar with a radially symmetric distribution of finite possibly anisotropic eigenstrains, we show that the anisotropy of these eigenstrains at the center (the center of the sphere or the axis of the cylinder) controls the stress singularity. If they are equal at the center no stress singularity develops but if they are not equal then stress always develops a logarithmic singularity. In both cases, the energy density and strains are everywhere finite. As a related problem, we consider annular inclusions for which the eigenstrains vanish in a core around the center. We show that even for an anisotropic distribution of eigenstrains, the stress inside the core is always hydrostatic. We show how these general results are connected to recent claims on similar problems in the limit of small eigenstrains.

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1. Introduction

The general problem of elastic inclusion (or more generally, inhomogeneity with eigenstrain) is to compute the stress generated by adding material within a given matrix. Mathematically, it can be formulated as a problem where eigenstrains, which represent the new included material, are given and for which the residual stress needs to be computed (see Yavari and Goriely, 2013 and references therein for a general introduction on the topic of inclusions, eigenstrains, and various extensions of the celebrated work of Eshelby (1957)).

In general, the eigenstrains do not need to be isotropic with respect to the symmetry of the underlying system. For instance in a ball, the spherical solution still exists even if different eigenstrains in the radial and angular directions are specified. More specifically, in Yavari and Goriely (2013), we analyzed a ball of radius R_o with a spherical inclusion of radius R_i with uniform radial and circumferential (finite) eigenstrains. The matrix and the inclusion both were assumed to be incompressible and isotropic with possibly different energy functions. It was observed that when the uniform radial and circumferential eigenstrains are not equal, i.e. an anisotropic eigenstrain distribution, the non-vanishing stress components all have a logarithmic singularity at the center of the ball $R=0$. However, the principal stretches and hence the strain energy density are finite everywhere.

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It has been known for a long time that certain anisotropies in elastic properties can lead to stress singularities even for bodies with smooth boundaries. The first such observations were made by Lekhnitskii (1957) and Reissner (1958). Lekhnitskii (1957) showed that the stress on the axis of a cylindrically uniform solid cylinder made of a monoclinic solid may become infinite under a finite uniform applied pressure. Here, cylindrically uniform means that the elastic constants in cylindrical coordinates are constant. Reissner (1958) observed similar singularities in the case of orthotropic shells of revolution. Later, Antman and Negrón-Marrero (1987) studied the radially symmetric equilibrium configurations of transversely isotropic solid cylinders and balls under constant pressure on their boundaries and showed that for applied pressure above a critical value, pressure at the center may become unbounded.

Avery and Herakovich (1986) analyzed a linear elastic cylindrically anisotropic circular cylindrical bar under uniform thermal load. They showed that in the case of radial orthotropy (radial stiffness larger than hoop stiffness) the stress develops a singularity on the axis of the bar. Gal and Dvorkin (1995) considered an anisotropic cylindrical bar with uniform finite tractions on the boundary. They showed that if the cylinder is stiffer radially than tangentially (radially orthotropic cylinder) stress on the axis of the cylinder becomes unbounded. Ting (1953) considered a spherically uniform (i.e. elastic constants in the spherical coordinates are constant) linear anisotropic spherical ball under uniform pressure on its boundary sphere and showed that for certain anisotropies the stress at the center of the ball is unbounded. Later Aguiar (2006) observed that in a neighborhood of the origin the Jacobian is negative in Ting's solution and hence the solution is unphysical.[1] Aguiar (2006) used Fosdick and Royer-Carfagni (2001)'s framework for avoiding self-intersection of matter and observed that the corresponding Lagrange multiplier has a logarithmic singularity at the center of the ball. It seems that in all these examples anisotropy in a neighborhood of the origin (of cylindrical or spherical coordinates) is responsible for stress singularities (see also Horgan and Baxter, 1996). More recently, Goriely et al. (2010) showed that, in morphoelasticity, non-isotropic growth in a ball or cylinder always leads to stress singularity and Sadik and Yavari (2014) showed that anisotropic thermal expansion induces logarithmic singularities as well.

At first sight, a singularity in the stress field may appear unphysical. It could be seen as an artifact of the mathematical model, related to the peculiar choice of coordinates. Although the stress field is an important physical construct, it is only through tractions that forces are exerted on the material. As long as the actual physical forces developed in the material remain finite, that the strain energy is bounded, and that the material does not interpenetrate, a solution with singularity is a valid physical solution for the problem at hand. Further, the setting in which these singularities develop may represent a challenge from a computational point of view. It is therefore particularly important to classify these solutions analytically so that their occurrence in a numerical scheme could be controlled locally.

The present work was motivated by two recent papers. First the paper by Shodja and Khorshidi (2013) where stress singularities are observed in the framework of linear elasticity. The question raised, by Markenscoff and Dundurs (2014), was whether these singularities can exist at all for small strains. To settle the matter, we will compute the exact nonlinear solution and show that indeed, in the limit of small strains, it is consistent with the solution of Shodja and Khorshidi (2013). We will further generalize the problem and identify the origin of stress singularity in cylindrical and spherical geometries. Second, the paper of Markenscoff and Dundurs (2014) who studied annular inhomogeneities with eigenstrains. The authors considered both spherical and cylindrical geometries and assumed that the eigenstrains in the inhomogeneities to be pure dilatational and positive. They showed that when the shear modulus of the annular inhomogeneity is larger than that of the core, tensile hydrostatic stress is created in the core. We revisit this problem by computing the exact solution. We show, among other results, that when the strain-energy density functions of the inhomogeneity and the core (and matrix) are identical, the stress inside the core does not necessarily vanish. However, it vanishes to first order in the eigenstrains where Markenscoff and Dundurs (2014)'s result is recovered. These problems of eigenstrains and singularity formation in elastic materials can be very subtle and their interpretation and validity may be clouded by the approximations made to obtain them. In such exceptional cases where an exact solution can be obtained and various limits explicitly computed, no such doubts persist.

2. Logarithmic stress singularities generated by finite anisotropic eigenstrains in a spherical ball

We first briefly review the problem solved in Yavari and Goriely (2013). Consider a spherical ball of radius R_0 made of an incompressible isotropic body with an energy function that may explicitly depend on R (in the spherical coordinates (R, Θ, Φ)). We assume that there are (finite) eigenstrains in the ball that may induce residual stresses. We assume that the radial and circumferential eigenstrains $e^{\omega_R(R)}$ and $e^{\omega_\Theta(R)}$ are given and that $\omega_R(R)$ and $\omega_\Theta(R)$ are analytic in a neighborhood of the origin. Yavari and Goriely (2013) assumed that the ball is stress free in the absence of eigenstrains for which the flat metric in the material manifold reads $\mathbf{G}_0(\mathbf{X}) = \mathbf{G}_0(R) = \mathrm{diag}(1, R^2, R^2\sin^2\Theta)$. In the presence of eigenstrains the material manifold (where the ball is stress free) has the Riemannian metric $\mathbf{G}(\mathbf{X}) = \mathbf{G}(R) = \mathrm{diag}(e^{2\omega_R(R)}, e^{2\omega_\Theta(R)}R^2, e^{2\omega_\Theta(R)}R^2\sin^2\Theta)$. Using the spherical coordinates (r, θ, ϕ) for the Euclidean ambient space, looking for solutions of the form $(r, \theta, \phi) = (r(R), \Theta, \Phi)$, assuming incompressibility, and $r(0) = 0$, one obtains $r(R) = (\int_0^R 3\xi^2 e^{\omega_R(\xi)+2\omega_\Theta(\xi)} d\xi)^{1/3}$. The principal stretches read $\lambda_1 = (R^2/r^2(R))e^{2\omega_\Theta(R)}$, $\lambda_2 = \lambda_3 = (r(R)/R)e^{-\omega_\Theta(R)}$. For an (inhomogeneous) isotropic solid the strain-energy density function

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[1] Note that in the examples that Yavari and Goriely (2013) solved $J=1$ everywhere and hence there is no interpenetration of matter anywhere.

depends only on R and the principal stretches, i.e. $W = W(R, \lambda_1, \lambda_2, \lambda_3)$ (Ogden, 1984). We further assume that W is finite for strictly positive stretches and that W has a single minimum at $\lambda_1 = \lambda_2 = \lambda_3 = 1$. These conditions are the general conditions valid for most elastic materials and can be interpreted physically as the boundedness of the energy for finite stretches, the absence of residual stress (no stress in the absence of deformation), and the absence of possible phase transformations. Note that some elastic materials with infinite strain-energy density in finite extension, such as the Gent model, are not covered by this description.

