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The twist-fit problem: finite torsional and shear eigenstrains in nonlinear elastic solids

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Eigenstrains in nonlinear elastic solids are created through defects, growth or other anelastic effects. These eigenstrains are known to be important as they can generate residual stresses and alter the overall response of the solid. Here, we study the residual stress fields generated by finite torsional or shear eigenstrains. This problem is addressed by considering a cylindrical bar made of an incompressible isotropic solid with an axisymmetric distribution of shear eigenstrains. As particular examples, we consider a cylindrical inhomogeneity and a double inhomogeneity with finite shear eigenstrains and study the effect of torsional shear eigenstrains on the axial and torsional stiffnesses of the circular cylindrical bar.

1. Introduction

A standard problem in elasticity is to consider the effect of an inclusion in the response of a solid under loads. The celebrated work of Eshelby [1] on the stress field generated by an ellipsoidal inclusion in a linear elastic material is the paradigm for such problems. Since this early work, the study of inclusions has overwhelmingly been restricted to linear elasticity (see the recent review [2]) with the exception of a handful of works dedicated to finding exact solutions in finite deformations. Among these, we should mention the recent two-dimensional solutions for harmonic materials [3–7]. The authors [8] studied the residual stress fields of finite radial and circumferential eigenstrains in the case of spherical

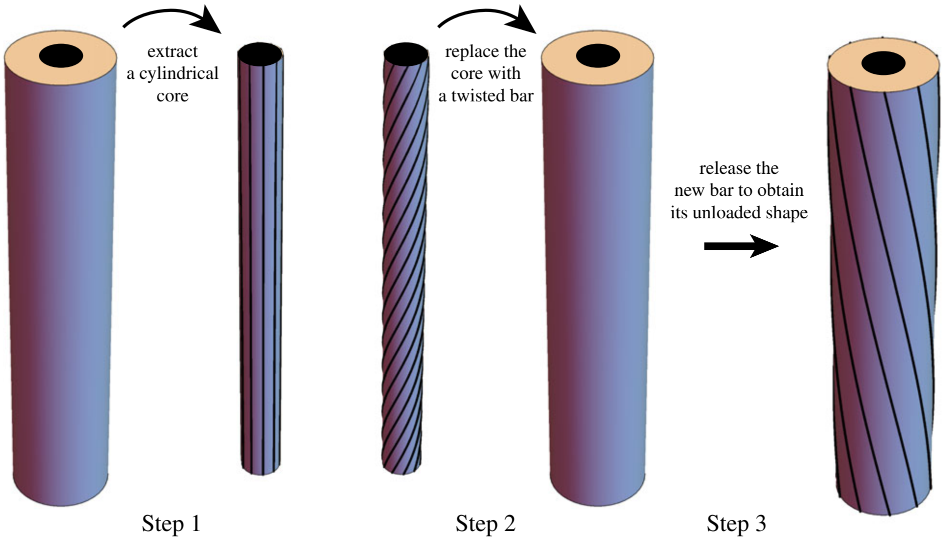

Figure 1. The twist-fit problem is the torsional analogue of the shrink-fit problem. A cylindrical bar is removed (Step 1) and replaced by a bar of the same height and radius but twisted (Step 2). When released (Step 3), the composite bar develops residual stress. (Online version in colour.)

balls and (finite and infinite) circular cylindrical bars made of arbitrary incompressible and isotropic solids. These studies provide exact solutions that can either be used as benchmark for numerical problems; provide bounds and justification for the approximations performed in a small-strain theory; or uncover new nonlinear effects absent or invisible in the linear regime [9]. We should mention that the problem of swelling in solids is closely related to the problem of inclusions with pure dilatational finite eigenstrains. Pence and his co-workers [10–13] have presented exact solutions for swelling in cylindrical and spherical geometries in the case of both incompressible and compressible isotropic solids.

Another important class of eigenstrains are obtained by considering shear strains. These type of eigenstrains can be easily generated by considering the *twist-fit problem*, that is the torsional analogue to the classical shrink-fit problem [14]: First, remove a cylindrical shell from a cylindrical bar and replace it by a twisted bar with the same height and radius. Then, release the new composite structure to obtain a new twisted configuration (figure 1). This new structure contains finite shear eigenstrains, that are referred to as *eigentwist*, and support, in general, a residual stress field. The problem is then to compute the residual stress field and understand how it affects the effective properties of the structure such as its axial or torsional stiffnesses. Despite their importance, there appears to be no studies in the literature on finite shear eigenstrains. In this paper, we study a generalized version of the torsional shrink-fit problem for a finite circular cylindrical bar made of an arbitrary isotropic incompressible solid.

In the nonlinear elasticity literature, problems of axial and azimuthal shear have been thoroughly studied using semi-inverse methods (see [15–18] and references therein) and an interesting and unusual application of these methods can be found in [19]. The traditional approach is to start with an unstressed reference configuration and compute the stresses and deformations under given external loads and boundary conditions. When considering eigenstrains, the problem is slightly different: Given a distribution of shear eigenstrains, what is the induced residual stress field? We obtain general results on the existence of such residual stresses in a circular cylindrical bar and identify shear eigenstrains that do not induce residual stresses.

In this paper, we consider the problem of a given arbitrary eigentwist distribution $\psi(R)$ in a circular cylindrical bar. For this problem, we calculate the deformation kinematics and hence the residual stresses. Kinematics is described by two constants: τ, the angle of twist per unit length, and λ, the stretch. In the case of traction-free lateral boundary and zero applied axial forces, these two constants must satisfy a system of nonlinear algebraic equations. As an example, we solve these equations for neo-Hookean and Mooney–Rivlin solids. In both cases, λ is larger than one, which is consistent with the celebrated Poynting effect [20]. We will further focus on two particular examples. In the first one, a cylindrical bar has a cylindrical inclusion with uniform eigentwist. In the second example, the cylindrical inclusion has a cylindrical inclusion and an outer annular inclusion, both with uniform eigentwists (but possibly different). In this case, we look at the relation between the two eigentwists so that the bar does not twist. Finally, for both problems, we study the effect of eigentwists on the axial and torsional stiffnesses of a circular cylindrical bar.

This paper is organized as follows. In §2, we briefly discuss shear deformations in nonlinear elasticity and the material manifold of a solid with a distribution of finite shear eigenstrains. We then find the impotent finite axisymmetric shear eigenstrains in a cylindrical bar. In §3, we calculate the residual stress field in a circular cylindrical bar made of an incompressible isotropic solid with an axisymmetric distribution of finite torsional shear eigenstrains. Two particular examples of a single and a double inclusion are worked out in detail. Section 4 discusses the effect of eigentwist on the axial and torsional rigidities of a circular cylindrical bar. Section 5 concludes the paper with some remarks.

2. Finite shear eigenstrains in a cylindrical bar

We consider a cylindrical bar of radius R_o. In the reference (undeformed) configuration, we use the cylindrical coordinates (R, Θ, Z), for which $0 \leq R \leq R_o$, $0 \leq \Theta \leq 2\pi$, $0 \leq Z \leq L$. Similarly, in the current configuration, we use the cylindrical coordinates (r, θ, z), for which $0 \leq r \leq r_o$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq \ell$, where r_o and ℓ are unknowns to be determined. In the following, we first look at two types of shear deformations that have been extensively studied in the literature of nonlinear elasticity. Our motivation is to understand the corresponding eigenstrains and their induced residual stresses.

