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The fourth mode of fracture in fractal fracture mechanics

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Abstract. This paper offers a systematic approach for obtaining the order of stress singularity for different self-similar and self-affine fractal cracks. Mode II and Mode III fractal cracks are studied and are shown to introduce the same order of stress singularity as Mode I fractal cracks do. In addition to these three classical modes, a Mode IV is discovered, which is a consequence of the fractal fracture. It is shown that, for this mode, stress has a weaker singularity than it does in the classical modes of fracture when self-affine fractal cracks are considered, and stress has the same order of singularity when self-similar cracks are considered. Considering this new mode of fracture, some single-mode problems of classical fracture mechanics could be mixed-mode problems in fractal fracture mechanics. By imposing a continuous transition from fractal to classical stress and displacement fields, the complete forms of the stress and displacement fields around the tip of a fractal crack are found. Then a universal relationship between fractal and classical stress intensity factors is derived. It is demonstrated that for a Mode IV fractal crack, only one of the stress components is singular; the other stress components are identically zero. Finally, stress singularity for three-dimensional bodies with self-affine fractal cracks is studied. As in the two-dimensional case, the fourth mode of fracture introduces a weaker stress singularity for self-affine fractal cracks than classical modes of fracture do.

1. Introduction

The word ‘fractal’ was coined by Benoit Mandelbrot in his foundational essay (Mandelbrot, 1983). In Latin *fractus* means broken. This word is used to describe the geometry of objects that are too irregular to be modeled by Euclidean geometry. If classical geometry is considered to be a first approximation to natural objects, fractal geometry is the next level of approximation. Fractal geometry offers a new scientific way of thinking about natural phenomena. According to Mandelbrot (1983), a fractal is a set whose Hausdorff–Besicovitch dimension is strictly larger than its topological dimension.

A variety of natural objects may be described mathematically as fractals; for example, cloud boundaries, coastlines, turbulence in fluids, fracture surfaces, or the rough surfaces in contact, rocks, and so forth. None of these is an actual fractal; fractal features disappear if an object is viewed at a sufficiently small scale. Nevertheless for a wide range of scales the natural objects appear very much like fractals, and in that case they may be regarded as fractals. There are no true fractals in nature and there are no true straight lines or circles, either. Clearly, fractals are better approximations for real objects than are straight lines or circles.

There have been many investigations regarding the contribution of the fractality of crack surfaces to the mechanics of fracture since the pioneering work of Mandelbrot et al. (1984). Recently, Saouma et al. (1994) showed experimentally that the fracture surfaces of concrete are fractals. Many researchers tried to find a relationship between the fractal dimension of cracks and the fracture toughness. But, so far, no universal relation has been found. A real crack on the mesoscale is different from the ideal smooth-edged crack; the fracture surfaces of most materials are very irregular. Experimental observations have proved a statistical fractality for fracture surfaces. Therefore, assuming that crack profiles are fractals is a more realistic model in fracture mechanics than assuming they are smooth.

The irregularities of crack surfaces, in contrast to mathematical fractals, are finite. Therefore crack profiles can be assumed to be fractals only in a range $\ell_0 \leq \mathbf{r} \leq \ell_1$ (Cherepanov et al., 1995). The lower cutoff ℓ_0. is related to the micromechanics of the cracked material and the upper cutoff is a function of the geometric size of the specimen, the crack size, and other factors.

Experimental studies have shown that fracture surfaces are self-affine rather than self-similar (Mandelbrot, 1985; Brown and Scholz, 1985; Wong et al., 1986. Power and Tullis, 199 1). Also, as pointed out by Bažant (1995, 1997) not all fractal curves are admissible crack trajectories: zones of material adjacent to the crack face must be able to move apart as rigid bodies. Consider a crack that is propagating along the x-axis: the crack trajectory deviates from the x-axis but the direction of propagation is the x-axis; the crack trajectory is an admissible fractal crack if and only if it is a single-valued function of the independent variable x, i.e., any line perpendicular to the x-axis must intersect the crack trajectory only once. Here, for the sake of completeness, we consider both self-similar and self-affine fractal cracks.

Bažant (1995, 1997) demonstrated that the fractality of cracks does not make an important contribution to size effect.

A method of defining fractals is to consider them as fixed points of iterated function systems (Bransley, 1986; Falconer, 1990). This idea was used by Panagiotopoulous et al. (1993), to define mechanical laws for fractal objects. Panagiotopoulous, et al. (1993), and Panagiotopoulous et al. (1995), worked on finite element and boundary element methods for bodies with fractal boundaries. Using iterated function systems, Panagiotopoulous et al. (1995), concluded that the $\mathrm{r}^{-1/2}$ stress singularity of linear elastic classical cracks still holds for fractal cracks. Their conclusion is incorrect because they did not consider the interaction of sharp corners in the limit case.

Xie and Sanderson (1995) studied the effects of fractal crack propagation on the dynamic stress intensity factor and on crack velocity. They were able to explain why experimentally observed terminal fracture speeds are only about half of the Rayleigh wave speed. Xie (1989) proposed a fractal model for crack branching in brittle materials. Using this model, he showed that the fracture toughness can be raised due to the fractality of fracture surfaces.

Borodich (1994, 1997) introduced the concept of specific energy for a unit measure of fractal in order to solve the paradox that fracture is impossible for a mathematical fractal crack.

The change of the order of stress singularity due to fractality of the crack surfaces was first studied by Mosolov (1991). Using Griffith's criterion and considering the fact that the true length of a fractal crack is larger than its apparent length, he obtained the correct asymptotic expression for a Mode I self-similar fractal crack:

$$\sigma \sim r^{\alpha}, \quad \alpha = \frac{D-2}{2} \tag{1}$$

where D is the fractal dimension of a self-similar crack. It is to be noted that there are many definitions for fractal dimension. All of these definitions yield the same fractal dimension for self-similar fractals. Gol'dshteĭn and Mosolov (1991, 1992) obtained the same singularity