The Cauchy stress is diagonal with the following nonzero components:

$$\sigma^{rr} = \frac{R^2}{r^2(R)} e^{2\omega_\Theta(R)} \frac{\partial W}{\partial \lambda_1} - p(R), \tag{1}$$

$$\sigma^{\theta\theta} = \frac{e^{-\omega_\Theta(R)}}{Rr(R)} \frac{\partial W}{\partial \lambda_2} - \frac{p(R)}{r^2(R)}, \tag{2}$$

$$\sigma^{\phi\phi} = \frac{1}{\sin^2\Theta} \sigma^{\theta\theta}, \tag{3}$$

where p is the Lagrange multiplier corresponding to the incompressibility constraint. The Cauchy equations for the equilibrium give $p'(R) = h(R)$, where

$$h(R) = \frac{2R}{r^2(R)} e^{\omega_\Theta(R)} \left(e^{\omega_\Theta(R)} \frac{\partial W}{\partial \lambda_1} - e^{\omega_R(R)} \frac{\partial W}{\partial \lambda_2} \right) + \omega'_\Theta(R) \left[\frac{2R^2 e^{2\omega_\Theta(R)}}{r^2(R)} \frac{\partial W}{\partial \lambda_1} + \frac{2R^4 e^{4\omega_\Theta(R)}}{r^4(R)} \frac{\partial^2 W}{\partial \lambda_1^2} - \frac{2R e^{\omega_\Theta(R)}}{r(R)} \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \right]$$
$$+ \left(\frac{2R^3}{r^4(R)} e^{4\omega_\Theta(R)} \frac{\partial^2 W}{\partial \lambda_1^2} - \frac{2e^{\omega_\Theta(R)}}{r(R)} \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \right) \left[1 - \frac{R^3}{r^3(R)} e^{\omega_R(R) + 2\omega_\Theta(R)} \right]. \tag{4}$$

If at the boundary $\sigma^{rr}(R_o) = -p_\infty$, then

$$p(R) = p_\infty - \frac{R_o^2 e^{2\omega_\Theta(R_o)}}{r^2(R_o)} \frac{\partial W}{\partial \lambda_1} \bigg|_{R=R_o} - \int_R^{R_o} h(\xi) \, d\xi. \tag{5}$$

Once the pressure field is known, the stress components can be easily obtained. Since we do not consider a possible change of topology such as cavitation, the stretches are finite and so is the energy density. Therefore, since all the stress components are linear in the pressure they will all exhibit the same singularity, if any. Expression (5) is the key quantity for further analysis.

Next we revisit the general inclusion problem solved in Yavari and Goriely (2013) for neo-Hookean solids. Consider a spherical inclusion of radius $R_i < R_o$ with the same center as the ball and the following uniform but anisotropic eigenstrain functions:

$$\omega_R(R) = \begin{cases} \omega_1, & 0 \le R \le R_i, \\ 0, & R_i < R \le R_o, \end{cases} \qquad \omega_\Theta(R) = \begin{cases} \omega_2, & 0 \le R \le R_i, \\ 0, & R_i < R \le R_o. \end{cases} \tag{6}$$

Note that for $R \le R_i$, $\lambda_1 = e^{-(2/3)\omega_1 + (2/3)\omega_2} = \lambda_0^{-2}$, $\lambda_2 = \lambda_3 = e^{(1/3)\omega_1 - (1/3)\omega_2} = \lambda_0$. The radial Cauchy stress has the following distribution:

$$\sigma^{rr}(R) = \begin{cases} h_0 \ln\left(\frac{R_i}{R}\right) + e^{-(2/3)(\omega_1 - \omega_2)} \frac{\partial W}{\partial \lambda_1} \bigg|_{\lambda_1 = \lambda_0^{-2}, \, \lambda_2 = \lambda_3 = \lambda_0} - p_\infty - \frac{R_o^2}{r^2(R_o)} \frac{\partial W}{\partial \lambda_1} \bigg|_{R=R_o} + A + \int_{R_i}^{R_o} \bar{h}(\xi) \, d\xi, & 0 \le R < R_i, \\[12pt] \frac{R^2}{r^2(R)} \frac{\partial W}{\partial \lambda_1} - p_\infty - \frac{R_o^2}{r^2(R_o)} \frac{\partial W}{\partial \lambda_1} \bigg|_{R=R_o} + \int_R^{R_o} \bar{h}(\xi) \, d\xi, & R_i < R \le R_o, \end{cases} \tag{7}$$

where[2]

$$h_0 = 2e^{-(1/3)(2\omega_1 + \omega_2)} \left(e^{\omega_2} \frac{\partial W}{\partial \lambda_1} - e^{\omega_1} \frac{\partial W}{\partial \lambda_2} \right) \bigg|_{\lambda_1 = \lambda_0^{-2}, \, \lambda_2 = \lambda_3 = \lambda_0}, \tag{8}$$

$$A = e^{-(1/3)(2\omega_1 + \omega_2)} \frac{\partial W}{\partial \lambda_1} \bigg|_{R=R_i^+} - e^{-(2/3)(\omega_1 - \omega_2)} \frac{\partial W}{\partial \lambda_1} \bigg|_{\lambda_1 = \lambda_0^{-2}, \, \lambda_2 = \lambda_3 = \lambda_0}, \tag{9}$$

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[2] Note that there is a typo in the h_0 expression given in Yavari and Goriely (2013).

$$\bar{h}(R) = \frac{2R}{r^2(R)}\left(\frac{\partial W}{\partial \lambda_1} - \frac{\partial W}{\partial \lambda_2}\right) + \frac{2R^3}{r^4(R)}\left(1 - \frac{R^3}{r^3(R)}\right)\frac{\partial^2 W}{\partial \lambda_1^2} - \frac{1}{r(R)}\left(1 - \frac{R^3}{r^3(R)}\right)\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}.$$
(10)

Inside the spherical inclusion, the radial stress has the following distribution:

$$\sigma^{rr}(R) = h_0 \ln\left(\frac{R_i}{R}\right) + \sigma_i,$$
(11)

where

$$\sigma_i = e^{-(1/3)(2\omega_1 + \omega_2)}\frac{\partial W}{\partial \lambda_1}\bigg|_{R=R_i^+} - p_\infty - \frac{R_o^2}{r^2(R_o)}\frac{\partial W}{\partial \lambda_1}\bigg|_{R=R_o} + \int_{R_i}^{R_o}\bar{h}(\xi)\,d\xi.$$
(12)

Based on the assumptions on the strain-energy density W, we observe that h_0 vanishes for finite eigenstrains if and only if $\lambda_0 = 1$. Therefore, we conclude that $h_0 \neq 0$ unless $\omega_1 = \omega_2$ and that for $\omega_1 \neq \omega_2$, the stress exhibits a logarithmic singularity.