(a) Helical shear

The kinematics of helical shear is fully described as follows [21]:

$$r = R, \quad \theta = \Theta + \phi(R), \quad z = Z + w(R). \tag{2.1}$$

If $\phi(R) \equiv 0$, helical shear reduces to axial shear and when $w(R) \equiv 0$, it reduces to circular (or azimuthal) shear [21]. In this case, $R\phi'(R)$ is the amount of local pure shear in the plane normal to $\mathbf{e}_z$ [17]. The deformation gradient for helical shear is

$$\mathbf{F}(\mathbf{X}) = \mathbf{F}(R) = \begin{pmatrix} 1 & 0 & 0 \\ \phi'(R) & 1 & 0 \\ w'(R) & 0 & 1 \end{pmatrix}. \tag{2.2}$$

The choice of cylindrical coordinates in both the reference and the ambient space induce the (flat) metrics $\mathbf{G}_0$ and $\mathbf{g}$: $\mathbf{G}_0 = \mathrm{diag}\{1, R^2, 1\}$, $\mathbf{g} = \mathrm{diag}\{1, r^2, 1\}$. The right Cauchy–Green strain $\mathbf{C} = \mathbf{F}^\mathsf{T}\mathbf{F}$ with components $C_{AB} = F^a{}_A F^b{}_B g_{ab}$ in matrix form reads

$$\mathbf{C}(\mathbf{X}) = \mathbf{C}(R) = \begin{pmatrix} 1 + r^2\phi'(R)^2 + w'(R)^2 & r^2\phi'(R) & w'(R) \\ r^2\phi'(R) & r^2 & 0 \\ w'(R) & 0 & 1 \end{pmatrix}. \tag{2.3}$$

(b) Torsional shear

Torsional shear is the deformation that corresponds to the intuitive notion of twisting a cylindrical shell along its axis. Its kinematics is described by

$$r = R, \quad \theta = \Theta + \psi Z, \quad z = Z, \tag{2.4}$$

where ψ is the angle of twist per unit length (a constant) [17]. The deformation gradient for this deformation reads

$$\mathbf{F}(\mathbf{X}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \psi \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.5}$$

and the associated metric tensor is

$$\mathbf{C}(\mathbf{X}) = \mathbf{C}(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & \psi R^2 \\ 0 & \psi R^2 & 1 + \psi^2 R^2 \end{pmatrix}. \tag{2.6}$$

(c) Material manifold of a cylinder with an axisymmetric distribution of finite shear eigenstrains

In a typical analysis of the stress-induced through shear, one assumes the kinematics (2.1) or (2.4), and by using the governing equations and boundary conditions one determines the unknown functions ($\phi(R)$ and $w(R)$) or constant (ψ). In an eigenstrain analysis, we assume that a distribution of shear eigenstrains is given and is part of the internal description of the structure. These eigenstrains, similar to the ones discussed in our previous works [8,9] on radial and circumferential eigenstrains, change the stress-free configuration of the body in the form of an eigenstrain-dependent material metric. That is, the material metric associated with the eigenstrains is no longer flat.

In a series of papers [8,9,22–28], we have demonstrated that a nonlinear anelasticity problem, i.e. a problem of deformation of a solid with some source of residual stresses, can be transformed to a nonlinear elasticity problem provided that one uses an appropriate material manifold, in which the body is stress-free by construction. The geometry of the material manifold (here the Riemannian metric) explicitly depends on the source of residual stresses. We use the formalism introduced in these papers and briefly review its key features.

We start with a stress-free body $\mathcal{B}$ without eigenstrains lying in the Euclidean space with metric $\mathbf{G}_0$. That is, the initial body is a Riemannian manifold $(\mathcal{B}, \mathbf{G}_0)$. The effect of the eigenstrains is to transform, locally, a line element $\mathrm{d}\mathbf{X}_0$ to $\mathrm{d}\mathbf{X} = \mathbf{K}\,\mathrm{d}\mathbf{X}_0$. Note that

$$\langle\!\langle \mathrm{d}\mathbf{X}_0, \mathrm{d}\mathbf{X}_0 \rangle\!\rangle_{\mathbf{G}_0} = \langle\!\langle \mathrm{d}\mathbf{X}, \mathrm{d}\mathbf{X} \rangle\!\rangle_{\mathbf{G}}, \tag{2.7}$$

where $\mathbf{G} = \mathbf{K}_* \mathbf{G}_0$ ($\mathbf{K}_*$ is push forward by $\mathbf{K}$) and $\langle\!\langle , \rangle\!\rangle_{\mathbf{G}_0}$ and $\langle\!\langle , \rangle\!\rangle_{\mathbf{G}}$ are the inner products induced by $\mathbf{G}_0$ and $\mathbf{G}$, respectively. In the Riemannian manifold $(\mathcal{B}, \mathbf{G})$, the body is stress-free as the distances have not changed compared to the initial stress-free configuration $(\mathcal{B}, \mathbf{G}_0)$. However, note that this manifold may not be flat. In components, $G_{AB} = K^{\alpha}_{A} K^{\beta}_{B} (G_0)_{\alpha\beta}$, where we have assumed the coordinate charts $\{\bar{X}^{\alpha}\}$ and $\{X^A\}$ in the initial and distorted reference configurations, respectively.

We illustrate this construction for the simple problem of a torsional eigenstrain (eigentwist) for a cylindrical bar. In this case, we choose $\mathbf{G}_0$ to be the metric associated with the usual cylindrical coordinates and $\mathbf{K}$ to have the same functional form as the deformation gradient given by (2.5) but with $\psi = \psi(R)$. Since ψ is a function of R, $\mathbf{K}$ is not, in general, the gradient of a deformation (or more precisely the tangent map of a deformation) but defines a local transformation of the metric on the material manifold. The metric of the material manifold is then

$$\mathbf{G}(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & \psi(R)R^2 \\ 0 & \psi(R)R^2 & 1+\psi^2R^2 \end{pmatrix}. \tag{2.8}$$

Similarly, in the case of helical eigenstrains using (2.2), the material manifold in cylindrical coordinates has the following representation:

$$\mathbf{G}(R) = \begin{pmatrix} 1+R^2A^2(R)+B^2(R) & R^2A(R) & B(R) \\ R^2A(R) & R^2 & 0 \\ B(R) & 0 & 1 \end{pmatrix}, \tag{2.9}$$

where $A(R) = \phi'(R)$ and $B(R) = w'(R)$.

(d) Zero-stress axial, azimuthal and torsional eigenstrains

Before studying the residual stresses induced from finite shear eigenstrains, we need to identify the non-trivial ones. In other words, we need to first find those shear eigenstrains that do not lead to residual stresses, i.e. zero-stress (impotent) shear eigenstrains. We use the method of Cartan's moving frames in cylindrical coordinates suitable for the geometry of our problem. To summarize this approach, one starts with a frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, or equivalently a coframe field $\{\vartheta^1, \vartheta^2, \vartheta^3\}$. This frame field is a set of three linearly independent vectors that span the tangent space of the material manifold at every point. The coframe field will depend on R through the distribution of shear eigenstrains. What is not known *a priori* is the connection ∇ of the material manifold that can be represented by three connection 1-forms (assuming vanishing non-metricity) $\{\omega^1_2, \omega^2_3, \omega^3_1\}$. The first structural equations relate the torsion 2-forms to the coframe field and the connection 1-forms and (in the absence of dislocations) read $\mathcal{T}^\alpha = \mathrm{d}\vartheta^\alpha + \omega^\alpha_\beta \wedge \vartheta^\beta = 0$. This set of equations uniquely determine the connection 1-forms. Note that as we have assumed that the metric is compatible with the connection (vanishing non-metricity), this connection is the Levi–Civita connection. Now the Riemannian curvature 2-forms are calculated using the second structural equations $\mathcal{R}^\alpha_\beta = \mathrm{d}\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta$. A shear eigenstrain field is impotent if and only if all the curvature 2-forms vanish. For more details on Cartan calculations, see [29] and our previous work.