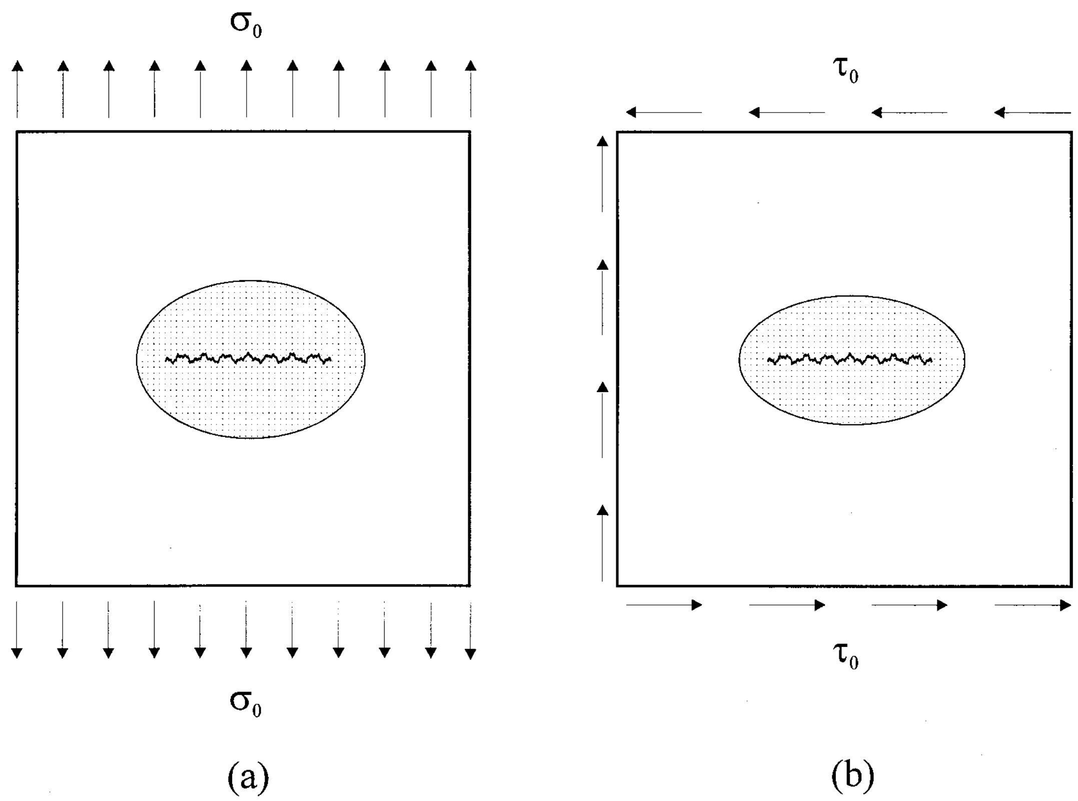

Figure 1. Unloaded regions for fractal cracks: (a) Mode I, (b) Mode II.

power using a cascade energy transfer. They showed that if there exists a dissipation in the transition from the nth to the $(n + 1)st$ micromechanics level, the fractal dimension of the crack increases and consequently the singularity order of the stresses decreases. Balankin (1997) found this stress singularity for self-affine fractal cracks using a dimensional analysis.

It should be mentioned that so far only Mode I fractal cracks have been studied. In this paper, we present a systematic approach for calculating the order of stress singularity of fractal cracks using the method of force lines, which is applicable for all modes of fracture. Here we consider all three classical modes of fracture for fractal cracks.

Mosolov (1993) tried to explain crack growth in compression using the fractality of cracks. He showed that for a fractal crack along a uniform compressive stress, stresses at the tip of the crack are singular. Later, this problem was discussed by Balankin (1997). In this article, we investigate this problem of stress singularity at the crack tip in more depth.

This paper is organized as follows. In Section 2, the method of force lines is used to calculate the order of stress singularity for Mode I and Mode II self-affine fractal cracks. In Section 3, a new mode of fracture is introduced for fractal cracks and its stress singularity is calculated. Some examples of mixed-mode fractal cracks are given in Section 4. In Section 5 the complete forms of stress and displacement fields around the tip of a fractal crack are obtained by imposing a continuous transition from fractal to classical stress and displacement fields. The stress field for Mode IV fractal cracks is found in Section 6. Conclusions are given in Section 7. Appendix 1 discusses self-affine fractals and briefly reviews different definitions of dimension. In Appendix 2 some properties of the Golden-section number are discussed.

2. A systematic method for calculating the order of stress singularity of Mode I and Mode II fractal cracks

In this section, we present a systematic approach for calculating the order of stress singularity of fractal cracks using the method of force lines. Here, we consider both Mode I and Mode II fractal cracks. To estimate the strain energy reduction due to a crack, the method of force lines is used (Yokobori, 1978., Kershtein et al., 1989., Bažant and Kazemi, 1990; Borodich, 1997). Consider a Mode I center–cracked plate. Without the crack, stress has a uniform distribution. The crack unloads a subdomain of the plate. It is postulated that the area of this unloaded domain is proportional to ℓ^2:

$$A = m\ell^2 \tag{2}$$

where 2ℓ is the apparent crack length and m is an unknown proportionality constant. Fig. 1 (a) shows a Mode I fractal crack. Here, the unloaded region has been approximated by an ellipse. The strain energy release due to this cut is:

$$\Delta U_e = \int_V U_0 \, \mathrm{d}V = \int_A t\frac{\sigma^2}{E} \, \mathrm{d}A = \int_\ell 2mt\frac{\sigma^2}{E}\ell \, \mathrm{d}\ell \sim 2mt\frac{\sigma^2}{E}\ell^2, \tag{3}$$

where $\sim$ means 'asymptotically equivalent' and t is the plate thickness. According to Griffith's criterion, the crack propagates if we have:

$$\Delta U_e = \Delta U_s, \tag{4}$$

where U_s is the surface energy of the fracture surfaces. The surface energy may be expressed as:

$$\Delta U_s = 4tL\gamma \sim L, \tag{5}$$

where γ is the specific surface energy and L is the true length of the crack. Note that $L = \ell$ for classical cracks but $L > \ell$ for fractal cracks. To justify the correctness of the estimation (2), a sharp crack is considered. From (3)–(5), we obtain:

$$\frac{2tm}{E}\sigma^2\ell^2 \sim 4t\gamma\ell \quad \text{or} \quad \sigma^2\ell^2 \sim \ell \tag{6}$$

At the crack tip, stress has a singularity, $\sigma(r) \sim r^{-\alpha}$. Hence:

$$\ell^{2-2\alpha} \sim \ell. \tag{7}$$

This gives $\alpha = \frac{1}{2}$, which is the correct order of stress singularity in classical fracture mechanics. Therefore, the assumption (2) is correct. Because the fractality of crack trajectories is a local phenomenon, the estimation (2) is still applicable for fractal cracks.

Now suppose that the crack trajectory is a self-affine fractal with Hurst exponent $H(O < H < 1)$. The asymptotic behavior of the true length of the crack, L, is:

$$L \sim \ell^{D_D} = \ell^{1/H}, \tag{8}$$

where D_D is the divider (latent) fractal dimension. In Appendix 1, we explain this fractal dimension in more detail. Hence, from (5) and (8), we obtain:

$$\Delta U_s \sim \ell^{1/H}. \tag{9}$$

Again, a stress singularity of the form $\sigma(r) \sim r^{-\alpha}$ is assumed. We know that:

$$r = \sqrt{x^2 + y^2} \sim \ell. \tag{10}$$

The released strain energy due to the fractal crack asymptotically behaves as:

$$\Delta U_e \sim (\ell)^{-2\alpha}\ell^2 = \ell^{2-2\alpha}. \tag{11}$$

From (9) and (11) and Griffith's criterion, we obtain:

$$\alpha = \frac{2H-1}{2H} > 0 \quad \text{for} \quad 1 > H > \tfrac{1}{2}. \tag{12}$$

As we know, $H > \frac{1}{2}$ corresponds to brittle fracture and $H < \frac{1}{2}$ corresponds to ductile fracture, for which $\alpha = 0$ (Balankin, 1997). In terms of the divider dimension $(D_D = 1/H)$:

$$\alpha = \frac{2-D_D}{2} < \tfrac{1}{2}. \tag{13}$$

Thus, the fractality of crack surfaces reduces the power of stress singularity. It should be noted that the specific energy of a fractal crack could be different from that of a smooth crack. But this difference has no effect on the order of stress singularity. The stress distribution may be expressed by:

$$\sigma_{ij}^f(r,\theta) = K_I^f r^{\frac{1-2H}{2H}} \varphi_{ij}(\theta, H), \tag{14a}$$

or

$$\sigma_{ij}^f(r,\theta) = K_I^f r^{\frac{D_D-2}{2}} \bar{\varphi}_{ij}(\theta, D_D). \tag{14b}$$

Here we have used φ_{ij} and $\bar{\varphi}_{ij}$ to emphasize that they have different functional forms. In (14) K_I^f is the fractal Mode I stress intensity factor. Among all the researchers into fractal fracture mechanics only Balankin (1997) considered the possibility that φ_{ij} may be a function of the fractal dimension. As we will see later, φ_{ij} cannot not be a function of H ($\bar{\varphi}_{ij}$ does not depend on D_D).