The particular case of a neo-Hookean solid: An easy way to recover the linear solution is to consider the limit of small eigenstrains for a neo-Hookean solid $W(\lambda_1, \lambda_2, \lambda_3) = (\mu/2)(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$, where μ is the shear modulus. In this case we have

$$h_0 = 2\mu e^{-(2/3)(\omega_1 + 2\omega_2)}(e^{\omega_2}\lambda_1 - e^{\omega_1}\lambda_2)\bigg|_{\lambda_1 = \lambda_0^{-2}, \lambda_2 = \lambda_3 = \lambda_0} = 2\mu\left(e^{(4/3)(\omega_2 - \omega_1)} - e^{-(2/3)(\omega_2 - \omega_1)}\right).$$
(13)

Small eigenstrain difference limit: Note that as $x \to 0$, $e^x = 1 + x + O(x^2)$. Thus, when the eigenstrain difference is small, i.e. $|\omega_1 - \omega_2| \ll 1$, we have

$$h_0 = 4\mu(\omega_2 - \omega_1) + O((\omega_2 - \omega_1)^2).$$
(14)

It is seen that even in the case of small eigenstrains the logarithmic singularity survives as shown in Shodja and Khorshidi (2013).

A natural question is then: Under what conditions can a hyperelastic incompressible isotropic material with anisotropic eigenstrains develop a singularity?

Asymptotic analysis of stress: Let $\omega_1 = \omega_R(0)$ and $\omega_2 = \omega_\Theta(0)$. As $R \to 0$, we have

$$\omega_R(R) = \omega_1 + O(R), \quad \omega_\Theta(R) = \omega_2 + O(R).$$
(15)

Note that $r(R) = \left(\int_0^R 3\xi^2 e^{\omega_R(\xi) + 2\omega_\Theta(\xi)}\,d\xi\right)^{1/3}$. Thus

$$r(R) = e^{(1/3)(\omega_1 + 2\omega_2)}R + O(R^2) \quad \text{as } R \to 0.$$
(16)

The principal stretches have the following asymptotic expansions for $R \to 0$:

$$\lambda_1(R) = \frac{R^2}{r^2(R)}e^{2\omega_\Theta(R)} = e^{-(2/3)(\omega_1 - \omega_2)} + O(R) = \lambda_0^{-2} + O(R), \quad \lambda_2(R) = \lambda_3(R) = \frac{r(R)}{R}e^{-\omega_\Theta(R)} = e^{(1/3)(\omega_1 - \omega_2)} + O(R) = \lambda_0 + O(R).$$
(17)

In the ODE, $p'(R) = h(R)$, we need to find an asymptotic expansion for $h(R)$ for small R. First, note that

$$\frac{\partial W}{\partial \lambda_1} = \frac{\partial W}{\partial \lambda_1}\bigg|_{\lambda_1 = \lambda_0^{-2}, \lambda_2 = \lambda_3 = \lambda_0} + O(R), \quad \frac{\partial W}{\partial \lambda_2} = \frac{\partial W}{\partial \lambda_2}\bigg|_{\lambda_1 = \lambda_0^{-2}, \lambda_2 = \lambda_3 = \lambda_0} + O(R),$$

$$\frac{\partial^2 W}{\partial \lambda_1^2} = \frac{\partial^2 W}{\partial \lambda_1^2}\bigg|_{\lambda_1 = \lambda_0^{-2}, \lambda_2 = \lambda_3 = \lambda_0} + O(R), \quad \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} = \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}\bigg|_{\lambda_1 = \lambda_0^{-2}, \lambda_2 = \lambda_3 = \lambda_0} + O(R).$$
(18)

Similarly, we have

$$1 - \frac{R^3}{r^3(R)}e^{\omega_R(R) + 2\omega_\Theta(R)} = O(R).$$
(19)

Therefore, using some elementary classical techniques of asymptotic analysis (Murray, 1984) we have

$$h(R) = \frac{h_0}{R} + O(1),$$
(20)

where h_0 is given in (8). Thus, $p(R) = h_0 \ln R + O(R)$ and, as before, $h_0 \neq 0$ unless $\omega_R(0) = \omega_\Theta(0)$. Therefore, we have proved the following proposition.

Proposition 1. *Consider an isotropic and incompressible hyperelastic spherical ball under uniform pressure on its boundary. Assume that a radially symmetric distribution of radial $e^{\omega_R(R)}$ and circumferential $e^{\omega_\Theta(R)}$ eigenstrains is given. Then, the Cauchy stress exhibits a logarithmic singularity at the center of the ball if and only if the radial and circumferential eigenstrains are not equal at the origin.*

In Appendix A we consider an infinitely long circular cylindrical bar with finite eigenstrains and study problems similar to those that were discussed in this section.

3. Hydrostatic stress generated by a spherical annular inhomogeneity with finite eigenstrains

In this section, we consider a finite ball with a spherical annular inhomogeneity with finite eigenstrains. The case of an infinitely long circular cylindrical bar with a cylindrical inhomogeneity is discussed in Appendix B. We assume that both the matrix and the inhomogeneity are made of incompressible isotropic solids. We assume that the annular inhomogeneity has a strain-energy density function different from that of the core and the matrix, see Fig. 1. We denote the strain-energy density function of the matrix and the core by $W^{(1)}$ and that of the annular inhomogeneity by $W^{(2)}$. We assume that the eigenstrains are non-zero only in the annular inhomogeneity.

The radii of the core and the ball are denoted as R_i and R_o, respectively, and the outer radius of the annular inclusion is R_a, see Fig. 1. We assume constant eigenstrains in the annulus. That is

$$\omega_R(R) = \begin{cases} 0, & 0 \leq R < R_i, \\ \omega_1, & R_i \leq R \leq R_a, \\ 0, & R_a < R \leq R_o, \end{cases} \quad \omega_\Theta(R) = \begin{cases} 0, & 0 \leq R < R_i, \\ \omega_2, & R_i \leq R \leq R_a, \\ 0, & R_a < R \leq R_o. \end{cases} \tag{21}$$

Thus

$$r(R) = \begin{cases} R, & 0 \leq R \leq R_i, \\ \left[e^{\omega_1 + 2\omega_2} R^3 + (1 - e^{\omega_1 + 2\omega_2}) R_i^3 \right]^{1/3}, & R_i \leq R \leq R_a, \\ \left[R^3 + (1 - e^{\omega_1 + 2\omega_2})(R_i^3 - R_a^3) \right]^{1/3}, & R_a \leq R \leq R_o. \end{cases} \tag{22}$$
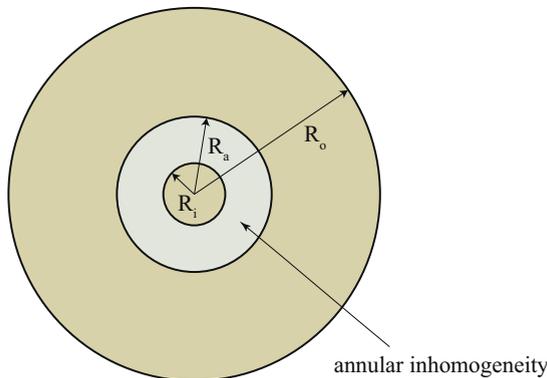
The principal stretches are

$$\lambda_1(R) = \begin{cases} 1, & 0 \leq R < R_i, \\ \dfrac{R^2}{r^2(R)} e^{2\omega_2}, & R_i \leq R \leq R_a, \\ \dfrac{R^2}{r^2(R)}, & R_a < R \leq R_o, \end{cases} \quad \lambda_2(R) = \lambda_3(R) = \begin{cases} 1, & 0 \leq R < R_i, \\ \dfrac{r(R)}{R} e^{-\omega_2}, & R_i \leq R \leq R_a, \\ \dfrac{r(R)}{R}, & R_a < R \leq R_o. \end{cases} \tag{23}$$

Note that $\omega_\Theta(R) = \omega_2[H(R - R_i) - H(R - R_a)]$, where $H(\cdot)$ is the Heaviside function. Therefore, $\omega'_\Theta(R) = \omega_2[\delta(R - R_i) - \delta(R - R_a)]$. Hence (Yavari and Goriely, 2013)

$$h(R) = A\delta(R - R_i) + B\delta(R - R_a) + \hat{h}(R), \tag{24}$$

where

Fig. 1. Cross section of a spherical ball or a cylindrical bar with an annular inhomogeneity. The core and the matrix are assumed to be made of the same incompressible isotropic solid.