We start with an arbitrary distribution of axial shear eigenstrains (corresponding to $A(R) = 0$ and an arbitrary function $B(R)$). We assume that the moving coframe field is orthonormal. That is $\mathbf{G} = \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3$, with

$$\vartheta^1 = \mathrm{d}R, \quad \vartheta^2 = R\,\mathrm{d}\Theta \quad \text{and} \quad \vartheta^3 = B(R)\,\mathrm{d}R + \mathrm{d}Z. \tag{2.10}$$

The material manifold is described by the above moving coframe field. Assuming that the material manifold is metric-compatible, there are three connection 1-forms $\{\omega^1_2, \omega^2_3, \omega^3_1\}$. We know that the material manifold is torsion-free and from Cartan's first structural equations $\mathcal{T}^\alpha = \mathrm{d}\vartheta^\alpha + \omega^\alpha_\beta \wedge \vartheta^\beta$ we obtain the connection 1-forms to be

$$\omega^1_2 = -\frac{1}{R}\vartheta^2 \quad \text{and} \quad \omega^2_3 = \omega^3_1 = 0. \tag{2.11}$$

The curvature 2-forms are calculated from Cartan's second structural equations $\mathcal{R}^\alpha_\beta = \mathrm{d}\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta$. It is straightforward to show that all the curvature 2-forms identically vanish. Therefore, any distribution of axial shear eigenstrains is stress-free.

Next, we consider an arbitrary distribution of azimuthal shear eigenstrains (corresponding to $B(R) = 0$ and an arbitrary function $A(R)$). It is described by the orthonormal moving coframe field

$$\vartheta^1 = \mathrm{d}R, \quad \vartheta^2 = RA(R)\,\mathrm{d}R + R\,\mathrm{d}\Theta \quad \text{and} \quad \vartheta^3 = \mathrm{d}Z. \tag{2.12}$$

One can easily show that the above moving coframe field defines a flat manifold for any choice of $A(R)$. Therefore, any distribution of azimuthal shear eigenstrains is stress-free as well. We conclude that helical shear eigenstrains are always impotent, which is consistent with the fact that there exists a motion compatible with any choice of functions A and B.

We now turn our attention to the case of a cylindrical bar with torsional eigenstrain distribution. It is described by the following orthonormal moving coframe field:

$$\vartheta^1 = \mathrm{d}R, \quad \vartheta^2 = R\,\mathrm{d}\Theta + R\psi(R)\,\mathrm{d}Z \quad \text{and} \quad \vartheta^3 = \mathrm{d}Z, \tag{2.13}$$

where $\psi(R)$ is the radial density of the angle of twist per unit length of the bar. We know, by construction, that the case $\psi(R)$ constant is impotent since it corresponds to the classical torsion problem. Using Cartan's first structural equations, the connection 1-forms are

$$\omega^1{}_2 = -\frac{1}{R}\vartheta^2 - \frac{R}{2}\psi'(R)\vartheta^3, \quad \omega^2{}_3 = -\frac{R}{2}\psi'(R)\vartheta^1 \quad \text{and} \quad \omega^3{}_1 = \frac{R}{2}\psi'(R)\vartheta^2. \tag{2.14}$$

Cartan's second structural equations give us the following torsion 2-forms:

$$\mathcal{R}^1{}_2 = \tfrac{1}{4}[R\psi'(R)]^2\vartheta^1 \wedge \vartheta^2, \tag{2.15}$$

$$\mathcal{R}^2{}_3 = \tfrac{1}{4}[R\psi'(R)]^2\vartheta^2 \wedge \vartheta^3 \tag{2.16}$$

and

$$\mathcal{R}^3{}_1 = \left[\frac{R}{2}\psi''(R) + \psi'(R)\right]\vartheta^1 \wedge \vartheta^2 - \frac{3}{4}[R\psi'(R)]^2\vartheta^3 \wedge \vartheta^1. \tag{2.17}$$

This manifold is flat if and only if $\psi'(R) = 0$ or $\psi(R) = \psi_0$ is a constant. Therefore, any non-uniform distribution of torsional shear eigenstrains induces residual stresses.

Remark 2.1. In the case of combined helical and torsional eigenstrains, one has the following moving coframe field

$$\vartheta^1 = \mathrm{d}R, \quad \vartheta^2 = RA(R)\,\mathrm{d}R + R\,\mathrm{d}\Theta + R\psi(R)\,\mathrm{d}Z \quad \text{and} \quad \vartheta^3 = B(R)\,\mathrm{d}R + \mathrm{d}Z. \tag{2.18}$$

One can show that again the material manifold in this case is flat if and only if $\psi'(R) = 0$. In other words, axial and azimuthal shear eigenstrains are always impotent even in combination with torsional shear eigenstrains. For this reason, in what follows we consider a bar with a distribution of only torsional shear eigenstrains.

(e) Constitutive laws

So far the discussion has been purely geometric. We showed that helical shear eigenstrains are always compatible and that torsional eigenstrains are incompatible, hence can create residual stresses. In order to compute these stresses, we need constitutive laws. We restrict our attention to hyperelastic, isotropic incompressible solids. Therefore, we can use the classical representation formula of Cauchy stress in terms of invariants by assuming the existence of a strain-energy density function W.

Let us assume that the ambient space is a Riemannian manifold $(\mathcal{S}, \mathbf{g})$. Motion is a mapping $\varphi : \mathcal{B} \to \mathcal{S}$. The left Cauchy–Green deformation tensor is defined as $\mathbf{B}^\sharp = \varphi^*(\mathbf{g}^\sharp)$ and has components $B^{AB} = (F^{-1})^A{}_a (F^{-1})^B{}_b\, g^{ab}$, where g^{ab} are components of $\mathbf{g}^\sharp$. The spatial analogues of $\mathbf{C}^\flat$ and $\mathbf{B}^\sharp$ are $\mathbf{c}^\flat = \varphi_*(\mathbf{G})$, $c_{ab} = (F^{-1})^A{}_a (F^{-1})^B{}_b\, G_{AB}$ and

$$\mathbf{b}^\sharp = \varphi_*(\mathbf{G}^\sharp), \quad b^{ab} = F^a{}_A F^b{}_B G^{AB}. \tag{2.19}$$

The tensor $\mathbf{b}^\sharp$ is called the Finger deformation tensor. $\mathbf{C}$ and $\mathbf{b}$ have the same principal invariants that are denoted by I_1, I_2 and I_3 [17]. For an isotropic material, the strain-energy function W depends only on the principal invariants of $\mathbf{b}$. It is known that for an incompressible and isotropic hyperelastic material with strain-energy density function $W = W(I_1, I_2)$, the Cauchy stress has the following representation [30]:

$$\boldsymbol{\sigma} = \left(-p + 2I_2 \frac{\partial W}{\partial I_2}\right) \mathbf{g}^\sharp + 2\frac{\partial W}{\partial I_1}\mathbf{b}^\sharp - 2\frac{\partial W}{\partial I_2}\mathbf{b}^{-1}, \tag{2.20}$$

where p is an arbitrary function of $\mathbf{X}$ (the Lagrange multiplier corresponding to the incompressibility constraint).