For a Mode II fractal crack (Figure 1(b)) we can show that the unloaded region may again be approximated by an ellipse that is, in general, different from the ellipse of a Mode I fractal crack. Following a similar procedure, we can show that stresses have the same power of singularity as Mode I fractal cracks do., i.e.:

$$\sigma_{ij}^f(r,\theta) = K_{II}^f r^{\frac{1-2H}{2H}} \psi_{ij}(\theta, H) \tag{15a}$$

or in terms of the divider dimension:

$$\sigma_{ij}^f(r,\theta) = K_{II}^f r^{\frac{D_D-2}{H}} \bar{\psi}_{ij}(\theta, D_D). \tag{15b}$$

For self-similar cracks $L \sim \ell^D$ and we have:

$$\alpha = \frac{2-D}{2} \tag{16}$$

as was obtained by Mosolov (1991) and others. It is worth noting that for self-similar curves x and y have the same asymptotic orders. Therefore, they are not suitable models for a crack, which is globally very much like a straight line. Cracks have a much smaller size perpendicular to the crack propagation than do in the direction of crack propagation.

3. The fourth mode of fracture for fractal cracks

Experiments have shown crack propagation under compression for brittle solids. As is known from classical fracture mechanics, a crack along a uniform compressive stress cannot propagate because there is no interaction between the crack and loading. In this case stresses do not have a singularity and classical fracture mechanics cannot explain the crack growth. One possibility is the process of interaction among the many load-parallel tensile fractures that are generated by pre- existing cracks. Griffith (1924) postulated that brittle fracture in compression is due to tensile microstresses. Lajtai (1974) investigated brittle fracture in compression and concluded that it is a complex process consisting of several stages of fracture development. He writes: ‘Since hairline-fractures parallel to the load change neither the strain energy nor the external work component of the total energy budget, the source for the surface energy is no longer obvious.’ Another explanation for this paradox is that in reality cracks are inclined and slightly kinked. Using this fact, Steif (1984) studied wing cracks under compression. Some other suggestions have been proposed to date. Nonuniformities in the stress field and in the structure of the material could be another reason for failure in compression (Obert, 1968; Gramanovich and Dyskin, 1988).

Mosolov (1993) tried to explain crack propagation under compression using the fractality of fracture surfaces. He showed that for an elastic body with a self-affine fractal crack under a uniform compressive stress along the axis of the crack, stress has a singularity. While introducing this worthy idea, Mosolov obtained the following incorrect asymptotic behavior of stresses:

$$\sigma \sim r^{\frac{D-\frac{3}{2}}{2-D}}. \tag{17}$$

As pointed out by Balankin (1997), instead of using the divider dimension ($D_D = 1/H$), Mosolov used the box dimension ($D_B = 2 - H$) in his calculations. Unfortunately, Balankin also made a mistake and obtained an incorrect order for stress singularity and again used this result to explain crack propagation in compression. His incorrect formula for the power of stress singularity yields the correct answer for two-dimensional cracked bodies by numerical coincidence. Here, we obtain the asymptotic stress behavior for self-affine fractal cracks using the systematic approach introduced in Section 2.

Consider a smooth-edged crack along a uniform stress σ_0. For this cracked structure, the area of unloaded region may be written as:

$$A = m'h\ell, \tag{18}$$

where m' is a constant and h is independent of ℓ. Assuming a stress singularity $\sigma \sim r^{-\beta}$, the released strain energy due to this cut may be expressed as:

$$\Delta U_e = \frac{t}{E}\int_A \sigma^2\,\mathrm{d}A = \frac{th}{E}\int_A \sigma^2\,\mathrm{d}\ell \sim \ell^{-2\beta+1}. \tag{19}$$

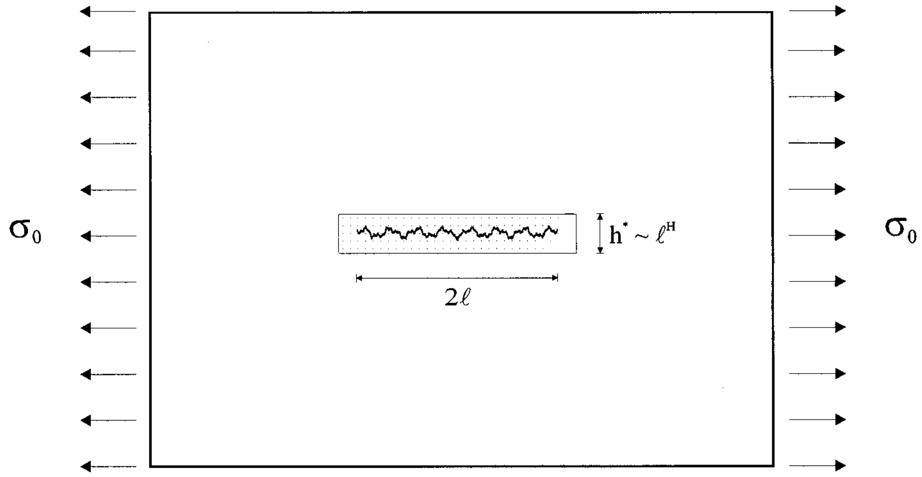

Figure 2. A Mode IV fractal crack with its unloaded region.

We know that $\Delta U_s \sim \ell$, hence $\beta = 0$ which is correct for a classical crack. This is a justification for the estimation (18). Fractal cracks are locally different from smooth cracks, therefore (18) can still be used with some modification for fractal cracks.

Consider a self-affine fractal crack along the uniform stress σ_0 (Figure 2). For this loaded cracked plate, the area of the unloaded region may be estimated as:

$$A = m'h^*\ell, h^* \sim \ell^H, h^* > h \tag{20}$$

where m' is a constant and h^* is a function of ℓ for a fractal crack. For this loading condition the unloaded region is more localized. Hence:

$$r = \sqrt{x^2 + y^2} = \sqrt{x^2 + x^{2H}} \sim \ell^H (x \ll 1, 0 < H < 1). \tag{21}$$

Again, assuming a stress singularity of the form $\sigma(r) \sim r^{-\beta}$, we have:

$$\Delta U_e \sim \ell^{-2\beta H}\ell\ell^H = \ell^{-2\beta H + H + 1}, \tag{22a}$$

$$\Delta U_s \sim \ell^{1/H}. \tag{22b}$$

According to Griffith's criterion:

$$\ell^{\frac{1}{H}} \sim \ell^{-2\beta H + H + 1} \quad \text{or} \quad \beta = \frac{H^2 + H - 1}{2H^2}. \tag{23}$$