$$\hat{h}(R) = \frac{2R}{r^2(R)}e^{\omega_\Theta(R)}\left(e^{\omega_\Theta(R)}\frac{\partial W}{\partial\lambda_1} - e^{\omega_R(R)}\frac{\partial W}{\partial\lambda_2}\right) + \frac{2R^3}{r^4(R)}e^{4\omega_\Theta(R)}\left[1 - \frac{R^3}{r^3(R)}e^{\omega_R(R)+2\omega_\Theta(R)}\right]\frac{\partial^2 W}{\partial\lambda_1^2}$$

$$- \frac{2e^{\omega_\Theta(R)}}{r(R)}\left[1 - \frac{R^3}{r^3(R)}e^{\omega_R(R)+2\omega_\Theta(R)}\right]\frac{\partial^2 W}{\partial\lambda_1\partial\lambda_2}. \tag{25}$$

The pressure in the ball has the following distribution:

$$p(R) = \begin{cases} p_0 - (A+B) - \int_{R_i}^{R_a} h_1(\xi)\,d\xi - \int_{R_a}^{R_o} h_2(\xi)\,d\xi, & 0 \le R < R_i, \\[2mm] p_0 - B - \int_{R}^{R_a} h_1(\xi)\,d\xi - \int_{R_a}^{R_o} h_2(\xi)\,d\xi, & R_i \le R \le R_a, \\[2mm] p_0 - \int_{R}^{R_o} h_2(\xi)\,d\xi, & R_a < R \le R_o, \end{cases} \tag{26}$$

where

$$p_0 = p_\infty + \frac{R_o^2}{r^2(R_o)}\frac{\partial W^{(1)}}{\partial\lambda_1}\bigg|_{R=R_o}, \tag{27}$$

$$h_1(R) = \frac{2Re^{\omega_2}}{r^2(R)}\left(e^{\omega_2}\frac{\partial W^{(2)}}{\partial\lambda_1} - e^{\omega_1}\frac{\partial W^{(2)}}{\partial\lambda_2}\right) + \left(\frac{2R^3 e^{4\omega_2}}{r^4(R)}\frac{\partial^2 W^{(2)}}{\partial\lambda_1^2} - \frac{2e^{\omega_2}}{r(R)}\frac{\partial^2 W^{(2)}}{\partial\lambda_1\partial\lambda_2}\right)\left[1 - \frac{R^3 e^{\omega_1+2\omega_2}}{r^3(R)}\right], \tag{28}$$

$$h_2(R) = \frac{2R}{r^2(R)}\left(\frac{\partial W^{(1)}}{\partial\lambda_1} - \frac{\partial W^{(1)}}{\partial\lambda_2}\right) + \left(\frac{2R^3}{r^4(R)}\frac{\partial^2 W^{(1)}}{\partial\lambda_1^2} - \frac{2}{r(R)}\frac{\partial^2 W^{(1)}}{\partial\lambda_1\partial\lambda_2}\right)\left[1 - \frac{R^3}{r^3(R)}\right]. \tag{29}$$

Note that for $R < R_i$, the pressure in the core is constant. The nonzero physical components of the Cauchy stress components (denoted by an overbar) read

$$\bar{\sigma}^{rr} = \sigma^{rr} = \frac{R^2}{r^2(R)}e^{2\omega_\Theta(R)}\frac{\partial W}{\partial\lambda_1} - p(R), \tag{30}$$

$$\bar{\sigma}^{\theta\theta} = r^2\sigma^{\theta\theta} = \frac{r(R)e^{-\omega_\Theta(R)}}{R}\frac{\partial W}{\partial\lambda_2} - p(R), \tag{31}$$

$$\bar{\sigma}^{\phi\phi} = r^2\sin^2\theta\sigma^{\phi\phi} = \bar{\sigma}^{\theta\theta}. \tag{32}$$

The radial Cauchy stress has the following distribution:

$$\sigma^{rr}(R) = \begin{cases} \dfrac{\partial W^{(1)}}{\partial\lambda_1}\bigg|_{\lambda_1=\lambda_2=\lambda_3=1} - p_0 + (A+B) + \int_{R_i}^{R_a} h_1(\xi)\,d\xi + \int_{R_a}^{R_o} h_2(\xi)\,d\xi, & 0 \le R < R_i, \\[3mm] \dfrac{R^2 e^{2\omega_2}}{r^2(R)}\dfrac{\partial W^{(2)}}{\partial\lambda_1} - p_0 + B + \int_{R}^{R_a} h_1(\xi)\,d\xi + \int_{R_a}^{R_o} h_2(\xi)\,d\xi, & R_i \le R \le R_a, \\[3mm] \dfrac{R^2}{r^2(R)}\dfrac{\partial W^{(1)}}{\partial\lambda_1} - p_0 + \int_{R}^{R_o} h_2(\xi)\,d\xi, & R_a < R \le R_o. \end{cases} \tag{33}$$

Continuity of traction at $R = R_i$ and R_a gives

$$A = e^{2\omega_2}\frac{\partial W^{(2)}}{\partial\lambda_1}\bigg|_{\lambda_1=e^{2\omega_2},\lambda_2=\lambda_3=e^{-\omega_2}} - \frac{\partial W^{(1)}}{\partial\lambda_1}\bigg|_{\lambda_1=\lambda_2=\lambda_3=1},$$

$$B = \frac{R_a^2}{r^2(R_a)}\left(\frac{\partial W^{(1)}}{\partial\lambda_1}\bigg|_{\lambda_1=R_a^2/r^2(R_a),\lambda_2=\lambda_3=r(R_a)/R_a} - e^{2\omega_2}\frac{\partial W^{(2)}}{\partial\lambda_1}\bigg|_{\lambda_1=R_a^2 e^{2\omega_2}/r^2(R_a),\lambda_2=\lambda_3=r(R_a)e^{-\omega_2}/R_a}\right). \tag{34}$$

The hoop stress has the following distribution:

$$\bar{\sigma}^{\theta\theta}(R) = \bar{\sigma}^{\phi\phi}(R) = \begin{cases} \dfrac{\partial W^{(1)}}{\partial \lambda_2}\bigg|_{\lambda_1=\lambda_2=\lambda_3=1} - p_0 + (A+B) + \displaystyle\int_{R_i}^{R_a} h_1(\xi)\,d\xi + \int_{R_a}^{R_o} h_2(\xi)\,d\xi, & 0 \le R < R_i, \\[2mm] \dfrac{r(R)e^{-\omega_2}}{R}\dfrac{\partial W^{(2)}}{\partial \lambda_2} - p_0 + B + \displaystyle\int_{R}^{R_a} h_1(\xi)\,d\xi + \int_{R_a}^{R_o} h_2(\xi)\,d\xi, & R_i \le R \le R_a, \\[2mm] \dfrac{r(R)}{R}\dfrac{\partial W^{(1)}}{\partial \lambda_2} - p_0 + \displaystyle\int_{R}^{R_o} h_2(\xi)\,d\xi, & R_a < R \le R_o. \end{cases} \tag{35}$$