3. Residual stress field in a circular cylindrical bar with eigentwist

Consider a cylindrical bar of initial length L and radius R_o made of an isotropic and incompressible material with an energy function $W = W(I_1, I_2)$. We assume that an eigentwist per unit length $\psi(R)$ is given and our objective is to calculate the resulting residual stress field. We use the cylindrical coordinates (r, θ, z) for the Euclidean ambient space with the flat metric $\mathbf{g} = \text{diag}\{1, r^2, 1\}$. The material metric depends on $\psi(R)$ and is given, in the cylindrical coordinates (R, Θ, Z) by (2.8). The computation of the residual stress amounts to finding a suitable embedding of the material manifold into the ambient space. This embedding can be accomplished by the semi-inverse method that suggests an ansatz of the form:

$$r = r(R), \quad \theta = \Theta + \tau Z \quad \text{and} \quad z = \lambda^2 Z, \tag{3.1}$$

where τ and λ are some unknown constants to be determined. Physically, we are looking for realization of our body as a stretched and twisted cylinder. The deformation gradient for this problem is

$$\mathbf{F} = \begin{pmatrix} r'(R) & 0 & 0 \\ 0 & 1 & \tau \\ 0 & 0 & \lambda^2 \end{pmatrix}, \tag{3.2}$$

and we observe that the incompressibility condition is given by

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{\lambda^2 r(R) r'(R)}{R} = 1. \tag{3.3}$$

This condition, together with $r(0) = 0$, imply that $r(R) = R/\lambda$.

According to (2.19), the Finger tensor is given by

$$\mathbf{b}^\sharp = \begin{pmatrix} \frac{1}{\lambda^2} & 0 & 0 \\ 0 & \frac{1}{R^2} + (\tau - \psi(R))^2 & \lambda^2(\tau - \psi(R)) \\ 0 & \lambda^2(\tau - \psi(R)) & \lambda^4 \end{pmatrix}. \tag{3.4}$$

The principal invariants of $\mathbf{b}$ are

$$I_1 = \frac{2 + \lambda^6 + R^2(\tau - \psi(R))^2}{\lambda^2} \quad \text{and} \quad I_2 = \frac{1 + 2\lambda^6 + R^2(\tau - \psi(R))^2}{\lambda^4}. \tag{3.5}$$

Note that $(b^{-1})^{ab} = c^{ab} = g^{am} g^{bn} c_{mn}$ and hence

$$\mathbf{b}^{-1} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \frac{\lambda^4}{R^2} & \psi(R) - \tau \\ 0 & \psi(R) - \tau & \frac{1 + R^2(\tau - \psi(R))^2}{\lambda^4} \end{pmatrix}. \tag{3.6}$$

We know that $\boldsymbol{\sigma} = (-p + I_2\beta)\mathbf{g}^\sharp + \alpha\mathbf{b}^\sharp - \beta\mathbf{b}^{-1}$, where $\alpha = 2(\partial W/\partial I_1)$ and $\beta = 2(\partial W/\partial I_2)$. Therefore, the Cauchy stress has the following representation:

$$\boldsymbol{\sigma} = \begin{pmatrix} -p + \dfrac{\alpha}{\lambda^2} + \dfrac{1+\lambda^6}{\lambda^4}\beta + \dfrac{R^2}{\lambda^4}\beta(\tau-\psi)^2 & 0 & 0 \\ 0 & \dfrac{\alpha\lambda^2(1+R^2(\tau-\psi)^2) - \lambda^4 p + \beta(1+\lambda^6+R^2(\tau-\psi)^2)}{R^2\lambda^2} & (\alpha\lambda^2+\beta)(\tau-\psi) \\ 0 & (\alpha\lambda^2+\beta)(\tau-\psi) & \alpha\lambda^4 + 2\beta\lambda^2 - p \end{pmatrix}. \tag{3.7}$$

The circumferential and axial equilibrium equations imply that $p = p(R)$ and the radial equilibrium equation reads

$$\frac{\partial \sigma^{rr}}{\partial r} + \frac{1}{r}\sigma^{rr} - r\sigma^{\theta\theta} = 0, \tag{3.8}$$

which can be written explicitly as

$$\frac{\mathrm{d}\sigma^{rr}}{\mathrm{d}R} - \frac{\alpha R(\tau-\psi)^2}{\lambda^2} = 0. \tag{3.9}$$

Substitution of σ^{rr} given by (3.7), gives

$$p'(R) = k(R), \tag{3.10}$$

where

$$k(R) = \frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{1}{\lambda^2}\alpha(R) + \frac{1+\lambda^6}{\lambda^4}\beta(R) + \frac{1}{\lambda^4}R^2\beta(R)(\tau-\psi(R))^2\right] - \frac{1}{\lambda^2}R\alpha(R)(\tau-\psi(R))^2. \tag{3.11}$$

We assume that the cylindrical boundary of the bar is traction free. In terms of the first Piola–Kirchhoff stress $P^{aA} = J\sigma^{ab}(F^{-1})^A_b$, this reads $P^{rR}|_{R_o} = 0$. Note that $P^{rR} = (1/\lambda^3)[\lambda^2(\alpha - \lambda^2 p) + \beta(1+\lambda^6+R^2(\tau-\psi)^2)]$. Let $\alpha_o = \alpha(R_o)$, $\beta_o = \beta(R_o)$ and $\psi_o = \psi(R_o)$. Therefore

$$p(R_o) = \frac{\alpha_o}{\lambda^2} + \frac{\beta_o}{\lambda^4}[1 + \lambda^6 + R_o^2(\tau-\psi_o)^2]. \tag{3.12}$$

Integrating (3.10) from R to R_o and using (3.12), the pressure field is calculated and reads

$$p(R) = \frac{1}{\lambda^2}\alpha(R) + \frac{1+\lambda^6}{\lambda^4}\beta(R) + \frac{1}{\lambda^4}R^2\beta(R)(\tau-\psi(R))^2 + \frac{1}{\lambda^2}\int_R^{R_o}\xi\alpha(\xi)(\tau-\psi(\xi))^2\,\mathrm{d}\xi. \tag{3.13}$$

We can now calculate the total force and torque. Since we evaluate these quantities with respect to the reference configuration, the area element in the material manifold must be used. On the two faces, the area element is simply $\boldsymbol{\mu} = R\,\mathrm{d}R \wedge \mathrm{d}\Theta$. To compute the residual stresses, we assume that at the two ends of the bar $Z = 0, L$, the axial force and torque are zero, i.e. $F = 0$, $M = 0$, where

$$F = 2\pi\int_0^{R_o} P^{zZ}(R)R\,\mathrm{d}R \quad \text{and} \quad M = 2\pi\int_0^{R_o}\bar{P}^{\theta Z}(R)R^2\,\mathrm{d}R, \tag{3.14}$$

and $\bar{P}^{\theta Z} = rP^{\theta Z}$ is the physical θZ component of the first Piola–Kirchhoff stress. Note that

$$P^{zZ}(R) = \lambda^2\alpha(R) + 2\beta(R) - \frac{1}{\lambda^2}p(R) \quad \text{and} \quad P^{\theta Z}(R) = \frac{1}{\lambda^2}(\lambda^2\alpha(R) + \beta(R))(\tau - \psi(R)). \tag{3.15}$$