Also, in terms of the divider dimension we have:

$$\beta = \frac{-D_D^2 + D_D + 1}{2}. \tag{24}$$

Clearly

$$\beta > 0 \quad \text{for} \quad 1 > H > \frac{\sqrt{5} - 1}{2} = \frac{1}{g} > \frac{1}{2}, 1 < D_D < \frac{\sqrt{5} + 1}{2} = g, \tag{25}$$

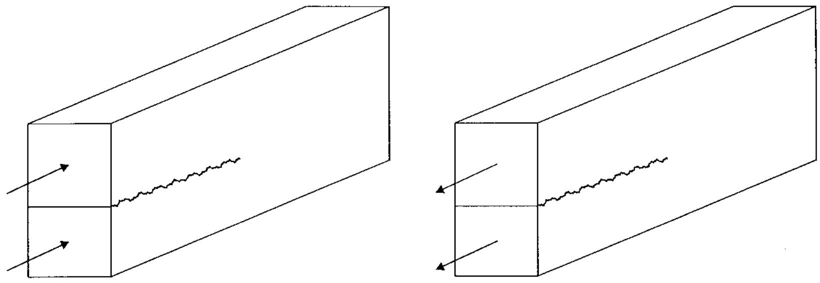

Figure 3. The fourth mode of fracture for fractal cracks (axial mode): (a) in compression, (b) in tension.

where g is the Golden Mean (Golden Ratio). Some properties of this number are given in Appendix 2.

Therefore, for a self-affine fractal crack, with divider fractal dimension less than the Golden Ratio or a Hurst exponent larger than the inverse of the Golden Ratio, there is a stress singularity at the tip of the fractal crack. This shows that strain energy release is possible and the crack can propagate. This result can be an explanation for failure in compression, as mentioned by Mosolov and Balankin. We should note that we did not use the assumption that the uniform stress is compressive. Thus this result remains valid even if the applied stress is tensile. Actually, this is more than an explanation for compressive failure.

Balankin (1997, p. 184) mentioned that '... for the problem with rough crack, the resultant stress field is generally a superposition of three basic modes of loading.' But the stress field of the cracked plate of Figure 2 is not a superposition of the three known fracture modes of classical fracture mechanics. In fact, it is a new mode of fracture for fractal cracks. We call it 'Mode IV' fractal fracture (or the axial mode). In other words, the irregularity of fractal cracks dictates the existence of a new mode of fracture. The axial mode of fracture in fractal fracture mechanics is shown in Figure 3.

It can be shown that the stress singularity of Mode IV self-affine fractal cracks is weaker than that of Mode I and II fractal cracks. Figure 4 compares the orders of singularity of self-affine Mode I and II fractal cracks with those of self-affine Mode IV fractal cracks.

For a self-similar fractal crack (such as a Koch curve), even when the uniform stress is along the crack propagation direction the area of the unloaded region is proportional to ℓ^2 (Figure 5). Therefore, for a Mode IV self-similar crack, stresses at the tip of the crack have the same order of stress singularity as classical modes of fracture do, i.e.:

$$\beta = \frac{2 - D}{2}. \tag{26}$$

4. Mixed-Mode problems of fractal fracture mechanics

With this new mode of fracture, some single-mode problems of classical fracture mechanics could now be described as mixed-mode problems in fractal fracture mechanics. For example, a center-cracked plate under biaxial tension and a three bend-point specimen are examples of Mode I fracture in classical fracture mechanics. But for both cases, in fractal fracture mechanics, both Mode II and Mode IV exist. Hence:

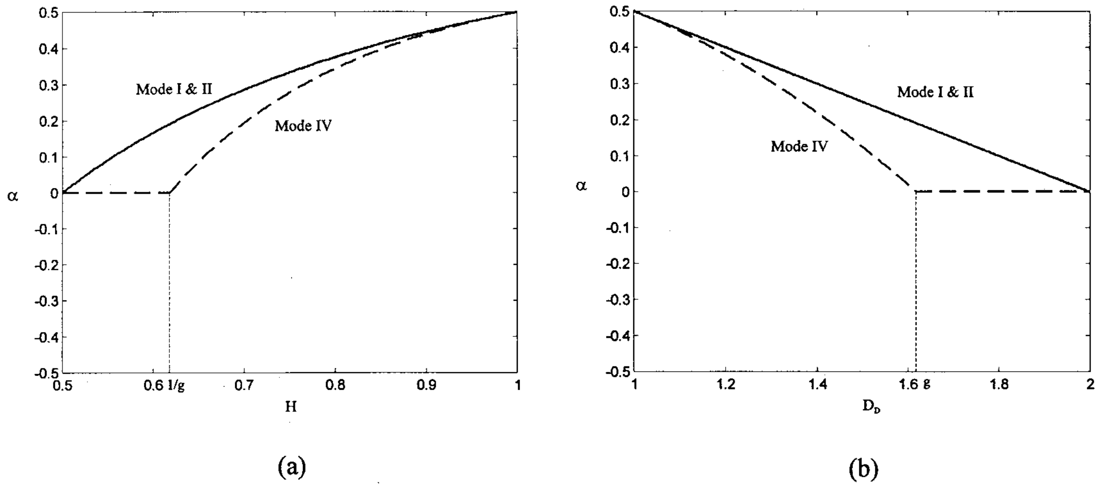

Figure 4. The variation of the order of stress singularity for Mode I and Mode II and Mode IV self-affine fractal cracks in terms of (a) the Hurst exponent, (b) latent fractal dimension.

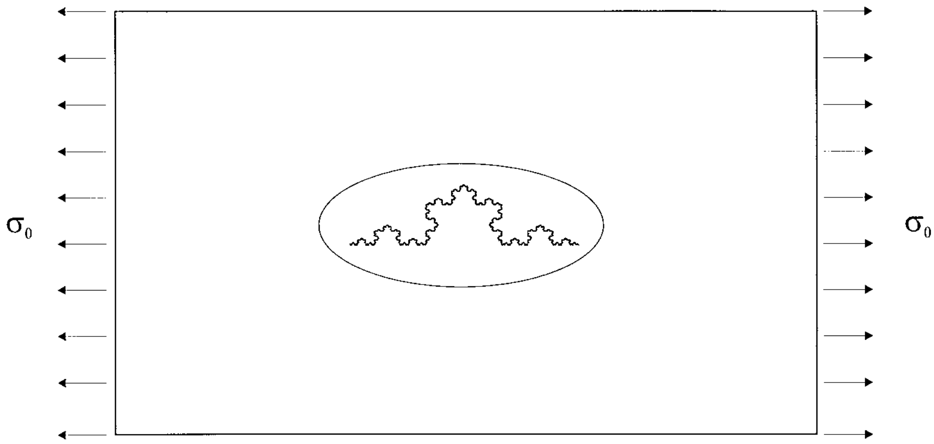

Figure 5. A self-similar fractal crack under a uniform stress along the crack and its unloaded region.

$$\sigma_{ij}^{f}(r,\theta) = K_I^f r^{-\frac{2H-1}{2H}} \varphi_{ij}(\theta, H) + K_{IV}^f r^{-\frac{H^2+H-1}{2H^2}} \psi_{ij}(\theta, H) \tag{27a}$$

or

$$\sigma_{ij}^{f}(r,\theta) = K_I^f r^{\frac{D_D-2}{2}} \bar{\varphi}_{ij}(\theta, D_D) + K_{IV}^f r^{\frac{D_D^2-D_D-1}{2}} \bar{\psi}_{ij}(\theta, D_D) \tag{27b}$$

An example is given in Figure 6. In Figures 6a and 6b a center-cracked plate is subjected to a uniform stress perpendicular to the crack axis. For classical and fractal cracks this is a pure Mode I. In Figures 6c and 6d the same center-cracked plate is subjected to a biaxial tension. For a classical crack (Figure 6c) this is again a pure Mode I. However, for a fractal crack (Figure 6d) both Mode I and Mode IV exist.