We know that (Yavari and Goriely, 2013)

$$\frac{\partial W^{(1)}}{\partial \lambda_2}\bigg|_{\lambda_1=\lambda_2=\lambda_3=1} = \frac{\partial W^{(1)}}{\partial \lambda_1}\bigg|_{\lambda_1=\lambda_2=\lambda_3=1}, \tag{36}$$

and hence inside the core stress is hydrostatic $\boldsymbol{\sigma} = \sigma_c \mathbf{1}$ with magnitude σ_c:

$$\sigma_c = +e^{2\omega_2}\frac{\partial W^{(2)}}{\partial \lambda_1}\bigg|_{\lambda_1=e^{2\omega_2},\lambda_2=\lambda_3=e^{-\omega_2}} - \frac{R_o^2}{r^2(R_o)}\frac{\partial W^{(1)}}{\partial \lambda_1}\bigg|_{\lambda_1=R_o^2/r^2(R_o),\lambda_2=\lambda_3=r(R_o)/R_o}$$

$$+ \frac{R_a^2}{r^2(R_a)}\left(\frac{\partial W^{(1)}}{\partial \lambda_1}\bigg|_{\lambda_1=R_a^2/r^2(R_a),\lambda_2=\lambda_3=r(R_a)/R_a} - e^{2\omega_2}\frac{\partial W^{(2)}}{\partial \lambda_1}\bigg|_{\lambda_1=R_a^2 e^{2\omega_2}/r^2(R_a),\lambda_2=\lambda_3=r(R_a)e^{-\omega_2}/R_a}\right) + H, \tag{37}$$

where

$$H = \int_{R_i}^{R_a} h_1(\xi)\,d\xi + \int_{R_a}^{R_o} h_2(\xi)\,d\xi. \tag{38}$$

Note that the stress inside the core is hydrostatic even when the eigenstrain in the inhomogeneity is anisotropic, i.e. $\omega_1 \neq \omega_2$. When the ball is infinitely large ($R_o \to \infty$), we have $r(R_o)/R_o \to 1$. In this case

$$\sigma_c = e^{2\omega_2}\frac{\partial W^{(2)}}{\partial \lambda_1}\bigg|_{\lambda_1=e^{2\omega_2},\lambda_2=\lambda_3=e^{-\omega_2}} - \frac{\partial W^{(1)}}{\partial \lambda_1}\bigg|_{\lambda_1=\lambda_2=\lambda_3=1}$$

$$+ \frac{R_a^2}{r^2(R_a)}\left(\frac{\partial W^{(1)}}{\partial \lambda_1}\bigg|_{\lambda_1=R_a^2/r^2(R_a),\lambda_2=\lambda_3=r(R_a)/R_a} - e^{2\omega_2}\frac{\partial W^{(2)}}{\partial \lambda_1}\bigg|_{\lambda_1=R_a^2 e^{2\omega_2}/r^2(R_a),\lambda_2=\lambda_3=r(R_a)e^{-\omega_2}/R_a}\right) + H, \tag{39}$$

where

$$H = \int_{R_i}^{R_a} h_1(\xi)\,d\xi + \int_{R_a}^{\infty} h_2(\xi)\,d\xi. \tag{40}$$

The sign of the hydrostatic stress: Next we study the sign of the hydrostatic stress σ_c in the core to determine if the core can be under hydrostatic tension when the eigenstrains are purely dilational, i.e. $\omega_1 = \omega_2 = \omega$. We assume that both materials are neo-Hookean with the following energy functions for the three regions (core and matrix are made from the same material):

$$W^{(1)}(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu_1}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right), \quad W^{(2)}(\lambda_1, \lambda_2, \lambda_3) = \frac{\mu_2}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right). \tag{41}$$

In this case, all the integrals can be computed exactly:

$$h_1(R) = 2\mu_2 e^{\omega}\left(-\frac{1}{r(R)} + 2\frac{R^3 e^{3\omega}}{r^4(R)} - \frac{R^6 e^{6\omega}}{r^7(R)}\right) = -\frac{2\mu_2 e^{\omega}(1-e^{3\omega})^2 R_i^6}{\left[R_i^3 + e^{3\omega}(R^3 - R_i^3)\right]^{7/3}} = -3\mu_2(1 - s^{-6})\omega^2 + O(\omega^3),$$

$$h_2(R) = 2\mu_1\left(-\frac{1}{r(R)} + 2\frac{R^3}{r^4(R)} - \frac{R^6}{r^7(R)}\right) = -\frac{2\mu_1(1-e^{3\omega})^2\left(R_a^3 - R_i^3\right)^2}{\left[R^3 + \left(e^{3\omega} - 1\right)(R_a^3 - R_i^3)\right]^{\frac{7}{3}}} = -3\mu_1(1 - s^{-3})^2\omega^2 + O(\omega^3). \tag{42}$$

Therefore, $H(\omega) = O(\omega^2)$. Denoting $s = R_a/R_i > 1$, we have

$$\frac{R_a^4}{r^4(R_a)} = 1 + 4(1 - s^{-3})\omega + (8 - 10s^{-3} + 2s^{-6})\omega^2 + O(\omega^3),\tag{43}$$

so that

$$\sigma_c = 4(1 - s^{-3})(\mu_2 - \mu_1)\omega + (1 - s^{-3})[(5 + 11s^{-3})(\mu_2 - \mu_1) + 10\mu_1]\omega^2 + O(\omega^3).\tag{44}$$

We observe that to first order in ω, the sign of σ_c is the same as that of $\mu_2 - \mu_1$. However, this result is not true at higher orders or when $\mu_2 - \mu_1$ is small enough. In particular, when $\mu_1 = \mu_2$, the stress inside the core is zero only to first order in ω and interestingly, the hydrostatic stress is always tensile to second order (independently from the sign of ω, that is it is tensile whether the inhomogeneity is larger or smaller than the original annulus).

4. Concluding remarks

Inclusions in nonlinear solids generate particular features in the residual stress field. Here, we studied two of these defining features, the generation of singularity when the eigenstrains are anisotropic and the creation of a state of hydrostatic stress when the eigenstrains are restricted to an annulus. Both the case of spheres and cylinders have been considered and up to slight differences in the actual expressions, the two geometries exhibit exactly the same behaviors, that is (i) stress singularities are generated by the anisotropy of (finite) eigenstrains, and (ii) the residual stress inside a core due to anisotropic (finite) eigenstrains in an annular inhomogeneity is hydrostatic. Our general analysis fully confirms the results of Shodja and Khorshidi (2013), which are recovered here in the limit of small strains despite the strong disagreement expressed in Markenscoff and Dundurs (2014). Further, an explicit analysis for the case of neo-Hookean solids reveals that the sign of this stress depends on both the shear moduli difference between core and matrix but also on the value of the (finite) dilatational eigenstrains. Interestingly, the sign of the hydrostatic stress is only fixed by this difference of moduli in the case of small eigenstrains (as obtained by Markenscoff and Dundurs, 2014). However, we also showed that even when the moduli are equal, a hydrostatic stress is generated due to the nonlinear coupling between the eigenstrains and the material nonlinearities. Both situations underline the subtle coupling between anisotropy, inhomogeneities, and nonlinearities.

Acknowledgments

Appendix A. Logarithmic stress singularities generated by finite anisotropic eigenstrains in an infinite circular cylindrical bar

Next, we consider an infinitely long circular cylindrical bar with a radially symmetric distribution of eigenstrains, which was also studied in Yavari and Goriely (2013). This case follows closely the spherical case and we provide here only the key results. In the cylindrical coordinates (R, Θ, Z) the Riemannian material metric reads $\mathbf{G} = \text{diag}(e^{2\omega_R(R)}, R^2 e^{2\omega_\Theta(R)}, 1)$. The bar is assumed to be made of an arbitrary incompressible isotropic solid as in the case of the sphere. Assuming incompressibility and $r(0) = 0$ we have $r(R) = (\int_0^R 2\xi e^{\omega_R(\xi) + \omega_\Theta(\xi)} d\xi)^{1/2}$. The principal stretches read $\lambda_1 = r'(R)e^{-\omega_R(R)} = (R/r(R))e^{\omega_\Theta(R)}$, $\lambda_2 = (r(R)/R)e^{-\omega_\Theta(R)}$, $\lambda_3 = 1$. The non-vanishing components of Cauchy stress are

$$\sigma^{rr} = \frac{Re^{\omega_\Theta(R)}}{r(R)}\frac{\partial W}{\partial \lambda_1} - p(R),\tag{A.1}$$