Explicitly, $(3.14)_2$ reads

$$\lambda^2\tau\int_0^{R_o}R^3\alpha(R)\,\mathrm{d}R - \lambda^2\int_0^{R_o}R^3\alpha(R)\psi(R)\,\mathrm{d}R + \tau\int_0^{R_o}R^3\beta(R)\,\mathrm{d}R = \int_0^{R_o}R^3\beta(R)\psi(R)\,\mathrm{d}R. \tag{3.16}$$

Whereas, equation $(3.14)_1$ gives

$$-\int_0^{R_o}Rp(R)\,\mathrm{d}R + \lambda^4\int_0^{R_o}R\alpha(R)\,\mathrm{d}R + 2\lambda^2\int_0^{R_o}R\beta(R)\,\mathrm{d}R = 0. \tag{3.17}$$

The last expression can be further simplified to read

$$\lambda^8\int_0^{R_o} R\alpha(R)\,dR+\lambda^6\int_0^{R_o} R\beta(R)\,dR-\lambda^2\left(\int_0^{R_o} R\alpha(R)\,dR+\int_0^{R_o} R\int_R^{R_o}\xi\alpha(\xi)\psi^2(\xi)\,d\xi\,dR\right)$$
$$-\tau^2\lambda^2\int_0^{R_o} R\int_R^{R_o}\xi\alpha(\xi)\,d\xi\,dR+2\tau\lambda^2\int_0^{R_o} R\int_R^{R_o}\xi\alpha(\xi)\psi(\xi)\,d\xi\,dR-\tau^2\int_0^{R_o} R^3\beta(R)\,dR$$
$$+2\tau\int_0^{R_o} R^3\beta(R)\psi(R)\,dR=\int_0^{R_o} R\beta(R)\,dR+\int_0^{R_o} R^3\beta(R)\psi^2(R)\,dR. \quad (3.18)$$

The unknown constants λ and τ are solutions of the nonlinear system of equations (3.16) and (3.18).

For neo-Hookean solids $\alpha(R)=\mu(R)>0$ and $\beta(R)=0$. In this case, we can find λ and τ analytically. Equation (3.16) gives us

$$\tau=\frac{\int_0^{R_o} R^3\mu(R)\psi(R)\,dR}{\int_0^{R_o} R^3\mu(R)\,dR}. \quad (3.19)$$

Note that the residual twist will be zero if

$$\int_0^{R_o} R^3\mu(R)\psi(R)\,dR=0. \quad (3.20)$$

In this configuration, the residual stretch is

$$\lambda=1+\frac{\int_0^{R_o} R\int_R^{R_o}\xi\mu(\xi)\psi^2(\xi)\,d\xi\,dR}{\int_0^{R_o} R\mu(R)\,dR}>1. \quad (3.21)$$

(a) A circular cylindrical inclusion with eigentwist

In this example, we study a bar with a cylindrical inhomogeneity with uniform eigentwist. Physically, it corresponds to removing the core of a cylinder and replacing it with a core of the same material and of the same height but twisted uniformly. We solve the problem for both Mooney–Rivlin and neo-Hookean solids. Consider a cylindrical inhomogeneity with a region of eigenstrain with radius R_i given by a uniform eigentwist ψ_i, i.e.

$$\psi(R)=\begin{cases}\psi_i & 0\le R<R_i\\ 0 & R_i<R\le R_o\end{cases}. \quad (3.22)$$

We assume that α and β are piecewise constants (Mooney–Rivlin material), i.e.

$$\alpha(R)=\begin{cases}\alpha_i & 0\le R<R_i\\ \alpha_0 & R_i<R\le R_o\end{cases} \quad\text{and}\quad \beta(R)=\begin{cases}\beta_i & 0\le R<R_i\\ \beta_0 & R_i<R\le R_o\end{cases}. \quad (3.23)$$

We introduce the following dimensionless parameters:

$$s=\frac{R_i}{R_o}<1,\quad k=\frac{\beta_0}{\alpha_0},\quad a=\frac{\alpha_i}{\alpha_0},\quad b=\frac{\beta_i}{\beta_0},\quad \chi=R_o\tau\quad\text{and}\quad m=R_o\psi_i. \quad (3.24)$$

Equations (3.16) and (3.18), written in terms of λ and χ read

$$[1+(a-1)s^4]\lambda^2\chi-(ams^4)\lambda^2+k[1+(b-1)s^4]\chi=bkms^4$$
$$\text{and}\quad [1+(a-1)s^2]\lambda^8+k[1+(b-1)s^2]\lambda^6$$
$$-[1+(a-1)s^2+\tfrac{1}{4}am^2s^4]\lambda^2-\tfrac{1}{4}[1+(a-1)s^4]\lambda^2\chi^2$$
$$+(\tfrac{1}{2}ams^4)\lambda^2\chi-\tfrac{1}{2}k[1+(b-1)s^4]\chi^2+(bkms^4)\chi=k\{[1+(b-1)s^2]+\tfrac{1}{2}bm^2s^4\}. \quad (3.25)$$

Note that in the above system of equations when $m\to-m$, we have $\{\lambda,\chi\}\to\{\lambda,-\chi\}$. Once m and χ are known, the residual stress is easily obtained from (3.13) and (3.7).

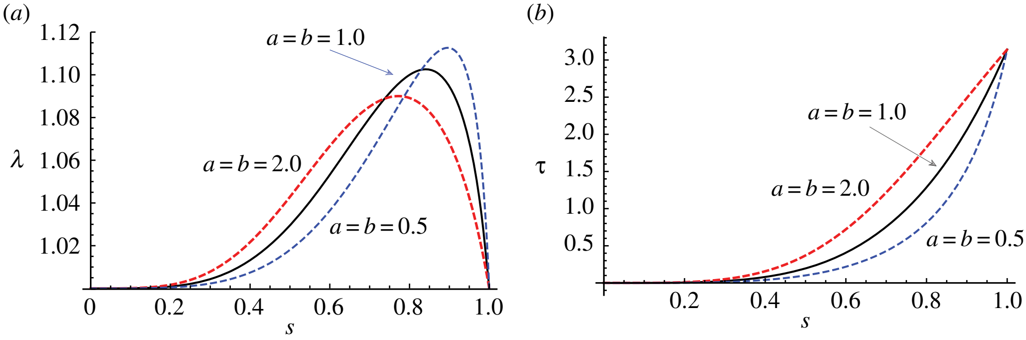

Figure 2. In (*a*,*b*), we have considered a cylindrical shaft with torsional shear eigenstrains for which $k = 0.5$, $m = \pi$. (*a*) Variation of stretch λ as a function of $s \in [0, 1]$ for three choices of parameters *a* and *b*. Note that $a = b = 1$ corresponds to an inclusion. (*b*) Variation of twist per unit length τ as a function of $s \in [0, 1]$ for three choices of parameters *a* and *b*. (Online version in colour.)

Figure 2*a*,*b* shows the variation of stretch λ and twist τ for a shaft with an inhomogeneity for which $k = 0.5$ and eigentwist $m = \pi$ for different values of the pair (a, b). Note that $s = 0$ corresponds to a bar without eigenstrain and hence $(\lambda, \tau) = (1, 0)$. When $s = 1$, the entire bar has a uniform eigentwist m. This corresponds to $(\lambda, \tau) = (1, m)$ as expected. Note that in this extreme case, the shear eigenstrain distribution is stress-free as was discussed earlier.