It should be noted that Mode I has a stronger stress singularity than Mode IV for self-affine fractal cracks and hence is the dominant mode. Mode IV can dominate only in special cases where the loads along the axis of the crack are much higher than the loads perpendicular to the crack. As we mentioned earlier, Mode IV self-similar fractal cracks introduce the same order of stress singularity as Mode I self-similar fractal cracks do.

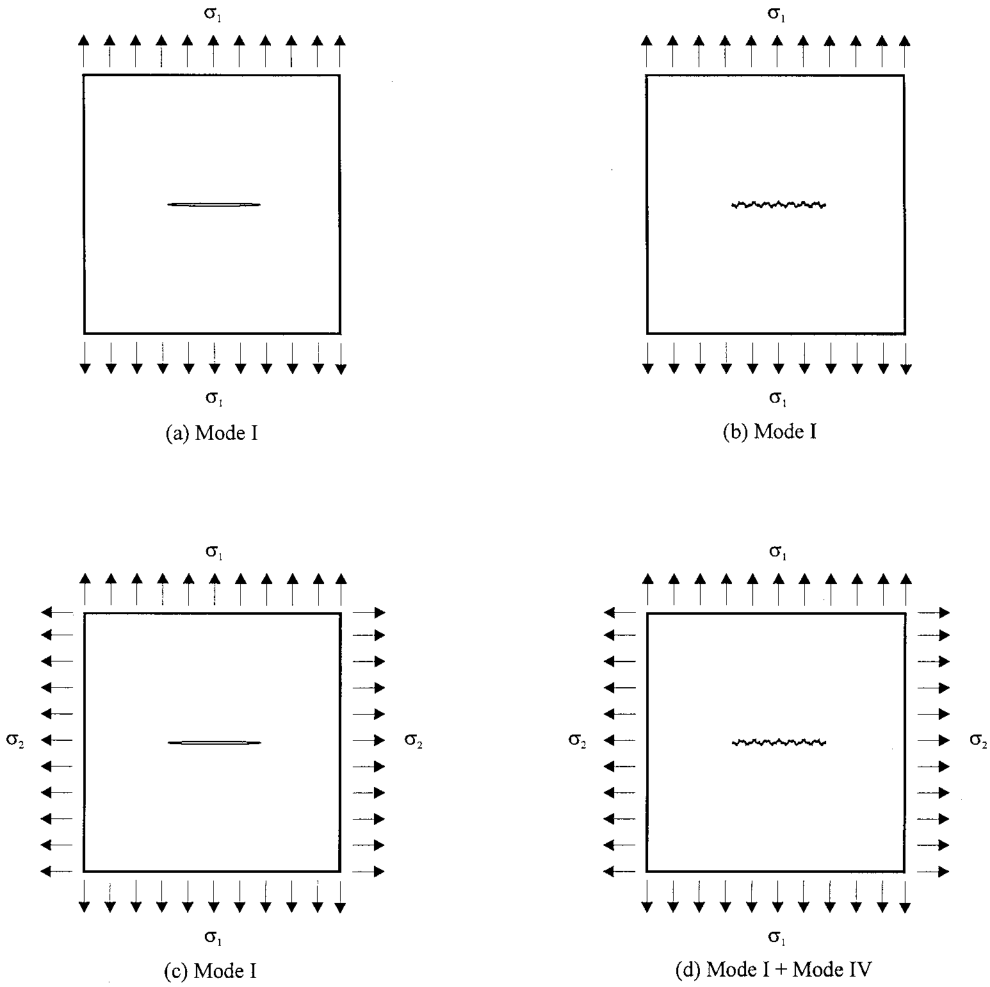

Figure 6. (a) a classical crack under a uniform tension perpendicular to the axis of the crack, (b) a fractal crack under a uniform tension perpendicular to the axis of the crack, (c) a classical crack under a biaxial tension, and (d) a fractal crack under a biaxial tension.

5. Stress and displacement fields for Mode I and Mode II fractal cracks

At first glance, it seems that to find the complete forms of stress and displacement fields we have to solve a boundary value problem with fractal boundaries. But these field quantities can be obtained by a much simpler method, as follows. In Balankin (1997, p. 189) we read: 'The explicit expression for $\Phi_{ij}(\theta, \nu, D_D)$ depends on the specific crack geometry.' As we see in this section, this is not the case.

As we mentioned earlier, it is known that fracture surfaces are fractals in the range $\ell_0 \leq r \leq \ell_1$. For $r \geq \ell_1$, the classical fracture mechanics solutions remain valid. We use this known fact to obtain displacement and stress fields around the tip of a fractal crack. It is to be noted that fracture phenomena in the range $r \leq \ell_0$ are governed by nanofracture mechanics (Cherepanov, et al., 1995).

For a Mode I fracture, stresses may be expressed as (Irwin, 1958 and Sih and Liebowitz, 1968):

$$\sigma_{xx}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right], \tag{28a}$$

$$\sigma_{yy}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\left[1 + \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right], \tag{28b}$$

$$\sigma_{xy}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right) \tag{28c}$$

Thus:

$$\sigma_{ij}(r,\theta) = K_I r^{-1/2} f_{ij}(\theta) \qquad r \geq \ell_1. \tag{29}$$

The stress distribution in the range of fractality of the crack can be written as:

$$\sigma_{ij}^f(r,\theta) = K_I^f r^{-\frac{2H-1}{2H}} \varphi_{ij}(\theta, H) \qquad \ell_0 \leq r \leq \ell_1. \tag{30}$$

From theory of elasticity, it is known that the stress field is continuous. Therefore, we must have a continuous transition from fractal to classical stress distributions, i.e.:

$$\lim_{r\to\ell_1^-} \sigma_{ij}^f(r,\theta) = \lim_{r\to\ell_1^+} \sigma_{ij}(r,\theta) \qquad \forall\theta \in [0, 2\pi) \tag{31}$$

Therefore:

$$K_I^f \varphi_{ij}(\theta, H) = K_I \ell_1^{(H-1)/(2H)} f_{ij}(\theta) \qquad \forall\theta \in [0, 2\pi) \tag{32}$$

From (32) we conclude that:

$$K_I^f = C\ell_1^{\frac{H-1}{2H}} K_I, \qquad \varphi_{ij}(\theta, H) = \frac{1}{C} f_{ij}(\theta), \theta \in [0, 2\pi), \tag{33}$$

where C is a positive constant. But we know that $K_I^f(H = 1) = K_I$. Hence, by substituting $H = 1$ in (33), it is concluded that $C = 1$. Therefore, the angular variation of stresses must be the same:

$$K_I^f = \ell_1^{\frac{H-1}{2H}} K_I, \quad \varphi_{ij}(\theta, H) = f_{ij}(\theta), \theta \in [0, 2\pi) \tag{34}$$

and in terms of the divider fractal dimension:

$$K_I^f = \ell_1^{\frac{1-D_D}{2}} K_I, \quad \varphi_{ij}(\theta, D_D) = f_{ij}(\theta), \theta \in [0, 2\pi). \tag{35}$$

It is to be noted that the above relations between classical and fractal stress intensity factors are universal. As can be seen, a length scale (ℓ_1) appears in this relationship.