$$\sigma^{\theta\theta} = \frac{e^{-\omega_\Theta(R)}}{Rr(R)}\frac{\partial W}{\partial \lambda_2} - \frac{p(R)}{r^2(R)},\tag{A.2}$$

$$\sigma^{zz} = \frac{\partial W}{\partial \lambda_3} - p(R).\tag{A.3}$$

Equilibrium gives again an ODE $p'(R) = k(R)$ for the Lagrange multiplier $p = p(R)$, where

$$k(R) = \frac{1}{r(R)} \left(e^{\omega_\Theta(R)} \frac{\partial W}{\partial \lambda_1} - e^{\omega_R(R)} \frac{\partial W}{\partial \lambda_2} \right) + \omega'_\Theta(R) \left[\frac{R e^{\omega_\Theta(R)}}{r(R)} \frac{\partial W}{\partial \lambda_1} + \frac{R^2 e^{2\omega_\Theta(R)}}{r^2(R)} \frac{\partial^2 W}{\partial \lambda_1^2} - \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \right]$$
$$+ \left(\frac{R e^{2\omega_\Theta(R)}}{r^2(R)} \frac{\partial^2 W}{\partial \lambda_1^2} - \frac{1}{R} \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \right) \left[1 - \frac{R^2}{r^2(R)} e^{\omega_R(R) + \omega_\Theta(R)} \right]. \tag{A.4}$$

If at the boundary $\sigma^{rr}(R_o) = -p_\infty$, then

$$p(R) = p_\infty - \frac{R_o e^{\omega_\Theta(R_o)}}{r(R_o)} \frac{\partial W}{\partial \lambda_1} \bigg|_{R=R_o} - \int_R^{R_o} k(\xi) \, d\xi. \tag{A.5}$$

The pressure field determines all the stress components. Next we revisit the inclusion problem solved in Yavari and Goriely (2013) and analyze it in the case of small eigenstrain difference. Consider a cylindrical inclusion of radius $R_i < R_o$ with the same axis as the cylindrical bar with the following uniform but anisotropic eigenstrain functions:

$$\omega_R(R) = \begin{cases} \omega_1, & 0 \le R \le R_i, \\ 0, & R_i < R \le R_o, \end{cases} \qquad \omega_\Theta(R) = \begin{cases} \omega_2, & 0 \le R \le R_i, \\ 0, & R_i < R \le R_o. \end{cases} \tag{A.6}$$

For $R \le R_i$, $\lambda_1 = e^{(1/2)(\omega_2 - \omega_1)} = \lambda_0$, $\lambda_2 = e^{(1/2)(\omega_1 - \omega_2)} = \lambda_0^{-1}$, $\lambda_3 = 1$. The radial Cauchy stress has the following distribution:

$$\sigma^{rr}(R) = \begin{cases} k_0 \ln\left(\dfrac{R_i}{R}\right) + e^{(1/2)(\omega_2 - \omega_1)} \dfrac{\partial W}{\partial \lambda_1} \bigg|_{\lambda_1 = \lambda_0, \lambda_2 = \lambda_0^{-1}, \lambda_3 = 1} - p_\infty \quad - \dfrac{R_o}{r(R_o)} \dfrac{\partial W}{\partial \lambda_1} \bigg|_{R=R_o} + C + \displaystyle\int_{R_i}^{R_o} \bar{k}(\xi) \, d\xi, \quad 0 \le R < R_i, \\[2ex] \dfrac{R}{r(R)} \dfrac{\partial W}{\partial \lambda_1} - p_\infty - \dfrac{R_o}{r(R_o)} \dfrac{\partial W}{\partial \lambda_1} \bigg|_{R=R_o} + \displaystyle\int_R^{R_o} \bar{k}(\xi) \, d\xi, \hfill R_i < R \le R_o, \end{cases} \tag{A.7}$$

where

$$k_0 = e^{-(1/2)(\omega_1 + \omega_2)} \left(e^{\omega_2} \frac{\partial W}{\partial \lambda_1} - e^{\omega_1} \frac{\partial W}{\partial \lambda_2} \right) \bigg|_{\lambda_1 = \lambda_0, \lambda_2 = \lambda_0^{-1}, \lambda_3 = 1}, \tag{A.8}$$

$$C = e^{-(1/2)(\omega_1 + \omega_2)} \frac{\partial W}{\partial \lambda_1} \bigg|_{R=R_i^+} - e^{(1/2)(\omega_2 - \omega_1)} \frac{\partial W}{\partial \lambda_1} \bigg|_{\lambda_1 = \lambda_0, \lambda_2 = \lambda_0^{-1}, \lambda_3 = 1}, \tag{A.9}$$

$$\bar{k}(R) = \frac{1}{r(R)} \left(\frac{\partial W}{\partial \lambda_1} - \frac{\partial W}{\partial \lambda_2} \right) + \frac{R}{r^2(R)} \left[1 - \frac{R^2}{r^2(R)} \right] \frac{\partial^2 W}{\partial \lambda_1^2} + \frac{R}{r^2(R)} \left[1 - \frac{r^2(R)}{R^2} \right] \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}. \tag{A.10}$$

It is seen that the radial stress inside the inclusion has the following form:

$$\sigma^{rr}(R) = h_0 \ln\left(\frac{R_i}{R}\right) + \sigma_i, \tag{A.11}$$

where

$$\sigma_i = e^{-(1/2)(\omega_1 + \omega_2)} \frac{\partial W}{\partial \lambda_1} \bigg|_{R=R_i^+} - p_\infty - \frac{R_o}{r(R_o)} \frac{\partial W}{\partial \lambda_1} \bigg|_{R=R_o} + \int_{R_i}^{R_o} \bar{k}(\xi) \, d\xi. \tag{A.12}$$

Yavari and Goriely (2013) concluded that the radial stress has a logarithmic singularity unless $\omega_1 = \omega_2$. They observed the same singularity for the other non-vanishing stress components.

The particular case of a neo-Hookean solid: To easily recover the linear solution we consider a neo-Hookean solid. In this case we have

$$k_0 = \mu(e^{\omega_2 - \omega_1} - e^{\omega_1 - \omega_2}). \tag{A.13}$$

Small eigenstrain difference limit: When the difference between the two eigenstrains is small, i.e. $|\omega_1 - \omega_2| \ll 1$, we have

$$k_0 = 2\mu(\omega_2 - \omega_1) + O((\omega_2 - \omega_1)^2). \tag{A.14}$$

We see that even in the case of small eigenstrains the logarithmic singularity survives. We next characterize all those anisotropic radially symmetric eigenstrain distributions that induce a stress singularity on the axis of the bar.

Asymptotic analysis of stress: Let $\omega_1 = \omega_R(0)$ and $\omega_2 = \omega_\Theta(0)$. As $R \to 0$, we have $\omega_R(R) = \omega_1 + O(R)$, $\omega_\Theta(R) = \omega_2 + O(R)$. Note that $r(R) = (\int_0^R 2\xi e^{\omega_R(\xi) + \omega_\Theta(\xi)} \, d\xi)^{1/2}$. Thus, $r(R) = e^{(1/2)(\omega_1 + \omega_2)} R + O(R^2)$ as $R \to 0$. For the principal stretches ($\lambda_3(R) = 1$) we have

$$\lambda_1(R) = \frac{R}{r(R)}e^{\omega_\Theta(R)} = e^{-(1/2)(\omega_1 - \omega_2)} + O(R) = \lambda_0 + O(R), \quad \lambda_2(R) = \frac{r(R)}{R}e^{-\omega_\Theta(R)} = e^{(1/2)(\omega_1 - \omega_2)} + O(R) = \lambda_0^{-1} + O(R).$$
(A.15)

In the ODE, $p'(R) = k(R)$, we need to find an asymptotic expansion for $k(R)$ for small R. One can easily show that

$$1 - \frac{R^2}{r^2(R)}e^{\omega_R(R) + \omega_\Theta(R)} = O(R).$$
(A.16)

Therefore

$$k(R) = \frac{k_0}{R} + O(1),$$
(A.17)

where k_0 is given in (A.8). Note that if $\omega_R(0) = \omega_\Theta(0)$, then $k_0 = 0$. Therefore, we have proved the following proposition.