In the case of neo-Hookean solids, we have

$$\chi = \frac{ams^4}{1 + (a-1)s^4} \quad \text{and} \quad \lambda^6 = 1 + \frac{am^2 s^4 (1 - s^4)}{4[1 + (a-1)s^2][1 + (a-1)s^4]}. \tag{3.26}$$

For illustrative purposes, if the bar is made out of a uniform material ($a = 1$), the only non-zero shear stress $\bar{\sigma}^{\theta z}$ has the following distribution:

$$\bar{\sigma}^{\theta z} = \begin{cases} \mu\lambda\psi_{\mathrm{i}} \left[\left(\dfrac{R_{\mathrm{i}}}{R_{\mathrm{o}}} \right)^4 - 1 \right] R & 0 \le R < R_{\mathrm{i}} \\ \mu\lambda\psi_{\mathrm{i}} \left(\dfrac{R_{\mathrm{i}}}{R_{\mathrm{o}}} \right)^4 R & R_{\mathrm{i}} < R \le R_{\mathrm{o}} \end{cases}, \tag{3.27}$$

where $\lambda = [1 + \frac{1}{4} m^2 s^4 (1 - s^4)]^{1/6}$. The radial stress has the following distribution:

$$\bar{\sigma}^{rr} = \begin{cases} -\dfrac{\mu\psi_{\mathrm{i}}^2}{2\lambda^2} \left(\dfrac{R_{\mathrm{i}}}{R_{\mathrm{o}}} \right)^8 (R_{\mathrm{o}}^2 - R_{\mathrm{i}}^2) + \dfrac{\mu\psi_{\mathrm{i}}^2}{2\lambda^2} \left[\left(\dfrac{R_{\mathrm{i}}}{R_{\mathrm{o}}} \right)^4 - 1 \right]^2 (R_{\mathrm{i}}^2 - R^2) & 0 \le R < R_{\mathrm{i}} \\ -\dfrac{\mu\psi_{\mathrm{i}}^2}{2\lambda^2} \left(\dfrac{R_{\mathrm{i}}}{R_{\mathrm{o}}} \right)^8 (R_{\mathrm{o}}^2 - R^2) & R_{\mathrm{i}} < R \le R_{\mathrm{o}} \end{cases}. \tag{3.28}$$

It is seen that when $s = 1$, stresses are identically zero as expected.

(i) Comparison with the linear elasticity solution

For small $m = R_{\mathrm{o}}\psi_{\mathrm{i}}$, one has

$$\bar{\sigma}^{\theta z} = \begin{cases} \mu\psi_{\mathrm{i}} \left[\left(\dfrac{R_{\mathrm{i}}}{R_{\mathrm{o}}} \right)^4 - 1 \right] R + O(m^3) & 0 \le R < R_{\mathrm{i}} \\ \mu\psi_{\mathrm{i}} \left(\dfrac{R_{\mathrm{i}}}{R_{\mathrm{o}}} \right)^4 R + O(m^3) & R_{\mathrm{i}} < R \le R_{\mathrm{o}} \end{cases}, \tag{3.29}$$

while the other stress components are of order $O(m^2)$ and hence can be neglected for small m. It is instructive to compare this solution with the classical linear solution for a twist inclusion in circular shafts. For a bar of radius R_{i} inside a hollow shaft of inner radius R_{i} and outer radius R_{o}, the eigenstrain distribution (3.22) implies that the inner shaft tends to twist by $\psi_{\rm i}$ per unit length if it is detached from the hollow shaft. To construct this configuration in the Euclidean ambient space we first apply a torque $M_{\rm i}$ to the inner shaft to twist it (per unit length) by $-\psi_{\rm i}$ (loading). Note that

$$-\psi_{\rm i}=\frac{M_{\rm i}}{\mu J_{\rm i}},\tag{3.30}$$

where $J_{\rm i}=(\pi/2)R_{\rm i}^4$ is the torsional rigidity of the inner shaft (when detached from the hollow shaft). Removing this torque, the shaft would twist (per unit length) by $\psi_{\rm i}$. In the unloading stage, we imagine gluing the inner shaft to the hollow shaft and applying $-M_{\rm i}$ to the whole system. The residual twist (per unit length) τ is calculated as follows:

$$\tau=\frac{-M_{\rm i}}{\mu J}=\frac{J_{\rm i}}{J}\psi_{\rm i}=\left(\frac{R_{\rm i}}{R_{\rm o}}\right)^4\psi_{\rm i},\tag{3.31}$$

where $J=(\pi/2)R_{\rm o}^4$ is the torsional rigidity of the cylindrical bar. Note that this is identical to $(3.26)_1$. To calculate the residual stress distribution, we use superposition. In the loading stage, the stress is non-zero only for $R\le R_{\rm i}$ and the only non-zero stress component reads

$$\bar{\sigma}^{\theta z}_{\rm loading}(R)=\frac{M_{\rm i}R}{J_{\rm i}}=-\mu\psi_{\rm i}R.\tag{3.32}$$

In the unloading stage, throughout the shaft the only non-zero stress component is

$$\bar{\sigma}^{\theta z}_{\rm unloading}(R)=\frac{-M_{\rm i}R}{J}=\mu\psi_{\rm i}\left(\frac{R_{\rm i}}{R_{\rm o}}\right)^4R.\tag{3.33}$$

By the principle of linear superposition, we can add the stresses in the loading and unloading stages to obtain

$$\bar{\sigma}^{\theta z}_{\rm linear}=\begin{cases}\mu\psi_{\rm i}\left[\left(\dfrac{R_{\rm i}}{R_{\rm o}}\right)^4-1\right]R & 0\le R<R_{\rm i}\\ \mu\psi_{\rm i}\left(\dfrac{R_{\rm i}}{R_{\rm o}}\right)^4R & R_{\rm i}<R\le R_{\rm o}\end{cases},\tag{3.34}$$

which is identical to (3.29) to first order in m.

(b) A double cylindrical inhomogeneity with eigentwist

In this example, we consider a cylindrical bar with a cylindrical double inhomogeneity with shear eigenstrains. Consider a cylindrical inhomogeneity of radius $R_{\rm i}$ covered by an annular inhomogeneity of outer radius $R_{\rm a}$ with the following distribution of eigenstrains:

$$\psi(R)=\begin{cases}\psi_{\rm i} & 0\le R<R_{\rm i}\\ \psi_a & R_{\rm i}<R\le R_{\rm a}\\ 0 & R_{\rm a}<R<R_{\rm o}\end{cases}.\tag{3.35}$$

We assume that (Mooney–Rivlin material)

$$\alpha(R)=\begin{cases}\alpha_{\rm i} & 0\le R<R_{\rm i}\\ \alpha_{\rm a} & R_{\rm i}<R\le R_{\rm a}\\ \alpha_0 & R_{\rm a}<R<R_{\rm o}\end{cases}\quad\text{and}\quad\beta(R)=\begin{cases}\beta_{\rm i} & 0\le R<R_{\rm i}\\ \beta_{\rm a} & R_{\rm i}<R\le R_{\rm a}\\ \beta_0 & R_{\rm a}<R<R_{\rm o}\end{cases}.\tag{3.36}$$