As an example, consider an infinite plate with a center crack of length 2a under two opposite point forces P. For this cracked structure $K_I = P/(\sqrt{\pi a})$. Lei and Chen (1994, 1995), used this formula for a self-similar fractal crack and obtained a relation between toughnesses of classical and fractal cracks, which is not correct, as was pointed out by Xie (1994). Using (35), the stress field for the self-affine crack may be expressed by ($\ell_0 \leq r \leq \ell_1$):

$$\sigma_{xx}^f(r,\theta) = \frac{P}{\pi\sqrt{2a}} \ell_1^{\frac{H-1}{2H}} r^{\frac{2H-1}{2H}} \cos\left(\frac{\theta}{2}\right)\left[1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right], \tag{36a}$$

$$\sigma_{yy}^{f}(r,\theta) = \frac{P}{\pi\sqrt{2a}}\ell_1^{\frac{H-1}{2H}} r^{\frac{2H-1}{2H}} \cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\right], \quad (36b)$$

$$\sigma_{xy}^{f}(r,\theta) = \frac{P}{\pi\sqrt{2a}}\ell_1^{\frac{H-1}{2H}} \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right). \quad (36c)$$

For a Mode I classical crack, the displacement field can be written as (Irwin, 1958 and Sih Liebowitz, 1968):

$$u_x(r,\theta) = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left[\kappa - 1 + 2\sin^2\left(\frac{\theta}{2}\right)\right], \quad (37a)$$

$$u_y(r,\theta) = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left[\kappa + 1 - 2\cos^2\left(\frac{\theta}{2}\right)\right], \quad (37b)$$

where $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress. Hence

$$u_i(r,\theta) = \frac{K_I}{\sqrt{8\pi\mu}} r^{\frac{1}{2}} g_i(\theta) \quad (38)$$

For a fractal crack:

$$u_i^{f}(r,\theta) = {}^{u}K_I^{f} r^{1-\alpha}\lambda_i(\theta,H), \quad \alpha = \frac{2H-1}{2H}, \quad \ell_0 \le r \le \ell_1. \quad (39)$$

Here we use ${}^{u}K_I^{f}$ to emphasize that it is not equal to K_I^{f}. From the theory of elasticity we know that the displacement field is continuous. Therefore, there is a continuous transition from fractal to classical displacement fields. Imposing this continuous transition condition yields:

$$ {}^{u}K_I^{f} = \frac{\ell_1^{\frac{H-1}{2H}}}{\sqrt{8\pi\mu}}K_I, \quad \lambda_i(\theta,H) = g_i(\theta) \quad \forall\theta \in [0,2\pi), \quad (40)$$

and in terms of the divider fractal dimension:

$$ {}^{u}K_I^{f} = \frac{\ell_1^{\frac{1-D_D}{2}}}{\sqrt{8\pi\mu}}K_I, \quad \bar{\lambda}_i(\theta,D_D) = g_i(\theta) \quad \forall\theta \in (0,2\pi). \quad (41)$$

In obtaining (40) and (41), we have considered the fact that ${}^{u}K_I^{f}(H = 1) = {}^{u}K_I^{f}(D_D = 1) = K_I$. The same conclusions hold for Mode II fractal cracks.

6. Stress field for Mode IV fractal cracks

In this section, the stress field around the tip of a Mode IV self-affine fractal crack is obtained and it is shown that only one of the stress components has a singularity. For a Mode IV self-affine fractal crack, the stress field may be expressed as:

$$\sigma_{ij}^{f}(r,\theta) = K_{IV}^{f} r^{-\frac{H^2+H-1}{2H^2}}\chi_{ij}(\theta,H), \quad \ell_0 \le r \le \ell_1, \quad (42a)$$

$$\sigma_{11}(r,\theta) = \sigma_0, \sigma_{12}(r,\theta) = \sigma_{22}(r,\theta) = 0, \quad r \ge \ell_1. \quad (42b)$$

Again, imposing a continuous transition from fractal to classical stress distributions, we obtain:

$$K_{IV}^{f} = \sigma_0 \ell_1^{\frac{H^2+H-1}{2H^2}}, \quad \chi_{11}(\theta, H) = 1, \quad \chi_{12}(\theta, H) = \chi_{22}(\theta, H) = 0 \tag{43}$$

and in terms of the divider dimension:

$$K_{IV}^{f} = \sigma_0 \ell_1^{\frac{D_D^2+D_D-1}{2}}, \quad \bar{\chi}_{11}(\theta, D_D) = 1, \quad \bar{\chi}_{12}(\theta, D_D) = \bar{\chi}_{22}(\theta, D_D) = 0. \tag{44}$$

As can be seen, only one of the stress components has a singularity; other stress components are identically zero. Hence, when the uniform applied stress is parallel to the x-axis, the stress distribution has the following form:

$$\sigma_{xx}^{f}(r, \theta) = \sigma_0 \ell_1^{\frac{H^2+H-1}{2H^2}} r^{-\frac{H^2+H-1}{2H^2}}, \sigma_{xy}^{f}(r, \theta) = \sigma_{yy}^{f}(r, \theta) = 0, \ell_0 \le r \le \ell_1. \tag{45a}$$

Or

$$\sigma_{xx}^{f}(r, \theta) = \sigma_0 \ell_1^{\frac{D_D^2+D_D-1}{2}} r^{-\frac{(D_D^2+D_D-1)}{2}}, \sigma_{xy}^{f}(r, \theta) = \sigma_{yy}^{f}(r, \theta) = 0, \quad \ell_0 \le r \le \ell_1. \tag{45b}$$

7. Three- dimensional, solid bodies with fractal cracks

A fractal crack in a 3-D body is a fractal surface with fractal dimension of between two and three. Due to the crack, a volume of the body is unloaded. The volume of this unloaded region may be written as:

$$V = m\ell^3, \tag{46}$$

where ℓ is the characteristic length of the disk shaped crack and m is a constant. This estimation is valid for all the classical fracture modes, i.e., Modes I, II, and III. Suppose that the fracture surface is a self-affine fractal surface with Hurst exponent $H(0 < H < 1)$. Denoting the plane of the crack by (x, y), it is assumed that the (x, y) plane is isotropic. For this fractal surface the divider fractal dimension is (Mandelbrot, 1985):

$$D_D = \frac{2}{H}. \tag{47}$$

The surface energy required for crack propagation can be written as:

$$\Delta U_s = 2A\gamma \sim \ell^{D_D} = \ell^{2/H}. \tag{48}$$

Assuming a stress singularity in the form $\sigma(r) \sim r^{-\alpha}$ and considering the fact that $r \sim \ell$, the released strain energy may be expressed by:

$$\Delta U_e = \int_V \frac{\sigma^2}{2E}\, \mathrm{d}V \sim \ell^{-2\alpha}\ell^3 = \ell^{3-2\alpha}. \tag{49}$$

Applying Griffith's criterion yields.