Proposition 2. *Consider an isotropic and incompressible hyperelastic circular cylindrical bar under uniform pressure on its boundary cylinder. Assume that there is a given radially symmetric distribution of radial $e^{\omega_R(R)}$ and circumferential $e^{\omega_\Theta(R)}$ eigenstrains. Then, unless the radial and circumferential eigenstrains are equal on the axis of the cylinder, the Cauchy stress exhibits a logarithmic singularity.*

Appendix B. Hydrostatic stress generated by a cylindrical annular inhomogeneity with finite eigenstrains

In this appendix we consider an annular inhomogeneity with finite eigenstrain in an infinitely long circular cylindrical bar. We assume that the radii of the core and the cylinder are R_i and R_o, respectively, and that the outer radius of the annular inclusion is R_a, see Fig. 1. We assume the following ω_R and ω_Θ distributions:

$$\omega_R(R) = \begin{cases} 0, & 0 \le R < R_i, \\ \omega_1, & R_i \le R \le R_a, \\ 0, & R_a < R \le R_o, \end{cases} \qquad \omega_\Theta(R) = \begin{cases} 0, & 0 \le R < R_i, \\ \omega_2, & R_i \le R \le R_a, \\ 0, & R_a < R \le R_o. \end{cases}$$
(B.1)

Thus

$$r(R) = \begin{cases} R, & 0 \le R \le R_i, \\ \left[e^{\omega_1 + \omega_2}R^2 + (1 - e^{\omega_1 + \omega_2})R_i^2\right]^{1/2}, & R_i \le R \le R_a, \\ \left[R^2 + (1 - e^{\omega_1 + \omega_2})(R_i^2 - R_a^2)\right]^{1/2}, & R_a \le R \le R_o. \end{cases}$$
(B.2)

The principal stretches read ($\lambda_3(R) = 1$)

$$\lambda_1(R) = \begin{cases} 1, & 0 \le R < R_i, \\ \frac{R}{r(R)}e^{\omega_2}, & R_i \le R \le R_a, \\ \frac{R}{r(R)}, & R_a < R \le R_o, \end{cases} \qquad \lambda_2(R) = \begin{cases} 1, & 0 \le R < R_i, \\ \frac{r(R)}{R}e^{-\omega_2}, & R_i \le R \le R_a, \\ \frac{r(R)}{R}, & R_a < R \le R_o. \end{cases}$$
(B.3)

Note that $\omega_\Theta(R) = \omega_2 H(R - R_i) - \omega_2 H(R - R_a)$ and hence $\omega'_\Theta(R) = \omega_2 \delta(R - R_i) - \omega_2 \delta(R - R_a)$. Therefore (Yavari and Goriely, 2013).

$$k(R) = C\delta(R - R_i) + D\delta(R - R_a) + \hat{k}(R),$$
(B.4)

where

$$\hat{k}(R) = \frac{1}{r(R)}\left(e^{\omega_\Theta(R)}\frac{\partial W}{\partial \lambda_1} - e^{\omega_R(R)}\frac{\partial W}{\partial \lambda_2}\right) + \left(\frac{Re^{2\omega_\Theta(R)}}{r^2(R)}\frac{\partial^2 W}{\partial \lambda_1^2} - \frac{1}{R}\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}\right)\left[1 - \frac{R^2 e^{\omega_R(R) + \omega_\Theta(R)}}{r^2(R)}\right].$$
(B.5)

Note that for $R < R_i$, $k(R) = 0$ and the pressure is constant inside the core. In the cylindrical bar, the pressure has the following distribution:

$$
p(R) = \begin{cases} p_0 - (C + D) - \int_{R_i}^{R_a} k_1(\xi)\, d\xi - \int_{R_a}^{R_o} k_2(\xi)\, d\xi, & 0 \le R < R_i, \\[2mm] p_0 - D - \int_{R}^{R_a} k_1(\xi)\, d\xi - \int_{R_a}^{R_o} k_2(\xi)\, d\xi, & R_i \le R \le R_a, \\[2mm] p_0 - \int_{R}^{R_o} k_2(\xi)\, d\xi, & R_a < R \le R_o, \end{cases}
$$

(B.6)

where

$$
p_0 = p_\infty + \frac{R_o}{r(R_o)} \frac{\partial W^{(1)}}{\partial \lambda_1}\bigg|_{R=R_o},
$$

(B.7)

$$
k_1(R) = \frac{1}{r(R)}\left(e^{\omega_2}\frac{\partial W^{(2)}}{\partial \lambda_1} - e^{\omega_1}\frac{\partial W^{(2)}}{\partial \lambda_2}\right) + \left(\frac{Re^{2\omega_2}}{r^2(R)}\frac{\partial^2 W^{(2)}}{\partial \lambda_1^2} - \frac{1}{R}\frac{\partial^2 W^{(2)}}{\partial \lambda_1 \partial \lambda_2}\right)\left[1 - \frac{R^2 e^{\omega_1+\omega_2}}{r^2(R)}\right],
$$

(B.8)

$$
k_2(R) = \frac{1}{r(R)}\left(\frac{\partial W^{(1)}}{\partial \lambda_1} - \frac{\partial W^{(1)}}{\partial \lambda_2}\right) + \left(\frac{R}{r^2(R)}\frac{\partial^2 W^{(1)}}{\partial \lambda_1^2} - \frac{1}{R}\frac{\partial^2 W^{(1)}}{\partial \lambda_1 \partial \lambda_2}\right)\left[1 - \frac{R^2}{r^2(R)}\right].
$$

(B.9)

The nonzero physical components of the Cauchy stress read

$$
\bar{\sigma}^{rr} = \sigma^{rr} = \frac{R}{r(R)}e^{\omega_\Theta(R)}\frac{\partial W}{\partial \lambda_1} - p(R),
$$

$$
\bar{\sigma}^{\theta\theta} = r^2(R)\sigma^{\theta\theta} = \frac{r(R)e^{-\omega_\Theta(R)}}{R}\frac{\partial W}{\partial \lambda_2} - p(R),
$$

$$
\bar{\sigma}^{zz} = \sigma^{zz} = \frac{\partial W}{\partial \lambda_3} - p(R).
$$

(B.10)

The radial Cauchy stress has the following distribution:

$$
\sigma^{rr}(R) = \begin{cases} \dfrac{\partial W^{(1)}}{\partial \lambda_1}\bigg|_{\lambda_1=\lambda_2=\lambda_3=1} - p_0 + (C + D) + \int_{R_i}^{R_a} k_1(\xi)\, d\xi + \int_{R_a}^{R_o} k_2(\xi)\, d\xi, & 0 \le R < R_i, \\[3mm] \dfrac{Re^{\omega_2}}{r(R)}\dfrac{\partial W^{(2)}}{\partial \lambda_1} - p_0 + D + \int_{R}^{R_a} k_1(\xi)\, d\xi + \int_{R_a}^{R_o} k_2(\xi)\, d\xi, & R_i \le R \le R_a, \\[3mm] \dfrac{R}{r(R)}\dfrac{\partial W^{(1)}}{\partial \lambda_1} - p_0 + \int_{R}^{R_o} k_2(\xi)\, d\xi, & R_a < R \le R_o. \end{cases}
$$

(B.11)