As before, we introduce the dimensionless parameters

$$\left.\begin{aligned}s_{\rm i}=\frac{R_{\rm i}}{R_{\rm o}},\quad s_{\rm a}=\frac{R_{\rm a}}{R_{\rm o}},\quad k=\frac{\beta_0}{\alpha_0},\quad a_{\rm i}=\frac{\alpha_{\rm i}}{\alpha_0},\quad a_{\rm a}=\frac{\alpha_{\rm a}}{\alpha_0},\\ b_{\rm i}=\frac{\beta_{\rm i}}{\beta_0},\quad b_{\rm a}=\frac{\beta_{\rm a}}{\beta_0},\quad \chi=R_{\rm o}\tau,\quad M_{\rm i}=R_{\rm o}\psi_{\rm i}\quad\text{and}\quad m_{\rm a}=R_{\rm o}\psi_a.\end{aligned}\right\}\tag{3.37}$$

The unknowns λ and χ must satisfy the following system of nonlinear equations:

$$[a_i s_i^4 + a_a(s_a^4 - s_i^4) + 1 - s_a^4]\lambda^2\chi - [a_i M_i s_i^4 + a_a m_a(s_a^4 - s_i^4)]\lambda^2 + k[b_i s_i^4 + b_a(s_a^4 - s_i^4) + 1 - s_a^4]\chi = k[b_i M_i s_i^4 + b_a m_a(s_a^4 - s_i^4)] \tag{3.38}$$

and

$$\begin{aligned}[] [a_i s_i^2 + a_a(s_a^2 - s_i^2) + 1 - s_a^2]\lambda^8 + k[b_i s_i^2 + b_a(s_a^2 - s_i^2) + 1 - s_a^2]\lambda^6 \\ - [a_i s_i^2 + a_a(s_a^2 - s_i^2) + 1 - s_a^2 + \tfrac{1}{4}a_i M_i^2 s_i^4 + \tfrac{1}{4}a_a m_a^2(s_a^4 - s_i^4)]\lambda^2 \\ - \tfrac{1}{4}[a_i s_i^4 + a_a(s_a^4 - s_i^4) + 1 - s_a^4]\lambda^2\chi^2 + k[b_i M_i s_i^4 + b_a m_a(s_a^4 - s_i^4)]\chi \\ + \tfrac{1}{2}[a_i M_i s_i^4 + a_a m_a(s_a^4 - s_i^4)]\lambda^2\chi - \tfrac{1}{2}k[b_i s_i^4 + b_a(s_a^4 - s_i^4) + 1 - s_a^4]\chi^2 \\ = k[b_i s_i^2 + b_a(s_a^2 - s_i^2) + 1 - s_a^2 + \tfrac{1}{2}b_i M_i^2 s_i^4 + \tfrac{1}{2}b_a m_a^2(s_a^4 - s_i^4)]. \end{aligned} \tag{3.39}$$

If the inhomogeneities and the matrix are all made of neo-Hookean solids, we have

$$\chi = \frac{a_i M_i s_i^4 + a_a m_a(s_a^4 - s_i^4)}{a_i s_i^4 + a_a(s_a^4 - s_i^4) + 1 - s_a^4}. \tag{3.40}$$

If the bar is made of the same material (the case of a double inclusion), this is further simplified to read

$$\tau = \psi_i \left(\frac{R_i}{R_o}\right)^4 + \psi_a\left[\left(\frac{R_a}{R_o}\right)^4 - \left(\frac{R_i}{R_o}\right)^4\right]. \tag{3.41}$$

In this case, we have the following expression for stretch:

$$\begin{aligned} \lambda^6 = 1 + \frac{R_o^2}{4}\left\{\psi_i^2\left(\frac{R_i}{R_o}\right)^4 + \psi_a^2\left[\left(\frac{R_a}{R_o}\right)^4 - \left(\frac{R_i}{R_o}\right)^4\right]\right\} \\ - \frac{R_o^2}{4}\left\{\psi_i\left(\frac{R_i}{R_o}\right)^4 + \psi_a\left[\left(\frac{R_a}{R_o}\right)^4 - \left(\frac{R_i}{R_o}\right)^4\right]\right\}^2. \end{aligned} \tag{3.42}$$

Example 3.1. Note that the residual twist τ vanishes when

$$\psi_i = -\left[\left(\frac{R_a}{R_i}\right)^4 - 1\right]\psi_a. \tag{3.43}$$

In this case

$$\lambda = \left[1 + \frac{1}{4}\frac{\psi_a^2 R_a^4}{R_o^2}\left(\frac{R_a^4}{R_i^4} - 1\right)\right]^{1/6} > 1. \tag{3.44}$$

The only non-vanishing shear stress has the following distribution:

$$\bar{\sigma}^{\theta z}(R) = \begin{cases} -\mu\lambda\psi_i R & 0 \le R < R_i \\ -\mu\lambda\psi_a R & R_i < R \le R_a \\ 0 & R_a < R < R_o \end{cases}. \tag{3.45}$$

The radial stress has the following distribution:

$$\bar{\sigma}^{rr}(R) = \begin{cases} -\dfrac{\mu}{2\lambda^2}[\psi_i^2(R_i^2 - R^2) + \psi_a^2(R_a^2 - R_i^2)] & 0 \le R < R_i \\ -\dfrac{\mu}{2\lambda^2}\psi_a^2(R_a^2 - R^2) & R_i < R \le R_a \\ 0 & R_a < R < R_o \end{cases}. \tag{3.46}$$

Note that the stress in the matrix is not identically zero; the axial stress has the constant value $\mu(\lambda^6 - 1)/\lambda^2$ in the matrix.

4. The effective stiffnesses of a cylindrical bar with eigentwist

The eigentwist distribution considered so far maintains the cylindrical geometry. It is therefore natural to compare the response of a cylinder with eigentwist to a stress-free cylinder. To do so, we compute the effective axial and torsional stiffnesses of a cylinder with eigentwist. We consider both a single and double inhomogeneity with eigenstrain and assume, for simplicity, that they are neo-Hookean solids. For an arbitrary strain-energy density function, the effective stiffnesses must be obtained numerically.

(a) Effect of a cylindrical inclusion with eigentwist

We consider a circular cylindrical bar of radius R_o with a cylindrical inhomogeneity of radius R_i. We assume that both the inhomogeneity and the matrix are made of the same incompressible neo-Hookean material with shear modulus μ and assume an eigenstrain distribution given by (3.22).

(i) Effective axial stiffness

We assume that the bar is under an axial force F at its two ends while there is no external torque, i.e. $M=0$. We are interested in calculating F as a function of λ. Using $\chi = ms^4$ (3.14)$_1$ gives

$$\frac{F}{\pi R_o^2 \mu} = \lambda^2 - \left[1 + \frac{1}{4} m^2 s^4 (1 - s^4)\right] \lambda^{-4}. \tag{4.1}$$

We refer to the stretch corresponding to $F=0$ as the residual stretch and denote it by $\lambda_0 = [1 + \frac{1}{4} m^2 s^4 (1 - s^4)]^{1/6}$. Equation (4.1) can now be written as

$$F = \pi R_o^2 \mu \lambda_0^2 \left[\left(\frac{\lambda}{\lambda_0}\right)^2 - \left(\frac{\lambda}{\lambda_0}\right)^{-4}\right]. \tag{4.2}$$

Therefore

$$\frac{\partial F}{\partial \lambda} = 2\pi R_o^2 \mu \lambda_0 \left[\left(\frac{\lambda}{\lambda_0}\right) + 2\left(\frac{\lambda}{\lambda_0}\right)^{-5}\right]. \tag{4.3}$$

In particular, we define axial stiffness as

$$K_a(m) := \left.\frac{\partial F}{\partial \lambda}\right|_{\lambda=\lambda_0} = 6\mu\pi R_o^2 \lambda_0. \tag{4.4}$$

Hence

$$\frac{K_a(m)}{K_a(0)} = \lambda_0 = \left[1 + \frac{1}{4} m^2 s^4 (1 - s^4)\right]^{1/6} > 1. \tag{4.5}$$

We observe that an inclusion with eigentwist always increases the axial stiffness.