$$\alpha = \frac{3H-2}{2H} > 0 \quad \text{for} \quad \frac{2}{3} < H < 1 \tag{50a}$$

and in terms of the latent fractal dimension:

$$\alpha = \frac{3 - D_D}{2}. \tag{50b}$$

For a Mode IV fractal crack the unloaded region is more localized. In this case:

$$V = m\ell^2 h^*, \tag{51}$$

where h^* is a function of ℓ. For a self-affine surface:

$$\Delta z^2 \sim (\Delta x^2 + \Delta y^2)^H \quad \text{or} \quad \Delta z \sim \ell^H. \tag{52}$$

Hence:

$$h^* \sim \ell^H \quad \text{or} \quad V \sim \ell^{2+H}. \tag{53}$$

Also

$$r = \sqrt{x^2 + y^2 + z^2} \sim \sqrt{x^2 + y^2 + (x^2 + y^2)^H} \sim \ell^H \quad (x, y \ll 1, 0 < H < 1). \tag{54}$$

The surface energy has the same asymptotic form as (48). The released strain energy has the following asymptotic form:

$$\Delta U_e \sim (\ell^H)^{-2\alpha} \ell^{2+H} = \ell^{2+H-2\alpha H}. \tag{55}$$

Applying Griffith's criterion, we obtain:

$$\alpha = \frac{H^2 + 2H - 2}{2H^2} > 0 \quad \text{for} \quad \sqrt{3} - 1 < H < 1, \tag{56}$$

which is different from what was derived by Balankin (1997). Balankin (1997, p. 195) proposed the following formula for the order of stress singularity:

$$\alpha = \frac{(d-1)(H^2 + H - 1)}{2H^2}, \tag{57}$$

where d is the dimension of the embedding space. It can be seen that in Balankin's formula for some values of H the stress singularity power is higher than one-half, which is not possible in linear elastic fracture mechanics. For a fractal crack, the reentrant corner interactions change the order of stress singularity. This stress singularity cannot be stronger than that of a smooth crack. Also, for $d = 2$, by numerical coincidence Balankin's formula and Equation (56) give the same order of stress singularity. Balankin's formula may be corrected to read:

$$\alpha = \frac{H^2 + (d-1)(H-1)}{2H^2} \tag{58}$$

In summary, for a fractal crack in a three-dimensional body with Hurst exponent larger than $\sqrt{3} - 1$, a stress singularity exists at the crack front when a uniform stress is applied parallel to the crack surface. Figure 7 compares the power of stress singularity for classical modes of fracture with that of Mode IV fractal cracks. As can be seen, for all values of the Hurst dimension or the latent fractal dimension, classical modes of fracture introduce a stronger order of stress singularity. All the results obtained for two-dimensional bodies with fractal cracks can easily be generalized to the three-dimensional case.

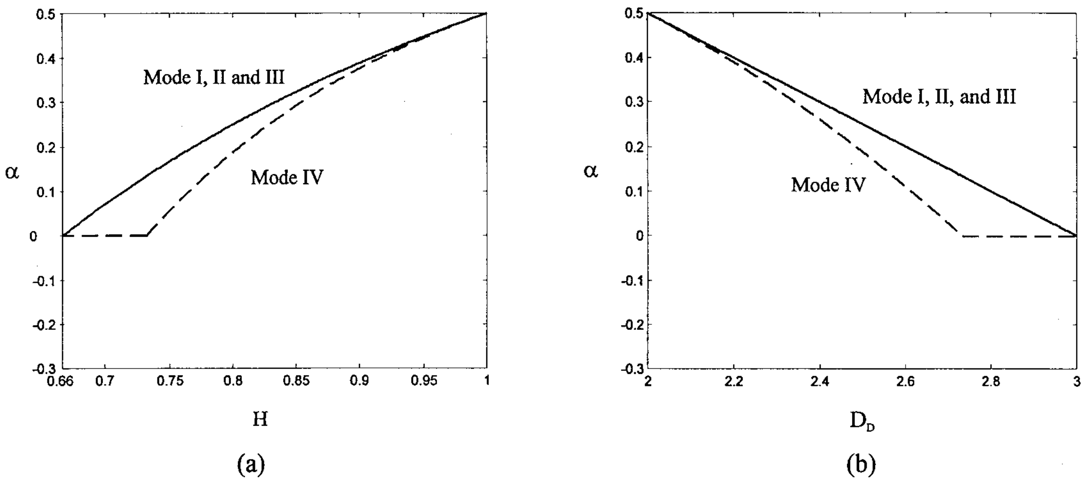

Figure 7. The variation of the order of stress singularity for Modes I, II, III, and IV self-affine fractal cracks in three-dimensional bodies in terms of (a) the Hurst exponent, (b) latent fractal, dimension.

8. Conclusions

A systematic method for computing the power of stress singularity for fractal cracks is offered in this article. It is shown that Mode II and Mode III fractal cracks introduce the same order of stress singularity as Mode I fractal cracks do.

It is pointed out that, in contrast to classical fracture mechanics, in fractal fracture mechanics situations arise for which the stress distribution cannot be written as a superposition of the three known fracture modes. In other words, the fractality of cracks in fractal fracture mechanics dictates the existence of a new mode of fracture. It is found that this new mode of fracture introduces a weaker stress singularity for self-affine fractal cracks compared to the stress singularity that Mode I and Mode II self-affine fractal cracks introduce. For two-dimensional problems, this mode exists only for self-affine fractal cracks with a Hurst exponent larger than the inverse of the Golden-section number, or the latent fractal dimension larger than the Golden Ratio. For self-similar cracks the fourth mode of fracture introduces the same order of stress singularity as the other modes do.

When the fourth mode of fracture is taken into consideration, some single-mode problems of classical fracture mechanics could be characterized as mixed-mode problems in fractal fracture mechanics.

Real cracks are fractal curves only in a limited interval of scales $\ell_0 \leq r \leq \ell_1$. By imposing a continuous transition from fractal to classical stress and displacement fields, it is demonstrated that the angular variation of stresses and displacements for fractal and smooth cracks must be the same. A universal relationship between the stress intensity factors of classical and fractal cracks is found. Using a continuous transition from fractal to classical stresses, it is shown that for Mode IV cracks, only one of the stress components is singular; the other stress components are identically zero.

Three-dimensional bodies with fractal cracks are investigated and it is observed that all three classical modes of fracture introduce the same stress singularity, whereas Mode IV self-affine fractal cracks in three-dimensional solids introduce a comparatively weaker stress singularity which is similar to those attributed to the two-dimensional case.

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Appendix 1. Self-affine fractals

This appendix offers a brief description of self-affine fractals. The reader may refer to Mandelbrot (1985, 1986a, b), Feder (1988), Viesek (1989), and Gouyet (1996) for more detail and examples.

Definition 1: An affine transformation transforms a point $x = (x_1, x_2, \dots, x_d)$ in $\mathbb{R}^d$ into a point $x' = (r_1 x_1, r_2 x_2, \dots, r_d x_d)$, where the scaling factors r_i are not all equal.

Definition 2: A bounded set B is self-affine with respect to a ratio vector $r = (r_1, r_2, \dots, r_d)$ if B is the union of N nonoverlapping subsets $B_1, B_2, \dots, B_N$, such that each of these subsets is congruent to the set $r(B)$ obtained by applying r to B.