The continuity of traction at $R = R_i$ and R_a gives

$$
C = e^{\omega_2}\frac{\partial W^{(2)}}{\partial \lambda_1}\bigg|_{\lambda_1=e^{\omega_2},\lambda_2=e^{-\omega_2},\lambda_3=1} - \frac{\partial W^{(1)}}{\partial \lambda_1}\bigg|_{\lambda_1=\lambda_2=\lambda_3=1},
$$

$$
D = \frac{R_a}{r(R_a)}\left(\frac{\partial W^{(1)}}{\partial \lambda_1}\bigg|_{\lambda_1=R_a/r(R_a),\lambda_2=r(R_a)/R_a,\lambda_3=1} - e^{\omega_2}\frac{\partial W^{(2)}}{\partial \lambda_1}\bigg|_{\lambda_1=R_a e^{\omega_2}/r(R_a),\lambda_2=r(R_a)e^{-\omega_2}/R_a,\lambda_3=1}\right).
$$

(B.12)

The hoop stress has the following distribution:

$$
\bar{\sigma}^{\theta\theta}(R) = \begin{cases} \dfrac{\partial W^{(1)}}{\partial \lambda_2}\bigg|_{\lambda_1=\lambda_2=\lambda_3=1} - p_0 + (C + D) + \int_{R_i}^{R_a} k_1(\xi)\, d\xi + \int_{R_a}^{R_o} k_2(\xi)\, d\xi, & 0 \le R < R_i, \\[3mm] \dfrac{r(R)e^{-\omega_2}}{R}\dfrac{\partial W^{(2)}}{\partial \lambda_2} - p_0 + D + \int_{R}^{R_a} k_1(\xi)\, d\xi + \int_{R_a}^{R_o} k_2(\xi)\, d\xi, & R_i \le R \le R_a, \\[3mm] \dfrac{r(R)}{R}\dfrac{\partial W^{(1)}}{\partial \lambda_2} - p_0 + \int_{R}^{R_o} k_2(\xi)\, d\xi, & R_a < R \le R_o. \end{cases}
$$

(B.13)

Finally, the axial stress can also be obtained explicitly as

$$\bar{\sigma}^{zz}(R) = \begin{cases} \left.\dfrac{\partial W^{(1)}}{\partial \lambda_3}\right|_{\lambda_1=\lambda_2=\lambda_3=1} - p_0 + (C+D) + \displaystyle\int_{R_i}^{R_a} k_1(\xi)\,d\xi + \int_{R_a}^{R_o} k_2(\xi)\,d\xi, & 0 \le R < R_i, \\[2ex] \dfrac{\partial W^{(2)}}{\partial \lambda_3} - p_0 + D + \displaystyle\int_{R}^{R_a} k_1(\xi)\,d\xi + \int_{R_a}^{R_o} k_2(\xi)\,d\xi, & R_i \le R \le R_a, \\[2ex] \dfrac{\partial W^{(1)}}{\partial \lambda_3} - p_0 + \displaystyle\int_{R}^{R_o} k_2(\xi)\,d\xi, & R_a < R \le R_o. \end{cases} \tag{B.14}$$

Since we know that (Yavari and Goriely, 2013)

$$\left.\frac{\partial W^{(1)}}{\partial \lambda_1}\right|_{\lambda_1=\lambda_2=\lambda_3=1} = \left.\frac{\partial W^{(1)}}{\partial \lambda_2}\right|_{\lambda_1=\lambda_2=\lambda_3=1} = \left.\frac{\partial W^{(1)}}{\partial \lambda_3}\right|_{\lambda_1=\lambda_2=\lambda_3=1}, \tag{B.15}$$

the stress inside the core is hydrostatic $\boldsymbol{\sigma} = \sigma_c \mathbf{1}$ with magnitude

$$\sigma_c = +e^{\omega_2}\left.\frac{\partial W^{(2)}}{\partial \lambda_1}\right|_{\lambda_1=e^{\omega_2},\lambda_2=e^{-\omega_2},\lambda_3=1} - \frac{R_o}{r(R_o)}\left.\frac{\partial W^{(1)}}{\partial \lambda_1}\right|_{\lambda_1=R_o/r(R_o),\lambda_2=r(R_o)/R_o,\lambda_3=1}$$

$$+ \frac{R_a}{r(R_a)}\left(\left.\frac{\partial W^{(1)}}{\partial \lambda_1}\right|_{\lambda_1=R_a/r(R_a),\lambda_2=r(R_a)/R_a,\lambda_3=1} - e^{\omega_2}\left.\frac{\partial W^{(2)}}{\partial \lambda_1}\right|_{\lambda_1=R_a e^{\omega_2}/r(R_a),\lambda_2=r(R_a)e^{-\omega_2}/R_a,\lambda_3=1}\right) + K, \tag{B.16}$$

where

$$K = \int_{R_i}^{R_a} k_1(\xi)\,d\xi + \int_{R_a}^{R_o} k_2(\xi)\,d\xi. \tag{B.17}$$

Note that the stress inside the core is hydrostatic even when the eigenstrain in the inhomogeneity is anisotropic, i.e. $\omega_1 \ne \omega_2$.

The sign of the hydrostatic stress: Next, we study the sign of the hydrostatic stress in the core in the limit when the circular cross section of the bar is infinitely large: $R_o \to \infty$, and $r(R_o)/R_o \to 1$. In this case

$$\sigma_c = e^{\omega_2}\left.\frac{\partial W^{(2)}}{\partial \lambda_1}\right|_{\lambda_1=e^{\omega_2},\lambda_2=e^{-\omega_2},\lambda_3=1} - \left.\frac{\partial W^{(1)}}{\partial \lambda_1}\right|_{\lambda_1=\lambda_2=\lambda_3=1}$$

$$+ \frac{R_a}{r(R_a)}\left(\left.\frac{\partial W^{(1)}}{\partial \lambda_1}\right|_{\lambda_1=R_a/r(R_a),\lambda_2=r(R_a)/R_a,\lambda_3=1} - e^{\omega_2}\left.\frac{\partial W^{(2)}}{\partial \lambda_1}\right|_{\lambda_1=R_a e^{\omega_2}/r(R_a),\lambda_2=r(R_a)e^{-\omega_2}/R_a,\lambda_3=1}\right) + K, \tag{B.18}$$

where

$$K = \int_{R_i}^{R_a} k_1(\xi)\,d\xi + \int_{R_a}^{\infty} k_2(\xi)\,d\xi. \tag{B.19}$$

We further assume that $\omega_1 = \omega_2 = \omega$ and that the material under consideration is a neo-Hookean solid with the following energy functions for the three regions (core and matrix are made from the same material):

$$W^{(1)}(\lambda_1,\lambda_2,\lambda_3) = \frac{\mu_1}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right), \quad W^{(2)}(\lambda_1,\lambda_2,\lambda_3) = \frac{\mu_2}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right). \tag{B.20}$$

We have

$$k_1(R) = \mu_2\left(-\frac{1}{R} + 2\frac{Re^{2\omega}}{r^2(R)} - \frac{R^3 e^{4\omega}}{r^4(R)}\right) = -(1 - s^{-4})\mu_2 \omega^2 + O(\omega^3),$$

$$k_2(R) = \mu_1\left(-\frac{1}{R} + 2\frac{R}{r^2(R)} - \frac{R^3}{r^4(R)}\right) = -(1 - s^{-2})^2 \mu_1 \omega^2 + O(\omega^3). \tag{B.21}$$

Therefore, $K(\omega) = O(\omega^3)$ and hence

$$\sigma_c = 2(1 - s^{-2})(\mu_2 - \mu_1)\omega + (1 - s^{-2})[\mu_1 + \mu_2 + 3s^{-2}(\mu_2 - \mu_1)]\omega^2 + O(\omega^3). \tag{B.22}$$

It is seen that for small enough ω, the sign of σ_c is the same as that of $\mu_2 - \mu_1$. However, this result is not true for larger eigenstrains or in the case when $\mu_2 - \mu_1$ is small enough. In particular, when $\mu_1 = \mu_2$, the stress inside the core is tensile for small eigenstrains since in that case $\sigma_c = 2(1 - s^{-2})\mu_1\omega^2 + O(\omega^3)$.

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