(ii) Effective torsional stiffness

We assume that the bar is under an external torque M at its two ends while there is no external axial force, i.e. $F=0$. We are interested in calculating M as a function of τ. From (3.14), we have

$$M = \left(\mu \frac{\pi}{2} R_o^3\right) \frac{1}{\lambda} (\chi - ms^4), \quad \lambda^6 = 1 + \frac{1}{4} m^2 s^4 + \frac{1}{4}\chi^2 - \frac{1}{2} ms^4 \chi. \tag{4.6}$$

Let us denote $\chi_0 = ms^4$, which corresponds to the case of no applied force/moment configuration. Now M can be rewritten as

$$M = \mu \frac{\pi}{2} R_o^3 \frac{\chi - \chi_0}{[1 + (1/4)\chi_0^2 (s^{-4} - 1) + (1/4)(\chi - \chi_0)^2]^{1/6}} = \mu \frac{\pi}{2} R_o^3 \frac{\chi - \chi_0}{[\lambda_0^6 + (1/4)(\chi - \chi_0)^2]^{1/6}}. \tag{4.7}$$

The torsional stiffness is defined as

$$K_t(m) := \left.\frac{\partial M}{\partial \chi}\right|_{\chi=\chi_0} = \frac{\pi}{2} R_o^3 \mu \frac{1}{\lambda_0}. \tag{4.8}$$

Hence, we have

$$\frac{K_t(m)}{K_t(0)} = \frac{1}{\lambda_0} = \left[1 + \frac{1}{4}m^2 s^4(1 - s^4)\right]^{-1/6} < 1. \tag{4.9}$$

We observe that an inclusion with eigentwist always decreases the torsional stiffness.

Remark 4.1. Note that axial and torsional stiffnesses are defined with respect to the relaxed configuration under no external force and torque, i.e. the configuration (λ_0, χ_0). Given an external force and torque, the material manifold still has the metric (2.8). This enabled us to find the pair (F, M) as functions of (λ, χ) and calculate their derivatives evaluated at the (intermediate) configuration (λ_0, χ_0).

(b) Effect of a double cylindrical inclusion with shear eigenstrains

Next, we find the axial and torsional stiffnesses of the bar with a double inclusion. Again, we can assume different shear moduli for the inhomogeneities and the matrix but as our goal is to understand the effect of shear eigenstrains we will restrict ourselves to inclusions. It is straightforward to show that the results for a double inhomogeneity are similar to those of a single inclusion, i.e.

$$\frac{K_a(m)}{K_a(0)} = \lambda_0 \quad \text{and} \quad \frac{K_t(m)}{K_t(0)} = \frac{1}{\lambda_0}, \tag{4.10}$$

with the only difference that now, we have

$$\lambda_0 = 1 + \frac{1}{4R_o^2}[\psi_i^2 R_i^4 + \psi_a^2(R_a^4 - R_i^4)] - \frac{1}{4R_o^6}[\psi_i R_i^4 + \psi_a(R_a^4 - R_i^4)]^2. \tag{4.11}$$

Note that $\lambda_0 = 1 + \frac{1}{4}f(M_i, m_a)$, where

$$f(M_i, m_a) = M_i^2 s_i^4 + m_a^2(s_a^4 - s_i^4) - [M_i s_i^4 + m_a(s_a^4 - s_i^4)]^2. \tag{4.12}$$

We think of s_i and s_a as parameters. Looking for extrema of f, the relations $\partial f/\partial M_i = \partial f/\partial m_a = 0$ give us $M_i = m_a = 0$. The Hessian matrix at this point reads

$$\mathbf{H} = 2\begin{pmatrix} s_i^4(1 - s_i^4) & -s_i^4(s_a^4 - s_i^4) \\ -s_i^4(s_a^4 - s_i^4) & s_a^4 - s_i^4 - (s_a^4 - s_i^4)^2 \end{pmatrix}. \tag{4.13}$$

Note that $\det \mathbf{H} = 4s_i^4(1 - s_a^4)(s_a^4 - s_i^4) > 0$ and $\mathrm{tr}\mathbf{H} = 4s_i^4(s_a^4 - s_i^4) + 2s_a^4(1 - s_a^4) > 0$. Therefore, $\mathbf{H}$ is positive-definite and hence $f(M_i, m_a) > f(0, 0) = 0$. Therefore, $\lambda_0 > 1$. We observe that similar to a single inclusion, a double inclusion with arbitrary shear eigenstrains always makes the cylindrical bar axially stiffer but torsionally softer.

5. Concluding remarks

The problem of dilatational eigenstrains, both in small and large deformations, is well appreciated. The analogue problem for shear eigenstrains has yet to be properly addressed, despite the known importance of shear stresses in solids. To address this issue, we consider various distributions of shear eigenstrains in cylindrical geometry where semi-inverse methods can be combined with differential geometric techniques to obtain exact solutions. We studied in detail, helical and torsional eigenstrains and showed that in a circular cylindrical bar any axisymmetric distribution of helical shear eigenstrains is impotent (stress-free). We then showed that any non-uniform axisymmetric distribution of torsional shear eigenstrains (eigentwist) induces residual stresses.

As illustrative examples of these general results, we studied the residual stress field of an axially symmetric distribution of finite eigentwist induced by a single or double inclusion. We showed that a bar with such inclusions always elongates independently of the value of the eigentwists. In the case of a single inclusion, we compared the residual stress field with that of linear elasticity solution using the classical mechanics of materials approach. We finally studied the effect of these torsional eigenstrains for the effective axial and torsional stiffnesses of a cylindrical bar. In the case of neo-Hookean solids, we showed that in both cases, independent of the values of the eigenstrains, the bar becomes stiffer axially but torsionally softer.

Eigenstrains can be used to model a host of different effects in mechanical biology (such as growth, remodelling and active stresses) and engineering (thermal stresses, defects, magneto-elastic effects). In particular, many biological structures, such as roots, stems, fungi, and arteries, oesophagus, trachea, the elephant trunk are tubular. They are typically composed of different cylindrical layers and their overall function as a structure is controlled by their internal eigenstrains. Torsional eigenstrains are particularly important in these structures. In plants stems for instance, different layers have different twists in order to tune the stiffness of the structure [31]. The rotation of phycomyces is created by the internal balance of hydrostatic pressure and helical fibre contraction that induces eigentwist [32]. In twisting vines, differential growth produces dilatational and shear eigenstrains that create both the intrinsic curvature and torsion necessary for helical shapes [33,34]. Similarly, in muscular hydrostats, the combined contraction of helical fibres and lateral muscle fibres along a tubular structure create eigenstrains that change the body's three-dimensional shape [35].

The presence of eigentwist in a solid can have a profound effect on the response of a structure under loads. In finite deformations, these effects can be highly non-trivial as they couple the nonlinear response of the material, the geometry of the structure and a combination of eigenstrains. The geometric framework that we have developed allows for a systematic analysis of these effects.

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