Single-valued, nowhere differentiable functions are good examples of self-affine fractals. It is to be noted that there are self-affine fractals which are not single-valued functions. When for a function in the two-dimensional Euclidean space the following scaling holds, the function is called self-affine:

$$F(t) = r^{-H} F(rt), \quad 0 < H < 1. \tag{1.1}$$

where H is the Hurst exponent and is sometimes called the roughness exponent. In contrast to self-similar fractals, self-affine fractals do not have a unique dimension. Their global behavior is characterized by an integer dimension smaller than that of the embedding space, while the local properties can be described by local fractal dimensions.

For fractal curves, the relation between approximate length L and length of the yardstick ε is written as $L \sim \varepsilon^{1-D_D}$, where D_D is the divider (latent) dimension. The number of boxes of size ε needed to cover the curve is written as $N(\varepsilon) \sim \varepsilon^{-D_B}$, where D_B is the box dimension. When the fractal is self-similar, all of these fractal dimensions have the same value.

Consider a self-affine fractal curve $F(x)$ in a two-dimensional Euclidean space. We consider an interval $\Delta x = 1$ corresponding to a vertical variation $\Delta y = 1$. F is self-affine; therefore the transformation $\Delta x \to \lambda \Delta x$ transforms $\Delta y \to \lambda^H \Delta y (0 < H < 1)$.

Box dimension: If Δx is divided into n parts, the length of each part is $\Delta x' = 1/n$. When $\delta y/\delta x$ is sufficiently large, Δy will be divided into n^H parts and the length of each part is equal to $\Delta y' = 1/n^H$. The curve can be covered by covering each portion $\Delta y'$ by $(1/n^H)/(1/n)$ square boxes of side $1/n$ along the y-axis, and repeating this process n times along the x-axis. The total number of boxes is:

$$N(n) = n \times \frac{n}{n^H} = n^{2-H} = n^{D_B}. \tag{1.2}$$

Therefore, the local box dimension is $D_B = 2 - H$. However if $\Delta y/\Delta x$ is not large, only one square box is needed along the y-axis to cover the portion $\Delta y'$. In other words, if we use boxes of size $\varepsilon > x_c$, where x_c is such that $x_c \approx x_c^H$, the effect of varying the altitude is no longer significant. Here, x_c is called the crossover scale. The crossover scale is not intrinsic: it depends on the units we choose for x and y. In this case:

$$N(n) = n = n^{D_B} \tag{1.3}$$

Therefore, the global box dimension is $D_B = 1$.

The divider dimension: If the divider method is used for calculating the fractal dimension of self-affine curves, a completely different value is found. For yardsticks of length ε we have:

$$\varepsilon^2 = \lambda^2 \Delta x^2 + \lambda^{2H} \Delta y^2. \tag{1.4}$$

When $\Delta y / \Delta x$ is sufficiently large, $\varepsilon \propto \lambda^H$. The total length of the curve with step length ε is $N\varepsilon$, where N is the total number of steps. For the curve in the interval $[0, X]$, we have $N = X/(\lambda \Delta x)$, and $L = N \propto \lambda^{H-1} \propto \varepsilon^{1-(1/H)}$, hence, $D_c = 1/H$. Here, D_c is called the divider (latent) dimension of the self-affine fractal. Again, if we calculate the length by yardsticks larger than the crossover scale, we have $\varepsilon \propto \lambda$. Therefore:

$$N = \frac{X}{\lambda \Delta x}, L = N \propto \frac{\lambda}{\lambda} \propto \varepsilon^0 = \varepsilon^{1-D_D} \tag{1.5}$$

Hence, the global divider dimension is $D_D = 1$.

In general, for self-affine fractals embedded in a d-dimensional Euclidean space, the divider and box dimensions are locally related to the Hurst exponent by:

$$D_B = d - H, \tag{1.6a}$$

$$D_D = \begin{cases} \frac{d-1}{H}, & \dfrac{d-1}{d} \le H < 1, \\ d, & 0 < H \le \dfrac{d-1}{d}. \end{cases} \tag{1.6b}$$

And globally, $D_B = D_D = d - 1$.

Appendix 2. The golden section number

As was shown for self-affine fractal cracks with Hurst exponent larger than the inverse of the Golden Ratio (Golden Mean), a uniform stress along the crack introduces stress singularity at the crack tip. In this appendix some interesting properties of the Golden Ratio are presented. The interested reader may refer to Huntley (1970), Vojda (1989), Schroeder (1997), and Dunlap (1997).

This number has been called the golden mean, the golden section, the golden cut, the divine proportion, the Fibonacci number, and the mean of Phidias. The Golden Ratio possesses a number of interesting and important properties that make it unique among the set of irrational numbers.

Definition 1: The *Golden Ratio (section)* is defined, geometrically, by sectioning a straight line segment in such a way that the ratio of the total length to the longer segment equals the ratio of the longer to the shorter segment. Supposing the total length of the line segment is L and the longer segment has length b, g is determined by:

$$g = \frac{L}{b} = \frac{L}{L - b}. \tag{2.1}$$

Substituting g for L/b yields:

$$g = \frac{1}{g-1} \quad \text{or} \quad g^2 - g - 1 = 0. \tag{2.2}$$

The only positive solution is:

$$g = \frac{1+\sqrt{5}}{2}. \tag{2.3}$$

The ancient Greeks believed that a rectangle, with the length-to-width ratio equal to the Golden Ratio, was the most aesthetically pleasing shape.

Definition 2: A *Continued Fraction*

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}} \tag{2.4}$$

is written in the compact form as $[b_0, b_1, b_2, b_3, \ldots]$. Continued fractions are often much more efficient in approximating irrational numbers than ordinary fractions. From (2.2), we have:

$$g = 1 + \frac{1}{g}. \tag{2.5}$$

Hence,

$$g = 1 + \cfrac{1}{1 + \cfrac{1}{g}}. \tag{2.6}$$

Repeating this process yields:

$$g = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = [1; 1, 1, 1, \ldots]. \tag{2.7}$$

Actually, the continued fraction of the Golden-section number is the most slowly converging continued fraction. Therefore, g is called the 'most irrational' number: for a given order of rational approximation, the approximation to g is the worst.

Definition 3: Fibonacci numbers are defined by the recursion relation:

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0 \quad \text{and} \quad F_1 = 1. \tag{2.8}$$

The first Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, ... where each number is the sum of its two predecessors. The ratio of two successive F_n, approaches the Golden Ratio because from (2.8):

$$\frac{F_{n+1}}{F_n} = [1; \underbrace{1, \ldots, 1}_{n-1\ 1's}] \quad (n > 1), \tag{2.9}$$

where the right side of (2.9) is the approximating fraction to the Golden Ratio.

Another interesting representation of g is obtained as follows. From (2.2) we have:

$$g^2 = g + 1 \quad \text{or} \quad g = \sqrt{1+g} \tag{2.10}$$

Hence,

$$g = \sqrt{1+\sqrt{1+g}}. \tag{2.11}$$

Repeating this process, we obtain:

$$g = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}. \tag{2.12}$$

Similarly:

$$\frac{1}{g} = \sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\cdots}}}}. \tag{2.13}$$

The Golden Ratio has some other interesting features. This number plays a prominent role in the dimensions of all objects that exhibit fivefold symmetry. Using the Golden Ratio, sequences can be constructed that are both additive and geometric. The Golden Ratio has applications in optimization problems and search algorithms.

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